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Research Article

The Atangana-Fabrizio Hybrid Fractional Operator: Formulation, Analysis, and Application

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Abstract: In this paper, we introduce the Atangana-Fabrizio (AF) fractional operator as a novel hybrid operator designed to bridge the gap between power-law and exponential-decay memory kernels in complex system modeling. The AF operator, governed by a hybridization parameter λ, is formulated as a convex combination of the Caputo-Fabrizio (CF) and Atangana-Baleanu-Caputo (ABC) operators. We establish its fundamental mathematical properties, including the Laplace transform, linearity, and the conditions defining the associated fractional integral, while also verifying the operator's validity with several examples. Existence and uniqueness of solutions to nonlinear AF-type fractional systems are ensured through Banach and Schaefer fixed-point theorems, and both Ulam-Hyers (UH) and Generalized Ulam-Hyers (GUH) stability results are obtained. Two illustrative examples are provided to validate and clarify the theoretical analysis. Finally, we apply the AF operator to a fractional-order drug release model governed by a Michaelis-Menten-type equation and develop a generalized Adams-Bashforth Method (ABM). Simulations demonstrate the influence of the hybridization parameter λ on the dynamics.

Keywords: Fractional Calculus (FC), Atangana-Baleanu operator, Caputo-Fabrizio (CF) operator, hybrid operator, existence and uniqueness, Ulam-Hyers (UH) stability, drug release model

MSC: 26A33, 34A12, 34D20

1. Introduction

Fractional Calculus (FC) is a powerful tool for simulating complex systems across various scientific and engineering fields. It is an extension of differentiation and integration to non-integer orders. Fractional-order models naturally incorporate memory and hereditary properties, making them particularly well-suited for describing phenomena where the future state depends on the system's entire history, in contrast to classical integer-order models, which are inherently local in nature. This capability has been effectively exploited in fields such as viscoelasticity, anomalous diffusion, signal processing, epidemiology, and finance [1–6]. Artificial intelligence and deep neural networks have been widely applied to study fractional-order systems, epidemic models, and delayed differential equations, as well as to analyze psychological and biological dynamics [7–10]. FC has been effectively used in physical and biomedical applications, including falling

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objects with friction, hyperthermia cancer therapy, fluid flow, heat transfer, and efficient numerical methods for Fractional Differential Equations (FDEs) [11–14].

The classical Riemann-Liouville (RL) and Caputo Fractional Derivatives (Caputo Frac Derivs) [1, 3], which are defined by integrals with a power-law kernel, marked the beginning of the development of FC. Despite being basic, these operators only have one kernel, which can cause computational issues and make them less suitable for characterizing different kinds of memory effects. A new class of fractional operators with non-singular kernels was created as a result of this restriction. The Caputo-Fabrizio (CF)-Frac Deriv [15], which makes use of an exponential decay kernel, was introduced, marking a major advancement in this field. This operator works well when modeling systems with decaying memory effects or short-range memory, where the effects of previous events fade quickly over time. Its non-singularity provides a different physical interpretation of memory as well as mathematical advantages.

In line with this pattern, Atangana and Baleanu introduced a different non-singular operator based on the generalized Mittag-Leffler (ML) function [16]. This operator is now referred to as the Atangana-Baleanu-Caputo (ABC)-Frac Deriv. A more complex description of memory that encompasses both initial rapid decay and long-tail, power-law behavior is offered by this kernel, which interpolates between a stretched exponential and a power-law function. The ABC operator is frequently better than both the classical and CF operators, as evidenced by its extensive and successful application to a wide range of real-world problems.

Frac Derivs like RL, Caputo, CF, and ABC have been used in many recent studies to establish basic theoretical results, such as existence, uniqueness, and different kinds of UH stability [17–22]. Significant advancements have also been made in the numerical domain in the creation of effective simulation schemes and approximation techniques for fractional models, we refer the readers to these works [23–26]. On the theoretical side, in particular, recent research has established existence, uniqueness, and stability results for fractional models involving CF, ABC, and generalized operators (e.g., [27–29]). Significant progress has been made in the numerical domain, offering effective schemes and error estimates for simulating fractional systems (e.g., [30–33]).

There is still a big gap in spite of the individual achievements of the CF and ABC operators. In real-world systems, it is common to see memory properties that are neither purely exponential nor purely ML. For example, a biological tissue may exhibit both long-term structural memory (like a power law) and short-term relaxation (like an exponential). Similarly, in the financial industry, a combination of long-term historical trends and recent high-impact news can influence market sentiment. Existing operators may impose a restrictive modeling assumption by requiring a choice between a single type of memory kernel. This observation serves as the main motivation for the current study, which is to create a more flexible fractional operator capable of combining these disparate memory behaviors into a single mathematical framework.

The behavior of any fractional operator is fundamentally determined by its kernel, which encodes the memory structure of the model. The CF-Frac Deriv employs the non-singular exponential kernel $K_{\mathrm{CF}}(t-s)=\exp\left[-\dfrac{\alpha}{1-\alpha}(t-s)\right]$, which produces short-memory effects with rapid decay. In contrast, the ABC-Frac Deriv is defined through the ML kernel $K_{\mathrm{ABC}}(t-s)=E_{\alpha}\left(-\dfrac{\alpha}{1-\alpha}(t-s)^{\alpha}\right)$, whose slow decay captures long-memory and anomalous diffusion dynamics. To unify these two regimes, we introduce the hybrid AF-Frac Deriv kernel $K_{\mathrm{AF}}(t-s)=(1-\lambda)\,K_{\mathrm{CF}}(t-s)+\lambda\,K_{\mathrm{ABC}}(t-s)$, $\lambda\in[0,\ 1]$, which enables a smooth interpolation between exponential and ML memory. The parameter λ acts as a memory tuner, allowing the operator to represent intermediate or mixed memory behaviors commonly observed in complex biological and drug-release processes. This hybrid structure preserves the computational advantages of the CF kernel while incorporating the realistic long-term memory of the ABC kernel.

This study primarily contributes by introducing and analyzing a novel mixed fractional operator, the Atangana-Fabrizio (AF) operator, defined as a convex combination of the ABC and CF operators. In particular, we introduce a hybrid fractional operator defined by

$$ {}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}\,\varphi(t)=(1-\lambda)\left({}^{\mathrm{ABC}}_{\mathrm{C}}D_t^{\rho}\,\varphi(t)\right)+\lambda\left({}^{\mathrm{CF}}_{\mathrm{C}}D_t^{\rho}\,\varphi(t)\right). $$

The hybridization parameter λ enables smooth interpolation between the ABC kernel ($\lambda = 0$) and the CF kernel ($\lambda = 1$), providing flexibility to capture a wide range of memory effects. We establish the operator's key mathematical properties, proving existence, uniqueness, and UH stability for a class of nonlinear FDEs. As an application, the AF operator is used to model fractional-order drug release from a polymer matrix governed by a Michaelis-Menten-type equation, with a generalized Adams-Bashforth Method (ABM) employed to demonstrate its effectiveness.

The rest of the paper is organized as follows: Section 2 provides the core definitions and critical results necessary for the subsequent analysis. Section 3 introduces the new AF operator, presents its definition, and establishes its fundamental mathematical properties. Section 4 develops the theoretical framework for AF-governed nonlinear FDEs, where we prove the existence, uniqueness, and UH stability of solutions using fixed-point theorems. Section 5 applies the AF operator to simulate a fractional-order drug release model based on a Michaelis-Menten-type kinetic equation. Section 6 concludes the paper with a summary of the main findings and outlines possible directions for future research.

2. Preliminaries

Here, we outline the basic definitions and lemmas of FC that serve as the theoretical cornerstone of our investigation. These well-established ideas, such as the Caputo, CF, and ABC operators, provide the necessary tools for the subsequent analysis of our proposed hybrid operator.

Definition 1 [3] Let $\varphi(t)$ be a function whose (Derivs) up to order $n-1$ are continuous, with $\rho > 0$ and $n = \lceil \rho \rceil$. The Caputo Frac Deriv of order ρ is described as

$$ {}^{\mathrm{C}}D_t^{\rho}\varphi(t) = \frac{1}{\Gamma(n-\rho)} \int_a^t (t-\upsilon)^{n-\rho-1} \varphi^{(n)}(\upsilon) d\upsilon, \tag{1} $$

where $\Gamma(\cdot)$ is the Euler Gamma function. For $0 < \rho < 1$, the above relation reduces to

$$ {}^{\mathrm{C}}D_t^{\rho}\varphi(t) = \frac{1}{\Gamma(1-\rho)} \int_a^t (t-\upsilon)^{-\rho} \varphi'(\upsilon) d\upsilon. \tag{2} $$

Definition 2 [15] Assume that $b > a$, $\varphi \in H^1(a,\ b)$, and $\rho \in (0,\ 1)$. The CF-Frac Deriv of order ρ for the function φ is defined as

$$ {}^{\mathrm{CF}}_{\mathrm{C}}D_t^{\rho}\varphi(t) = \frac{\mathscr{CF}(\rho)}{1-\rho} \int_a^t \exp\left(-\frac{\rho}{1-\rho}(t-\upsilon)\right) \varphi'(\upsilon) d\upsilon, \tag{3} $$

where $\mathscr{CF}(\rho)$ is a normalization function satisfying $\mathscr{CF}(0) = \mathscr{CF}(1) = 1$. The exponential kernel used in this operator is nonsingular. In addition, the right fractional integral for the CF operator is

$$ {}^{\mathrm{CF}}I_t^{\rho}\varphi(t) = \frac{1-\rho}{\mathscr{CF}(\rho)}\varphi(t) + \frac{\rho}{\mathscr{CF}(\rho)} \int_0^t \varphi(\upsilon) d\upsilon. \tag{4} $$

Definition 3 [16] Let $b > a$, $\varphi \in H^1(a,\ b)$, and $\rho \in (0,\ 1)$. The ABC-Frac Deriv of order ρ is given by

$$ {}^{\mathrm{AB}}_{\mathrm{C}}D_t^{\rho}\varphi(t) = \frac{\mathscr{AB}(\rho)}{1-\rho} \int_a^t \mathscr{E}_{\rho}\left(-\frac{\rho}{1-\rho}(t-\upsilon)^{\rho}\right) \varphi'(\upsilon) d\upsilon. \tag{5} $$

Here, $\mathscr{E}_\rho(\cdot)$ denotes the ML function, defined as

$$\mathscr{E}_\rho(\varpi) = \sum_{r=0}^{\infty} \frac{\varpi^r}{\Gamma(\rho r+1)},$$

and $\mathscr{AB}(\rho)$ is a normalization function satisfying $\mathscr{AB}(0) = \mathscr{AB}(1) = 1$. This operator is constructed using the ML kernel, which is both nonlocal and nonsingular. Furthermore, the right fractional integral corresponding to the ABC operator is given by

$${}^{\mathrm{AB}}I_t^\rho \varphi(t) = \frac{1-\rho}{\mathscr{AB}(\rho)}\varphi(t) + \frac{\rho}{\mathscr{AB}(\rho)\Gamma(\rho)} \int_0^t (t-\upsilon)^{\rho-1}\varphi(\upsilon)d\upsilon. \tag{6}$$

The following lemmas regarding the Laplace transforms of these operators are crucial for solving FDEs.

Lemma 1 [15] The Laplace transform of the CF-Frac Derivs is given by

$$\mathscr{L}\{{}^{\mathrm{CF}}_{\mathrm{C}}D_t^\rho \varphi(t)\}(s) = \frac{\mathscr{CF}(\rho)}{1-\rho} \frac{s\Phi(s)-\varphi(0)}{s+\dfrac{\rho}{1-\rho}},$$

where $\Phi(s) = \mathscr{L}\{\varphi(t)\}(s)$.

Lemma 2 [16] The Laplace transform of the ABC-Frac Deriv is given by

$$\mathscr{L}\{{}^{\mathrm{AB}}_{\mathrm{C}}D_t^\rho \varphi(t)\}(s) = \frac{\mathscr{AB}(\rho)}{1-\rho} \frac{s^\rho\Phi(s)-s^{\rho-1}\varphi(0)}{s^\rho+\dfrac{\rho}{1-\rho}}.$$

Finally, we present the fixed-point theorems that are essential for our proofs of existence, uniqueness, and stability.

Theorem 1 [34] Consider a complete metric space $(X,\ d)$ and a self-map $T : X \to X$. Assume that T satisfies a contractive condition: there exists a constant $k \in [0,\ 1)$ such that

$$d(Tx,\ Ty) \le k\,d(x,\ y) \quad \text{for all } x,\ y \in X.$$

Under this assumption, there exists exactly one point $x^* \in X$ for which $T(x^*) = x^*$; in other words, T has a unique fixed point.

Theorem 2 [34] Let X be a Banach space and let $\mathscr{F} : X \to X$ be an operator that is continuous and maps bounded subsets of X into relatively compact ones. Define

$$\mathscr{E} = \{\, y \in X : y = \sigma\mathscr{F}y \quad \text{for some } \sigma \in (0,\ 1)\,\}.$$

If the set $\mathscr{E}$ is bounded in X, then the operator $\mathscr{F}$ possesses at least one fixed point.

3. The AF hybrid fractional operator

This section introduces a new mixed fractional-order operator, the AF hybrid operator, which is defined as a convex combination of the ABC and CF operators. We establish its linearity, derive the corresponding fractional integral, and prove the fundamental theorem of FC. Additional properties are also shown in parallel with those of the ABC and CF operators. Then, we provide some illustrative examples.

Definition 4 Let $\rho \in (0,\ 1)$, $b > a$, and $\varphi \in H^1(a,\ b)$. In the Caputo sense, we define the AF-Frac Deriv as a convex combination that is governed by a parameter $\lambda \in [0,\ 1]$

$$ {}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}\varphi(t) = (1-\lambda)\left({}^{\mathrm{ABC}}_{\mathrm{C}}D_t^{\rho}\varphi(t)\right) + \lambda\left({}^{\mathrm{CF}}_{\mathrm{C}}D_t^{\rho}\varphi(t)\right), \tag{7} $$

where λ is the hybridization parameter, ${}^{\mathrm{ABC}}_{\mathrm{C}}D_t^{\rho}$ and ${}^{\mathrm{CF}}_{\mathrm{C}}D_t^{\rho}$ are the ABC operator and the CF operator, defined as in (2) and (3) respectively.

Definition 5 The AF fractional integral, ${}^{\mathrm{AF}}I_t^{\rho}$, which serves as the right inverse to the Frac Deriv, is given by

$$ \begin{aligned} {}^{\mathrm{AF}}I_t^{\rho}\varphi(t) &= (1-\lambda)\,{}^{\mathrm{AB}}I_t^{\rho}\varphi(t) + \lambda\,{}^{\mathrm{CF}}I_t^{\rho}\varphi(t) \\ &= (1-\lambda)\left[\frac{1-\rho}{\mathscr{A}\mathscr{B}(\rho)}\varphi(t) + \frac{\rho}{\mathscr{A}\mathscr{B}(\rho)\Gamma(\rho)}\int_0^t (t-\upsilon)^{\rho-1}\varphi(\upsilon)d\upsilon\right] \\ &\quad + \lambda\left[\frac{1-\rho}{\mathscr{C}\mathscr{F}(\rho)}\varphi(t) + \frac{\rho}{\mathscr{C}\mathscr{F}(\rho)}\int_0^t \varphi(\upsilon)d\upsilon\right]. \end{aligned} $$

One has,

$$ \begin{aligned} {}^{\mathrm{AF}}I_t^{\rho}\varphi(t) &= \left(\frac{(1-\lambda)(1-\rho)}{\mathscr{A}\mathscr{B}(\rho)} + \frac{\lambda(1-\rho)}{\mathscr{C}\mathscr{F}(\rho)}\right)\varphi(t) \\ &\quad + \frac{(1-\lambda)\rho}{\mathscr{A}\mathscr{B}(\rho)\Gamma(\rho)}\int_0^t (t-\upsilon)^{\rho-1}\varphi(\upsilon)\,d\upsilon + \frac{\lambda\rho}{\mathscr{C}\mathscr{F}(\rho)}\int_0^t \varphi(\upsilon)\,d\upsilon. \end{aligned} \tag{8} $$

Remark 1 The proposed AF-Frac Deriv and its corresponding AF fractional integral are defined as convex combinations of the ABC operator and the CF operator, governed by the hybrid parameter $\lambda \in [0,\ 1]$. These operators satisfy 7 and 8. Hence, the AF operators reduce to known, well-established operators

- For $\lambda = 0$,

$$ {}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}\varphi(t) = {}^{\mathrm{ABC}}_{\mathrm{C}}D_t^{\rho}\varphi(t), \qquad {}^{\mathrm{AF}}I_t^{\rho}\varphi(t) = {}^{\mathrm{AB}}I_t^{\rho}\varphi(t). $$

That is, the AF operator coincides exactly with the ABC-Frac Deriv and AB integral.

- For $\lambda = 1$,

$$ {}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}\varphi(t) = {}^{\mathrm{CF}}_{\mathrm{C}}D_t^{\rho}\varphi(t), \qquad {}^{\mathrm{AF}}I_t^{\rho}\varphi(t) = {}^{\mathrm{CF}}I_t^{\rho}\varphi(t). $$

Thus, our operator reproduces the CF-Frac Deriv exactly.

Therefore, the AF operator is a valid fractional operator that continuously interpolates between the ABC and CF operators and inherits their limiting structures without contradiction.

Proposition 1 For all $\varphi \in H^1(a,\ b)$:

$$\lambda = 0 \Rightarrow {}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}\varphi = {}^{\mathrm{ABC}}_{\mathrm{C}}D_t^{\rho}\varphi, \qquad \lambda = 1 \Rightarrow {}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}\varphi = {}^{\mathrm{CF}}_{\mathrm{C}}D_t^{\rho}\varphi.$$

Proof. Substituting $\lambda = 0$ or $\lambda = 1$ in (7) gives the result. □

Proposition 2 For fixed φ and t, the mapping $F(\lambda) = {}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}\varphi(t)$ is continuous on $\lambda \in [0,\ 1]$.

Proof. From (7), $F(\lambda) = (1-\lambda)A(t) + \lambda C(t)$, where $A,\ C$ are constants in λ. Hence F is a linear and continuous function of λ. □

Proposition 3 Let $\|\cdot\|$ be any norm for which $|{}^{\mathrm{ABC}}_{\mathrm{C}}D_t^{\rho}\varphi| \le M_1\|\varphi\|$ and $|{}^{\mathrm{CF}}_{\mathrm{C}}D_t^{\rho}\varphi| \le M_2\|\varphi\|$. Then

$$|{}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}\varphi| \le M_3\|\varphi\|.$$

Proof. By the triangle inequality applied to (7), that is

$$|{}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}\varphi| \le (1-\lambda)|{}^{\mathrm{ABC}}_{\mathrm{C}}D_t^{\rho}\varphi| + \lambda|{}^{\mathrm{CF}}_{\mathrm{C}}D_t^{\rho}\varphi| \le ((1-\lambda)M_1 + \lambda M_2)\|\varphi\| := M_3\|\varphi\|.$$

□

Proposition 4 The kernel of the AF operator is non-singular.

Proof. The integral representation of the AF operator is

$${}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}\varphi(t) = \int_a^t K_{AF}(t-\tau,\ \rho,\ \lambda)\varphi'(\tau)d\tau,$$

where the kernel K_{AF} is a convex combination of the ABC and CF kernels:

$$K_{AF}(t-\tau) = (1-\lambda)\frac{B(\rho)}{1-\rho}E_{\rho}\left(-\frac{\rho(t-\tau)^{\rho}}{1-\rho}\right) + \lambda\frac{M(\rho)}{1-\rho}\exp\left(-\frac{\rho(t-\tau)}{1-\rho}\right).$$

The kernel of the CF operator, $\exp(\cdot)$, is non-singular, and the kernel of the ABC operator, $E_{\rho}(\cdot)$, is likewise non-singular. Consequently, since the AF kernel is a linear combination of these two non-singular functions, it remains non-singular for all $t,\ \tau$ within the domain. □

Remark 2 The validity of the proposed AF operator is further supported by the fact that all the main analytical results of this work reduce to previously known results when $\lambda = 0$ or $\lambda = 1$. In particular, Propositions 3-6, reduce to the corresponding properties of the ABC operator (for $\lambda = 0$) and CF operator (for $\lambda = 1$).

Corollary 1 If $\varphi \in C^1([a,\ b])$ and $K_{\mathrm{AF}} \in L^1([0,\ T])$, then

$$\left|{}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}\varphi(t)\right| \le \|K_{\mathrm{AF}}\|_{L^1([0,\ T])}\|\varphi'\|_{L^{\infty}([a,\ b])}.$$

Proof. The AF-Frac Deriv has the integral representation

$$ {}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}\,\varphi(t) = \int_a^t K_{\mathrm{AF}}(t-\tau)\,\varphi'(\tau)\,d\tau, $$

which implies

$$ \left|{}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}\,\varphi(t)\right| \le \int_a^t |K_{\mathrm{AF}}(t-\tau)|\,|\varphi'(\tau)|\,d\tau. $$

Since $\varphi \in C^1([a,\ b])$ there exists a constant $\tau \in [a,\ b]$ such that $|\varphi'(\tau)| \le \|\varphi'\|_{L^\infty([a,\ b])}$. It follows that

$$ \left|{}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}\,\varphi(t)\right| \le \|\varphi'\|_{L^\infty([a,\ b])} \int_a^t |K_{\mathrm{AF}}(t-\tau)|\,d\tau. $$

Finally, noting that $t-\tau \in [0,\ T]$ for $t \in [a,\ b]$ and $K_{\mathrm{AF}} \in L^1([0,\ T])$, we conclude

$$ \left|{}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}\,\varphi(t)\right| \le \|\varphi'\|_{L^\infty([a,\ b])}\,\|K_{\mathrm{AF}}\|_{L^1([0,\ T])}. $$

□

Remark 3 Corollary 1 shows explicit identities involving the AF integral and AF-Frac Deriv. These identities coincide with the known identities for the ABC and CF cases when $\lambda = 0$ and $\lambda = 1$, respectively.

Theorem 3 The Laplace transform of the AF-Frac Deriv ${}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}\,\varphi(t)$ can be expressed in closed form as

$$ \mathscr{L}\{{}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}\,\varphi(t)\}(s) = \big(s\Phi(s) - \varphi(0)\big)\left[\frac{(1-\lambda)\,\mathscr{AB}(\rho)}{1-\rho}\cdot\frac{s^{\rho-1}}{s^{\rho}+\dfrac{\rho}{1-\rho}} + \frac{\lambda\,\mathscr{CF}(\rho)}{1-\rho}\cdot\frac{1}{s+\dfrac{\rho}{1-\rho}}\right]. $$

Proof. We begin with the definition of the AF-Frac Deriv as stated in Equation (7)

$$ {}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}\,\varphi(t) = (1-\lambda)\left({}^{\mathrm{ABC}}_{\mathrm{C}}D_t^{\rho}\,\varphi(t)\right) + \lambda\left({}^{\mathrm{CF}}_{\mathrm{C}}D_t^{\rho}\,\varphi(t)\right). \tag{9} $$

Applying the Laplace transform to both sides of (9), we get

$$ \mathscr{L}\{{}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}\,\varphi(t)\}(s) = (1-\lambda)\mathscr{L}\{{}^{\mathrm{ABC}}_{\mathrm{C}}D_t^{\rho}\,\varphi(t)\}(s) + \lambda\mathscr{L}\{{}^{\mathrm{CF}}_{\mathrm{C}}D_t^{\rho}\,\varphi(t)\}(s). \tag{10} $$

Now, we substitute the known Laplace transforms of the ABC and CF operators from Lemma 1 and Lemma 2 into Equation (10):

$$\mathscr{L}\{{}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}\varphi(t)\}(s) = (1-\lambda)\frac{\mathscr{AB}(\rho)}{1-\rho}\frac{s^{\rho}\Phi(s)-s^{\rho-1}\varphi(0)}{s^{\rho}+\dfrac{\rho}{1-\rho}} + \lambda\frac{\mathscr{CF}(\rho)}{1-\rho}\frac{s\Phi(s)-\varphi(0)}{s+\dfrac{\rho}{1-\rho}},$$

which implies

$$\mathscr{L}\{{}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}\varphi(t)\}(s) = \big(s\Phi(s)-\varphi(0)\big)\left[\frac{(1-\lambda)\mathscr{AB}(\rho)}{1-\rho}\cdot\frac{s^{\rho-1}}{s^{\rho}+\dfrac{\rho}{1-\rho}} + \frac{\lambda\mathscr{CF}(\rho)}{1-\rho}\cdot\frac{1}{s+\dfrac{\rho}{1-\rho}}\right].$$

□

Theorem 4 The AF fractional operator ${}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}$ is a linear operator.

Proof. Let $\varphi_1,\ \varphi_2 \in H^1(a,\ b)$ and let $c_1,\ c_2 \in \mathbb{R}$ be arbitrary constants. Then, we will show that

$${}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}[c_1\varphi_1(t)+c_2\varphi_2(t)] = c_1\left({}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}\varphi_1(t)\right) + c_2\left({}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}\varphi_2(t)\right).$$

Applying the definition from Equation (7),

$${}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}[c_1\varphi_1(t)+c_2\varphi_2(t)] = (1-\lambda)\left({}^{\mathrm{ABC}}_{\mathrm{C}}D_t^{\rho}[c_1\varphi_1(t)+c_2\varphi_2(t)]\right) + \lambda\left({}^{\mathrm{CF}}_{\mathrm{C}}D_t^{\rho}[c_1\varphi_1(t)+c_2\varphi_2(t)]\right).$$

The linearity of the ABC and CF operators is well established. As a result, we can write

$$\begin{aligned}
{}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}[c_1\varphi_1(t)+c_2\varphi_2(t)] &= (1-\lambda)\left[c_1({}^{\mathrm{ABC}}_{\mathrm{C}}D_t^{\rho}\varphi_1(t)) + c_2({}^{\mathrm{ABC}}_{\mathrm{C}}D_t^{\rho}\varphi_2(t))\right] \\
&\quad + \lambda\left[c_1({}^{\mathrm{CF}}_{\mathrm{C}}D_t^{\rho}\varphi_1(t)) + c_2({}^{\mathrm{CF}}_{\mathrm{C}}D_t^{\rho}\varphi_2(t))\right] \\
&= c_1\left[(1-\lambda)({}^{\mathrm{ABC}}_{\mathrm{C}}D_t^{\rho}\varphi_1(t)) + \lambda({}^{\mathrm{CF}}_{\mathrm{C}}D_t^{\rho}\varphi_1(t))\right] \\
&\quad + c_2\left[(1-\lambda)({}^{\mathrm{ABC}}_{\mathrm{C}}D_t^{\rho}\varphi_2(t)) + \lambda({}^{\mathrm{CF}}_{\mathrm{C}}D_t^{\rho}\varphi_2(t))\right].
\end{aligned}$$

According to the AF operator's definition (Equation (7)), this reduces to

$$= c_1\left({}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}\varphi_1(t)\right) + c_2\left({}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}\varphi_2(t)\right).$$

□

Theorem 5 Let $\rho \in (0,\ 1)$, $\lambda \in [0,\ 1]$, and let f be such that its Laplace transform $F(s)$ exists (for s large). Assume the normalizations $\mathscr{AB}(\rho) = \mathscr{CF}(\rho) = 1-\rho$ hold. Then

$$ {}^{\mathrm{AF}}I_t^{\rho}\left({}_{\mathrm{C}}^{\mathrm{AF}}D_t^{\rho} f(t)\right) = f(t) - f(0). \tag{11}$$

Proof. Our aim is to establish the existence of a right-inverse operator ${}^{\mathrm{AF}}I_t^{\rho}$ for the equation ${}_{C}^{\mathrm{AF}}D_t^{\rho} f(t) = g(t)$, with the property that ${}^{\mathrm{AF}}I_t^{\rho} g(t) = f(t) - f(0)$.

In view of Definition 4, we have

$$(1-\lambda)\left({}_{\mathrm{C}}^{\mathrm{ABC}}D_t^{\rho} f(t)\right) + \lambda\left({}_{\mathrm{C}}^{\mathrm{CF}}D_t^{\rho} f(t)\right) = g(t).$$

Upon applying the Laplace transform, we obtain

$$(1-\lambda)\mathscr{L}\{{}_{\mathrm{C}}^{\mathrm{ABC}}D_t^{\rho} f(t)\} + \lambda\mathscr{L}\{{}_{\mathrm{C}}^{\mathrm{CF}}D_t^{\rho} f(t)\} = G(s), \tag{12}$$

where $\mathscr{L}\{f(t)\} = F(s)$ and $\mathscr{L}\{g(t)\} = G(s)$.

Using Lemmas 1 and 2, we get

$$(1-\lambda)\left[\frac{s^{\rho}F(s) - s^{\rho-1}f(0)}{s^{\rho} + \dfrac{\rho}{1-\rho}}\right] + \lambda\left[\frac{sF(s) - f(0)}{s + \dfrac{\rho}{1-\rho}}\right] = G(s),$$

which implies

$$F(s)\left[(1-\lambda)\frac{s^{\rho}}{s^{\rho} + \dfrac{\rho}{1-\rho}} + \lambda\frac{s}{s + \dfrac{\rho}{1-\rho}}\right] = G(s) + f(0)\left[(1-\lambda)\frac{s^{\rho-1}}{s^{\rho} + \dfrac{\rho}{1-\rho}} + \frac{\lambda}{s + \dfrac{\rho}{1-\rho}}\right].$$

Let

$$\Psi(s,\ \rho,\ \lambda) = (1-\lambda)\frac{s^{\rho}}{s^{\rho} + \dfrac{\rho}{1-\rho}} + \lambda\frac{s}{s + \dfrac{\rho}{1-\rho}}, \quad \Omega(s,\ \rho,\ \lambda) = (1-\lambda)\frac{s^{\rho-1}}{s^{\rho} + \dfrac{\rho}{1-\rho}} + \frac{\lambda}{s + \dfrac{\rho}{1-\rho}}.$$

Hence,

$$F(s)\cdot\Psi(s,\ \rho,\ \lambda) = G(s) + f(0)\cdot\Omega(s,\ \rho,\ \lambda),$$

which gives

$$F(s) = \frac{G(s)}{\Psi(s,\ \rho,\ \lambda)} + f(0)\frac{\Omega(s,\ \rho,\ \lambda)}{\Psi(s,\ \rho,\ \lambda)}. \tag{13}$$

Let's analyze the term $\dfrac{\Omega(s,\,\rho,\,\lambda)}{\Psi(s,\,\rho,\,\lambda)}$.

$$\frac{\Omega(s,\,\rho,\,\lambda)}{\Psi(s,\,\rho,\,\lambda)} = \frac{(1-\lambda)\dfrac{s^{\rho-1}}{D_{\mathrm{AB}}} + \dfrac{\lambda}{D_{(\mathrm{CF})}}}{(1-\lambda)\dfrac{s^{\rho}}{D_{\mathrm{AB}}} + \lambda\dfrac{s}{D_{(\mathrm{CF})}}} = \frac{\dfrac{(1-\lambda)s^{\rho-1}D_{(\mathrm{CF})} + \lambda D_{\mathrm{AB}}}{D_{\mathrm{AB}}D_{(\mathrm{CF})}}}{\dfrac{(1-\lambda)s^{\rho}D_{(\mathrm{CF})} + \lambda s D_{\mathrm{AB}}}{D_{\mathrm{AB}}D_{(\mathrm{CF})}}}$$

$$= \frac{(1-\lambda)s^{\rho-1}\left(s+\dfrac{\rho}{1-\rho}\right) + \lambda\left(s^{\rho}+\dfrac{\rho}{1-\rho}\right)}{(1-\lambda)s^{\rho}\left(s+\dfrac{\rho}{1-\rho}\right) + \lambda s\left(s^{\rho}+\dfrac{\rho}{1-\rho}\right)}$$

$$= \frac{(1-\lambda)\left(s^{\rho}+\dfrac{\rho}{1-\rho}s^{\rho-1}\right) + \lambda s^{\rho} + \lambda\dfrac{\rho}{1-\rho}}{(1-\lambda)\left(s^{\rho+1}+\dfrac{\rho}{1-\rho}s^{\rho}\right) + \lambda s^{\rho+1} + \lambda s\dfrac{\rho}{1-\rho}}$$

$$= \frac{1}{s}.$$

Consequently, Equation (13) reduces to

$$F(s) - \frac{f(0)}{s} = \frac{G(s)}{\Psi(s,\,\rho,\,\lambda)}. \tag{14}$$

The left side of Equation (14) is the Laplace transform of $f(t) - f(0)$. Hence

$$\mathscr{L}\{f(t) - f(0)\} = \frac{G(s)}{\Psi(s,\,\rho,\,\lambda)}.$$

Invoking the inverse Laplace transform on both sides gives

$$f(t) - f(0) = \mathscr{L}^{-1}\left\{\frac{\mathscr{L}\{g(t)\}(s)}{\Psi(s,\,\rho,\,\lambda)}\right\}.$$

Thus, a right-inverse operator exists, and its Laplace-domain expression serves as the definition of the AF integral as

$${}^{\mathrm{AF}}I_t^{\rho}\,g(t) \equiv \mathscr{L}^{-1}\left\{\frac{G(s)}{\Psi(s,\,\rho,\,\lambda)}\right\}.$$

Applying this operator to $g(t) = {}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho} f(t)$ yields exactly $f(t) - f(0)$. □

Remark 4 All theoretical results align with and extend the established frameworks of the ABC and CF operators. For instance,

• Theorem 3 (Laplace transform) and Theorem 4 (linear operator) reduce directly to the established results in the literature for the ABC operator when $\lambda = 0$ and for the CF operator when $\lambda = 1$.

• For Theorem 5 (composition property). This composition reduces to the familiar right-inverse relation $f(t) - f(0)$ in the limiting cases $\lambda = 0$ and $\lambda = 1$, where the AF operator coincides with the ABC and CF operators, respectively.

3.1 *Illustrative examples*

To provide a more thorough comprehension of the suggested AF operator, we offer several straightforward examples. For simplicity, we assume that the normalization functions $\mathscr{AB}(\rho) = \mathscr{CF}(\rho) = 1$.

Example 1 Consider a constant function, $y(t) = C$.

From the definition of the AF operator, we have

$$ {}^{\mathrm{AF}}_{\mathrm{C}}D^{\rho}_{t}\, y(t) = (1-\lambda)({}^{\mathrm{ABC}}_{\mathrm{C}}D^{\rho}_{t} C) + \lambda({}^{\mathrm{CF}}_{\mathrm{C}}D^{\rho}_{t} C). $$

Since the definitions of both the ABC and CF operators involve an integral of $y'(\upsilon)$, and $y'(\upsilon) = 0$ for all υ, because $y'(t) = \dfrac{d}{dt}(C) = 0$. Hence, ${}^{\mathrm{ABC}}_{\mathrm{C}}D^{\rho}_{t} C = 0$ ${}^{\mathrm{CF}}_{\mathrm{C}}D^{\rho}_{t} C = 0$. Therefore, ${}^{\mathrm{AF}}_{\mathrm{C}}D^{\rho}_{t} C = 0$.

Example 2 Consider a linear function, $y(t) = ct$, where c is a constant. First, we compute the derivative: $y'(t) = \dfrac{d}{dt}(ct) = c$. Then, we compute the CF and ABC operators on $y(t) = ct$.

For CF Part:

$$ {}^{\mathrm{CF}}_{\mathrm{C}}D^{\rho}_{t}(ct) = \frac{1}{1-\rho}\int_0^t \exp\left(-\frac{\rho(t-\upsilon)}{1-\rho}\right)\cdot c\, d\upsilon $$

$$ = \frac{c}{1-\rho}\left[\frac{1-\rho}{\rho}\exp\left(-\frac{\rho(t-\upsilon)}{1-\rho}\right)\right]_{\upsilon=0}^{\upsilon=t} $$

$$ = \frac{c}{\rho}\left(\exp(0) - \exp\left(-\frac{\rho t}{1-\rho}\right)\right) $$

$$ = \frac{c}{\rho}\left(1 - \exp\left(-\frac{\rho t}{1-\rho}\right)\right). $$

For ABC Part, we set $\theta = \dfrac{\rho}{1-\rho}$ and substitute $u = t - \upsilon$. Then

$$ {}^{\mathrm{ABC}}_{\mathrm{C}}D^{\rho}_{t}(ct) = \frac{1}{1-\rho}\int_a^t \mathscr{E}_{\rho}\left(-\frac{\rho}{1-\rho}(t-\upsilon)^{\rho}\right)\cdot c\, d\upsilon $$

$$ = \frac{c}{1-\rho}\int_0^{t-a} \mathscr{E}_{\rho}\left(-\theta u^{\rho}\right) du. $$

Using the standard ML integral identity, we obtain

$$ {}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}(ct) = \frac{ct}{1-\rho} E_{\rho,\,2}\left(-\frac{\rho}{1-\rho}t^{\rho}\right). $$

Here $\mathscr{E}_{\rho,\,\beta}(\varpi) = \sum_{r=0}^{\infty} \frac{\varpi^r}{\Gamma(\rho r+\beta)}$ is the two-parameter ML function.
Finally, we combine these results using the definition of the AF operator:

$$ {}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}(ct) = (1-\lambda)\left[\frac{ct}{1-\rho} E_{\rho,\,2}\left(-\frac{\rho}{1-\rho}t^{\rho}\right)\right] + \lambda\frac{c}{\rho}\left[1-\exp\left(-\frac{\rho t}{1-\rho}\right)\right]. $$

As $t \to \infty$, we have

$$ E_{\rho,\,2}\left(-\frac{\rho}{1-\rho}t^{\rho}\right) \sim \frac{(1-\rho)t^{-\rho}}{\rho\Gamma(2-\rho)} \text{ and } \exp\left(-\frac{\rho t}{1-\rho}\right) \sim 0. $$

Therefore the asymptotic behaviour as $t \to \infty$ is

$$ {}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}(ct) \sim (1-\lambda)\frac{ct^{1-\rho}}{\rho\Gamma(2-\rho)} + \lambda\frac{c}{\rho}. $$

In Figure 1, solid curves represent the exact solution, while dashed curves illustrate the large-t asymptotic behavior.

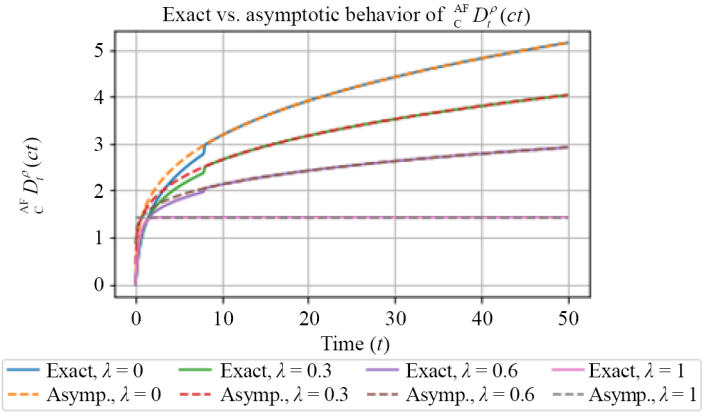


Figure 1. Exact versus asymptotic behavior of ${}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}(ct)$ for different values of λ, with parameters $\rho = 0.7$ and $c = 1$

Example 3 Consider the relaxation-type FDE:

$$ {}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho} y(t) = -ky(t), \text{ with } y(0) = y_0. \tag{15} $$

We will find the analytical solution in the Laplace domain and then simulate the time-domain solution numerically. Using the derived Laplace transform for the AF operator,

$$(1-\lambda)\frac{s^{\rho}Y(s)-s^{\rho-1}y_0}{s^{\rho}+\dfrac{\rho}{1-\rho}}+\lambda\frac{sY(s)-y_0}{s+\dfrac{\rho}{1-\rho}}=-kY(s).$$

It follows that

$$Y(s)\left[(1-\lambda)\frac{s^{\rho}}{s^{\rho}+\dfrac{\rho}{1-\rho}}+\lambda\frac{s}{s+\dfrac{\rho}{1-\rho}}+k\right]=y_0\left[(1-\lambda)\frac{s^{\rho-1}}{s^{\rho}+\dfrac{\rho}{1-\rho}}+\lambda\frac{1}{s+\dfrac{\rho}{1-\rho}}\right].$$

This gives the explicit analytical solution for the Laplace transform $Y(s)$,

$$Y(s)=y_0\frac{(1-\lambda)\dfrac{s^{\rho-1}}{s^{\rho}+\dfrac{\rho}{1-\rho}}+\lambda\dfrac{1}{s+\dfrac{\rho}{1-\rho}}}{(1-\lambda)\dfrac{s^{\rho}}{s^{\rho}+\dfrac{\rho}{1-\rho}}+\lambda\dfrac{s}{s+\dfrac{\rho}{1-\rho}}+k}. \tag{16}$$

Set $a=\dfrac{\rho}{1-\rho}$, we have

$$D(s)=(1-\lambda)\frac{s^{\rho}}{s^{\rho}+a}+\lambda\frac{s}{s+a}+k,\qquad N(s)=(1-\lambda)\frac{s^{\rho-1}}{s^{\rho}+a}+\lambda\frac{1}{s+a}.$$

Then

$$Y(s)=y_0\frac{N(s)}{D(s)}.$$

Define

$$B(s)=a\Big(\frac{1-\lambda}{s^{\rho}+a}+\frac{\lambda}{s+a}\Big),\ b(t)=\mathscr{L}^{-1}\{B(s)\}=a(1-\lambda)t^{\rho-1}E_{\rho,\,\rho}(-at^{\rho})+a\lambda e^{-at}.$$

Using the Neumann expansion $\dfrac{1}{D(s)}=\dfrac{1}{1+k}\displaystyle\sum_{n\geq 0}\left(\frac{B(s)}{1+k}\right)^n$ one obtains

$$y(t)=y_0-\frac{y_0k}{1+k}\sum_{n=0}^{\infty}\frac{1}{(1+k)^n}\int_0^t b^{*n}(\upsilon)\,d\upsilon,$$

where $b^{*0} = \delta$ and b^{*n} is the n-fold convolution of b (this series is explicit and convergent under the usual Neumann condition). Note that,

- $\lambda = 1$ (pure CF):

$$Y(s) = \frac{y_0}{(1+k)s+ka} \quad \Longrightarrow \quad y(t) = \frac{y_0}{1+k}\exp\Big(-\frac{ka}{1+k}t\Big).$$

- $\lambda = 0$ (pure ABC):

$$Y(s) = \frac{y_0}{1+k}\cdot\frac{s^{\rho-1}}{s^{\rho}+\dfrac{ka}{1+k}} \quad \Longrightarrow \quad y(t) = \frac{y_0}{1+k}\,E_{\rho,\,1}\Big(-\frac{ka}{1+k}t^{\rho}\Big).$$

To visualize the solution, we develop a numerical scheme based on the ABM. The integral form of the relaxation equation is

$$y(t) = y_0 + {}^{\mathrm{AF}}I_t^{\rho}[-ky(t)].$$

Discretizing this equation leads to a recursive formula for $y_{n+1} = y(t_{n+1})$. We simulate this for three cases: pure ABC ($\lambda = 0$), pure CF ($\lambda = 1$), and hybrid AF ($\lambda = 0.5$). The simulation Parameters are $y_0 = 1.0$, $k = 0.5$, $\rho = 0.8$, $T = 15$ and $h = 0.05$.

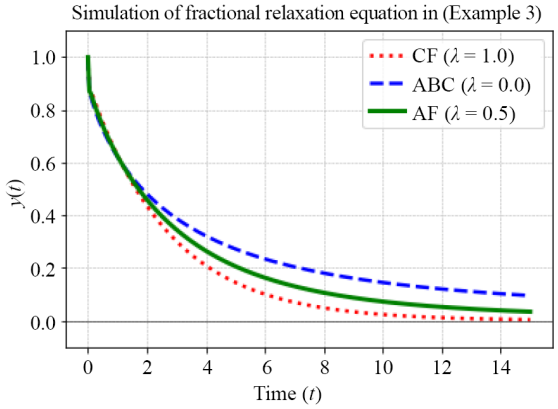


Figure 2. Numerical simulation of the fractional relaxation equation for different fractional operators

Figure 2 highlights the effect of the hybridization parameter λ. The CF case ($\lambda = 1$) exhibits the fastest exponential-type decay, reflecting short memory. In contrast, the ABC case ($\lambda = 0$) shows the slowest ML decay, capturing persistent long-range memory. The AF case ($\lambda = 0.5$) lies in between, demonstrating tunable relaxation dynamics that bridge the two extremes.

Remark 5 The validity and correctness of the proposed AF operator are demonstrated through the illustrative examples in Subsection 3.1, which confirm its consistency with fundamental requirements and with the known behavior of the ABC and CF operators.

• Example 1 shows that ${}^{\mathrm{AF}}_{C}D^{\rho}_{t}(C) = 0$, verifying that the AF operator satisfies the essential property that the derivative of a constant must vanish. Moreover, for $\lambda = 0$ and $\lambda = 1$, the result coincides with the ABC and CF operators, respectively.

• Example 2 demonstrates that ${}^{\mathrm{AF}}_{C}D^{\rho}_{t}(ct)$ produces a weighted combination of the ABC and CF outputs, with λ controlling the transition between long-range power-law memory ($\lambda = 0$) and short-range exponential memory ($\lambda = 1$). Figure 1 further confirms that both the exact formula and its large-t asymptotic behavior match the expected limiting case ${}^{\mathrm{AF}}_{C}D^{1}_{t}(ct) = \dfrac{d}{dt}(ct) = c$.

• Example 3 and Figure 2 show that the AF operator correctly interpolates between the ABC and CF relaxation dynamics. For $\lambda = 0$, the solution reproduces the long-memory ABC behavior; for $\lambda = 1$, it matches the exponential CF decay; and intermediate values of λ yield the expected hybrid relaxation profile.

4. Existence and stability analysis

In this section, we prove the uniqueness and existence of a solution to the nonlinear Initial Value Problem (IVP) involving the proposed hybrid operator AF. We then discuss the stability of the UH sense.

Let us consider the following general nonlinear fractional IVP governed by the AF operator,

$$\begin{cases} {}^{\mathrm{AF}}_{\mathrm{C}}D^{\rho}_{t}y(t) = h(t,\, y(t)), \quad t \in [0,\, T], \\ \\ y(0) = y_0, \end{cases} \tag{17}$$

where ${}^{\mathrm{AF}}_{\mathrm{C}}D^{\rho}_{t}$ is the AF-Frac Deriv of order $\rho \in (0,\, 1)$, and $h : [0,\, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function for some $T > 0$.

4.1 *Convert the IVP to an integral equation*

Lemma 3 Let $\rho \in (0,\, 1)$, $\lambda \in [0,\, 1]$ and assume that $h : [0,\, T] \times \mathbb{R} \to \mathbb{R}$ is continuous with $h(0,\, y(0)) = 0$. Then a function $y \in C([0,\, T])$ is a solution of the IVP (17), if and only if y satisfies

$$y(t) = y_0 + \Big((1-\lambda)\frac{1-\rho}{\mathscr{AB}(\rho)} + \lambda\frac{1-\rho}{\mathscr{CF}(\rho)}\Big)h(t,\, y(t))$$

$$+ \frac{(1-\lambda)\rho}{\mathscr{AB}(\rho)\Gamma(\rho)}\int_0^t (t-\upsilon)^{\rho-1}h(\upsilon,\, y(\upsilon))\,d\upsilon$$

$$+ \frac{\lambda\rho}{\mathscr{CF}(\rho)}\int_0^t h(\upsilon,\, y(\upsilon))\,d\upsilon.$$

Proof. Applying ${}^{\mathrm{AF}}I^{\rho}_{t}$ to both sides of (17) and using the Theorem 5 with the initial condition, we get

$$y(t) = y_0 + {}^{\mathrm{AF}}I^{\rho}_{t}\left[h(t,\, y(t))\right]. \tag{18}$$

The AB and CF integrals are hybridized in the AF integral operator ${}^{\mathrm{AF}}I^{\rho}_{t}$ defined by

$$
\begin{aligned}
{}^{\mathrm{AF}}I_t^{\rho} h(t,\, y(t)) &= \left((1-\lambda)\,{}^{\mathrm{AB}}I_t^{\rho} + \lambda\,{}^{\mathrm{CF}}I_t^{\rho}\right) h(t,\, y(t)) \\
&= (1-\lambda)\left[\frac{1-\rho}{\mathscr{A}\mathscr{B}(\rho)} h(t,\, y(t)) + \frac{\rho}{\mathscr{A}\mathscr{B}(\rho)\Gamma(\rho)} \int_0^t (t-\upsilon)^{\rho-1} h(\upsilon,\, y(\upsilon))\, d\upsilon\right] \\
&\quad + \lambda\left[\frac{1-\rho}{\mathscr{C}\mathscr{F}(\rho)} h(t,\, y(t)) + \frac{\rho}{\mathscr{C}\mathscr{F}(\rho)} \int_0^t h(\upsilon,\, y(\upsilon))\, d\upsilon\right] \\
&= \left((1-\lambda)\frac{1-\rho}{\mathscr{A}\mathscr{B}(\rho)} + \lambda\frac{1-\rho}{\mathscr{C}\mathscr{F}(\rho)}\right) h(t,\, y(t)) \\
&\quad + \frac{(1-\lambda)\rho}{\mathscr{A}\mathscr{B}(\rho)\Gamma(\rho)} \int_0^t (t-\upsilon)^{\rho-1} h(\upsilon,\, y(\upsilon))\, d\upsilon + \frac{\lambda\rho}{\mathscr{C}\mathscr{F}(\rho)} \int_0^t h(\upsilon,\, y(\upsilon))\, d\upsilon.
\end{aligned}
$$

Substituting into (18), we get the final integral equation

$$
\begin{aligned}
y(t) &= y_0 + \left((1-\lambda)\frac{1-\rho}{\mathscr{A}\mathscr{B}(\rho)} + \lambda\frac{1-\rho}{\mathscr{C}\mathscr{F}(\rho)}\right) h(t,\, y(t)) \\
&\quad + \frac{(1-\lambda)\rho}{\mathscr{A}\mathscr{B}(\rho)\Gamma(\rho)} \int_0^t (t-\upsilon)^{\rho-1} h(\upsilon,\, y(\upsilon))\, d\upsilon + \frac{\lambda\rho}{\mathscr{C}\mathscr{F}(\rho)} \int_0^t h(\upsilon,\, y(\upsilon))\, d\upsilon. \qquad (19)
\end{aligned}
$$

Conversely, if y satisfies (19), then applying the AF-Frac Deriv (which annihilates the constant y_0) yields

$$
\begin{aligned}
{}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho} y(t) &= {}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho} y_0 + {}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}\left[\left((1-\lambda)\frac{1-\rho}{\mathscr{A}\mathscr{B}(\rho)} + \lambda\frac{1-\rho}{\mathscr{C}\mathscr{F}(\rho)}\right) h(t,\, y(t))\right. \\
&\quad \left. + \frac{(1-\lambda)\rho}{\mathscr{A}\mathscr{B}(\rho)\Gamma(\rho)} \int_0^t (t-\upsilon)^{\rho-1} h(\upsilon,\, y(\upsilon))\, d\upsilon + \frac{\lambda\rho}{\mathscr{C}\mathscr{F}(\rho)} \int_0^t h(\upsilon,\, y(\upsilon))\, d\upsilon\right] \\
&= {}^{\mathrm{AF}}_{\mathrm{C}}D_t^{\rho}\left((1-\lambda)\,{}^{\mathrm{AB}}I_t^{\rho} + \lambda\,{}^{\mathrm{CF}}I_t^{\rho}\right) h(t,\, y(t)) \\
&= \left((1-\lambda) + \lambda\right) h(t,\, y(t)) = h(t,\, y(t)).
\end{aligned}
$$

Since $h(0,\, y(0)) = 0$, $y(0) = y_0$. □

Before proving our basic theorems, we introduce the following terminology.

Let $(\mathscr{C}[0,\, T],\, \|\cdot\|_\infty)$ be the Banach space of all continuous functions on $[0,\, T]$ equipped with $\|y\|_\infty = \sup_{t\in[0,\, T]} |y(t)|$. This is a complete Banach space. Also, we define the operator $\mathscr{F} : \mathscr{C}[0,\, T] \to \mathscr{C}[0,\, T]$ as

$$(\mathscr{F}y)(t) = y_0 + \left((1-\lambda)\frac{1-\rho}{\mathscr{A}\mathscr{B}(\rho)} + \lambda\frac{1-\rho}{\mathscr{C}\mathscr{F}(\rho)}\right)h(t,\, y(t))$$

$$+ \frac{(1-\lambda)\rho}{\mathscr{A}\mathscr{B}(\rho)\Gamma(\rho)}\int_0^t (t-\upsilon)^{\rho-1}h(\upsilon,\, y(\upsilon))\,d\upsilon$$

$$+ \frac{\lambda\rho}{\mathscr{C}\mathscr{F}(\rho)}\int_0^t h(\upsilon,\, y(\upsilon))\,d\upsilon. \tag{20}$$

This simplifies the problem to identifying a unique fixed point of $\mathscr{F}$, that is $(\mathscr{F}y)(t) = y(t)$.
Here, we impose the following assumptions.
(A1) $h : [0,\, T] \times \mathbb{R} \to \mathbb{R}$ is continuous.
(A2) There exists a constant $L > 0$ such that

$$|h(t,\, y_1(t)) - h(t,\, y_2(t))| \leq L|y_1(t) - y_2(t)|,\ \forall t \in [0,\, T] \text{ and } y_1,\, y_2 \in \mathscr{C}[0,\, T].$$

(A3) There exist functions

$$\psi : [0,\, \infty) \to (0,\, \infty) \ \text{ and } \ p : [0,\, T] \to [0,\, \infty)$$

such that ψ is continuous and non-decreasing, p is continuous, and they satisfy

$$|h(t,\, y)| \leq p(t)\psi(|y|) \text{ for all } (t,\, y) \in [0,\, T] \times \mathbb{R}.$$

(A4) There exists a constant $K > 0$ such that

$$\frac{K}{\|p\|_\infty \psi(K)\left[\left|\frac{(1-\lambda)(1-\rho)}{\mathscr{A}\mathscr{B}(\rho)} + \frac{\lambda(1-\rho)}{\mathscr{C}\mathscr{F}(\rho)}\right| + \frac{|1-\lambda|T^\rho}{|\mathscr{A}\mathscr{B}(\rho)|\Gamma(\rho+1)} + \frac{|\lambda|\rho T}{|\mathscr{C}\mathscr{F}(\rho)|}\right] + |y_0|} > 1.$$

4.2 *Uniqueness theorem*

Here, we introduce our first theorem for proving uniqueness using Theorem 1.

Theorem 6 Assume assumptions (A1) and (A2) hold. Then the IVP (17) has a unique solution on $[0,\, T]$ provided that

$$L\left[\left|\frac{(1-\lambda)(1-\rho)}{\mathscr{A}\mathscr{B}(\rho)} + \frac{\lambda(1-\rho)}{\mathscr{C}\mathscr{F}(\rho)}\right| + \frac{|1-\lambda|\rho T^\rho}{|\mathscr{A}\mathscr{B}(\rho)|\Gamma(\rho+1)} + \frac{|\lambda|\rho T}{|\mathscr{C}\mathscr{F}(\rho)|}\right] < 1. \tag{21}$$

Proof. We shall prove that the operator $\mathscr{F}$ defined in Equation (20) is a contraction mapping on $(\mathscr{C}[0,\, T],\, \|\cdot\|_\infty)$. Let $y_1,\, y_2 \in \mathscr{C}[0,\, T]$. Then

$$
\begin{aligned}
|(\mathscr{F}y_1)(t)-(\mathscr{F}y_2)(t)| &= \left|\left(\frac{(1-\lambda)(1-\rho)}{\mathscr{A}\mathscr{B}(\rho)}+\frac{\lambda(1-\rho)}{\mathscr{C}\mathscr{F}(\rho)}\right)[h(t,\,y_1)-h(t,\,y_2)]\right.\\
&\quad+\frac{(1-\lambda)\rho}{\mathscr{A}\mathscr{B}(\rho)\Gamma(\rho)}\int_0^t(t-\upsilon)^{\rho-1}[h(\upsilon,\,y_1)-h(\upsilon,\,y_2)]\,d\upsilon\\
&\quad\left.+\frac{\lambda\rho}{\mathscr{C}\mathscr{F}(\rho)}\int_0^t[h(\upsilon,\,y_1)-h(\upsilon,\,y_2)]\,d\upsilon\right|\\
&\le\left|\frac{(1-\lambda)(1-\rho)}{\mathscr{A}\mathscr{B}(\rho)}+\frac{\lambda(1-\rho)}{\mathscr{C}\mathscr{F}(\rho)}\right||h(t,\,y_1)-h(t,\,y_2)|\\
&\quad+\frac{|1-\lambda|\rho}{|\mathscr{A}\mathscr{B}(\rho)|\Gamma(\rho)}\int_0^t(t-\upsilon)^{\rho-1}|h(\upsilon,\,y_1)-h(\upsilon,\,y_2)|\,d\upsilon\\
&\quad+\frac{|\lambda|\rho}{|\mathscr{C}\mathscr{F}(\rho)|}\int_0^t|h(\upsilon,\,y_1)-h(\upsilon,\,y_2)|\,d\upsilon.
\end{aligned}
$$

Now, we apply the Lipschitz condition (A2) to each term,

$$
\begin{aligned}
|(\mathscr{F}y_1)(t)-(\mathscr{F}y_2)(t)| &\le\left|\frac{(1-\lambda)(1-\rho)}{\mathscr{A}\mathscr{B}(\rho)}+\frac{\lambda(1-\rho)}{\mathscr{C}\mathscr{F}(\rho)}\right|L|y_1(t)-y_2(t)|\\
&\quad+\frac{|1-\lambda|\rho L}{|\mathscr{A}\mathscr{B}(\rho)|\Gamma(\rho)}\int_0^t(t-\upsilon)^{\rho-1}|y_1(\upsilon)-y_2(\upsilon)|\,d\upsilon\\
&\quad+\frac{|\lambda|\rho L}{|\mathscr{C}\mathscr{F}(\rho)|}\int_0^t|y_1(\upsilon)-y_2(\upsilon)|\,d\upsilon\\
&\le\left|\frac{(1-\lambda)(1-\rho)}{\mathscr{A}\mathscr{B}(\rho)}+\frac{\lambda(1-\rho)}{\mathscr{C}\mathscr{F}(\rho)}\right|L\|y_1-y_2\|_\infty\\
&\quad+\frac{|1-\lambda|\rho L\|y_1-y_2\|_\infty}{|\mathscr{A}\mathscr{B}(\rho)|\Gamma(\rho+1)}\frac{t^\rho}{\rho}\\
&\quad+\frac{|\lambda|\rho L\|y_1-y_2\|_\infty}{|\mathscr{C}\mathscr{F}(\rho)|}t,
\end{aligned}
$$

which implies

$$
\|\mathscr{F}y_1-\mathscr{F}y_2\|_\infty\le L\|y_1-y_2\|_\infty\left[\left|\frac{(1-\lambda)(1-\rho)}{\mathscr{A}\mathscr{B}(\rho)}+\frac{\lambda(1-\rho)}{\mathscr{C}\mathscr{F}(\rho)}\right|+\frac{|1-\lambda|T^\rho}{|\mathscr{A}\mathscr{B}(\rho)|\Gamma(\rho+1)}+\frac{|\lambda|\rho T}{|\mathscr{C}\mathscr{F}(\rho)|}\right].
$$

According to (21), $\mathscr{F}$ is contraction mapping. Thus, Theorem 1 guarantees that $\mathscr{F}$ has a unique fixed point. This unique fixed point is the unique solution to the integral equation (20) and therefore to IVP (17). □

4.3 *Existence theorem*

In this part, we present our second theorem, which proves the existence theory using Theorem 2.

Theorem 7 Let (A1), (A3), and (A4) hold. Then, IVP (17) admits at least one solution on $[0,\ T]$.

Proof. Step 1: $\mathscr{F}$ maps $\mathscr{C}[0,\ T]$ to $\mathscr{C}[0,\ T]$.

Clearly, each component of $(\mathscr{F}y)(t)$ (the constant term, the compositions $h(t,\ y(t))$, and the integral terms) is continuous on $[0,\ T]$. their sum is also continuous. Therefore, $\mathscr{F}y \in \mathscr{C}[0,\ T]$. Hence, $\mathscr{F}$ maps $\mathscr{C}[0,\ T]$ into itself.

Step 2: $\mathscr{F}$ is Continuous.

Here, we need to show that if $y_n \to y$ in $\mathscr{C}[0,\ T]$ (i.e., $\|y_n - y\| \to 0$), then $\mathscr{F}y_n \to \mathscr{F}y$ in $\mathscr{C}[0,\ T]$ (i.e., $\|\mathscr{F}y_n - \mathscr{F}y\| \to 0$).

For $y_n,\ y \in \mathscr{C}[0,\ T]$, we have

$$|(\mathscr{F}y_n)(t) - (\mathscr{F}y)(t)| \leq \left((1-\lambda)\frac{1-\rho}{\mathscr{A}\mathscr{B}(\rho)} + \lambda\frac{1-\rho}{\mathscr{C}\mathscr{F}(\rho)}\right)|(h(t,\ y_n(t)) - h(t,\ y(t))|$$

$$+ \frac{(1-\lambda)\rho}{\mathscr{A}\mathscr{B}(\rho)\Gamma(\rho)}\int_0^t (t-\upsilon)^{\rho-1}\,|(h(\upsilon,\ y_n(\upsilon)) - h(\upsilon,\ y(\upsilon))|\,d\upsilon$$

$$+ \frac{\lambda\rho}{\mathscr{C}\mathscr{F}(\rho)}\int_0^t |(h(\upsilon,\ y_n(\upsilon)) - h(\upsilon,\ y(\upsilon))|\,d\upsilon.$$

It follows that

$$\|\mathscr{F}y_n - \mathscr{F}y\| \leq \left((1-\lambda)\frac{1-\rho}{\mathscr{A}\mathscr{B}(\rho)} + \lambda\frac{1-\rho}{\mathscr{C}\mathscr{F}(\rho)}\right)\|(h(\cdot,\ y_n(\cdot)) - h(\cdot,\ y(\cdot))\|$$

$$+ \frac{(1-\lambda)T^\rho}{\mathscr{A}\mathscr{B}(\rho)\Gamma(\rho)}\|(h(\cdot,\ y_n(\cdot)) - h(\cdot,\ y(\cdot))\|$$

$$+ \frac{\lambda T\rho}{\mathscr{C}\mathscr{F}(\rho)}\int_0^t \|(h(\cdot,\ y_n(\cdot)) - h(\cdot,\ y(\cdot))\|\,.$$

Since $y_n \to y$, $\|y_n - y\| \to 0$, as $n \to \infty$ and using the Lebesgue-dominated theorem, $\|\mathscr{F}y_n - \mathscr{F}y\| \to 0$ as $n \to \infty$. Thus, $\mathscr{F}$ is continuous.

Step 3: $\mathscr{F}$ is Compact.

It is necessary to demonstrate that $\mathscr{F}$ maps bounded sets to relatively compact sets to establish its compactness. The Arzelà-Ascoli theorem states that if a set of functions in $\mathscr{C}[0,\ T]$ is equicontinuous and uniformly bounded, it is relatively compact.

Let $B_R = \{y \in \mathscr{C}[0,\ T] : \|y\| \leq R\}$ be a bounded set in $\mathscr{C}[0,\ T]$. We need to show that $\mathscr{F}(B_R)$ is uniformly bounded and equicontinuous.

For any $y \in B_R$, we have $\|y\| \leq R$. Using the definition of $\mathscr{F}y$ and assumption (A3), we obtain

$$|(\mathscr{F}y)(t)| \leq |y_0| + \left|(1-\lambda)\frac{1-\rho}{\mathscr{A}\mathscr{B}(\rho)} + \lambda\frac{1-\rho}{\mathscr{C}\mathscr{F}(\rho)}\right| |h(t,\, y(t))|$$

$$+ \frac{|1-\lambda|\rho}{|\mathscr{A}\mathscr{B}(\rho)|\Gamma(\rho)} \int_0^t (t-\upsilon)^{\rho-1} |h(\upsilon,\, y(\upsilon))|\, d\upsilon$$

$$+ \frac{|\lambda|\rho}{|\mathscr{C}\mathscr{F}(\rho)|} \int_0^t |h(\upsilon,\, y(\upsilon))|\, d\upsilon$$

$$\leq |y_0| + \left|(1-\lambda)\frac{1-\rho}{\mathscr{A}\mathscr{B}(\rho)} + \lambda\frac{1-\rho}{\mathscr{C}\mathscr{F}(\rho)}\right| \|p\|_\infty \psi(R)$$

$$+ \frac{|1-\lambda|\rho}{|\mathscr{A}\mathscr{B}(\rho)|\Gamma(\rho)} \int_0^t (t-\upsilon)^{\rho-1} \|p\|_\infty \psi(R)\, d\upsilon$$

$$+ \frac{|\lambda|\rho}{|\mathscr{C}\mathscr{F}(\rho)|} \int_0^t \|p\|_\infty \psi(R)\, d\upsilon$$

$$\leq |y_0| + \|p\|_\infty \psi(R) \left[\left|(1-\lambda)\frac{1-\rho}{\mathscr{A}\mathscr{B}(\rho)} + \lambda\frac{1-\rho}{\mathscr{C}\mathscr{F}(\rho)}\right|\right.$$

$$\left. + \frac{|1-\lambda|\rho}{|\mathscr{A}\mathscr{B}(\rho)|\Gamma(\rho)} \frac{T^\rho}{\rho} + \frac{|\lambda|\rho}{|\mathscr{C}\mathscr{F}(\rho)|} T\right].$$

Let $M_R = |y_0| + \|p\|_\infty \psi(R) \left[\left|(1-\lambda)\frac{1-\rho}{\mathscr{A}\mathscr{B}(\rho)} + \lambda\frac{1-\rho}{\mathscr{C}\mathscr{F}(\rho)}\right| + \frac{|1-\lambda|T^\rho}{|\mathscr{A}\mathscr{B}(\rho)|\Gamma(\rho+1)} + \frac{|\lambda|\rho T}{|\mathscr{C}\mathscr{F}(\rho)|}\right]$. Then $|(\mathscr{F}y)(t)| \leq M_R$ for all $t \in [0,\, T]$ and for all $y \in B_R$. Thus, $\|\mathscr{F}y\| \leq M_R$, which means $\mathscr{F}(B_R)$ is uniformly bounded.

We need to show that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $y \in B_R$ and for all $t_1,\, t_2 \in [0,\, T]$ with $|t_1 - t_2| < \delta$, we have $|(\mathscr{F}y)(t_2) - (\mathscr{F}y)(t_1)| < \varepsilon$. Asumme $t_1 < t_2$. Then

$$|(\mathscr{F}y)(t_2) - (\mathscr{F}y)(t_1)| = \left|\left((1-\lambda)\frac{1-\rho}{\mathscr{A}\mathscr{B}(\rho)} + \lambda\frac{1-\rho}{\mathscr{C}\mathscr{F}(\rho)}\right)(h(t_2,\, y(t_2)) - h(t_1,\, y(t_1)))\right.$$

$$+ \frac{(1-\lambda)\rho}{\mathscr{A}\mathscr{B}(\rho)\Gamma(\rho)} \int_0^{t_2} (t_2-\upsilon)^{\rho-1} h(\upsilon,\, y(\upsilon))\, d\upsilon$$

$$- \frac{(1-\lambda)\rho}{\mathscr{A}\mathscr{B}(\rho)\Gamma(\rho)} \int_0^{t_1} (t_1-\upsilon)^{\rho-1} h(\upsilon,\, y(\upsilon))\, d\upsilon$$

$$+ \frac{\lambda\rho}{\mathscr{C}\mathscr{F}(\rho)} \int_0^{t_2} h(\upsilon,\, y(\upsilon))\, d\upsilon$$

$$- \frac{\lambda\rho}{\mathscr{C}\mathscr{F}(\rho)} \int_0^{t_1} h(\upsilon,\, y(\upsilon))\, d\upsilon \Bigg|.$$

Since $h(t,\, y(t))$ and $y(t)$ are continuous, the first term tends to 0 as $t_2 \to t_1$. As $t_2 \to t_1$, the third and fifth terms are differences of integrals of continuous functions, which likewise tend to 0.

For the integral with $(t-\upsilon)^{\rho-1}$,

$$\left| \int_0^{t_2} (t_2-\upsilon)^{\rho-1} h(\upsilon,\, y(\upsilon))\, d\upsilon - \int_0^{t_1} (t_1-\upsilon)^{\rho-1} h(\upsilon,\, y(\upsilon))\, d\upsilon \right|$$

$$\leq \left| \int_0^{t_1} [(t_2-\upsilon)^{\rho-1} - (t_1-\upsilon)^{\rho-1}] h(\upsilon,\, y(\upsilon))\, d\upsilon \right| + \left| \int_{t_1}^{t_2} (t_2-\upsilon)^{\rho-1} h(\upsilon,\, y(\upsilon))\, d\upsilon \right|$$

$$\leq \|p\|_\infty \psi(R) \left[\int_0^{t_1} |(t_2-\upsilon)^{\rho-1} - (t_1-\upsilon)^{\rho-1}|\, d\upsilon + \int_{t_1}^{t_2} (t_2-\upsilon)^{\rho-1}\, d\upsilon \right].$$

The uniform continuity of $(t-\upsilon)^{\rho-1}$ on compact sets (for $\rho > 1$) or careful analysis for $0 < \rho \leq 1$ indicate that the first integral tends to 0 as $t_2 \to t_1$. The second integral, $\frac{(t_2-t_1)^\rho}{\rho}$, likewise tends to 0 as $t_2 \to t_1$.

For the integral without $(t-\upsilon)^{\rho-1}$:

$$\left| \int_0^{t_2} h(\upsilon,\, y(\upsilon))\, d\upsilon - \int_0^{t_1} h(\upsilon,\, y(\upsilon))\, d\upsilon \right| = \left| \int_{t_1}^{t_2} h(\upsilon,\, y(\upsilon))\, d\upsilon \right| \leq \|p\|_\infty \psi(R)(t_2-t_1).$$

This also tends to 0 as $t_2 \to t_1$.

Hence, the family $\mathscr{F}(B_R)$ is equicontinuous. By the Arzelà-Ascoli theorem, it follows that $\mathscr{F}$ is compact.

Step 4: Boundedness of the Set of Fixed Points.

Let $y \in \mathscr{C}[0,\, T]$ be such that $y = \lambda \mathscr{F} y$ for some $0 < \lambda < 1$. We need to show that there exists a constant K_0 such that $\|y\| \leq K_0$.

From $y = \lambda \mathscr{F} y$, we have

$$|y(t)| = \lambda \Bigg| y_0 + \left((1-\lambda) \frac{1-\rho}{\mathscr{A}\mathscr{B}(\rho)} + \lambda \frac{1-\rho}{\mathscr{C}\mathscr{F}(\rho)} \right) h(t,\, y(t))$$

$$+ \frac{(1-\lambda)\rho}{\mathscr{A}\mathscr{B}(\rho)\Gamma(\rho)} \int_0^t (t-\upsilon)^{\rho-1} h(\upsilon,\, y(\upsilon))\, d\upsilon$$

$$+ \frac{\lambda\rho}{\mathscr{C}\mathscr{F}(\rho)} \int_0^t h(\upsilon,\, y(\upsilon))\, d\upsilon \Bigg|.$$

Assumption (A3) shows that

$$|y(t)| \leq \lambda \left[|y_0| + \left| (1-\lambda)\frac{1-\rho}{\mathscr{A}\mathscr{B}(\rho)} + \lambda \frac{1-\rho}{\mathscr{C}\mathscr{F}(\rho)} \right| \|p\|_\infty \psi(\|y\|) \right.$$

$$+ \frac{|1-\lambda|\rho}{|\mathscr{A}\mathscr{B}(\rho)|\Gamma(\rho)} \int_0^t (t-\upsilon)^{\rho-1} \|p\|_\infty \psi(\|y\|)\, d\upsilon$$

$$\left. + \frac{|\lambda|\rho}{|\mathscr{C}\mathscr{F}(\rho)|} \int_0^t \|p\|_\infty \psi(\|y\|)\, d\upsilon \right].$$

Consequently,

$$\|y\| \leq \lambda \left[|y_0| + \|p\|_\infty \psi(\|y\|) \left(\left| (1-\lambda)\frac{1-\rho}{\mathscr{A}\mathscr{B}(\rho)} + \lambda \frac{1-\rho}{\mathscr{C}\mathscr{F}(\rho)} \right| \right.\right.$$

$$\left.\left. + \frac{|1-\lambda|T^\rho}{|\mathscr{A}\mathscr{B}(\rho)|\Gamma(\rho+1)} + \frac{|\lambda|\rho T}{|\mathscr{C}\mathscr{F}(\rho)|} \right) \right].$$

Let $C_0 = \left| (1-\lambda)\frac{1-\rho}{\mathscr{A}\mathscr{B}(\rho)} + \lambda \frac{1-\rho}{\mathscr{C}\mathscr{F}(\rho)} \right| + \frac{|1-\lambda|T^\rho}{|\mathscr{A}\mathscr{B}(\rho)|\Gamma(\rho+1)} + \frac{|\lambda|\rho T}{|\mathscr{C}\mathscr{F}(\rho)|}$. Then, we have

$$\|y\| \leq \lambda \left[|y_0| + \|p\|_\infty \psi(\|y\|) C_0 \right].$$

Since $0 < \lambda < 1$, we can write

$$\|y\| < |y_0| + \|p\|_\infty \psi(\|y\|) C_0.$$

Let $K = \|y\|$. Then $K < |y_0| + \|p\|_\infty \psi(K) C_0$. This can be rewritten as:

$$K - \|p\|_\infty \psi(K) C_0 - |y_0| < 0.$$

Let $g(K) = K - \|p\|_\infty \psi(K) C_0 - |y_0|$ be considered. We are trying to find a bound K_0 such that $g(K_0) < 0$. According to assumption (A4), we have a constant $K^* > 0$ such that

$$\frac{K^*}{\|p\|_\infty \psi(K^*) \left[\left| \frac{(1-\lambda)(1-\rho)}{\mathscr{A}\mathscr{B}(\rho)} + \frac{\lambda(1-\rho)}{\mathscr{C}\mathscr{F}(\rho)} \right| + \frac{|1-\lambda|T^\rho}{|\mathscr{A}\mathscr{B}(\rho)|\Gamma(\rho+1)} + \frac{|\lambda|\rho T}{|\mathscr{C}\mathscr{F}(\rho)|} \right] + |y_0|} > 1.$$

It follows that

$$K^* - \|p\|_\infty \psi(K^*)C_0 - |y_0| > 0.$$

This contradicts our inequality $K - \|p\|_\infty \psi(K)C_0 - |y_0| < 0$ if we assume $K = K^*$. This indicates that some K_0, which is not K^*, must surround the fixed points. The inequality in (A4) suggests that if $\|y\| > K_0$, then $\|y\| > \lambda \mathscr{F} y$. This implies that there is a bound K_0. Therefore, any fixed point $y = \lambda \mathscr{F} y$ must satisfy $\|y\| \leq K_0$. This proves the set E is bounded.

All requirements of Theorem 2 are satisfied. As a result, there is at least one fixed point in $\mathscr{C}[0,\ T]$ for the operator $\mathscr{F}$. This fixed point is a solution to the problem defined by the operator. □

4.4 *UH and GUH stability*

In this part, we discuss the Ulam-Hyers (UH) and Generalized Ulam-Hyers (GUH) stability for the nonlinear IVP (17). The analysis is predicated on applying concepts associated with Theorem 1. The findings prove that approximate solutions to the equation must inevitably resemble true solutions.

Definition 6 [35] The IVP (17) is said to be UH stable if there exists a constant $C_f > 0$ such that for each $\varepsilon > 0$ and for each function $z \in \mathscr{C}[0,\ T]$ satisfying the inequality

$$|{}^{\mathrm{AF}}_{\mathrm{C}}D^{\rho}_t z(t) - h(t,\ z(t))| \leq \varepsilon, \quad t \in [0,\ T], \tag{22}$$

there exists a unique solution $y \in \mathscr{C}[0,\ T]$ of the IVP (17) with $y(0) = z(0)$ such that

$$|z(t) - y(t)| \leq C_f \varepsilon, \quad t \in [0,\ T]. \tag{23}$$

Definition 7 [35] The IVP (17) is said to be generalized UH stable if there exists a continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that for each function $z \in \mathscr{C}[0,\ T]$ satisfying the inequality (22), there exists a unique solution $y \in \mathscr{C}[0,\ T]$ of the IVP (17) with $y(0) = z(0)$ such that

$$|z(t) - y(t)| \leq \phi(\varepsilon), \quad t \in [0,\ T]. \tag{24}$$

Remark 6 The inequality (22) can equivalently be expressed by asserting the existence of a function $g(t) \in \mathscr{C}[0,\ T]$ (depending on z) such that

1. $|g(t)| \leq \varepsilon$ for all $t \in [0,\ T]$.
2. ${}^{\mathrm{AF}}_{\mathrm{C}}D^{\rho}_t z(t) = h(t,\ z(t)) + g(t)$.

To establish stability, we start with a key lemma that connects the approximate solution $z(t)$ to a corresponding integral inequality.

Lemma 4 Let $z(t)$ be a function satisfying the inequality (22). Then $z(t)$ satisfies

$$|z(t) - z_0 - \mathscr{I}^{\rho}[h(t,\ z(t))]| \leq \varepsilon \Lambda, \tag{25}$$

where $z_0 = z(0)$, $\mathscr{I}^{\rho}[h(t,\ z(t))]$ is the integral part of the operator, and Λ is a constant defined as

$$\Lambda = \left[\left|\frac{(1-\lambda)(1-\rho)}{\mathscr{A}\mathscr{B}(\rho)} + \frac{\lambda(1-\rho)}{\mathscr{C}\mathscr{F}(\rho)}\right| + \frac{|1-\lambda|T^{\rho}}{|\mathscr{A}\mathscr{B}(\rho)|\Gamma(\rho+1)} + \frac{|\lambda|\rho T}{|\mathscr{C}\mathscr{F}(\rho)|}\right].$$

Proof. From the Remark 6, and applying the AF integral operator ${}^{\mathrm{AF}}I_t^{\rho}$, we obtain

$$z(t) - z(0) - {}^{\mathrm{AF}}I_t^{\rho}[h(t,\, z(t))] = {}^{\mathrm{AF}}I_t^{\rho}[g(t)].$$

Thus

$$\begin{aligned}
|z(t) - z_0 - \mathscr{I}^{\rho}[h(t,\, z(t))]| &= |{}^{\mathrm{AF}}I_t^{\rho}[g(t)]| \\
&\leq \left|\frac{(1-\lambda)(1-\rho)}{\mathscr{A}\mathscr{B}(\rho)} + \frac{\lambda(1-\rho)}{\mathscr{C}\mathscr{F}(\rho)}\right| |g(t)| \\
&\quad + \frac{|1-\lambda|\rho}{|\mathscr{A}\mathscr{B}(\rho)|\Gamma(\rho)} \int_0^t (t-\upsilon)^{\rho-1} |g(\upsilon)|\, d\upsilon \\
&\quad + \frac{|\lambda|\rho}{|\mathscr{C}\mathscr{F}(\rho)|} \int_0^t |g(\upsilon)|\, d\upsilon \\
&\leq \varepsilon \left[\left|\frac{(1-\lambda)(1-\rho)}{\mathscr{A}\mathscr{B}(\rho)} + \frac{\lambda(1-\rho)}{\mathscr{C}\mathscr{F}(\rho)}\right| + \frac{|1-\lambda|\rho}{|\mathscr{A}\mathscr{B}(\rho)|\Gamma(\rho)} \int_0^t (t-\upsilon)^{\rho-1} d\upsilon + \frac{|\lambda|\rho}{|\mathscr{C}\mathscr{F}(\rho)|} \int_0^t d\upsilon\right] \\
&\leq \varepsilon \left[\left|\frac{(1-\lambda)(1-\rho)}{\mathscr{A}\mathscr{B}(\rho)} + \frac{\lambda(1-\rho)}{\mathscr{C}\mathscr{F}(\rho)}\right| + \frac{|1-\lambda|T^{\rho}}{|\mathscr{A}\mathscr{B}(\rho)|\Gamma(\rho+1)} + \frac{|\lambda|\rho T}{|\mathscr{C}\mathscr{F}(\rho)|}\right] \\
&= \varepsilon\Lambda.
\end{aligned}$$

□

Theorem 8 (UH Stability) Under the assumptions of Theorem 6. Then the IVP (17) is UH stable.

Proof. Based on Theorem 6, the condition $L\Lambda < 1$ ensures the existence of a unique solution $y(t)$ to the IVP (17) with $y(0) = z(0) = z_0$. That is

$$y(t) = z_0 + \mathscr{I}^{\rho}[h(t,\, y(t))]. \tag{26}$$

Hence,

$$
\begin{aligned}
|z(t)-y(t)| &= |z(t)-z_0-\mathscr{I}^{\rho}[h(t,\,y(t))]| \\
&= |z(t)-z_0-\mathscr{I}^{\rho}[h(t,\,z(t))]+\mathscr{I}^{\rho}[h(t,\,z(t))]-\mathscr{I}^{\rho}[h(t,\,y(t))]| \\
&\leq |z(t)-z_0-\mathscr{I}^{\rho}[h(t,\,z(t))]|+|\mathscr{I}^{\rho}[h(t,\,z(t))]-\mathscr{I}^{\rho}[h(t,\,y(t))]| \\
&\leq \varepsilon\Lambda+|\mathscr{I}^{\rho}[h(t,\,z(t))]-\mathscr{I}^{\rho}[h(t,\,y(t))]| \\
&\leq \varepsilon\Lambda+L\Lambda\|z-y\|_{\infty}.
\end{aligned}
$$

For all $t\in[0,\,T]$, we get

$$
\|z-y\|_{\infty}\leq \varepsilon\Lambda+L\Lambda\|z-y\|_{\infty}.
$$

Since $L\Lambda\neq 1$,

$$
\|z-y\|_{\infty}\leq \frac{\Lambda}{1-L\Lambda}\varepsilon.
$$

Let $C_f=\dfrac{\Lambda}{1-L\Lambda}$. Then

$$
\|z-y\|_{\infty}\leq C_f\varepsilon.
$$

This proves that the IVP (17) is UH stable. □

Corollary 2 (GUH Stability) Under the assumptions of Theorem 8, the IVP (17) is also GUH stable.

Proof. Set $\phi(\varepsilon)=C_f\varepsilon$. Since $\phi(0)=0$, this function is continuous. It is true that $\|z-y\|_{\infty}\leq\phi(\varepsilon)$. As such, the system is UH stable in a generalized sense. □

4.5 *Examples*

This part provides numerical examples of a nonlinear IVP (17) to verify the conditions required by the key theorems established previously.

Example 4 Consider the following nonlinear FDEs:

$$
\begin{cases}
{}^{\mathrm{AF}}_{\mathrm{C}}D_t^{0.8}y(t)=\dfrac{t}{20}\dfrac{y(t)}{1+|y(t)|}, \quad t\in[0,\,1], \\
\\
y(0)=0.5,
\end{cases}
\tag{27}
$$

where $h(t,\ y(t)) = \dfrac{t}{20}\left(\dfrac{y(t)}{1+|y(t)|}\right)$, $\rho = 0.8$, $\lambda = 0.5$, $y_0 = 0.5$, $\mathscr{AB}(\rho) = 1$ and $\mathscr{CF}(\rho) = 1$.

Clearly, the function $h(t,\ y)$ is continuous function on $[0,\ 1] \times \mathbb{R}$. For all $t \in [0,\ 1]$, and $y_1,\ y_2 \in \mathbb{R}$, we have

$$|(h(t,\ y_1(t)) - (h(t,\ y_2(t))| \leq \left|\frac{1}{20}\left(\frac{y_1(t)}{1+|y_1(t)|}\right) - \frac{1}{20}\left(\frac{y_2(t)}{1+|y_2(t)|}\right)\right|$$

$$\leq \frac{1}{20}\left|y_1(t) - y_2(t)\right|.$$

Thus, the Lipschitz constant is $L = 0.05 > 0$. Thus, (A1) and (A2) are satisfied. By the given parameters above, we conclude that the constant $\Lambda = 1.1368$. Hence, the contraction condition is $L\Lambda = 0.05684 < 1$. Since $L\Lambda < 1$, the contraction condition is satisfied.

Since (A1) and (A2) hold and $LC_0 < 1$, the uniqueness theorem (Theorem 6) applies.

Corollary 3 The IVP (27) has a unique solution on $[0,\ 1]$.

To calculate the UH constant C_f, the condition for UH stability is identical to the contraction condition. As calculated above, $L\Lambda = 0.05684 < 1$. Also, we have

$$C_f = \frac{\Lambda}{1 - L\Lambda} = \frac{1.1368}{1 - 0.05684} \approx 1.2053.$$

Since uniqueness is guaranteed, the UH stability theorem (Theorem 8) also applies.

Corollary 4 The IVP (27) is UH stable. For any approximate solution $z(t)$ such that $|{}^{\mathrm{AF}}_{\mathrm{C}}D^{0.8}z(t) - h(t,\ z(t))| \leq \varepsilon$, there exists a unique true solution $y(t)$ such that

$$|z(t) - y(t)| \leq 1.2053\,\varepsilon, \quad \forall t \in [0,\ 1].$$

To verify the conditions for the existence theorem, firstly, we check the growth condition (A3). We need to find functions $p(t)$ and $\psi(|y|)$ such that $|h(t,\ y)| \leq p(t)\psi(|y|)$.

$$|h(t,\ y)| = \left|\frac{t}{20}\frac{y}{1+|y|}\right| \leq \frac{t}{20} \cdot 1.$$

Choose $p(t) = t/20$ and $\psi(|y|) = 1$. Both are continuous, and ψ is non-decreasing. Thus, the condition is satisfied. Here, $\|p\|_\infty = 1/20 = 0.05$.

Next, we check the a priori bound condition (A4), i.e., we show there exists a constant $K > 0$ such that

$$\frac{K}{|y_0| + \|p\|_\infty \psi(K)\Lambda} > 1.$$

Since $|y_0| = 0.5$, $\|p\|_\infty = 0.05$, $\psi(K) = 1$, and $\Lambda = 1.1368$, $\implies \dfrac{K}{0.55684} > 1$. This inequality holds for any $K > 0.55684$. For instance, choosing $K = 1$ gives $1/0.55684 \approx 1.796 > 1$. The condition is satisfied. Since (A1), (A3), and (A4) hold, the existence theorem (Theorem 7) applies.

Corollary 5 Under the assumptions of Theorem 7, the IVP (27) admits at least one solution.

Example 5 Considering the problem in the previous example, we put

$$h(t,\, y) = \frac{\cos t}{1+t} + \frac{e^{-\sin^2 t}}{\sqrt{25+t}\,(2+|y|)}\,|y|,$$

$\rho = 0.7,\ \lambda = 0.3,\ T = 1,\ y_0 = 0.01$, and $\mathscr{AB}(\rho) = \mathscr{CF}(\rho) = 1$.

(A1) Continuity: $h(t,\, y)$ is continuous on $[0,\, 1] \times \mathbb{R}$, since each term involves continuous functions of t and y.

(A2) Lipschitz condition: For all $t \in [0,\, 1]$, and $y_1,\, y_2 \in \mathbb{R}$, we have

$$|h(t,\, y_1) - h(t,\, y_2)| = \left| \frac{e^{-\sin^2 t}|y_1|}{\sqrt{25+t}\,(2+|y_1|)} - \frac{e^{-\sin^2 t}|y_2|}{\sqrt{25+t}\,(2+|y_2|)} \right| \leq \frac{1}{5}|y_1 - y_2| = L|y_1 - y_2|.$$

(A3) Boundedness condition: Define $\psi(|y|) = |y| + 1$, and $p(t) = \dfrac{1}{1+t} + \dfrac{1}{\sqrt{25+t}}$. Then one checks that

$$|h(t,\, y)| \leq p(t)\,\psi(|y|), \qquad \forall (t,\, y) \in [0,\, 1] \times \mathbb{R}.$$

(A4) Fixed point bound: Using the definition of C_0 from the general theory,

$$C_0 = \left| \frac{(1-\lambda)(1-\rho)}{\mathscr{AB}(\rho)} + \frac{\lambda(1-\rho)}{\mathscr{CF}(\rho)} \right| + \frac{|1-\lambda|T^\rho}{|\mathscr{AB}(\rho)|\Gamma(\rho+1)} + \frac{|\lambda|\rho T}{|\mathscr{CF}(\rho)|}.$$

Substituting $\rho = 0.7,\ \lambda = 0.3,\ T = 1,\ \mathscr{AB}(\rho) = \mathscr{CF}(\rho) = 1$, one obtains $C_0 \approx 1.280383$. Then for a suitable constant $K > 0$, the inequality $\dfrac{K}{\|p\|_\infty \psi(K)} C_0 + |y_0| > 1$ holds, so assumption (A4) is satisfied.

4.6 *Verification of theorems*

- Since (A1) and (A2) are satisfied and $LC_0 < 1$, the uniqueness theorem (Theorem 6) applies.
- Since (A1), (A3), and (A4) are satisfied, the existence theorem (Theorem 7) applies.
- Since uniqueness holds, the UH stability theorem (Theorem 8) also applies.

5. Application on fractional drug release model

Recent advances in FC have revolutionized the modeling of drug release kinetics, with numerous studies demonstrating its superiority in capturing anomalous diffusion, memory effects, and non-Markovian processes in complex drug delivery systems [36–38]. This section presents an application of the proposed fractional approach to model the drug release process. The model describes how the concentration of a drug changes over time within a medium. Numerical results are provided to illustrate the behavior and effectiveness of the fractional model. Consider the following FDE:

$$\begin{cases} {}^{\mathrm{AF}}_{C}D^{\rho}_{t}\, y(t) = -\dfrac{\kappa y(t)}{K + y(t)}, \quad t \in [0,\, T], \\ \\ y(0) = y_0, \end{cases} \tag{28}$$

where ${}^{\mathrm{AF}}_{C}D^{\rho}_{t}$ denotes the AF-Frac Deriv of order $\rho \in (0,\ 1)$ in the Caputo sense, $y(t)$ is the concentration of drug remaining in the polymer matrix at time t, $\kappa > 0$ is the maximum release rate constant, governing the maximum speed of the drug release process, $K > 0$ is the saturation constant, representing the drug concentration at which the release rate is half of its maximum value κ, $y_0 > 0$ is the initial condition, signifying the initial drug loading concentration in the polymer matrix at time $t = 0$ and $T > 0$ is the final time defining the total duration of the studied release process.

Using the fundamental lemma (Lemma 3), the analytical solution to the problem (28) is

$$\begin{aligned} y(t) = y_0 &+ \Big((1-\lambda)\frac{1-\rho}{\mathscr{A}\mathscr{B}(\rho)} + \lambda\frac{1-\rho}{\mathscr{C}\mathscr{F}(\rho)}\Big)g(t,\ y(t)) \\ &+ \frac{(1-\lambda)\rho}{\mathscr{A}\mathscr{B}(\rho)\Gamma(\rho)}\int_0^t (t-\upsilon)^{\rho-1} g(\upsilon,\ y(\upsilon))\,d\upsilon + \frac{\lambda\rho}{\mathscr{C}\mathscr{F}(\rho)}\int_0^t g(\upsilon,\ y(\upsilon))\,d\upsilon, \end{aligned} \tag{29}$$

where $g(t,\ y(t)) = -\dfrac{\kappa y(t)}{K + y(t)}$.

Now, we use the two-step ABM approximation to solve the nonlinear fractional IVP governed by the AF operator, defined as a convex combination of the ABC and CF operators,

$${}^{\mathrm{AF}}_{C}D^{\rho}_{t}\,y(t) = (1-\lambda)\,{}^{\mathrm{ABC}}_{C}D^{\rho}_{t}\,y(t) + \lambda\,{}^{\mathrm{CF}}_{C}D^{\rho}_{t}\,y(t),$$

with nonlinear dynamics (28). To obtain a recursive formula for numerical simulation, we will discretize the integral equation (29). Then, we discretize the interval $[0,\ T]$ with uniform step h, $t_{p+1} = (p+1)h$ for $p = 0,\ 1,\ \ldots,\ N$, where $h = T/N$ is the step size, and set $y^p \approx y(t_p)$, $g_p := g(t_p,\ y^p) = g(t_p,\ y(t_p))$, where $g(t,\ y) = -\dfrac{\kappa y}{K+y}$. Based on the Newton polynomial approximation (see [39]), the two-step ABMs corresponding to the CF and ABC operators are given, respectively, by

$$y^{p+1}_{CF} = y^p + \frac{1-\rho}{\mathscr{C}\mathscr{F}(\rho)}\big[g_p - g_{p-1}\big] + \frac{\rho}{\mathscr{C}\mathscr{F}(\rho)}\left[\frac{23}{12}g_p h - \frac{4}{3}g_{p-1}h + \frac{5}{12}g_{p-2}\right],$$

and

$$\begin{aligned} y^{p+1}_{ABC} = y^p &+ \frac{1-\rho}{\mathscr{A}\mathscr{B}(\rho)}g_p \\ &+ \frac{\rho h^\rho}{\mathscr{A}\mathscr{B}(\rho)\Gamma(\rho+1)}\sum_{r=2}^{p} p_{r-2}\left[(p-r+1)^\rho - (p-r)^\rho\right] \\ &+ \frac{\rho h^\rho}{\mathscr{A}\mathscr{B}(\rho)\Gamma(\rho+2)}\sum_{r=2}^{p}\left[g_{r-1} - g_{r-2}\right]\times A_3 \\ &+ \frac{\rho h^\rho}{2\mathscr{A}\mathscr{B}(\rho)\Gamma(\rho+3)}\sum_{r=2}^{p}\left[g_r - 2g_{r-1} + g_{r-2}\right]\times A_4, \end{aligned}$$

where

$$A_3 = [(p-r+1)^{\rho}(p-r+3+2\rho-(p-r)^{\rho}(p-r+3+3\rho]$$

$$A_4 = \left[(p-r+1)^{\rho}\left(2(p-r)^2+(3\rho+10)(p-r)+2\rho^2+9\rho+12\right)\right.$$

$$\left.-(p-r)^{\rho}\left(2(p-r)^2+(5\rho+10)(p-r)+6\rho^2+18\rho+12\right)\right].$$

Hence,

$$y_{AF}^{p+1} = y^p + (1-\lambda)y_{ABC}^{p+1} + \lambda y_{CF}^{p+1}. \tag{30}$$

Thus, Equation (30) provides the numerical solution to problem (28). For the simulation, we set the parameters as $\rho = 0.8$, $\lambda = 0.5$, $\kappa = 0.8$, $K = 1$, $y_0 = 1$, $T = 8$, and $N = 200$. The corresponding results are presented in Figures 3-6.

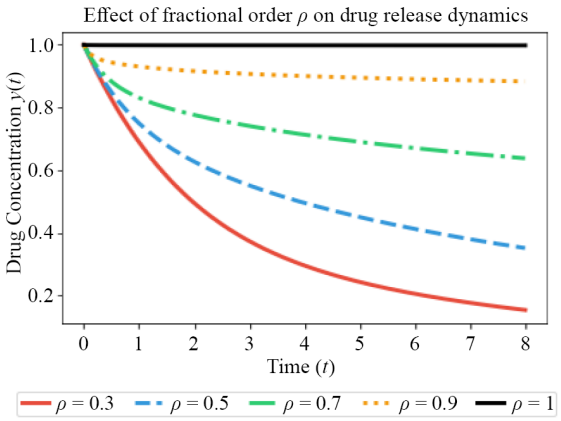


Figure 3. The effect of fractional order ρ on drug release dynamics

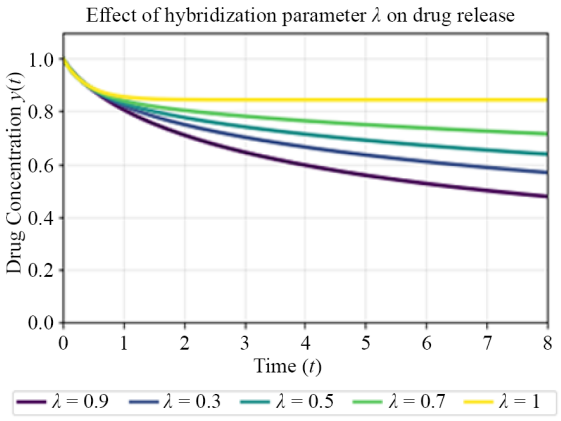


Figure 4. The hybridization parameter λ affects drug release

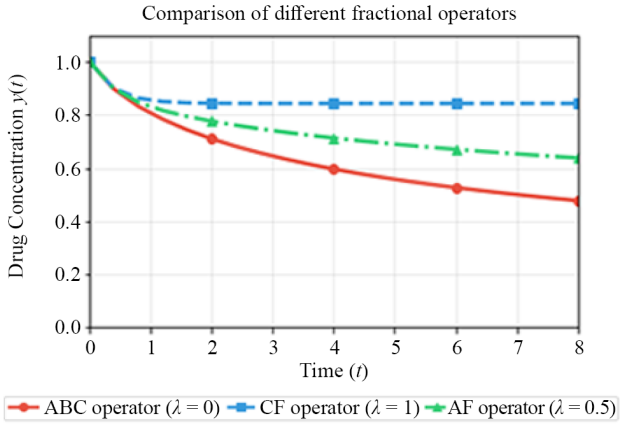


Figure 5. Compares the three fractional operators at $\rho = 0.7$

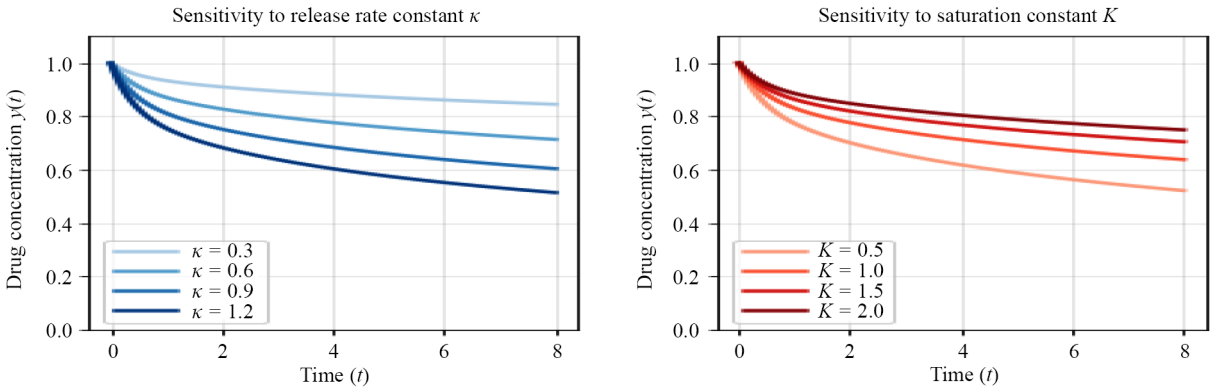


Figure 6. Sensitivity to physical parameters κ and K

Figure 3 illustrates how varying the fractional order ρ affects drug release, where smaller values ($\rho = 0.3,\ 0.5$) yield stronger memory effects and sustained release, while larger values ($\rho = 0.7,\ 0.9$) approach classical exponential decay with the classical case ($\rho \to 1$) showing the fastest release.

Figure 4 compares the ABC, CF, and AF operators at $\rho = 0.7$, where ABC gives the slowest release with the longest half-life, CF gives the fastest release with the shortest half-life, and AF provides intermediate balanced behavior.

Figure 5 demonstrates the effect of the hybridization parameter λ, where $\lambda = 0$ ABC yields the strongest memory and slowest release, $\lambda = 1$ CF gives the fastest release, and intermediate values gradually transition between the two extremes with $\lambda = 0.5$ offering balanced AF dynamics.

Figure 6 shows sensitivity to the physical parameters κ and K, where larger κ accelerates release speed, while higher K slows the initial release due to nonlinear saturation, thus jointly shaping the overall release kinetics.

6. Conclusion

In this work, we introduced and analyzed a new hybrid fractional operator, the AF operator, which seamlessly unifies the memory characteristics of the CF (exponential-decay) and ABC ML derivatives. Through the hybridization parameter λ, the operator allows smooth modulation between short-memory and long-memory behaviors, making it well-suited for modeling complex systems exhibiting mixed or non-standard decay patterns.

We have established a rigorous mathematical framework for the AF operator by deriving its Laplace transform, proving key analytical properties, and ensuring well-posedness via existence, uniqueness, and GUH stability results. Its practical relevance was demonstrated through a fractional drug-release model, where a generalized ABM numerical scheme produced reliable simulations that clearly illustrated the role of λ in shaping release kinetics, interpolating between CF-like rapid decay and ABC-like persistent memory effects.

Future research will focus on expanding the theoretical foundations of the AF operator through deeper comparisons with existing fractional operators, as well as developing additional numerical schemes, broadening real-world applications, and exploring experimental approaches for estimating optimal values of λ.

Author contributions

Jabr Aljedani: Conceptualization, Writing-review & editing, prepared figures, investigation, supervision. Mohammed S. Abdo: Writing-original draft preparation, methodology, formal analysis, software, investigation. All authors reviewed the manuscript.

Conflict of interest

The authors declare that they have no competing interests.

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