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Research Article

A Conformable Fractional Composition Operator and Its Boundedness with Applications

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Abstract: It is well known that there is a strong connection between operator theory and complex analysis. The concept of conformable operators is also employed in fractional calculus to define fractional differential and integral operators. In this study, we develop conformable composition operators in a complex domain acting on spaces of analytic functions. The boundedness of weighted composition operators has been extensively studied on various analytic function spaces. The present work investigates the boundedness of two weighted operators with several variable differences on different analytic function spaces, particularly those involving Banach spaces of analytic functions. Our approach is based on the use of Möbius transformations and Green's functions defined in terms of the pseudo-hyperbolic distance.

***Keywords*:** operators in complex plan, fractional operators, Möbius transformation, conformable operator, weighted composition operators

MSC: 30C35, 30C45, 30C99

1. Introduction

Mathematical functions known as composition operators combine two or more functions to create a single new function. Composition is a basic process in mathematics, especially in relation to functions and rotations [1–3]. The composition of two functions is a new function that is produced through converting one function to the outcome and then the other way around. Calculus and Linear Algebra are two areas of mathematics that heavily rely on composition operators [4–6]. By mixing smaller transformations, they offer a means of representing more complicated changes. Furthermore, function composition is a potent idea in Computer Science and Programming that is applied in Functional Programming Languages [7–9]. In order to represent and analyze mathematical associations and transformations between different mathematical structures, composition operators are essential [10–14].

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The concept of conformable operators is used in the field of fractional calculus to define fractional differential and integral operators, while these operators are not likely of the usual fractional operators. For the first time this class was defined by Abdeljawad [15]. Later, Anderson and Ulness [16] presented a new types of conformable operators which are extended to a complex domain using the class of normalized analytic functions defined by Ibrahim and Jahangiri [17]. In terms of a quantum calculus, Ibrahim and Baleanu [18] introduced a quantum conformable operator.

Classical conformable operators act pointwise on a single function and typically introduce fractionality by weighting the first derivative, a mechanism that breaks holomorphy, destroys subordination, and is incompatible with geometric subclasses such as starlike, convex, or Bazilevic families. In contrast, many contemporary problems in geometric function theory, fractional modeling, and multidimensional signal analysis require operators that deform several interacting variables simultaneously while preserving analyticity and the underlying geometric structure. Introducing fractionality through a holomorphic deformation of the independent variables, rather than through local derivative weights, provides a stable framework in which multidimensional chain rules, boundedness on Hardy-type spaces, and analytic invariance are retained. Moreover, practical models including anisotropic diffusion, multidimensional memory effects, and quantum-deformed mappings depend on variable-dependent fractional behavior that cannot be captured by traditional conformable derivatives. These theoretical and applied challenges collectively motivate the development of a multi-variable conformable composition operator capable of encoding fractional deformation in a way that is geometrically meaningful, analytically consistent, and adaptable to complex multidimensional systems.

We address conformable composition operators of analytic function spaces in the open unit disk. We examine the boundedness of the weighted composition of these operators. We examine the two many variable differences weighted composition operators with two dimensional parametric framework and their boundedness on multi smooth function spaces. The paper is devoted as follows: the knowledge and facts that we will utilize for the successor is presented in Section 2. Section 3 gives the meaning for the recommended operator, and the outcomes of boundedness are shown in the final part of this section. The last Section 4 concludes our findings and recommendations along with potential directions for further research.

2. Preliminaries

We have the following concepts, assuming that:
(i) $\mathscr{O} := \{\xi \in \mathbb{C}: |\xi| < 1\}$ presents the open unit disk;
(ii) $\mathscr{H}_{\mathscr{O}}$ is the space of all holomorphic functions on $\mathscr{O}$;
(iii) $\mathscr{B}_{\mathscr{O}}$ is the Banach space of bounded analytic functions with the sup. norm;
(iv) $L_{\varsigma,\rho}: \mathscr{H}_{\mathscr{O}} \to \mathscr{H}_{\mathscr{O}}$ is a linear map satisfying

$$(P_{\varsigma,\rho}g)(\xi) = \rho(\xi)(g \circ \varsigma)(\xi),$$

where $g \in \mathscr{H}_{\mathscr{O}}$, $\rho: \mathscr{O} \to \mathbb{C}$ is analytic in $\mathscr{O}$ and $\varsigma: \mathscr{O} \to \mathscr{O}$ is a *Schwarz self mapping function* on $\mathscr{O}$ $\big($i.e. $\varsigma(\mathscr{O}) \subset \mathscr{O}$, $\varsigma(0) = 0\big)$. This class of smooth functions is denoted by $\Sigma_{\mathscr{O}}$. The function $P_{\varsigma,\rho}g$ is known as *the weighted composition operator*. Obviously,

$$(P_{\varsigma,\rho}g)(\xi) = \Pi_\rho \Omega_\varsigma(g), \quad \Pi_\rho = \rho \cdot g, \quad \Omega_\varsigma = g \circ \varsigma.$$

In this paper, we will use the following functional. For $\zeta \in \mathscr{O}$ the *Möbius transformation* is defined by

$$\Lambda_\zeta(\xi) = \frac{\xi - \zeta}{1 - \overline{\zeta}\xi}, \quad \xi \in \mathscr{O}.$$

For $\zeta \in \mathscr{O}$ the *Green function* is defined by

$$G_\zeta(\xi) = \log\left(\frac{1}{|\Lambda_\xi(\xi)|}\right),$$

where $|\Lambda_\zeta(\xi)|$ is the *pseudohyperbolic distance*. Also, let us define the difference transform

$$\Upsilon(g(\xi), h(\xi)) = \frac{g(\xi) - h(\xi)}{1 - \overline{g(\xi)}h(\xi)}, \quad \xi \in \mathscr{O}.$$

In the study of analytic function spaces, including the Hardy and Bergman spaces, this distinction $(\Upsilon(g(\xi), h(\xi)))$ is a frequently employed metric, particularly in the unit disk.

In this article, we deal with the following special spaces on $\mathscr{H}(\mathscr{O})$:

(1) For a non-negative real number, i.e. $\eta \geq 0$, the *Bers-type space* is defined by

$$\mathscr{B}_\eta := \left\{\psi \in \mathscr{H}_\mathscr{O} : \sup_{\xi \in \mathscr{O}} \left(\left(1 - |\xi|^2\right)^\eta |\psi|\right) < +\infty\right\},$$

therefore, the function $\psi \in \mathscr{B}_\eta$ whenever it have the following norm

$$\|\psi\|_{\mathscr{B}_\eta} = \sup_{\xi \in \mathscr{O}} \left(\left(1 - |\xi|^2\right)^\eta |\psi|\right).$$

Clearly, $\mathscr{B}_\eta$ represents a *Banach space* as we can see from [11].

(2) The space of integrable analytic functions over $\mathscr{O}$ is given by [12]

$$\mathscr{J}_K := \left\{\psi \in \mathscr{H}_\mathscr{O} : \sup_{\xi \in \mathscr{O}} \left(\int_\mathscr{O} |\psi|^2 K\left(\phi(\xi, \zeta)\right) dA(\xi)\right) < +\infty\right\},$$

where K: $[0, \infty) \to (0, \infty)$ is a continuous and nondecreasing function, ϕ is an analytic function in $\mathscr{O}$, and "dA" denotes the *Lebesgue area measure*. Thus, $\psi \in \mathscr{J}_K$ whenever the next norm is achieved

$$\|\psi\|^2_{\mathscr{J}_K} = \sup_{\xi \in \mathscr{O}} \left(\int_\mathscr{O} |\psi|^2 K(\phi(\xi, \zeta))\, dA(\xi)\right),$$

and obviously $\mathscr{J}_K \subset \mathscr{B}_1$ (see [6]).

We are convinced every single one of the product spaces meets the "sup" norm. Let consider the following analytic functions ς_n, σ_n: $\mathscr{O} \to \mathscr{O}$, and ρ_n, ρ_n: $\mathscr{O} \to \mathbb{C}$, $n \in \mathbb{N}$. If we define the following Conformable Composition Operator (CCO)

$$\mathbb{P}^{\alpha_n,\beta_n}_{\rho_n,\rho_n}(g) = \frac{\alpha_n}{\alpha_n+\beta_n}\rho_n\Omega_{\sigma_n} + \frac{\beta_n}{\alpha_n+\beta_n}\rho_n\Omega_{\varsigma_n}$$

where $\Omega_\sigma := g \circ \sigma = g(\sigma)$.

Let $\mathfrak{U} = [0, 1]$ and

$$\varsigma = (\varsigma_1, \ldots, \varsigma_n) \in [\Sigma_{\mathscr{O}}]^n, \qquad \sigma = (\sigma_1, \ldots, \sigma_n) \in [\Sigma_{\mathscr{O}}]^n;$$

$$\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathfrak{U}^n, \qquad \beta = (\beta_1, \ldots, \beta_n) \in \mathfrak{U}^n;$$

$$\rho = (\rho_1, \ldots, \rho_n) \in [\mathscr{H}_{\mathscr{O}}]^n, \qquad \rho = (\rho_1, \ldots, \rho_n) \in [\mathscr{H}_{\mathscr{O}}]^n,$$

and

$$K = (K_1, \ldots, K_n) \in \mathbb{R}^n_+, \quad \eta = (\eta_1, \ldots, \eta_n) \in \mathbb{R}^n_+,$$

then the multiplication operator

$$\mathbf{P}^{\alpha,\beta}_{\rho,\rho} := \left(\mathbb{P}^{\alpha_1,\beta_1}_{\rho_1,\rho_1}, \ldots, \mathbb{P}^{\alpha_n,\beta_n}_{\rho_n,\rho_n}\right) : \prod_{j=1}^{n}\left[\mathscr{J}_{K_j}\right] \to \prod_{j=1}^{n}\left[\mathscr{B}_{\eta_j}\right]$$

becomes

$$\begin{aligned}\mathbf{P}^{\alpha,\beta}_{\rho,\rho}(g) &= \left(\mathbb{P}^{\alpha_1,\beta_1}_{\rho_1,\rho_1}(g_1), \ldots, \mathbb{P}^{\alpha_n,\beta_n}_{\rho_n,\rho_n}(g_n)\right) \qquad (1)\\ &= \left(\frac{\alpha_1}{\alpha_1+\beta_1}\rho_1\Omega_{\sigma_1} + \frac{\beta_1}{\alpha_1+\beta_1}\rho_1\Omega_{\varsigma_1}, \ldots, \frac{\alpha_n}{\alpha_n+\beta_n}\rho_n\Omega_{\sigma_n} + \frac{\beta_n}{\alpha_n+\beta_n}\rho_n\Omega_{\varsigma_n}\right)\\ &= \left(\frac{\alpha_1}{\alpha_1+\beta_1}\rho_1 g_1(\sigma_1) + \frac{\beta_1}{\alpha_1+\beta_1}\rho_1 g_1(\varsigma_1), \ldots, \frac{\alpha_n}{\alpha_n+\beta_n}\rho_n g_n(\sigma_n) + \frac{\beta_n}{\alpha_n+\beta_n}\rho_n g_n(\varsigma_n)\right),\end{aligned}$$

where g_j is not a constant analytic function for all $j = 1, \ldots, n$, and the product spaces is given by the norm

$$\left\|\mathbf{P}^{\alpha,\beta}_{\rho,\rho}(g)\right\|_{\mathscr{B}=\prod_{j=1}^{n}\left[\mathscr{B}_{\eta_j}\right]} = \sum_{j=1}^{n}\left\|\mathbb{P}^{\alpha_j,\beta_j}_{\rho_j,\rho_j}(g_j)\right\|_{\mathscr{B}_{\eta_j}}.$$

Physical and geometric meaning. The operator $\mathbf{P}^{\alpha,\beta}_{\rho,\rho}$ performs a fractional blending of two analytic deformation paths. Physically, the coefficients $\alpha_j/(\alpha_j+\beta_j)$ and $\beta_j/(\alpha_j+\beta_j)$ represent fractional mixing parameters, similar to proportional derivative control weights, while the analytic functions ρ_j and ρ_j act as position dependent gain functions describing inhomogeneous media, damping, or amplification. The maps σ_j and ς_j correspond to two analytic geometric deformations of the unit disk, so that $\mathbb{P}^{\alpha_j,\beta_j}_{\rho_j,\rho_j}(g_j)$ represents a fractional convex combination of two weighted composition operators. Geometrically, the operator interpolates between two conformal self-maps of the disk, governed by the pseudohyperbolic metric, and thus encodes a hyperbolically consistent deformation of analytic functions. In the special case $\sigma_j=\varsigma_j=\xi$, the operator reduces to the well-known conformable fractional derivative, showing that $\mathbf{P}^{\alpha,\beta}_{\rho,\rho}$ generalizes both fractional memory effects and conformal geometric deformations.

2.1 *Special cases*

We have the following special cases of the operator (1) [17, 19].

Proposition 1 [17] Let $g_n\in\mathscr{H}_{\mathscr{O}}$ be a non-constant analytic function and let $\xi_n\in\mathscr{O}$ be such that $g_n(\xi_n)\neq 0$. Consider the operator $\mathbb{P}^{\alpha_n,\beta_n}_{\rho_n,\rho_n}$ defined by

$$\mathbb{P}^{\alpha_n,\beta_n}_{\rho_n,\rho_n}(g_n)=\frac{\alpha_n}{\alpha_n+\beta_n}\rho_n\,\Omega_{\sigma_n}(g_n)+\frac{\beta_n}{\alpha_n+\beta_n}\rho_n\,\Omega_{\varsigma_n}(g_n),$$

where $\Omega_{\sigma_n}(g_n)=g_n\circ\sigma_n$ and $\Omega_{\varsigma_n}(g_n)=g_n\circ\varsigma_n$. If we choose

$$\sigma_n(\xi_n)=\varsigma_n(\xi_n)=\xi_n,\qquad \rho_n(\xi_n)=1,\qquad \rho_n(\xi_n)=\frac{\xi_n g_n'(\xi_n)}{g_n(\xi_n)},$$

then, for every $\xi_n\in\mathscr{O}$ with $g_n(\xi_n)\neq 0$, we have

$$\mathbb{P}^{\alpha_n,\beta_n}_{1,\frac{\xi_n g_n'(\xi_n)}{g_n(\xi_n)}}(g_n)(\xi_n)=\frac{\alpha_n}{\alpha_n+\beta_n}g_n(\xi_n)+\frac{\beta_n}{\alpha_n+\beta_n}\xi_n g_n'(\xi_n). \tag{2}$$

In particular, (2) coincides with the conformable fractional combination of g_n and its first derivative at the point ξ_n.

Proof. By definition of the weighted composition operator, for each fixed $\xi_n\in\mathscr{O}$ we have

$$\big(\mathbb{P}^{\alpha_n,\beta_n}_{\rho_n,\rho_n}(g_n)\big)(\xi_n)=\frac{\alpha_n}{\alpha_n+\beta_n}\rho_n(\xi_n)\big(\Omega_{\sigma_n}(g_n)\big)(\xi_n)+\frac{\beta_n}{\alpha_n+\beta_n}\rho_n(\xi_n)\big(\Omega_{\varsigma_n}(g_n)\big)(\xi_n).$$

By the definition of the composition operators,

$$\big(\Omega_{\sigma_n}(g_n)\big)(\xi_n)=g_n\big(\sigma_n(\xi_n)\big),\qquad \big(\Omega_{\varsigma_n}(g_n)\big)(\xi_n)=g_n\big(\varsigma_n(\xi_n)\big).$$

Using the special choices

$$\sigma_n(\xi_n)=\varsigma_n(\xi_n)=\xi_n,\qquad \rho_n(\xi_n)=1,\qquad \rho_n(\xi_n)=\frac{\xi_n g_n'(\xi_n)}{g_n(\xi_n)},$$

we obtain

$$\left(\mathbb{P}^{\alpha_n,\beta_n}_{1,\frac{\xi_n g_n'(\xi_n)}{g_n(\xi_n)}}(g_n)\right)(\xi_n) = \frac{\alpha_n}{\alpha_n+\beta_n}\cdot 1\cdot g_n(\sigma_n(\xi_n)) + \frac{\beta_n}{\alpha_n+\beta_n}\cdot\frac{\xi_n g_n'(\xi_n)}{g_n(\xi_n)}\cdot g_n(\varsigma_n(\xi_n))$$

$$= \frac{\alpha_n}{\alpha_n+\beta_n}\, g_n(\xi_n) + \frac{\beta_n}{\alpha_n+\beta_n}\,\frac{\xi_n g_n'(\xi_n)}{g_n(\xi_n)}\, g_n(\xi_n)$$

$$= \frac{\alpha_n}{\alpha_n+\beta_n}\, g_n(\xi_n) + \frac{\beta_n}{\alpha_n+\beta_n}\,\xi_n g_n'(\xi_n).$$

This is exactly the desired identity (2). The assumption $g_n(\xi_n)\neq 0$ is needed only to ensure that $\rho_n(\xi_n) = \xi_n g_n'(\xi_n)/g_n(\xi_n)$ is well-defined. □

The function g: $\mathscr{O}\to\mathscr{O}$ of the following power series

$$g(\xi) = \xi + \sum_{m=2}^{\infty} a_m \xi^m, \quad \xi\in\mathscr{O}$$

is called the normalized class $\big(g(0)=0 \text{ and } g'(0)=1\big)$ and it is denoted by $\mathscr{A}$. The subclass $\mathscr{S}\subset\mathscr{A}$ ($\mathscr{S}$ consists on injective functions) plays a role in the theory of univalent functions. Numerous fields of mathematics and physics, such as the study of conformal mappings and their applications in fluid dynamics, heat conduction, and electrostatics difficulties, are addressed by univalent function theory. In addition, this theory is relevant to the geometry of Riemann surfaces and Teichmuller spaces (see [20]). The *Conformalization Theorem* is a fundamental finding related to Teichmuller spaces. It asserts that any Riemann surface may be conformally represented by one of three types of surfaces: the unit disk, the plane, or the sphere. The moduli space of Riemann surfaces is crucial to Algebraic Geometry and the theory of algebraic curves, and it may be well understand by using the Teichmuller spaces.

The operator (1) contains the conformable formula to be handled in the computations. A Proportional-Derivative (PD) controller with two tuning parameters for controller output υ at time τ has the expression

$$\upsilon(\tau) = \alpha_{\wp}\Xi(\tau) + \beta_{\flat}\Xi'(\tau),$$

where $\wp$ is the proportional gain, $\flat$ is the derivative gain, and Ξ is the error between the present state variable and the process variable (see, for example, [21]). This is the driving force behind the preceding formula.

Moreover, we have the following special case.

Proposition 2 (Symmetric conformable operator) [19] Let $g_n\in\mathscr{H}_{\mathscr{O}}$ be analytic and normalized, and let $\mathbb{P}^{\alpha_n,\beta_n}_{\rho_n,\rho_n}$ be as above. Assume

$$\sigma_n(\xi_n) = \varsigma_n(\xi_n) = \xi_n, \qquad \rho_n(\xi_n) = \frac{\xi_n g_n'(\xi_n)}{g_n(\xi_n)}, \qquad \rho_n(\xi_n) = -\frac{\xi_n g_n'(-\xi_n)}{g_n(\xi_n)},$$

for all $\xi_n\in\mathscr{O}$ such that $g_n(\xi_n)\neq 0$. Then the associated symmetric conformable operator satisfies

$$\mathbb{P}^{\alpha_n,\beta_n}_{\frac{\xi_n g_n'(\xi_n)}{g_n(\xi_n)},-\frac{\xi_n g_n'(-\xi_n)}{g_n(\xi_n)}}(g_n)(\xi_n)=\frac{\alpha_n}{\alpha_n+\beta_n}\,\xi_n g_n'(\xi_n)-\frac{\beta_n}{\alpha_n+\beta_n}\,\xi_n g_n'(-\xi_n), \tag{3}$$

for all $\xi_n \in \mathscr{O}$ with $g_n(\xi_n)\neq 0$.

Proof. Using the definition of $\mathbb{P}^{\alpha_n,\beta_n}_{\rho_n,\rho_n}$ and the given choices of σ_n, ς_n, ρ_n and ρ_n, we compute

$$\begin{aligned}\mathbb{P}^{\alpha_n,\beta_n}_{\frac{\xi_n g_n'(\xi_n)}{g_n(\xi_n)},-\frac{\xi_n g_n'(-\xi_n)}{g_n(\xi_n)}}(g_n)(\xi_n)=&\frac{\alpha_n}{\alpha_n+\beta_n}\,\frac{\xi_n g_n'(\xi_n)}{g_n(\xi_n)}\,g_n\big(\sigma_n(\xi_n)\big)+\frac{\beta_n}{\alpha_n+\beta_n}\left(-\frac{\xi_n g_n'(-\xi_n)}{g_n(\xi_n)}\right)g_n\big(\varsigma_n(\xi_n)\big)\\ =&\frac{\alpha_n}{\alpha_n+\beta_n}\,\frac{\xi_n g_n'(\xi_n)}{g_n(\xi_n)}\,g_n(\xi_n)-\frac{\beta_n}{\alpha_n+\beta_n}\,\frac{\xi_n g_n'(-\xi_n)}{g_n(\xi_n)}\,g_n(\xi_n),\end{aligned}$$

because again $\sigma_n(\xi_n)=\varsigma_n(\xi_n)=\xi_n$. Since $g_n(\xi_n)\neq 0$, we can cancel $g_n(\xi_n)$ in both terms and obtain

$$\mathbb{P}^{\alpha_n,\beta_n}_{\frac{\xi_n g_n'(\xi_n)}{g_n(\xi_n)},-\frac{\xi_n g_n'(-\xi_n)}{g_n(\xi_n)}}(g_n)(\xi_n)=\frac{\alpha_n}{\alpha_n+\beta_n}\,\xi_n g_n'(\xi_n)-\frac{\beta_n}{\alpha_n+\beta_n}\,\xi_n g_n'(-\xi_n),$$

which is exactly (3). This proves the claim. □

It refers to the notion of harmonic and appealing proportion, and to the equilibrium as a symmetric operator. In mathematics, it refers to an object that is invariant when subjected to an expected transformation, rolling or growing. In geometry, an element possesses symmetry if an operator or transformation maps it onto itself.

As an example of the operators (2) and (3), if we select $\xi/(1-\xi)$ and $\xi/(1-\xi)^2$ with $\alpha_n=\beta_n=1/2$, $n\in\mathbb{N}$, then we obtain in the first case that

$$\begin{aligned}\mathbb{P}^{\alpha_n,\beta_n}_{\rho_n,\rho_n}\left(\frac{\xi}{1-\xi}\right)=&\frac{\alpha_n}{\alpha_n+\beta_n}\,\frac{\xi}{1-\xi}+\frac{\beta_n}{\alpha_n+\beta_n}\,\xi\left(\frac{\xi}{1-\xi}\right)'=\frac{2\xi-\xi^2}{2\xi^2-4\xi+2}\\ =&\xi+\frac{3\xi^2}{2}+2\xi^3+\frac{5\xi^4}{2}+3\xi^5+O\left(\xi^6\right)\in\mathscr{A},\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}^{\alpha_n,\beta_n}_{\rho_n,\rho_n}\left(\frac{\xi}{(1-\xi)^2}\right)=&\frac{\alpha_n}{\alpha_n+\beta_n}\,\frac{\xi}{(1-\xi)^2}+\frac{\beta_n}{\alpha_n+\beta_n}\,\xi\left(\frac{\xi}{(1-\xi)^2}\right)'\\ =&-\frac{\xi}{(\xi-1)^3}=\xi+3\xi^2+6\xi^3+10\xi^4+15\xi^5+O(\xi^6)\in\mathscr{A}.\end{aligned}$$

As more examples for the symmetric conformable operator defined by (3) we have

$$\mathbb{P}_{\rho_n,\rho_n}^{\alpha_n,\beta_n}\left(\frac{\xi}{1-\xi}\right)=\frac{\alpha_n}{\alpha_n+\beta_n}\,\xi\left(\frac{\xi}{1-\xi}\right)'-\frac{\beta_n}{\alpha_n+\beta_n}\,\xi\left(\frac{-\xi}{1+\xi}\right)'$$

$$=\frac{\xi^3+\xi}{\xi^4-2\xi^2+1}=\xi+3\xi^3+5\xi^5+7\xi^7+O\left(\xi^9\right)\in\mathscr{A},$$

and

$$\mathbb{P}_{\rho_n,\rho_n}^{\alpha_n,\beta_n}\left(\frac{\xi}{(1-\xi)^2}\right)=\frac{\alpha_n}{\alpha_n+\beta_n}\,\xi\left(\frac{\xi}{(1-\xi)^2}\right)'-\frac{\beta_n}{\alpha_n+\beta_n}\,\xi\left(\frac{-\xi}{(1+\xi)^2}\right)'$$

$$=\frac{-\xi^5-6\xi^3-\xi}{\xi^6-3\xi^4+3\xi^2-1}=\xi+9\xi^3+25\xi^5+49\xi^7+O\left(\xi^9\right)\in\mathscr{A}.$$

In the next sections we will deal with the boundedness properties.

3. Boundedness property

In Functional Analysis, the bounded operators are helpful, particularly when examining the linear transformations across the normed vector spaces. They are significant in many mathematical domains, including Operator Theory, Functional Analysis, and Linear Algebra, and have a lot of appealing qualities. If a complex operator satisfies the boundedness condition for vectors in the complex vector space it acts on, it can be referred to as bounded. We argue about and find various features of the proposed operator $\mathbf{P}_{\rho,\rho}^{\alpha,\beta}(g)$, including its boundedness.

First, we will discuss two cases when the operator $\mathbf{P}_{\rho,\rho}^{\alpha,\beta}$ is bounded in $\prod_{j=1}^{n}\left[\mathscr{B}_{\eta_j}\right]$.

Theorem 1 Let $g\in\prod_{j=1}^{n}\left[\mathscr{J}_{K_j}\right]$. If $\alpha_j\geq\beta_j$ for all $j=1,\ldots,n$, and

$$\text{(A1)}\quad \sup_{1\leq j\leq n}\frac{\alpha_j}{\alpha_j+\beta_j}\left(\sup_{\xi\in\mathscr{O}}\left(\left(1+|\rho_j||\rho_j|\right)\left(1-|\xi|^2\right)^{\eta_j}\left|\Upsilon\left(\rho_j(\xi);-\rho_j(\xi)\right)\right|\right)\right)<+\infty,$$

where

$$\Upsilon(\varphi(\xi);-\psi(\xi)):=\frac{\varphi(\xi)+\psi(\xi)}{1+\overline{\varphi(\xi)}\psi(\xi)},\quad \xi\in\mathscr{O},\tag{4}$$

then $\mathbf{P}_{\rho,\rho}^{\alpha,\beta}$ is bounded in $\prod_{j=1}^{n}\left[\mathscr{B}_{\eta_j}\right]$.

Proof. Letting $g\in\prod_{j=1}^{n}\left[\mathscr{J}_{K_j}\right]$ and $\mathbf{P}_{\rho,\rho}^{\alpha,\beta}(g)$ be defined by (1), then the operator $\mathbf{P}_{\rho,\rho}^{\alpha,\beta}\colon\prod_{j=1}^{n}\left[\mathscr{J}_{K_j}\right]\to\prod_{j=1}^{n}\left[\mathscr{B}_{\eta_j}\right]$ achieves the next inequality

$$\|\mathbf{P}_{\rho,\rho}^{\alpha,\beta}\|_{\mathscr{B}=\prod_{j=1}^{n}[\mathscr{B}_{\kappa_j}]}=\sum_{j=1}^{n}\|\mathbb{P}_{\rho_j,\rho_j}^{\alpha_j,\beta_j}(g_j)\|_{\mathscr{B}_{\eta_j}}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \left(\sup_{\xi \in \mathscr{O}} \left(1-|\xi|^2\right)^{\eta_j} \left| \frac{\alpha_j}{\alpha_j+\beta_j} \rho_j \Omega_{\sigma_j} + \frac{\beta_j}{\alpha_j+\beta_j} \rho_j \Omega_{\varsigma_j} \right| \right) \\
&= \sum_{j=1}^{n} \left(\sup_{\xi \in \mathscr{O}} \left(1-|\xi|^2\right)^{\eta_j} \left| \frac{\alpha_j}{\alpha_j+\beta_j} \rho_j g_j(\sigma_j) + \frac{\beta_j}{\alpha_j+\beta_j} \rho_j g_j(\varsigma_j) \right| \right) \\
&\leq \sum_{j=1}^{n} \left(\sup_{\xi \in \mathscr{O}} \left(1-|\xi|^2\right)^{\eta_j} \left| \frac{\alpha_j}{\alpha_j+\beta_j} \rho_j g_j(\sigma_j) + \frac{\alpha_j}{\alpha_j+\beta_j} \rho_j g_j(\varsigma_j) \right| \right) \\
&\leq \sum_{j=1}^{n} \frac{\alpha_j}{\alpha_j+\beta_j} \left(\sup_{\xi \in \mathscr{O}} \left(1-|\xi|^2\right)^{\eta_j} \left| \rho_j g_j(\sigma_j) + \rho_j g_j(\varsigma_j) \right| \right) \\
&\leq \sum_{j=1}^{n} \frac{\alpha_j}{\alpha_j+\beta_j} \|g_j\|_{\mathscr{J}_{K_j}} \left(\sup_{\xi \in \mathscr{O}} \left(1-|\xi|^2\right)^{\eta_j} \left| \rho_j + \rho_j \right| \right) \\
&\leq \|g\|_{\prod_{j=1}^{n}\left[\mathscr{J}_{K_j}\right]} \sum_{j=1}^{n} \frac{\alpha_j}{\alpha_j+\beta_j} \left(\sup_{\xi \in \mathscr{O}} \left(1-|\xi|^2\right)^{\eta_j} |1+\overline{\rho_j}\rho_j| \left| \frac{\rho_j+\rho_j}{1+\overline{\rho_j}\rho_j} \right| \right) \\
&\leq \|g\|_{\prod_{j=1}^{n}\left[\mathscr{J}_{K_j}\right]} \left(\sum_{j=1}^{n} \frac{\alpha_j}{\alpha_j+\beta_j} \left(\sup_{\xi \in \mathscr{O}} \left(1-|\xi|^2\right)^{\eta_j} \left| \Upsilon(\rho_j, -\rho_j) \right| \right) \right. \\
&\quad \left. + \sum_{j=1}^{n} \frac{\alpha_j}{\alpha_j+\beta_j} \left(\sup_{\xi \in \mathscr{O}} \left(1-|\xi|^2\right)^{\eta_j} |\rho_j||\rho_j| \left| \Upsilon(\rho_j, -\rho_j) \right| \right) \right) \\
&= \|g\|_{\prod_{j=1}^{n}\left[\mathscr{J}_{K_j}\right]} \sup_{1\leq j\leq n} \frac{\alpha_j}{\alpha_j+\beta_j} \left(\sup_{\xi \in \mathscr{O}} \left(\left(1+|\rho_j||\rho_j|\right) \left(1-|\xi|^2\right)^{\eta_j} \left| \Upsilon\left(\rho_j(\xi); -\rho_j(\xi)\right) \right| \right) \right).
\end{aligned}
$$

From the assumption (A1), by letting

$$
\mathscr{M} := \sup_{1\leq j\leq n} \frac{\alpha_j}{\alpha_j+\beta_j} \left(\sup_{\xi \in \mathscr{O}} \left(\left(1+|\rho_j||\rho_j|\right) \left(1-|\xi|^2\right)^{\eta_j} \left| \Upsilon\left(\rho_j(\xi); -\rho_j(\xi)\right) \right| \right) \right) < +\infty,
$$

it follows

$$
\|\mathbf{P}_{\rho,\rho}^{\alpha,\beta}(g)\|_{\mathscr{B}=\prod_{j=1}^{n}\left[\mathscr{B}_{\eta_j}\right]} = \sum_{j=1}^{n} \|\mathbb{P}_{\rho_j,\rho_j}^{\alpha_j,\beta_j}(g_j)\|_{\mathscr{B}_{\eta_j}} \leq \mathscr{M} \|g\|_{\prod_{j=1}^{n}\left[\mathscr{J}_{K_j}\right]},
$$

where $\mathscr{M}$ is a positive constant. Hence, $\mathbf{P}_{\rho,\rho}^{\alpha,\beta}: \prod_{j=1}^{n}\left[\mathscr{J}_{K_j}\right] \to \prod_{j=1}^{n}\left[\mathscr{B}_{\eta_j}\right]$ is bounded. □

In this place, we note that when $n=1,$ we obtain the conformable fractional operator given in [17].

Example 1 1. If we consider the following parameters $\alpha=\beta=1/2$, $\rho(\xi)=\rho(\xi)=1$, $\varsigma(\xi)=-\xi$, $\sigma(\xi)=\xi$ and $\eta=1/2$ for all $j=1,\ldots,n$, then a straight calculation implies

$$\frac{1}{2}\sup_{1\leq j\leq n}\left(\sup_{\xi\in\mathscr{O}}\left(\left(1+|\rho_j||\rho_j|\right)\left(1-|\xi|^2\right)^{1/2}\left|\Upsilon\left(\rho_j(\xi);-\rho_j(\xi)\right)\right|\right)\right)=1<+\infty,$$

where $\left|\Upsilon\left(\rho_j(\xi);-\rho_j(\xi)\right)\right|=1.$ Thus, in view of Theorem 1, we conclude that the operator $\mathbf{P}_{1,1}^{0.5,0.5}\colon\prod_{j=1}^{n}\left[\mathscr{J}_{K_j}\right]\to\prod_{j=1}^{n}\left[\mathscr{B}_{0.5_j}\right]$ is bounded.

It is noted that when $n=1,$ the conformable fractional operator of Ibrahim-Jahangiri [17] is bounded.

2. Taking $\alpha=\beta=1/2$, $\rho(\xi)=\rho(\xi)=\xi$, $\varsigma(\xi)=-\xi$, $\sigma(\xi)=\xi$ and $\eta=1$ for all $j=1,\ldots,n$, then this brings the next inequality:

$$\frac{1}{2}\sup_{1\leq j\leq n}\left(\sup_{\xi\in\mathscr{O}}\left(\left(1+|\rho_j||\rho_j|\right)\left(1-|\xi|^2\right)\left|\Upsilon\left(\rho_j(\xi);-\rho_j(\xi)\right)\right|\right)\right)=\frac{3}{5}<+\infty,$$

where $|\xi|=1/2$ and $\left|\Upsilon\left(\rho_j(\xi);-\rho_j(\xi)\right)\right|=\left|\frac{2\xi}{1+\overline{\xi}\xi}\right|=\frac{4}{5}.$ Thus, in view of Theorem 1 we conclude that the operator $\mathbf{P}_{\xi,\xi}^{0.5,0.5}\colon\prod_{j=1}^{n}\left[\mathscr{J}_{K_j}\right]\to\prod_{j=1}^{n}\left[\mathscr{B}_{1_j}\right]$ is bounded. Similarly, as a special case when $n=1,$ we conclude that the conformable fractional operator of Ibrahim-Jahangiri is bounded.

3. For the values $\alpha=\beta=1/2$, $\rho(\xi)=\rho(\xi)=\frac{\xi}{1-\xi}$, $\varsigma(\xi)=-\xi$, $\sigma(\xi)=\xi$ and $\eta=1$ for all $j=1,\ldots,n$, we obtain

$$\frac{1}{2}\sup_{1\leq j\leq n}\left(\sup_{\xi\in\mathscr{O}}\left(\left(1+|\rho_j||\rho_j|\right)\left(1-|\xi|^2\right)\left|\Upsilon\left(\rho_j(\xi);-\rho_j(\xi)\right)\right|\right)\right)$$

$$=\frac{1}{2}\sup_{1\leq j\leq n}\left(\sup_{\xi\in\mathscr{O}}\left(\left(1+\left|\frac{\xi}{1-\xi}\right|\left|\frac{\xi}{1-\xi}\right|\right)\left(1-|\xi|^2\right)\left|\frac{2\frac{\xi}{1-\xi}}{1+\frac{\overline{\xi}}{1-\xi}\frac{\xi}{1-\xi}}\right|\right)\right)$$

$$=\sup_{1\leq j\leq n}\left(\sup_{\xi\in\mathscr{O}}\left(\left(1+\left|\frac{\xi}{1-\xi}\right|^2\right)\left(1-|\xi|^2\right)\left|\frac{\frac{\xi}{1-\xi}}{1+\frac{\overline{\xi}}{1-\xi}\frac{\xi}{1-\xi}}\right|\right)\right)=2<+\infty.$$

Hence, by Theorem 1 the operator $\mathbf{P}_{\frac{\xi}{1-\xi},\frac{\xi}{1-\xi}}^{0.5,0.5}:\prod_{j=1}^{n}\left[\mathscr{J}_{K_j}\right]\to\prod_{j=1}^{n}\left[\mathscr{B}_{1_j}\right]$ is bounded. Likewise, we deduce that the conformable fractional operator of Ibrahim-Jahangiri is bounded in the particular situation of $n=1.$

4. The next example shows that Theorem 1 is not covered the boundedness for some analytic functions. Thus, for the particular case $\alpha=\beta=1/2$, $\rho(\xi)=\rho(\xi)=\frac{\xi}{(1-\xi)^2}$, $\varsigma(\xi)=-\xi$, $\sigma(\xi)=\xi$ and $\eta=1$ for all $j=1,\ldots,n$, a simple computation gives

$$\frac{1}{2}\sup_{1\leq j\leq n}\left(\sup_{\xi\in\mathscr{O}}\left(\left(1+|\rho_j||\rho_j|\right)\left(1-|\xi|^2\right)\left|\Upsilon\left(\rho_j(\xi);-\rho_j(\xi)\right)\right|\right)\right)$$

$$=\frac{1}{2}\sup_{1\leq j\leq n}\left(\sup_{\xi\in\mathscr{O}}\left(\left(1+\left|\frac{\xi}{(1-\xi)^2}\right|\left|\frac{\xi}{(1-\xi)^2}\right|\right)\left(1-|\xi|^2\right)\left|\frac{\frac{\xi}{(1-\xi)^2}}{1+\overline{\frac{\xi}{(1-\xi)^2}}\frac{\xi}{(1-\xi)^2}}\right|\right)\right)$$

$$=\frac{1}{2}\sup_{1\leq j\leq n}\left(\sup_{\xi\in\mathscr{O}}\left(\left(1+\left|\frac{\xi}{(1-\xi)^2}\right|^2\right)\left(1-|\xi|^2\right)\left|\frac{\frac{\xi}{(1-\xi)^2}}{1+\overline{\frac{\xi}{(1-\xi)^2}}\frac{\xi}{(1-\xi)^2}}\right|\right)\right)=+\infty,\quad \text{for } \xi\to 1.$$

Consequently, the Theorem 1 does not address for the boundedness of the operator $\mathbf{P}^{\alpha,\beta}_{\frac{\xi}{(1-\xi)^2},\frac{\xi}{(1-\xi)^2}}:\prod_{j=1}^{n}\left[\mathscr{J}_{K_j}\right]\to\prod_{j=1}^{n}\left[\mathscr{B}_{1_j}\right]$ (for all $j=1,...,n$).

5. For the values $\alpha=\beta=1/2$, $\rho(\xi)=\rho(\xi)=\dfrac{1+\xi}{1-\xi}$, $\varsigma(\xi)=-\xi$, $\sigma(\xi)=\xi$ and $\eta=1$ for all $j=1,\ldots,n$, we obtain

$$\frac{1}{2}\sup_{1\leq j\leq n}\left(\sup_{\xi\in\mathscr{O}}\left(\left(1+|\rho_j||\rho_j|\right)\left(1-|\xi|^2\right)\left|\Upsilon\left(\rho_j(\xi);-\rho_j(\xi)\right)\right|\right)\right)$$

$$=\frac{1}{2}\sup_{1\leq j\leq n}\left(\sup_{\xi\in\mathscr{O}}\left(\left(1+\left|\frac{1+\xi}{1-\xi}\right|\left|\frac{1+\xi}{1-\xi}\right|\right)\left(1-|\xi|^2\right)\left|\frac{2\frac{1+\xi}{1-\xi}}{1+|\overline{\frac{\xi}{1-\xi}}\frac{1+\xi}{1-\xi}}\right|\right)\right)$$

$$=\sup_{1\leq j\leq n}\left(\sup_{\xi\in\mathscr{O}}\left(\left(1+\left|\frac{1+\xi}{1-\xi}\right|^2\right)\left(1-|\xi|^2\right)\left|\frac{\frac{1+\xi}{1-\xi}}{1+\overline{\frac{\xi}{1-\xi}}\frac{1+\xi}{1-\xi}}\right|\right)\right)=4<+\infty.$$

Hence, according to Theorem 1, the operator $\mathbf{P}^{0.5,0.5}_{\frac{1+\xi}{1-\xi},\frac{1+\xi}{1-\xi}}:\prod_{j=1}^{n}\left[\mathscr{J}_{K_j}\right]\to\prod_{j=1}^{n}\left[\mathscr{B}_{1_j}\right]$ is bounded. Likewise, in the particular situation of $n=1$, we determine that the conformable fractional operator of Ibrahim-Jahangiri is bounded.

Remark 1 The function $\dfrac{\xi}{1-\xi}=\xi+\sum_{n=2}^{\infty}\xi^n$ plays the role of the extremal convex function in $\mathscr{O}$. In addition, the Koebe function $\dfrac{\xi}{(1-\xi)^2}=\xi+\sum_{n=2}^{\infty}n\xi^n$ represents the extremal starlike function in $\mathscr{O}$ in many problems of Geometric Function Theory.

Proposition 3 (Necessary condition for Theorem 1) Let $\mathbf{P}^{\alpha,\beta}_{\rho,\rho}:\prod_{j=1}^{n}[\mathscr{J}_{K_j}]\to\prod_{j=1}^{n}[\mathscr{B}_{\eta_j}]$ be bounded and assume $\alpha_j\geq\beta_j$ for all $j=1,\ldots,n$. Then necessarily

$$\sup_{1\le j\le n}\frac{\alpha_j}{\alpha_j+\beta_j}\sup_{\xi\in\mathscr{O}}\left((1+|\rho_j(\xi)\rho_j(\xi)|)\,(1-|\xi|^2)^{\eta_j}\,\left|\Upsilon\big(\rho_j(\xi);-\rho_j(\xi)\big)\right|\right)<\infty,$$

that is, condition (A1) is necessary for boundedness.

Proof. Assume $\mathbf{P}^{\alpha,\beta}_{\rho,\rho}$ is bounded. Fix an index $j\in\{1,\dots,n\}$. Boundedness gives a constant $C>0$ such that

$$\|\mathbb{P}^{\alpha_j,\beta_j}_{\rho_j,\rho_j}(g_j)\|_{\mathscr{B}_{\eta_j}}\le C\|g_j\|_{\mathscr{J}_{K_j}}\qquad\text{for all } g_j\in\mathscr{J}_{K_j}. \tag{5}$$

Because $\mathscr{J}_{K_j}\subset\mathscr{B}_1$ and the weight $(1-|\xi|^2)^{-\eta_j}$ is admissible, we may select an extremal family of test functions which asymptotically saturate the norm of $\mathscr{B}_{\eta_j}$ at any chosen point $\xi_0\in\mathscr{O}$. Define

$$g_{j,\,\xi_0}(\xi)=\frac{1}{(1-|\sigma_j(\xi_0)|^2)^{\eta_j}}\,\Lambda_{\sigma_j(\xi_0)}(\xi),\qquad \xi\in\mathscr{O},$$

where $\Lambda_{\sigma_j(\xi_0)}$ is the Möbius transformation. Since $\Lambda_{\sigma_j(\xi_0)}$ is bounded and analytic in $\mathscr{O}$, there exists $A_j>0$ such that

$$\|g_{j,\,\xi_0}\|_{\mathscr{J}_{K_j}}\le A_j.$$

Moreover, $g_{j,\,\xi_0}$ satisfies the extremal property

$$(1-|\xi_0|^2)^{\eta_j}\,|g_{j,\,\xi_0}(\sigma_j(\xi_0))|=1. \tag{6}$$

Now evaluate $\mathbb{P}^{\alpha_j,\beta_j}_{\rho_j,\rho_j}(g_{j,\,\xi_0})$ at $\xi=\xi_0$. From definition,

$$\mathbb{P}^{\alpha_j,\beta_j}_{\rho_j,\rho_j}(g_{j,\,\xi_0})(\xi_0)=\frac{\alpha_j}{\alpha_j+\beta_j}\,\rho_j(\xi_0)\,g_{j,\,\xi_0}(\sigma_j(\xi_0))+\frac{\beta_j}{\alpha_j+\beta_j}\,\rho_j(\xi_0)\,g_{j,\,\xi_0}(\varsigma_j(\xi_0)).$$

Using the triangle inequality and $\alpha_j\ge\beta_j$,

$$\begin{aligned}&(1-|\xi_0|^2)^{\eta_j}\left|\mathbb{P}^{\alpha_j,\beta_j}_{\rho_j,\rho_j}(g_{j,\,\xi_0})(\xi_0)\right|\\ \ge&\frac{\alpha_j}{\alpha_j+\beta_j}(1-|\xi_0|^2)^{\eta_j}\left|\rho_j(\xi_0)\,g_{j,\,\xi_0}(\sigma_j(\xi_0))+\rho_j(\xi_0)\,g_{j,\,\xi_0}(\sigma_j(\xi_0))\right|\\ =&\frac{\alpha_j}{\alpha_j+\beta_j}(1-|\xi_0|^2)^{\eta_j}|g_{j,\,\xi_0}(\sigma_j(\xi_0))|\,|\rho_j(\xi_0)+\rho_j(\xi_0)|.\end{aligned}$$

Using (6),

$$(1-|\xi_0|^2)^{\eta_j}\left|\mathbb{P}^{\alpha_j,\beta_j}_{\rho_j,\rho_j}(g_{j,\xi_0})(\xi_0)\right| \geq \frac{\alpha_j}{\alpha_j+\beta_j}\,|\rho_j(\xi_0)+\rho_j(\xi_0)|.$$

By boundedness (5),

$$(1-|\xi_0|^2)^{\eta_j}\left|\mathbb{P}^{\alpha_j,\beta_j}_{\rho_j,\rho_j}(g_{j,\xi_0})(\xi_0)\right| \leq C\|g_{j,\xi_0}\|_{\mathscr{J}_{K_j}} \leq CA_j.$$

Hence, we have

$$\frac{\alpha_j}{\alpha_j+\beta_j}\,|\rho_j(\xi_0)+\rho_j(\xi_0)| \leq CA_j.$$

We now rewrite the numerator using

$$\rho_j(\xi_0)+\rho_j(\xi_0) = (1+\overline{\rho_j(\xi_0)}\rho_j(\xi_0))\,\Upsilon(\rho_j(\xi_0);-\rho_j(\xi_0)),$$

which gives

$$\frac{\alpha_j}{\alpha_j+\beta_j}\,(1+|\rho_j(\xi_0)\rho_j(\xi_0)|)\,\left|\Upsilon(\rho_j(\xi_0);-\rho_j(\xi_0))\right| \leq CA_j.$$

Since $\xi_0 \in \mathscr{O}$ was arbitrary, we obtain

$$\sup_{\xi\in\mathscr{O}}\frac{\alpha_j}{\alpha_j+\beta_j}(1+|\rho_j(\xi)\rho_j(\xi)|)\left|\Upsilon(\rho_j(\xi);-\rho_j(\xi))\right| < \infty.$$

Taking the supremum over $j=1,\ldots,n$ proves the necessary condition. □

Using similar reasons like in the proof of the Theorem 1 and Proposition 3, assuming that $\alpha_j \leq \beta_j$ for all $j=1,\ldots,n$ we can obtain the next boundedness result.

Theorem 2 Let $g \in \prod_{j=1}^n \left[\mathscr{J}_{K_j}\right]$, where $\alpha_j \leq \beta_j$ for all $j=1,\ldots,n$. Then

$$\text{(A2)}\quad \sup_{1\leq j\leq n}\frac{\beta_j}{\alpha_j+\beta_j}\left(\sup_{\xi\in\mathscr{O}}\left(\left(1+|\rho_j||\rho_j|\right)\left(1-|\xi|^2\right)^{\eta_j}\left|\Upsilon\left(\rho_j(\xi);-\rho_j(\xi)\right)\right|\right)\right) < +\infty,$$

where $\Upsilon(\varphi(\xi);-\psi(\xi))$ is given by (4) holds if and only if $\mathbf{P}^{\alpha,\beta}_{\rho,\rho}$ is bounded in $\prod_{j=1}^n \left[\mathscr{B}_{\eta_j}\right]$.

Other two sufficient boundedness conditions such that the operator $\mathbf{P}^{\alpha,\beta}_{\rho,\rho}$ is bounded in $\prod_{j=1}^n \left[\mathscr{B}_{\eta_j}\right]$ will we presented in the next two theorems.

Theorem 3 Let $g \in \prod_{j=1}^n \left[\mathscr{B}_{\eta_j}\right]$. If $\alpha_j \geq \beta_j$ for all $j=1,\ldots,n$, and

$$\text{(A3)}\quad \sup_{1\le j\le n}\frac{\alpha_j}{\alpha_j+\beta_j}\left(\sup_{\xi\in\mathscr{O}}\left(\left(1+|\rho_j||\rho_j|\right)\left|\Upsilon\left(\rho_j(\xi);-\rho_j(\xi)\right)\right|\right)\right)<+\infty,$$

with $\Upsilon(\varphi(\xi);-\psi(\xi))$ given by (4), then $\mathbf{P}_{\rho,\rho}^{\alpha,\beta}$ is bounded in $\prod_{j=1}^{n}\left[\mathscr{B}_{\eta_j}\right]$.

Proof. For $g\in\prod_{j=1}^{n}\left[\mathscr{B}_{\eta_j}\right]$ the operator $\mathbf{P}_{\rho,\rho}^{\alpha,\beta}\colon\prod_{j=1}^{n}\left[\mathscr{B}_{\eta_j}\right]\to\prod_{j=1}^{n}\left[\mathscr{B}_{\eta_j}\right]$ satisfies the next inequality

$$\begin{aligned}
\|\mathbf{P}_{\rho,\rho}^{\alpha,\beta}\|_{\mathscr{B}=\prod_{j=1}^{n}[\mathscr{B}_{\kappa_j}]}&=\sum_{j=1}^{n}\|\mathbb{P}_{\rho_j,\rho_j}^{\alpha_j,\beta_j}(g_j)\|_{\mathscr{B}_{\eta_j}}\\
&=\sum_{j=1}^{n}\left(\sup_{\xi\in\mathscr{O}}\left(1-|\xi|^2\right)^{\eta_j}\left|\frac{\alpha_j}{\alpha_j+\beta_j}\rho_j\Omega_{\sigma_j}+\frac{\beta_j}{\alpha_j+\beta_j}\rho_j\Omega_{\varsigma_j}\right|\right)\\
&=\sum_{j=1}^{n}\left(\sup_{\xi\in\mathscr{O}}\left(1-|\xi|^2\right)^{\eta_j}\left|\frac{\alpha_j}{\alpha_j+\beta_j}\rho_j g_j(\sigma_j)+\frac{\beta_j}{\alpha_j+\beta_j}\rho_j g_j(\varsigma_j)\right|\right)\\
&\le\sum_{j=1}^{n}\frac{\alpha_j}{\alpha_j+\beta_j}\left(\sup_{\xi\in\mathscr{O}}\left(1-|\xi|^2\right)^{\eta_j}\left|\rho_j g_j(\sigma_j)+\rho_j g_j(\varsigma_j)\right|\right)\\
&\le\sum_{j=1}^{n}\frac{\alpha_j}{\alpha_j+\beta_j}\|g_j\|_{\prod_{j=1}^{n}\left[\mathscr{B}_{\eta_j}\right]}\left(\sup_{\xi\in\mathscr{O}}\left|\rho_j+\rho_j\right|\right)\\
&\le\|g\|_{\prod_{j=1}^{n}\left[\mathscr{B}_{\eta_j}\right]}\sum_{j=1}^{n}\frac{\alpha_j}{\alpha_j+\beta_j}\left(\sup_{\xi\in\mathscr{O}}|1+\overline{\rho_j}\rho_j|\left|\frac{\rho_j+\rho_j}{1+\overline{\rho_j}\rho_j}\right|\right)\\
&\le\|g\|_{\prod_{j=1}^{n}\left[\mathscr{B}_{\eta_j}\right]}\sup_{1\le j\le n}\frac{\alpha_j}{\alpha_j+\beta_j}\left(\sup_{\xi\in\mathscr{O}}\left(\left(1+|\rho_j||\rho_j|\right)\left|\Upsilon\left(\rho_j(\xi);-\rho_j(\xi)\right)\right|\right)\right).
\end{aligned}$$

Using the assumption (A3) we have

$$\mathscr{L}:=\sup_{1\le j\le n}\left(\frac{\alpha_j}{\alpha_j+\beta_j}\right)\left(\sup_{\xi\in\mathscr{O}}\left(\left(1+|\rho_j||\rho_j|\right)\left|\Upsilon\left(\rho_j(\xi);-\rho_j(\xi)\right)\right|\right)\right)<+\infty,$$

and from the above inequality we get

$$\|\mathbf{P}_{\rho,\rho}^{\alpha,\beta}(g)\|_{\mathscr{B}=\prod_{j=1}^{n}\left[\mathscr{B}_{\eta_j}\right]}=\sum_{j=1}^{n}\|\mathbb{P}_{\rho_j,\rho_j}^{\alpha_j,\beta_j}(g_j)\|_{\mathscr{B}_{\eta_j}}\le\mathscr{L}\|g\|_{\prod_{j=1}^{n}\left[\mathscr{B}_{\eta_j}\right]},$$

where $\mathscr{L}$ is a positive constant. Hence, $\mathbf{P}^{\alpha,\beta}_{\rho,\rho}\colon \prod_{j=1}^{n}\left[\mathscr{B}_{\eta_j}\right] \to \prod_{j=1}^{n}\mathscr{B}_{\eta_j}$ is bounded. □

Proposition 4 (Necessary condition for Theorem 3) Assume that the operator

$$\mathbf{P}^{\alpha,\beta}_{\rho,\rho}\colon \prod_{j=1}^{n}[\mathscr{B}_{\eta_j}] \longrightarrow \prod_{j=1}^{n}[\mathscr{B}_{\eta_j}]$$

is bounded and that $\alpha_j \geq \beta_j$ for all $j = 1, \ldots, n$. Then the following condition is necessary:

$$\sup_{1\leq j\leq n}\frac{\alpha_j}{\alpha_j+\beta_j}\sup_{\xi\in\mathscr{O}}\left((1+|\rho_j(\xi)\rho_j(\xi)|)\left|\Upsilon(\rho_j(\xi);-\rho_j(\xi))\right|\right)<\infty,$$

that is, condition (A3) is required for the boundedness of $\mathbf{P}^{\alpha,\beta}_{\rho,\rho}$.

Proof. Suppose $\mathbf{P}^{\alpha,\beta}_{\rho,\rho}$ is bounded. Fix an index $j \in \{1, \ldots, n\}$. Then there exists a constant $C > 0$ such that

$$\|\mathbb{P}^{\alpha_j,\beta_j}_{\rho_j,\rho_j}(g_j)\|_{\mathscr{B}_{\eta_j}} \leq C\|g_j\|_{\mathscr{B}_{\eta_j}} \qquad \text{for all } g_j \in \mathscr{B}_{\eta_j}. \tag{7}$$

For any fixed $\xi_0 \in \mathscr{O}$, consider the extremal test function

$$g_{j,\xi_0}(\xi) = \frac{1}{(1-|\sigma_j(\xi_0)|^2)^{\eta_j}}\Lambda_{\sigma_j(\xi_0)}(\xi), \qquad \xi \in \mathscr{O},$$

where $\Lambda_{\sigma_j(\xi_0)}$ is the Möbius transformation. Since $\Lambda_{\sigma_j(\xi_0)}$ is bounded and analytic in $\mathscr{O}$, we have $\|g_{j,\xi_0}\|_{\mathscr{B}_{\eta_j}} = 1$, and, crucially,

$$(1-|\xi_0|^2)^{\eta_j}\,|g_{j,\xi_0}(\sigma_j(\xi_0))| = 1. \tag{8}$$

Now evaluate $\mathbb{P}^{\alpha_j,\beta_j}_{\rho_j,\rho_j}(g_{j,\xi_0})$ at $\xi = \xi_0$:

$$\mathbb{P}^{\alpha_j,\beta_j}_{\rho_j,\rho_j}(g_{j,\xi_0})(\xi_0) = \frac{\alpha_j}{\alpha_j+\beta_j}\rho_j(\xi_0)\,g_{j,\xi_0}(\sigma_j(\xi_0)) + \frac{\beta_j}{\alpha_j+\beta_j}\rho_j(\xi_0)\,g_{j,\xi_0}(\varsigma_j(\xi_0)).$$

Using $\alpha_j \geq \beta_j$ and the triangle inequality,

$$\begin{aligned}
&(1-|\xi_0|^2)^{\eta_j}\left|\mathbb{P}^{\alpha_j,\beta_j}_{\rho_j,\rho_j}(g_{j,\xi_0})(\xi_0)\right| \\
\geq&\frac{\alpha_j}{\alpha_j+\beta_j}(1-|\xi_0|^2)^{\eta_j}\left|\rho_j(\xi_0)\,g_{j,\xi_0}(\sigma_j(\xi_0)) + \rho_j(\xi_0)\,g_{j,\xi_0}(\sigma_j(\xi_0))\right| \\
=&\frac{\alpha_j}{\alpha_j+\beta_j}\,(1-|\xi_0|^2)^{\eta_j}|g_{j,\xi_0}(\sigma_j(\xi_0))|\,|\rho_j(\xi_0)+\rho_j(\xi_0)|.
\end{aligned}$$

Applying (8),

$$(1-|\xi_0|^2)^{\eta_j}\left|\mathbb{P}_{\rho_j,\rho_j}^{\alpha_j,\beta_j}(g_{j,\xi_0})(\xi_0)\right| \geq \frac{\alpha_j}{\alpha_j+\beta_j}\,|\rho_j(\xi_0)+\rho_j(\xi_0)|.$$

On the other hand, boundedness (7) gives

$$(1-|\xi_0|^2)^{\eta_j}\left|\mathbb{P}_{\rho_j,\rho_j}^{\alpha_j,\beta_j}(g_{j,\xi_0})(\xi_0)\right| \leq C\|g_{j,\xi_0}\|_{\mathscr{B}_{\eta_j}} = C.$$

Thus, we get

$$\frac{\alpha_j}{\alpha_j+\beta_j}\,|\rho_j(\xi_0)+\rho_j(\xi_0)| \leq C.$$

Using the identity

$$\rho_j(\xi_0)+\rho_j(\xi_0) = \left(1+\overline{\rho_j(\xi_0)}\rho_j(\xi_0)\right)\Upsilon(\rho_j(\xi_0);-\rho_j(\xi_0)),$$

we obtain

$$\frac{\alpha_j}{\alpha_j+\beta_j}\,(1+|\rho_j(\xi_0)\rho_j(\xi_0)|)\left|\Upsilon(\rho_j(\xi_0);-\rho_j(\xi_0))\right| \leq C.$$

Since $\xi_0 \in \mathscr{O}$ is arbitrary,

$$\sup_{\xi\in\mathscr{O}}\frac{\alpha_j}{\alpha_j+\beta_j}\,(1+|\rho_j(\xi)\rho_j(\xi)|)\left|\Upsilon(\rho_j(\xi);-\rho_j(\xi))\right| < \infty.$$

Finally, taking the supremum over $j = 1, \ldots, n$ proves the claim. □

Using a similar proof, we have the following outcome.

Theorem 4 Let $g \in \prod_{j=1}^n \left[\mathscr{B}_{\eta_j}\right]$, where $\alpha_j \leq \beta_j$ for all $j = 1, \ldots, n$. Then

$$\text{(A4)}\quad \sup_{1\leq j\leq n}\frac{\beta_j}{\alpha_j+\beta_j}\left(\sup_{\xi\in\mathscr{O}}\left(\left(1+|\rho_j||\rho_j|\right)\left|\Upsilon\left(\rho_j(\xi);-\rho_j(\xi)\right)\right|\right)\right) < +\infty,$$

where $\Upsilon(\varphi(\xi);-\psi(\xi))$ is given by (4) holds if and only if $\mathbf{P}_{\rho,\rho}^{\alpha,\beta}\colon \prod_{j=1}^n \left[\mathscr{B}_{\eta_j}\right] \to \prod_{j=1}^n \left[\mathscr{B}_{\eta_j}\right]$ is bounded.

Remark 2 (Further properties) Beyond the boundedness results established above, the operator $\mathbf{P}_{\rho,\rho}^{\alpha,\beta}$ enjoys several structural properties that are natural in the context of analytic function spaces. First, each component $\mathbb{P}_{\rho_j,\rho_j}^{\alpha_j,\beta_j}$ is *linear* in g_j, since both the composition maps Ω_{σ_j}, Ω_{ς_j} and the analytic weights ρ_j, ρ_j act linearly. Hence, $\mathbf{P}_{\rho,\rho}^{\alpha,\beta}$ is a linear operator on the product space. Second, the boundedness results in Theorems 1–3 imply that $\mathbf{P}_{\rho,\rho}^{\alpha,\beta}$ is *continuous* with

respect to the norms of the spaces $\mathscr{J}_{K_j}$ and $\mathscr{B}_{\eta_j}$. In particular, norm convergence $g^{(m)} \to g$ implies $\mathbf{P}^{\alpha,\beta}_{\rho,\rho}(g^{(m)}) \to \mathbf{P}^{\alpha,\beta}_{\rho,\rho}(g)$ uniformly on compact subsets of $\mathscr{O}$. Compactness, however, typically requires additional decay conditions on the symbols ρ_j and ρ_j or the maps σ_j and ς_j. In analogy with classical weighted composition operators, compactness can occur only if $\sigma_j(\xi)$ and $\varsigma_j(\xi)$ approach the boundary in such a way that the images of the unit ball become asymptotically small in the hyperbolic metric. Without such vanishing conditions, the operator $\mathbf{P}^{\alpha,\beta}_{\rho,\rho}$ is generally not compact, even when bounded. Thus, while linearity and continuity are built into the structure of $\mathbf{P}^{\alpha,\beta}_{\rho,\rho}$, compactness is a more delicate phenomenon that depends on additional geometric constraints on the defining analytic symbols. Thus, any compactness result for $\mathbf{P}^{\alpha,\beta}_{\rho,\rho}$ necessarily requires refined conditions involving the pseudohyperbolic metric, Carleson measure criteria, or vanishing-weight conditions, rather than any geometric property of the unit disk itself.

3.1 *Applications*

The study of bounded analytic functions on the unit disk, specifically in the Hardy space H^∞ (particular instances of Banach spaces of analytic functions) and associated interpolation problems, provides a significant illustration of the application of the pseudohyperbolic distance in function theory.

Example 2 For $f \in H^\infty$ such that

$$f(\xi_i) = \omega_i, \quad \xi_i, \omega_i \in \mathscr{O}, i = 1, ..., n,$$

where $\|f\| \leq 1$. The positivity of the Pick matrix, which is described as follows, determines whether the Nevanlinna-Pick problem is solvable:

$$\left[\frac{1-\omega_i\overline{\omega_j}}{1-\xi_i\overline{\xi_j}}\right]_{i,\,j=1}^{n},$$

where term $1-\omega_i\overline{\omega_j}$ is determined the pseudohyperbolic distance between ω_i and ω_j, and the term $1-\xi_i\overline{\xi_j}$ is determine the the pseudohyperbolic distance between nods ξ_i and ξ_j.

Assume $\xi_1 = 0$, $\xi_2 = 1/2$ and $\omega_1 = 0$, $\omega_2 = 1/3$. Our aim is to show that $f \in H^\infty \subset \prod_{j=1}^{n}[\mathscr{B}]$. The Pick matrix is

$$\begin{bmatrix} \frac{1-0\times\overline{0}}{1-0\times\overline{0}} & \frac{1-0\times\overline{1/3}}{1-0\times\overline{1/2}} \\ \\ \frac{1-1/3\times\overline{0}}{1-1/2\times\overline{0}} & \frac{1-1/3\times\overline{1/3}}{1-1/2\times\overline{1/2}} \end{bmatrix} = \begin{bmatrix} 1 & \frac{3}{2} \\ \\ \frac{3}{2} & \frac{8}{9} \end{bmatrix}.$$

Consider $\alpha_j = \beta_j = 1/2$. Then $\mathbf{P}^{\alpha,\beta}_{\rho,\rho}$ is bounded in $H^\infty \subset \prod_{j=1}^{n}[\mathscr{B}]$.

Example 3 Consider again the Nevanlinna-Pick interpolation problem on the unit disk. Let $f \in H^\infty$ satisfy

$$f(\xi_i) = \omega_i, \qquad \xi_i, \omega_i \in \mathscr{O}, \ i = 1, \ldots, n,$$

with $\|f\|_\infty \leq 1$. As before, the solvability of this problem is completely determined by the positivity of the Pick matrix

$$\mathscr{P} = \left[\frac{1-\omega_i\overline{\omega_j}}{1-\xi_i\overline{\xi_j}}\right]_{i,\,j=1}^{n},$$

where the numerator encodes the pseudohyperbolic distance between the target values ω_i, ω_j, and the denominator encodes the pseudohyperbolic distance between nodes ξ_i, ξ_j in the unit disk.

Let us choose a different dataset:

$$\xi_1 = \frac{1}{3}, \quad \xi_2 = -\frac{1}{2}, \qquad \omega_1 = \frac{1}{4}, \quad \omega_2 = -\frac{1}{5}.$$

Then the Pick matrix becomes

$$\mathscr{P} = \begin{bmatrix} \dfrac{1-|\omega_1|^2}{1-|\xi_1|^2} & \dfrac{1-\omega_1\overline{\omega_2}}{1-\xi_1\overline{\xi_2}} \\[2ex] \dfrac{1-\omega_2\overline{\omega_1}}{1-\xi_2\overline{\xi_1}} & \dfrac{1-|\omega_2|^2}{1-|\xi_2|^2} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1-\frac{1}{16}}{1-\frac{1}{9}} & \dfrac{1-\left(\frac{1}{4}\right)\left(-\frac{1}{5}\right)}{1-\left(\frac{1}{3}\right)\left(-\frac{1}{2}\right)} \\[2ex] \dfrac{1-\left(-\frac{1}{5}\right)\left(\frac{1}{4}\right)}{1-\left(-\frac{1}{2}\right)\left(\frac{1}{3}\right)} & \dfrac{1-\frac{1}{25}}{1-\frac{1}{4}} \end{bmatrix}.$$

After simplification, we obtain

$$\mathscr{P} = \begin{bmatrix} \frac{15}{8} & \frac{21}{14} \\[1ex] \frac{21}{14} & \frac{24}{19} \end{bmatrix} \approx \begin{bmatrix} 1.875 & 1.500 \\[1ex] 1.500 & 1.263 \end{bmatrix}.$$

The matrix $\mathscr{P}$ is easily seen to be positive definite (the determinant is positive and the diagonal entries are positive), so the Nevanlinna-Pick interpolation problem is solvable; hence there exists $f \in H^\infty$ with $\|f\|_\infty \leq 1$ satisfying the prescribed interpolation conditions. Since

$$H^\infty \subset \mathscr{B}_\eta \qquad \text{for every } \eta > 0,$$

we have $f \in \prod_{j=1}^{n}[\mathscr{B}_{\eta_j}]$. If we choose the parameters $\alpha_j = \beta_j = \frac{1}{2}$, then by Theorem 1 the conformable composition operator $\mathbf{P}_{\rho,\rho}^{\alpha,\beta}$ is bounded on $\prod_{j=1}^{n}[\mathscr{B}_{\eta_j}]$ whenever condition (A1) holds for the associated analytic weights ρ_j and ρ_j. In particular, for any admissible symbols satisfying (A1), we conclude:

$$f \in H^{\infty} \subset \prod_{j=1}^{n} \left[\mathscr{B}_{\eta_j}\right] \quad \Longrightarrow \quad \mathbf{P}_{\rho,\rho}^{0.5,0.5}(f) \in \prod_{j=1}^{n} \left[\mathscr{B}_{\eta_j}\right].$$

Thus, the boundedness of the conformable operator is compatible with the Nevanlinna-Pick interpolation framework.

4. Conclusion

A definition of the 2D Conformable Composition Operators (CCOs) with multi-complex variables was found through the aforementioned work. We offer the adequate requirements for boundedness, and special cases of well known operators are indicated, such as the conformable fractional and the symmetric conformable operators for the case $n = 1$. To cover the abstract findings, several examples are shown. Different kinds of operators, including the symmetric and pseudo-symmetric operators, might be proposed for further study.

Perspectives and future research. The present work establishes the boundedness of a new class of multivariable conformable composition operators acting on Bers-type and integral-type spaces of analytic functions. Several natural questions arise from these results and provide promising directions for future research. First, extending the boundedness analysis to compactness criteria constitutes an important continuation. In analogy with classical weighted composition operators, compactness is expected to depend on refined boundary behaviour of the analytic symbols σ_j, ς_j and on vanishing conditions for the weights ρ_j, ρ_j in the pseudohyperbolic metric. Establishing a Nevanlinna type characterization for compactness of $\mathbf{P}_{\rho,\rho}^{\alpha,\beta}$ would therefore be a natural and challenging step.

A second direction concerns the spectral theory of $\mathbf{P}_{\rho,\rho}^{\alpha,\beta}$. Determining point spectra, approximate point spectra, and essential spectra on various analytic function spaces may reveal new connections between conformable deformations and the classical theory of linear fractional operators. In particular, when the maps σ_j and ς_j are Möbius transformations, one expects explicit spectral radii expressed in terms of hyperbolic dilations.

Third, the multivariable structure of the operator suggests an extension to composition operators acting on analytic functions on the unit ball $\mathbb{B}^m$ or polydisc $\mathbb{D}^m$. The interaction between conformable deformations and several complex variables is largely unexplored; understanding boundedness, Carleson embeddings, and Bergman-type estimates in several dimensions constitutes an open research direction. Finally, applications to fractional differential equations in the complex domain, geometric function theory, and conformable integral operators provide an additional avenue for development. The operator $\mathbf{P}_{\rho,\rho}^{\alpha,\beta}$ may serve as a building block for defining conformable semigroups, studying stability of fractional flows, or constructing new subclasses of spirallike, convex, or Bazilevic functions under conformable deformation. These questions highlight the potential of the present framework and indicate that many analytical and geometric aspects of conformable composition operators remain to be explored.

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Conflict of interests

The authors declare no competing financial interest.

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