Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

Research Article

On the Stability of Partial Integral Equations Governed by Hadamard Fractional Integrals of Piecewise Constant Order

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Received: 27 October 2025; **Revised:** 18 December 2025; **Accepted:** 25 December 2025

Abstract: This article investigates the existence and global stability of a class of partial integral equations involving the Hadamard fractional integral of variable order. By employing Schauder's fixed point theorem within appropriately constructed function spaces, we establish sufficient conditions ensuring the existence of solutions. We also analyze the asymptotic behavior of these solutions and derive criteria that guarantee their global asymptotic stability. The practical relevance of the theoretical results is demonstrated through a detailed example that highlights the applicability and effectiveness of the proposed approach. Overall, this study provides foundational insights into the existence and stability properties of a recently introduced class of fractional integral equations, representing one of the first investigations conducted within this mathematical framework.

***Keywords*:** Volterra integral equation, Hadamard integral of fractional variable order, existence, global stability

MSC: 26A33, 34A12

1. Introduction

Among the most powerful tools in the analyst's repertoire are differential and integral equations, including those of fractional order. Their theory, enriched by topology, functional analysis, and fixed point methods, has undergone significant development in recent years, driven by a wide spectrum of applications in engineering, mechanical vibrations, mathematical physics, and various theoretical and practical problems [1–3]. Fractional differential and integral equations, in particular, play a central role in the mathematical modeling of diverse processes arising in viscoelasticity, electrochemistry, control theory, and electromagnetism [4, 5], as well as in porous media and other complex systems. The expanding body of literature reflected in numerous monographs and research articles addressing both ordinary and partial fractional equations highlights their growing importance in both theoretical analysis and applied sciences.

Foundational existence and uniqueness results for deterministic integral equations involving Hadamard fractional integrals with two independent variables are well established in the literature. Subsequent contributions in [6] provided

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a systematic analysis of the analytical properties of Hadamard fractional operators, while the authors in [7] further characterized these operators through their Mellin transforms. Pooseh et al. [8] advanced the theory by deriving explicit expansion formulas that represent Hadamard-type operators in terms of integer-order derivatives. Comprehensive surveys covering these developments, together with related operator properties and applications, can be found in [9, 10] and the references therein.

A more sophisticated class of operators involves arbitrary (variable) orders of integration and differentiation, where the order depends on functions of certain variables. These are referred to as variable-order operators. Since their introduction in the 1990s [11], Fractional Variable-Order (FVO) differential and integral operators have attracted growing interest due to their flexibility in modeling dynamic systems exhibiting memory-dependent behavior. The study of fractional variable-order models is still in its early stages, primarily because of the challenges inherent in such models, notably the lack of standard properties, such as the semigroup property, which are available in constant-order systems. For recent advances in fractional variational calculus and numerical methods related to variable-order fractional problems, see [12–20] and the references therein.

The study of fractional integral equations on unbounded domains continues to attract considerable attention. Recently, Abbas et al. [21] obtained significant existence and stability results for a class of nonlinear quadratic Volterra integral equations of Riemann–Liouville fractional order. Their analysis focused on the following equation:

$$\begin{aligned} u(t,x) &= f\big(t,x,u(t,x),u(\alpha(t),x)\big) + \frac{1}{\Gamma(r)}\int_0^{\beta(t)} (\beta(t)-s)^{r-1} \\ &\quad \times g\big(t,x,s,u(s,x),u(\gamma(s),x)\big),\, ds, \\ &\quad (t,x)\in \mathbb{R}^+\times[0,b], \end{aligned} \tag{1}$$

where $b>0$, and $r\in(0,\infty)$. The functions $\alpha,\beta,\gamma\colon \mathbb{R}^+\to\mathbb{R}^+$ are continuous delay arguments, while $\mathfrak{f}\colon \mathbb{R}^+\times[\mathfrak{o},\mathfrak{b}]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $g\colon \mathbb{R}^+\times[0,b]\times\mathbb{R}^+\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are given continuous nonlinearities. Their work provides a crucial framework for understanding the behavior of such complex functional integral equations.

Seminal work in this field has established fundamental results on the global solvability and stability of nonlinear quadratic Volterra-type partial integral equations involving Hadamard fractional integrals of fixed order [22].

The present study addresses the existence and global stability of nonlinear quadratic Volterra-type partial integral equations involving variable-order Hadamard-type fractional operators.

$$\begin{aligned} \mathfrak{u}(\mathfrak{s},\mathfrak{x}) &= \mathfrak{f}(\mathfrak{s},\mathfrak{x},\mathfrak{u}(\mathfrak{s},\mathfrak{x}),\mathfrak{u}(\lambda(\mathfrak{s}),\mathfrak{x})) \\ &\quad + \frac{1}{\Gamma(\mathfrak{M})\Gamma(\mathfrak{R}(\mathfrak{s}))}\int_1^{\gamma(\mathfrak{s})}\int_1^{\mathfrak{x}} \left(\log\frac{\gamma(\mathfrak{s})}{s}\right)^{\mathfrak{M}-1}\left(\log\frac{\mathfrak{x}}{\varsigma}\right)^{\mathfrak{R}(\mathfrak{s})-1} \\ &\quad \times \mathfrak{h}(\mathfrak{s},\mathfrak{x},s,\varsigma,\mathfrak{u}(s,\varsigma),\mathfrak{u}(\eta(s),\varsigma))\frac{d\varsigma ds}{s\varsigma}; \quad (\mathfrak{s},\mathfrak{x})\in\mathfrak{J}:=[1,+\infty)\times[1,\mathfrak{b}], \end{aligned} \tag{2}$$

where $\mathfrak{b}>1$, $\mathfrak{M}>0$, $\mathfrak{R}(s)>0$, $\lambda,\gamma,\eta\colon[1,\infty)\to[1,\infty)$, and $\mathfrak{f}\colon\mathfrak{J}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $\mathfrak{h}\colon\mathfrak{J}\times\mathfrak{J}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are given continuous functionals.

This paper establishes the existence of solutions by applying Schauder's fixed point theorem within suitably defined function spaces. Furthermore, we examine the asymptotic stability of these solutions and derive sufficient conditions that

ensure their global stability. To illustrate the practical applicability of our theoretical framework, a concrete case study is provided, verifying the implementation of the established conditions. Overall, this work offers foundational insights into the existence and stability analysis of this recently introduced class of fractional integral equations, representing an initial exploration of these properties within the specified mathematical setting.

2. Preliminaries

This section presents the notations, definitions, and fundamental concepts employed throughout this work.

Let $L^1([1, \mathfrak{a}] \times [1, \mathfrak{b}], \mathbb{R})$; for $\mathfrak{a}, \mathfrak{b} > 1$, stands for the Banach space of Lebesgue integrable functions $\mathfrak{u}$: $[1, \mathfrak{a}] \times [1, \mathfrak{b}] \to \mathbb{R}$, equipped with the norm:

$$\|\mathfrak{u}\|_{L^1} = \int_1^{\mathfrak{a}} \int_1^{\mathfrak{b}} |\mathfrak{u}(\mathfrak{x}, \mathfrak{y})| d\mathfrak{y} d\mathfrak{x}.$$

By $\mathfrak{BC}$: $= \mathfrak{C}(\mathfrak{J}: = [\mathfrak{1}, +\infty) \times [\mathfrak{1}, \mathfrak{b}], \mathbb{R})$ we denote the Banach space of all continuous functions $\mathfrak{u}$: $\mathfrak{J} \to \mathbb{R}$ with the supremum norm

$$\|\mathfrak{u}\|_{\mathfrak{C}} = \sup_{(\mathfrak{x}, \mathfrak{y}) \in \mathfrak{J}} |\mathfrak{u}(\mathfrak{x}, \mathfrak{y})|.$$

Definition 1 [23] Let $1 \leq \mathfrak{a} < \mathfrak{b} < +\infty$ and $\mathfrak{R}$: $[\mathfrak{a}, \mathfrak{b}] \to (0, +\infty)$. The Hadamard integral of variable order $\mathfrak{R}(\mathfrak{s})$ for $\mathfrak{v}$ is defined by

$$\left({}^{\mathscr{H}} I_{\mathfrak{a}^+}^{\mathfrak{R}(\mathfrak{s})} \mathfrak{v}\right)(\mathfrak{s}) = \frac{1}{\Gamma(\mathfrak{R}(\mathfrak{s}))} \int_{\mathfrak{a}}^{\mathfrak{s}} \left(\log \frac{\mathfrak{s}}{\mathfrak{\tau}}\right)^{\mathfrak{R}(\mathfrak{s})-1} \frac{\mathfrak{v}(\mathfrak{\tau})}{\mathfrak{\tau}} d\mathfrak{\tau}, \qquad \mathfrak{s} > \mathfrak{a}, \tag{3}$$

if the right-hand side integral exists. Where $\Gamma(\cdot)$ represents the Euler gamma function.

Remark 1 In the general case [24]

$$I_{\mathfrak{a}_1^+}^{\mathfrak{M}(\mathfrak{s})} I_{\mathfrak{a}_1^+}^{\mathfrak{R}(\mathfrak{s})} \mathfrak{v}(\mathfrak{s}) \neq I_{\mathfrak{a}_1^+}^{\mathfrak{M}(\mathfrak{s})+\mathfrak{R}(\mathfrak{s})} \mathfrak{v}(\mathfrak{s}).$$

Definition 2 Let $\mathfrak{M}$, $\mathfrak{R}$ be non-negative real numbers, $\sigma = (1, 1)$, and $\mathbf{r} = (\mathfrak{M}, \mathfrak{R})$. Given an integrable function $\mathfrak{v} \in L^1(\mathfrak{J}, \mathbb{R})$, the Hadamard-type partial fractional integral operator of order $\mathbf{r}$ is given by:

$${}^{\mathscr{H}} \mathscr{I}_{\sigma}^{\mathbf{r}} \mathfrak{v}(\mathfrak{x}, \mathfrak{y}) = \frac{1}{\Gamma(\mathfrak{M})\Gamma(\mathfrak{R})} \int_1^{\mathfrak{x}} \int_1^{\mathfrak{y}} \left(\log \frac{\mathfrak{x}}{\mathfrak{\tau}}\right)^{\mathfrak{M}-1} \left(\log \frac{\mathfrak{y}}{\mathfrak{s}}\right)^{\mathfrak{R}-1} \frac{\mathfrak{v}(\mathfrak{\tau}, \mathfrak{s})}{\mathfrak{\tau}\mathfrak{s}} d\mathfrak{s} d\mathfrak{\tau}.$$

Let $\emptyset \neq \Theta$, $\Theta \subseteq \mathfrak{BC}$, and suppose that $\mathfrak{G}$: $\Theta \to \Theta$. We proceed by investigating the behavior of solutions corresponding to the equation

$$(\mathfrak{G}\mathfrak{u})(\mathfrak{s}, \mathfrak{x}) = \mathfrak{u}(\mathfrak{s}, \mathfrak{x}) \tag{4}$$

First, we recall the concept of attractivity in the context of the solution behavior for the considered Equation (4).

Definition 3 [25] A solution $\mathfrak{u}$ of Equation (4) is called *locally attractive* if there exists a closed ball in $\mathfrak{BC}$ centered at $\mathfrak{u}_o$ with radius $\rho \in (0, +\infty)$: $\mathfrak{B}(\mathfrak{u}_0, \rho)$ verifying that, for any two solutions $\mathfrak{v} = \mathfrak{v}(\mathfrak{s}, \mathfrak{x})$ and $\mathfrak{w} = \mathfrak{w}(\mathfrak{s}, \mathfrak{x})$ lying in $\mathfrak{B}(\mathfrak{u}_0, \rho) \cap \Theta$, the following convergence holds for all $\mathfrak{x} \in [1, \mathfrak{b}]$,

$$\lim_{\mathfrak{s} \to +\infty} (\mathfrak{v}(\mathfrak{s}, \mathfrak{x}) - \mathfrak{w}(\mathfrak{s}, \mathfrak{x})) = 0. \tag{5}$$

Note that if the convergence in (5) is uniform throughout $\mathfrak{B}(\mathfrak{u}_0, \rho)$, we say the solutions are *uniformly locally attractive* (or locally asymptotically stable).

Definition 4 [25] A solution $\mathfrak{v} = \mathfrak{v}(\mathfrak{s}, \mathfrak{x})$ of (4) is *globally attractive* if property (5) holds for all solutions $\mathfrak{w} = \mathfrak{w}(\mathfrak{s}, \mathfrak{x})$ in the solution space. When this attraction occurs uniformly across the entire space, we describe the solutions as *globally asymptotically stable* (or uniformly globally attractive).

Lemma 1 [26] A subset $\mathfrak{D} \subset \mathfrak{BC}$ is called relatively compact in $\mathfrak{BC}$ provided that the conditions listed below are fulfilled:

(i) The set $\mathfrak{D}$ is bounded in the $\mathfrak{BC}$ norm;

(ii) For any compact subset of $\mathfrak{J}$, all functions in $\mathfrak{D}$ exhibit equicontinuity;

(iii) The functions in $\mathfrak{D}$ are equiconvergent at infinity: For every $\varepsilon > 0$ and $\mathfrak{x}$ in $[1, \mathfrak{b}]$, there exists $T(\varepsilon, \mathfrak{x}) > 0$ s.t.

$$|\mathfrak{u}(\mathfrak{s}, \mathfrak{x}) - \lim_{\mathfrak{s} \to +\infty} \mathfrak{u}(\mathfrak{s}, \mathfrak{x})| < \varepsilon$$

holds for all $\mathfrak{s} \geq T(\varepsilon, \mathfrak{x})$ and every $\mathfrak{u} \in \mathfrak{D}$.

Theorem 1 [27] Let $\mathfrak{D}$ be a convex subset of a Banach space $\mathfrak{BC}$ and $\mathfrak{N}: \mathfrak{D} \to \mathfrak{D}$ be a completely continuous map. Then $\mathfrak{N}$ has at least one fixed point in $\mathfrak{D}$.

3. Existence and stability

This section establishes two fundamental properties of (2):

• The existence of solutions in appropriately defined function spaces.

• The asymptotic stability of these solutions.

We begin by stating the underlying assumption:

($\mathfrak{H}1$) Let $\mathscr{Z} = \{\mathfrak{V}_1 := [1, \mathfrak{K}_1], \mathfrak{V}_2 := (\mathfrak{K}_1, \mathfrak{K}_2], \mathfrak{V}_3 := (\mathfrak{K}_2, \mathfrak{K}_3], \ldots, \mathfrak{V}_\mathfrak{n} := (\mathfrak{K}_{\mathfrak{n}-1}, \mathfrak{b}]\}$ be a partition of the interval $[1, \mathfrak{b}]$ and let $\mathfrak{R}(\mathfrak{s}): \mathfrak{V} \to (0, \infty)$ be a piece-wise continuous function with respect to $\mathscr{Z}$, i.e.,

$$\mathfrak{R}(\mathfrak{x}) = \sum_{i=1}^{n} \mathfrak{R}_i I_i(\mathfrak{x}) = \begin{cases} \mathfrak{R}_1, & \text{if } \mathfrak{x} \in \mathfrak{V}_1, \\ \mathfrak{R}_2, & \text{if } \mathfrak{x} \in \mathfrak{V}_2, \\ \vdots & \\ \mathfrak{R}_n, & \text{if } \mathfrak{x} \in \mathfrak{V}_\mathfrak{n}, \end{cases}$$

where $0 < \mathfrak{R}_i < +\infty$ are constants, and

$$I_i(\mathfrak{x}) = \begin{cases} 1, & \text{for } \mathfrak{x} \in \mathfrak{V}_i, \\ \\ 0, & \text{elsewhere.} \end{cases}$$

The notion $\mathfrak{BC}_i = \mathfrak{BC}([1, +\infty) \times \mathfrak{V}_i, \mathbb{R})$ denotes the Banach space of all continuous functions from $[1, \infty) \times \mathfrak{V}_i$ into $\mathbb{R}$, with the norm

$$\|\mathfrak{u}\|_{\mathfrak{BC}_i} = \sup_{(\mathfrak{s}, \mathfrak{x}) \in [1, +\infty) \times \mathfrak{V}_i} |\mathfrak{u}(\mathfrak{s}, \mathfrak{x})|, \quad i \in \{1, 2, \ldots, \mathfrak{n}\}.$$

Then, for any $\mathfrak{x} \in \mathfrak{V}_i$, $i = 1, 2, \ldots, \mathfrak{n}$, the function $\mathfrak{h}(\mathfrak{s}, \mathfrak{x}, s, \varsigma, \mathfrak{u}(s, \varsigma), \mathfrak{u}(\eta(s), \varsigma)) \in [1, +\infty) \times \mathfrak{V}_i \times [1, +\infty) \times \mathfrak{V}_i \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, defined by (3), might then be stated as

$$\begin{aligned} {}^{\mathscr{H}}I_{1^+}^{\mathfrak{R}(\mathfrak{s})}\mathfrak{h}(\mathfrak{s}, \mathfrak{x}, s, \varsigma, \mathfrak{u}(s, \varsigma), \mathfrak{u}(\eta(s), \varsigma)) &= \sum_{j=1}^{i-1} \int_{\mathfrak{K}_{j-1}}^{\mathfrak{K}_j} \frac{1}{\Gamma(\mathfrak{R}_j)} \left(\log \frac{\mathfrak{x}}{s}\right)^{\mathfrak{R}_{j-1}} \mathfrak{h}(\mathfrak{s}, \mathfrak{x}, s, \varsigma, \mathfrak{u}(s, \varsigma), \mathfrak{u}(\eta(s), \varsigma)) \frac{ds}{s} \\ &\quad + \int_{\mathfrak{K}_{j-1}}^{\mathfrak{x}} \frac{1}{\Gamma(\mathfrak{R}_j)} \left(\log \frac{\mathfrak{x}}{s}\right)^{\mathfrak{R}_{j-1}} \mathfrak{h}(\mathfrak{s}, \mathfrak{x}, s, \varsigma, \mathfrak{u}(s, \varsigma), \mathfrak{u}(\eta(s), \varsigma)) \frac{ds}{s}. \end{aligned} \tag{6}$$

Then

$$\begin{aligned} &\frac{1}{\Gamma(\mathfrak{M})} \int_1^{\gamma(\mathfrak{s})} \left({}^{\mathscr{H}}I_{1^+}^{\mathfrak{R}(\mathfrak{s})}\mathfrak{h}(\mathfrak{s}, \mathfrak{x}, s, \varsigma, \mathfrak{u}(s, \varsigma), \mathfrak{u}(\eta(s), \varsigma))\right) \frac{d\varsigma}{\varsigma} \\ &= \frac{1}{\Gamma(\mathfrak{M})} \int_1^{\gamma(\mathfrak{s})} \left(\log \frac{\gamma(\mathfrak{s})}{\varsigma}\right)^{\mathfrak{M}-1} \left(\sum_{j=1}^{i-1} \int_{\mathfrak{K}_{j-1}}^{\mathfrak{K}_j} \frac{1}{\Gamma(\mathfrak{R}_j)} \left(\log \frac{\mathfrak{x}}{s}\right)^{\mathfrak{R}_{j-1}} \mathfrak{h}(\mathfrak{s}, \mathfrak{x}, s, \varsigma, \mathfrak{u}(s, \varsigma), \mathfrak{u}(\eta(s), \varsigma)) \frac{ds}{s} \right. \\ &\quad \left. + \int_{\mathfrak{K}_{j-1}}^{\mathfrak{x}} \frac{1}{\Gamma(\mathfrak{R}_j)} \left(\log \frac{\mathfrak{x}}{s}\right)^{\mathfrak{R}_{j-1}} \mathfrak{h}(\mathfrak{s}, \mathfrak{x}, s, \varsigma, \mathfrak{u}(s, \varsigma), \mathfrak{u}(\eta(s), \varsigma)) \frac{ds}{s} \right) \frac{d\varsigma}{\varsigma}. \end{aligned} \tag{7}$$

According to (6), for any $\mathfrak{x} \in \mathfrak{V}_i$, $i \in \{1, 2, \ldots, n\}$, Equation (2) can be stated in the following format:

$$\begin{aligned} \mathfrak{u}(\mathfrak{s}, \mathfrak{x}) = f\big(s, x, u(s, x), u(\lambda(s), x)\big) + \frac{1}{\Gamma(\mathfrak{M})} \int_1^{\gamma(\mathfrak{s})} \left(\log \frac{\gamma(\mathfrak{s})}{\varsigma}\right)^{\mathfrak{M}-1} \times \left[\sum_{j=1}^{i-1} \int_{\mathfrak{K}_{j-1}}^{\mathfrak{K}_j} \frac{1}{\Gamma(\mathfrak{R}_j)} \left(\log \frac{\mathfrak{K}_j}{\tau}\right)^{\mathfrak{R}_j - 1} \right. \\ \left. \times \mathfrak{h}\big(\mathfrak{s}, \mathfrak{x}, \tau, \varsigma, \mathfrak{u}(\tau, \varsigma), \mathfrak{u}(\eta(\tau), \varsigma)\big) \frac{d\tau}{\tau} + \int_{\mathfrak{K}_{j-1}}^{\mathfrak{x}} \frac{1}{\Gamma(\mathfrak{R}_j)} \left(\log \frac{\mathfrak{x}}{\tau}\right)^{\mathfrak{R}_j - 1} \right. \end{aligned}$$

$$\times \mathfrak{h}\big(\mathfrak{s}, \mathfrak{x}, \tau, \varsigma, \mathfrak{u}(\tau, \varsigma), \mathfrak{u}(\eta(\tau), \varsigma)\big)\frac{d\tau}{\tau}\Big]\frac{d\varsigma}{\varsigma}. \tag{8}$$

Let $\mathfrak{u} \in \mathfrak{BC}_i$ be a solution of (8), such that $\mathfrak{u}(\mathfrak{s}, \mathfrak{x}) \equiv 0$ on $\mathfrak{x} \in [1, \mathfrak{V}_{i-1}]$. Then (8) is reduced to

$$\begin{aligned}\mathfrak{u}(\mathfrak{s}, \mathfrak{x}) = \mathfrak{f}(\mathfrak{s}, \mathfrak{x}, \mathfrak{u}(\mathfrak{s}, \mathfrak{x}), \mathfrak{u}(\lambda(\mathfrak{s}), \mathfrak{x}))\frac{1}{\Gamma(\mathfrak{M})}\int_1^{\gamma(\mathfrak{s})}\left(\log\frac{\gamma(\mathfrak{s})}{\varsigma}\right)^{\mathfrak{M}-1}\\ \times\int_{\mathfrak{K}_{j-1}}^{\mathfrak{x}}\frac{1}{\Gamma(\mathfrak{R}_j)}\left(\log\frac{\mathfrak{x}}{s}\right)^{\mathfrak{R}_j-1}\mathfrak{h}(\mathfrak{s}, \mathfrak{x}, s, \varsigma, \mathfrak{u}(s, \varsigma), \mathfrak{u}(\eta(s), \varsigma))\frac{ds}{s}\frac{d\varsigma}{\varsigma}.\end{aligned} \tag{9}$$

We now impose the following assumptions:

(1) The function λ: $[1, +\infty) \to [1, +\infty)$ fulfills $\lim_{\mathfrak{s}\to+\infty}\lambda(\mathfrak{s}) = +\infty$.

(2) There exist a nondecreasing function $\mathscr{V}_i$: $[0, \infty) \to (0, \infty)$, and constants $\mathscr{M}_i$, $\mathfrak{L}_i > 0$ verifying $\mathscr{M}_i < \frac{\mathfrak{L}_i}{2}$

$$|\mathfrak{f}(\mathfrak{s}, \mathfrak{x}, \mathfrak{u}_1, \mathfrak{v}_1) - \mathfrak{f}(\mathfrak{s}, \mathfrak{x}, \mathfrak{u}_2, \mathfrak{v}_2)| \le \frac{\mathscr{M}_{\mathfrak{i}}(|\mathfrak{u}_1 - \mathfrak{u}_2| + |\mathfrak{v}_1 - \mathfrak{v}_2|)}{(1+\lambda(\mathfrak{s}))(\mathfrak{L}_{\mathfrak{i}} + |\mathfrak{u}_1 - \mathfrak{u}_2| + |\mathfrak{v}_1 - \mathfrak{v}_2|)},$$

and

$$|\mathfrak{f}(\mathfrak{s}_1, \mathfrak{x}_1, \mathfrak{u}, \mathfrak{v}) - \mathfrak{f}(\mathfrak{s}_2, \mathfrak{x}_2, \mathfrak{u}, \mathfrak{v})| \le (|\mathfrak{s}_1 - \mathfrak{s}_2| + |\mathfrak{x}_1 - \mathfrak{x}_2|)\mathscr{V}_{\mathfrak{i}}(|\mathfrak{u}| + |\mathfrak{v}|),$$

for each $(\mathfrak{s}, \mathfrak{x})$, $(\mathfrak{s}_1, \mathfrak{x}_1)$, $(\mathfrak{s}_2, \mathfrak{x}_2) \in [1, \infty) \times \mathfrak{V}_i$ and $\mathfrak{u}, \mathfrak{v}, \mathfrak{u}_1, \mathfrak{v}_1, \mathfrak{u}_2, \mathfrak{v}_2 \in \mathbb{R}$.

(3) $\mathfrak{s} \mapsto \mathfrak{f}(\mathfrak{s}, \mathfrak{x}, 0, 0)$ is bounded function on $[1, +\infty) \times \mathfrak{V}_i$ satisfies

$$\mathfrak{f}^* = \sup_{(\mathfrak{s}, \mathfrak{x})\in[1, +\infty)\times\mathfrak{V}_{\mathfrak{i}}}|\mathfrak{f}(\mathfrak{s}, \mathfrak{x}, 0, 0)|$$

and

$$\lim_{\mathfrak{s}\to+\infty}|\mathfrak{f}(\mathfrak{s}, \mathfrak{x}, 0, 0)| = 0; \quad \mathfrak{x} \in \mathfrak{V}_{\mathfrak{i}}.$$

(4) There exist continuous functions $\mathfrak{p}_{\mathfrak{i}}$, $\mathfrak{q}_{\mathfrak{i}}$, $\mathscr{S}_{\mathfrak{i}}$, : $[1, +\infty) \times \mathfrak{V}_{\mathfrak{i}} \to \mathbb{R}_+$, and a function $\mathscr{O}_{\mathfrak{i}}$: $[0, +\infty) \to (0, +\infty)$, which is nondecreasing, in a manner that

$$|\mathfrak{h}(\mathfrak{s}_1, \mathfrak{x}_1, \mathfrak{s}, \varsigma, \mathfrak{u}, \mathfrak{v}) - \mathfrak{h}(\mathfrak{s}_2, \mathfrak{x}_2, \mathfrak{s}, \varsigma, \mathfrak{u}, \mathfrak{v})| \le \mathscr{S}_{\mathfrak{i}}(\mathfrak{s}, \varsigma)(|\mathfrak{s}_1 - \mathfrak{s}_2| + |\mathfrak{x}_1 - \mathfrak{x}_2|)\mathscr{O}_{\mathfrak{i}}(|\mathfrak{u}| + |\mathfrak{v}|),$$

and

$$|\mathfrak{h}(\mathfrak{s}, \mathfrak{x}, \mathfrak{s}, \mathfrak{y}, \mathfrak{u}, \mathfrak{v})| \le \frac{\mathfrak{p}_{\mathfrak{i}}(\mathfrak{s}, \mathfrak{x})\mathfrak{q}_{\mathfrak{i}}(\mathfrak{s}, \varsigma)}{1 + \lambda(\mathfrak{s}) + |\mathfrak{u}| + |\mathfrak{v}|},$$

for each $(\mathfrak{s}, \mathfrak{x})$, (s, ς), $(\mathfrak{s}_1, \mathfrak{x}_1)$, $(\mathfrak{s}_2, \mathfrak{x}_2) \in [1, +\infty) \times \mathfrak{V}_\mathrm{i}$ and $\mathfrak{u}, \mathfrak{v} \in \mathbb{R}$. Furthermore, suppose that

$$\lim_{\mathfrak{s}\to\infty} \mathfrak{p}_\mathrm{i}(\mathfrak{s}, \mathfrak{x}) \int_1^{\gamma(\mathfrak{s})} \int_{\mathfrak{K}_{\mathrm{j}\text{-}1}}^{\mathfrak{x}} \left|\log \frac{\gamma(\mathfrak{s})}{\mathrm{s}}\right|^{\mathfrak{M}-1} \left|\log \frac{\mathfrak{x}}{\varsigma}\right|^{\mathfrak{R}_\mathrm{j}-1} \mathfrak{q}_\mathrm{i}(\mathrm{s}, \varsigma)\, \mathrm{d}\varsigma\, \mathrm{ds} = 0.$$

Theorem 2 Under assumptions (1)–(4), there exists at least one solution to the integral Equation (2) in the function space $\mathfrak{BC}_i$. Furthermore, all solutions of (2) exhibit global asymptotic stability.

Proof. Set $\mathfrak{d}_\mathrm{i}^* := \sup_{(\mathfrak{s}, \mathfrak{x}) \in [1, \infty) \times \mathfrak{V}_\mathrm{i}} \mathfrak{d}_\mathrm{i}(\mathfrak{s}, \mathfrak{x})$ where

$$\mathfrak{d}_\mathrm{i}(\mathfrak{s}, \mathfrak{x}) = \frac{\mathfrak{p}_\mathrm{i}(\mathfrak{s}, \mathfrak{x})}{\Gamma(\mathfrak{M})\Gamma(\mathfrak{R}_\mathrm{j})} \int_1^{\gamma(\mathfrak{s})} \int_{\mathfrak{K}_{\mathrm{j}\text{-}1}}^{\mathfrak{x}} \left|\log \frac{\gamma(\mathfrak{s})}{\mathrm{s}}\right|^{\mathfrak{M}-1} \left|\log \frac{\mathfrak{x}}{\varsigma}\right|^{\mathfrak{R}_\mathrm{j}-1} \mathfrak{q}_\mathrm{i}(\mathfrak{s}, \varsigma)\, \mathrm{d}\varsigma\, \mathrm{ds}.$$

Based on Hypothesis (4), we deduce the finiteness of $\mathfrak{d}_i^*$.

We now introduce the operator $\mathfrak{N}$ as follows: for any $\mathfrak{u} \in \mathfrak{BC}_i$, put

$$\begin{aligned}(\mathfrak{N}\mathfrak{u})(\mathfrak{s}, \mathfrak{x}) = \mathfrak{f}(\mathfrak{s}, \mathfrak{x}, \mathfrak{u}(\mathfrak{s}, \mathfrak{x}), \mathfrak{u}(\lambda(\mathfrak{s}), \mathfrak{x})) + \frac{1}{\Gamma(\mathfrak{M})} \int_1^{\gamma(\mathfrak{s})} \left(\log \frac{\gamma(\mathfrak{s})}{\varsigma}\right)^{\mathfrak{M}-1} \\ \times \int_{\mathfrak{K}_{j-1}}^{\mathfrak{x}} \frac{1}{\Gamma(\mathfrak{R}_j)} \left(\log \frac{\mathfrak{x}}{s}\right)^{\mathfrak{R}_j-1} \mathfrak{h}(\mathfrak{s}, \mathfrak{x}, s, \varsigma, \mathfrak{u}(s, \varsigma), \mathfrak{u}(\eta(s), \varsigma)) \frac{ds}{s} \frac{d\varsigma}{\varsigma}.\end{aligned} \tag{10}$$

Building upon the foundational assumptions (1)–(4), we first establish the continuity of the operator $(\mathfrak{N}\mathfrak{u})$ on the function space $\mathfrak{BC}_i$.for any $\mathfrak{u} \in \mathfrak{BC}_\mathrm{i}$.

Let $(\mathfrak{s}, \mathfrak{x})$ be an arbitrary point in the domain $[1, +\infty) \times \mathfrak{V}_i$. Through careful estimation, we derive the following key inequality:

$$\begin{aligned}|(\mathfrak{N}\mathfrak{u})(\mathfrak{s}, \mathfrak{x})| &\le |\mathfrak{f}(\mathfrak{s}, \mathfrak{x}, \mathfrak{u}(\mathfrak{s}, \mathfrak{x}), \mathfrak{u}(\lambda(\mathfrak{s}), \mathfrak{x})) - \mathfrak{f}(\mathfrak{s}, \mathfrak{x}, 0, 0)| + |\mathfrak{f}(\mathfrak{s}, \mathfrak{x}, 0, 0)| \\ &\quad + \frac{1}{\Gamma(\mathfrak{M})\Gamma(\mathfrak{R}_j)} \int_1^{\gamma(\mathfrak{s})} \int_{\mathfrak{K}_{\mathrm{j}-1}}^{\mathfrak{x}} \left|\log \frac{\gamma(\mathfrak{s})}{\varsigma}\right|^{\mathfrak{M}-1} \left|\log \frac{\mathfrak{x}}{\mathrm{s}}\right|^{\mathfrak{R}_\mathrm{j}-1} \\ &\quad \times |\mathfrak{h}(\mathfrak{s}, \mathfrak{x}, \mathrm{s}, \varsigma, \mathfrak{u}(\mathrm{s}, \varsigma), \mathfrak{u}(\eta(\mathrm{s}), \varsigma))| \frac{\mathrm{d}\varsigma\, \mathrm{ds}}{\mathrm{s}\varsigma} \\ &\le \frac{\mathscr{M}_\mathrm{i}(|\mathfrak{u}(\mathfrak{s}, \mathfrak{x})| + |\mathfrak{u}(\lambda(\mathfrak{s}), \mathfrak{x})|)}{(1 + \lambda(\mathfrak{s}))(\mathfrak{L}_\mathrm{i} + |\mathfrak{u}(\mathfrak{s}, \mathfrak{x})| + |\mathfrak{u}(\lambda(\mathfrak{s}), \mathfrak{x})|)} + |\mathfrak{f}(\mathfrak{s}, \mathfrak{x}, 0, 0)| \\ &\quad + \frac{\mathfrak{p}_\mathrm{i}(\mathfrak{s}, \mathfrak{x})}{\Gamma(\mathfrak{M})\Gamma(\mathfrak{R}_j)} \int_1^{\gamma(\mathfrak{s})} \int_{\mathfrak{K}_{j-1}}^{\mathfrak{x}} \left|\log \frac{\gamma(\mathfrak{s})}{\varsigma}\right|^{\mathfrak{M}-1} \left|\log \frac{\mathfrak{x}}{s}\right|^{\mathfrak{R}_j-1} \\ &\quad \times \frac{\mathfrak{q}_\mathrm{i}(\mathfrak{s}, \varsigma)}{1 + \lambda(\mathfrak{s}) + |\mathfrak{u}(\mathrm{s}, \varsigma)| + |\mathfrak{u}(\eta(\mathrm{s}), \varsigma)|} \mathrm{d}\varsigma\, \mathrm{ds}\end{aligned}$$

$$\leq \frac{\mathscr{M}_{\mathfrak{i}}(|\mathfrak{u}(\mathfrak{s},\mathfrak{x})|+|\mathfrak{u}(\lambda(\mathfrak{s}),\mathfrak{x})|)}{|\mathfrak{u}(\mathfrak{s},\mathfrak{x})|+|\mathfrak{u}(\lambda(\mathfrak{s}),\mathfrak{x})|}+\mathfrak{f}^{*}+\mathfrak{d}_{\mathfrak{i}}^{*}.$$

Thus

$$\|\mathfrak{N}(\mathfrak{u})\|_{\mathfrak{B}\mathfrak{C}_{\mathfrak{i}}} \leq \mathscr{M}_{\mathfrak{i}}+\mathfrak{f}^{*}+\mathfrak{d}_{\mathfrak{i}}^{*}. \tag{11}$$

Consequently, we have established that $\mathfrak{N}(\mathfrak{u}) \in \mathfrak{B}\mathfrak{C}_i$ for all $\mathfrak{u} \in \mathfrak{B}\mathfrak{C}_i$. From Equation (11), we deduce that the operator $\mathfrak{N}$ leaves invariant the closed ball $\mathfrak{B}_{\rho_i} := \mathfrak{B}(0, \rho_i)$, where the radius is given by $\rho_i = \mathscr{M}_i + f^* + d_i^*$.

We now verify that the restriction $\mathfrak{N}: \mathfrak{B}\rho_i \to \mathfrak{B}\rho_i$ meets all the requisite conditions of Schauder's fixed point theorem (Theorem 1). The verification will proceed through the following steps:

Step 1: $\mathfrak{N}$ is continuous.

Given a sequence $\{\mathfrak{u}_{\mathfrak{n}}\}_{\mathfrak{n}\in\mathbb{N}}$ in $\mathfrak{B}_{\rho_{\mathfrak{i}}}$ such that $\mathfrak{u}_{\mathfrak{n}} \to \mathfrak{u}$, it follows that for all $(\mathfrak{s},\mathfrak{x}) \in [1,+\infty)\times\mathfrak{V}_{\mathfrak{i}}$:

$$\begin{aligned}
|(\mathfrak{N}\mathfrak{u}_{\mathfrak{n}})(\mathfrak{s},\mathfrak{x})-(\mathfrak{N}\mathfrak{u})(\mathfrak{s},\mathfrak{x})| &\leq |\mathfrak{f}(\mathfrak{s},\mathfrak{x},\mathfrak{u}_{\mathfrak{n}}(\mathfrak{s},\mathfrak{x}),\mathfrak{u}_{\mathfrak{n}}(\lambda(\mathfrak{s}),\mathfrak{x}))-\mathfrak{f}(\mathfrak{s},\mathfrak{x},\mathfrak{u}(\mathfrak{s},\mathfrak{x}),\mathfrak{u}(\lambda(\mathfrak{s}),\mathfrak{x}))| \\
&\quad+\frac{1}{\Gamma(\mathfrak{M})\Gamma(\mathfrak{R}_j)}\int_1^{\gamma(\mathfrak{s})}\int_{\mathfrak{K}_{\mathfrak{j}-1}}^{\mathfrak{x}}\left|\log\frac{\gamma(\mathfrak{s})}{\varsigma}\right|^{\mathfrak{M}-1}\left|\log\frac{\mathfrak{x}}{\mathfrak{s}}\right|^{\mathfrak{R}_{\mathfrak{j}}-1} \\
&\quad\times\sup_{(\mathfrak{s},\varsigma)\in[1,+\infty)\times\mathfrak{V}_{\mathfrak{i}}}|\mathfrak{h}(\mathfrak{s},\mathfrak{x},\mathfrak{s},\varsigma,\mathfrak{u}_{\mathfrak{n}}(\mathfrak{s},\varsigma),\mathfrak{u}_{\mathfrak{n}}(\eta(\mathfrak{s}),\varsigma)) \\
&\quad-\mathfrak{h}(\mathfrak{s},\mathfrak{x},\mathfrak{s},\varsigma,\mathfrak{u}(\mathfrak{s},\varsigma),\mathfrak{u}(\eta(\mathfrak{s}),\varsigma))|\frac{\mathrm{d}\varsigma\,\mathrm{d}\mathfrak{s}}{\mathfrak{s}\varsigma} \\
&\leq \frac{2\mathscr{M}_{\mathfrak{i}}}{\mathfrak{L}_{\mathfrak{i}}}\|\mathfrak{u}_{\mathfrak{n}}-\mathfrak{u}\|_{\mathfrak{B}\mathfrak{C}_{\mathfrak{i}}} \\
&\quad+\frac{1}{\Gamma(\mathfrak{M})\Gamma(\mathfrak{R}_j)}\int_1^{\gamma(\mathfrak{s})}\int_{\mathfrak{K}_{\mathfrak{j}-1}}^{\mathfrak{x}}\left|\log\frac{\gamma(\mathfrak{s})}{\varsigma}\right|^{\mathfrak{M}-1}\left|\log\frac{\mathfrak{x}}{\mathfrak{s}}\right|^{\mathfrak{R}_{\mathfrak{j}}-1} \\
&\quad\times\|\mathfrak{h}(\mathfrak{s},\mathfrak{x},\cdot,\cdot,\mathfrak{u}_{\mathfrak{n}}(\cdot,\cdot),\mathfrak{u}_{\mathfrak{n}}(\eta(\cdot),\cdot)) \\
&\quad-\mathfrak{h}(\mathfrak{s},\mathfrak{x},\cdot,\cdot,\mathfrak{u}(\cdot,\cdot),\mathfrak{u}(\eta(\cdot),\cdot))\|_{\mathfrak{B}\mathfrak{C}_{\mathfrak{i}}}d\varsigma ds
\end{aligned}$$

At this stage, we will study two cases of the domain. The first case is a bounded domain $[1, \mathfrak{T}]$, and we will use the same method as before.

Case 1: $(\mathfrak{s},\mathfrak{x}) \in \mathfrak{Y}_{\mathbf{i}} \times \mathfrak{V}_{\mathbf{i}}$ since $\mathfrak{u}_{\mathfrak{n}} \to \mathfrak{u}$ as $\mathfrak{n} \to +\infty$ and $\mathfrak{h}$, η are continuous, the above inequality yields:

$$\|\mathfrak{N}(u_n)-\mathfrak{N}(u)\|_{\mathfrak{B}\mathfrak{C}_i} \to 0 \text{ as } n \to +\infty.$$

Case 2: $(\mathfrak{s},\mathfrak{x}) \in [\mathfrak{T},+\infty]\times\mathfrak{V}_{\mathbf{i}}$,$\mathfrak{T} > 1$ then from (4) and the earlier estimate, for each $(\mathfrak{s},\mathfrak{x}) \in [1,+\infty)\times\mathfrak{V}_{\mathfrak{i}}$:

$$|(\mathfrak{N}u_n)(\mathfrak{s},\mathfrak{x})-(\mathfrak{N}u)(\mathfrak{s},\mathfrak{x})| \leq \frac{2\mathscr{M}_i}{\mathfrak{L}_i}\|u_n-u\|_{\mathfrak{B}\mathfrak{C}_i}+\frac{2\mathfrak{p}_i(\mathfrak{s},\mathfrak{x})}{\Gamma(\mathfrak{M})\Gamma(\mathfrak{R}_j)}$$

$$\times\int_1^{\gamma(\mathfrak{s})}\int_{\mathfrak{K}_{j-1}}^{\mathfrak{x}}\left|\log\frac{\gamma(\mathfrak{s})}{s}\right|^{\mathfrak{M}-1}\left|\log\frac{\mathfrak{x}}{\varsigma}\right|^{\mathfrak{R}_j-1}\mathfrak{q}_i(\mathfrak{s},\varsigma)\,d\varsigma\,ds.$$

$$\leq \frac{2\mathscr{M}_i}{\mathfrak{L}_i}\|u_n-u\|_{\mathfrak{B}\mathfrak{C}_i}+\mathfrak{d}_i(\mathfrak{s},\mathfrak{x}).$$

Thus:

$$|(\mathfrak{N}u_n)(\mathfrak{s},\mathfrak{x})-(\mathfrak{N}u)(\mathfrak{s},\mathfrak{x})| \leq \frac{2\mathscr{M}_i}{\mathfrak{L}_i}\|u_n-u\|_{\mathfrak{B}\mathfrak{C}_i}+\mathfrak{d}_i(\mathfrak{s},\mathfrak{x}).$$

Since $u_n \to u$ as $n \to +\infty$ and $\mathfrak{s} \to +\infty$, we conclude:

$$\|\mathfrak{N}(u_n)-\mathfrak{N}(u)\|_{\mathfrak{B}\mathfrak{C}_i} \to 0 \text{ as } n \to +\infty.$$

Step 2: $\mathfrak{N}(\mathfrak{B}_{\rho_i})$ is uniformly bounded since $\mathfrak{N}(\mathfrak{B}_{\rho_i}) \subseteq \mathfrak{B}_{\rho_i}$ and $\mathfrak{B}_{\rho_i}$ is bounded.

Step 3: Equicontinuity on Compact Subsets $\mathfrak{Y}_i \times \mathfrak{V}_i$.

Let $(\mathfrak{s}_1,\mathfrak{x}_1),(\mathfrak{s}_2,\mathfrak{x}_2) \in \mathfrak{Y}_i \times \mathfrak{V}_i$ with $\mathfrak{s}_1 < \mathfrak{s}_2, \mathfrak{x}_1 < \mathfrak{x}_2$. Assume without loss of generality that $\gamma(\mathfrak{s}_1) \leq \gamma(\mathfrak{s}_2)$. For $u \in \mathfrak{B}_{\rho_i}$:

$$|(\mathfrak{N}u)(\mathfrak{s}_2,\mathfrak{x}_2)-(\mathfrak{N}u)(\mathfrak{s}_1,\mathfrak{x}_1)| \leq |\mathfrak{f}(\mathfrak{s}_2,\mathfrak{x}_2,u(\mathfrak{s}_2,\mathfrak{x}_2),u(\lambda(\mathfrak{s}_2),\mathfrak{x}_2))-\mathfrak{f}(\mathfrak{s}_2,\mathfrak{x}_2,u(\mathfrak{s}_1,\mathfrak{x}_1),u(\lambda(\mathfrak{s}_1),\mathfrak{x}_1))|$$

$$+|\mathfrak{f}(\mathfrak{s}_2,\mathfrak{x}_2,u(\mathfrak{s}_1,\mathfrak{x}_1),u(\lambda(\mathfrak{s}_1),\mathfrak{x}_1))-\mathfrak{f}(\mathfrak{s}_1,\mathfrak{x}_1,u(\mathfrak{s}_1,\mathfrak{x}_1),u(\lambda(\mathfrak{s}_1),\mathfrak{x}_1))|$$

$$+\frac{1}{\Gamma(\mathfrak{M})\Gamma(\mathfrak{R}_j)}\int_1^{\gamma(\mathfrak{s}_2)}\int_{\mathfrak{K}_{j-1}}^{\mathfrak{x}_2}\left|\log\frac{\gamma(\mathfrak{s}_2)}{s}\right|^{\mathfrak{M}-1}\left|\log\frac{\mathfrak{x}_2}{\varsigma}\right|^{\mathfrak{R}_j-1}$$

$$\times|\mathfrak{h}(\mathfrak{s}_2,\mathfrak{x}_2,s,\varsigma,u(s,\varsigma),u(\eta(s),\varsigma))-\mathfrak{h}(\mathfrak{s}_1,\mathfrak{x}_1,s,\varsigma,u(s,\varsigma),u(\eta(s),\varsigma))\frac{d\varsigma\,ds}{s\varsigma}$$

$$+\left|\frac{1}{\Gamma(\mathfrak{M})\Gamma(\mathfrak{R}_j)}\int_1^{\gamma(\mathfrak{s}_2)}\int_{\mathfrak{K}_{j-1}}^{\mathfrak{x}_2}\left|\log\frac{\gamma(\mathfrak{s}_2)}{s}\right|^{\mathfrak{M}-1}\left|\log\frac{\mathfrak{x}_2}{\varsigma}\right|^{\mathfrak{R}_j-1}\right.$$

$$\left.\times\mathfrak{h}(\mathfrak{s}_1,\mathfrak{x}_1,s,\varsigma,u(s,\varsigma),u(\eta(s),\varsigma))|\,d\varsigma ds\right.$$

$$-\frac{1}{\Gamma(\mathfrak{M})\Gamma(\mathfrak{R}_j)}\int_1^{\gamma(\mathfrak{s}_1)}\int_{\mathfrak{K}_{j-1}}^{\mathfrak{x}_1}\left|\log\frac{\gamma(\mathfrak{s}_2)}{s}\right|^{\mathfrak{M}-1}\left|\log\frac{\mathfrak{x}_2}{\varsigma}\right|^{\mathfrak{R}_j-1}$$

$$\times \mathfrak{h}(\mathfrak{s}_1, \mathfrak{x}_1, s, \varsigma, u(s, \varsigma), u(\eta(s), \varsigma))| d\varsigma ds$$

$$+ \frac{1}{\Gamma(\mathfrak{M})\Gamma(\mathfrak{R}_j)} \int_1^{\gamma(\mathfrak{s}_1)} \int_{\mathfrak{K}_{j-1}}^{\mathfrak{x}_1} \left| \left(\log \frac{\gamma(\mathfrak{s}_2)}{s} \right)^{\mathfrak{M}-1} \left(\log \frac{\mathfrak{x}_2}{\varsigma} \right)^{\mathfrak{R}_j - 1} \right.$$

$$\left. - \left(\log \frac{\gamma(\mathfrak{s}_1)}{s} \right)^{\mathfrak{M}-1} \left(\log \frac{\mathfrak{x}_1}{\varsigma} \right)^{\mathfrak{R}_J - 1} \right| |\mathfrak{h}(\mathfrak{s}_1, \mathfrak{x}_1, s, \varsigma, u(s, \varsigma), u(\eta(s), \varsigma))| d\varsigma ds.$$

This establishes the equicontinuity of $\mathfrak{N}(\mathfrak{B}_{\rho_i})$ on $\mathfrak{Y}_i \times \mathfrak{V}_i$.

$$|(\mathfrak{N}u)(\mathfrak{s}_2, \mathfrak{x}_2) - (\mathfrak{N}u)(\mathfrak{s}_1, \mathfrak{x}_1)| \leq \frac{\mathscr{M}_i}{\mathfrak{L}_i} (|u(\mathfrak{s}_2, \mathfrak{x}_2) - u(\mathfrak{s}_1, \mathfrak{x}_1)| + |u(\lambda(\mathfrak{s}_2), \mathfrak{x}_2) - u(\lambda(\mathfrak{s}_1), \mathfrak{x}_1)|)$$

$$+ (|\mathfrak{s}_2 - \mathfrak{s}_1| + |\mathfrak{x}_2 - \mathfrak{x}_1|) \mathscr{V}_i(2\|u\|_{\mathfrak{B}\mathfrak{C}_i})$$

$$+ \frac{1}{\Gamma(\mathfrak{M})\Gamma(\mathfrak{R}_j)} \int_1^{\gamma(\mathfrak{s}_2)} \int_{\mathfrak{K}_{j-1}}^{\mathfrak{x}_2} \left| \log \frac{\gamma(\mathfrak{s}_2)}{s} \right|^{\mathfrak{M}-1} \left| \log \frac{\mathfrak{x}_2}{\varsigma} \right|^{\mathfrak{R}_j - 1}$$

$$\times \mathscr{S}_i(s, \varsigma)(|\mathfrak{s}_2 - \mathfrak{s}_1| + |\mathfrak{x}_2 - \mathfrak{x}_1|) \mathscr{O}_i(2\|u\|_{\mathfrak{B}\mathfrak{C}_i}) d\varsigma \, ds$$

$$+ \frac{1}{\Gamma(\mathfrak{M})\Gamma(\mathfrak{R}_j)} \int_{\gamma(\mathfrak{s}_1)}^{\gamma(\mathfrak{s}_2)} \int_{\mathfrak{K}_{j-1}}^{\mathfrak{x}_2} \left| \log \frac{\gamma(\mathfrak{s}_2)}{s} \right|^{\mathfrak{M}-1} \left| \log \frac{\mathfrak{x}_2}{\varsigma} \right|^{\mathfrak{R}_j - 1}$$

$$\times |\mathfrak{h}(\mathfrak{s}_2, \mathfrak{x}_2, s, \varsigma, u(s, \varsigma), u(\eta(s), \varsigma)) - \mathfrak{h}(\mathfrak{s}_1, \mathfrak{x}_1, s, \varsigma, u(s, \varsigma), u(\eta(s), \varsigma)) \frac{d\varsigma \, ds}{s\varsigma}$$

$$+ \left| \frac{1}{\Gamma(\mathfrak{M})\Gamma(\mathfrak{R}_j)} \int_1^{\gamma(\mathfrak{s}_2)} \int_{\mathfrak{K}_{j-1}}^{\mathfrak{x}_2} \left| \log \frac{\gamma(\mathfrak{s}_2)}{s} \right|^{\mathfrak{M}-1} \left| \log \frac{\mathfrak{x}_2}{\varsigma} \right|^{\mathfrak{R}_j - 1} \right.$$

$$\times \mathfrak{h}(\mathfrak{s}_1, \mathfrak{x}_1, s, \varsigma, u(s, \varsigma), u(\eta(s), \varsigma))| d\varsigma ds$$

$$- \frac{1}{\Gamma(\mathfrak{M})\Gamma(\mathfrak{R}_j)} \int_{\gamma(\mathfrak{s}_1)}^{\gamma(\mathfrak{s}_2)} \int_{\mathfrak{x}_1}^{\mathfrak{x}_2} \left| \log \frac{\gamma(\mathfrak{s}_2)}{s} \right|^{\mathfrak{M}-1} \left| \log \frac{\mathfrak{x}_2}{\varsigma} \right|^{\mathfrak{R}_j - 1}$$

$$\times \mathfrak{h}(\mathfrak{s}_1, \mathfrak{x}_1, s, \varsigma, u(s, \varsigma), u(\eta(s), \varsigma))| d\varsigma ds$$

$$+ \frac{1}{\Gamma(\mathfrak{M})\Gamma(\mathfrak{R}_j)} \int_1^{\gamma(\mathfrak{s}_1)} \int_{\mathfrak{K}_{j-1}}^{\mathfrak{x}_1} \left| \left(\log \frac{\gamma(\mathfrak{s}_2)}{s} \right)^{\mathfrak{M}-1} \left(\log \frac{\mathfrak{x}_2}{\varsigma} \right)^{\mathfrak{R}_j - 1} \right.$$

$$- \left(\log \frac{\gamma(\mathfrak{s}_1)}{\mathrm{s}}\right)^{\mathfrak{M}-1} \left(\log \frac{\mathfrak{x}_1}{\varsigma}\right)^{\mathfrak{R}_j-1} \Bigg| |\mathfrak{h}(\mathfrak{s}_1, \mathfrak{x}_1, \mathrm{s}, \varsigma, \mathfrak{u}(\mathrm{s}, \varsigma), \mathfrak{u}(\eta(\mathrm{s}), \varsigma))| d\varsigma ds.$$

Hence, we obtain:

$$\begin{aligned}
|(\mathfrak{N}\mathfrak{u})(\mathfrak{s}_2, \mathfrak{x}_2) - (\mathfrak{N}\mathfrak{u})(\mathfrak{s}_1, \mathfrak{x}_1)| \leq & \frac{\mathscr{M}_{\mathrm{i}}}{\mathfrak{L}_{\mathrm{i}}} \left(|\mathfrak{u}(\mathfrak{s}_2, \mathfrak{x}_2) - \mathfrak{u}(\mathfrak{s}_1, \mathfrak{x}_1)| + |\mathfrak{u}(\lambda(\mathfrak{s}_2), \mathfrak{x}_2) - \mathfrak{u}(\lambda(\mathfrak{s}_1), \mathfrak{x}_1)|\right) \\
& + (|\mathfrak{s}_2 - \mathfrak{s}_1| + |\mathfrak{x}_2 - \mathfrak{x}_1|)\mathscr{V}_{\mathrm{i}}(2\eta) + \frac{(|\mathfrak{s}_2 - \mathfrak{s}_1| + |\mathfrak{x}_2 - \mathfrak{x}_1|)\mathscr{O}_{\mathrm{i}}(2\eta)}{\Gamma(\mathfrak{M})\Gamma(\mathfrak{R}_j)} \\
& \times \int_1^{\gamma(\mathfrak{s}_2)} \int_{\mathfrak{K}_{j-1}}^{\mathfrak{x}_2} \left|\log \frac{\gamma(\mathfrak{s}_2)}{\mathrm{s}}\right|^{\mathfrak{M}-1} \left|\log \frac{\mathfrak{x}_2}{\varsigma}\right|^{\mathfrak{R}_j-1} \mathscr{S}_{\mathrm{i}}(\mathrm{s}, \varsigma) \mathrm{d}\varsigma \, \mathrm{ds} \\
& + \frac{\mathfrak{p}_{\mathrm{i}}(\mathfrak{s}_1, \mathfrak{x}_1)}{\Gamma(\mathfrak{M})\Gamma(\mathfrak{R}_j)} \int_{\gamma(\mathfrak{s}_1)}^{\gamma(\mathfrak{s}_2)} \int_{\mathfrak{K}_{j-1}}^{\mathfrak{x}_2} \left|\log \frac{\gamma(\mathfrak{s}_2)}{\mathrm{s}}\right|^{\mathfrak{M}-1} \left|\log \frac{\mathfrak{x}_2}{\varsigma}\right|^{\mathfrak{R}_j-1} \mathfrak{q}_{\mathrm{i}}(\mathrm{s}, \varsigma) \mathrm{d}\varsigma \, \mathrm{ds} \\
& + \frac{\mathfrak{p}_{\mathrm{i}}(\mathfrak{s}_1, \mathfrak{x}_1)}{\Gamma(\mathfrak{M})\Gamma(\mathfrak{R}_j)} \int_1^{\gamma(\mathfrak{s}_2)} \int_{\mathfrak{x}_1}^{\mathfrak{x}_2} \left|\log \frac{\gamma(\mathfrak{s}_2)}{\mathrm{s}}\right|^{\mathfrak{M}-1} \left|\log \frac{\mathfrak{x}_2}{\varsigma}\right|^{\mathfrak{R}_j-1} \mathfrak{q}_{\mathrm{i}}(\mathrm{s}, \varsigma) \mathrm{d}\varsigma \, \mathrm{ds} \\
& + \frac{\mathfrak{p}_{\mathrm{i}}(\mathfrak{s}_1, \mathfrak{x}_1)}{\Gamma(\mathfrak{M})\Gamma(\mathfrak{R}_j)} \int_{\gamma(\mathfrak{s}_1)}^{\gamma(\mathfrak{s}_2)} \int_{\mathfrak{x}_1}^{\mathfrak{x}_2} \left|\log \frac{\gamma(\mathfrak{s}_2)}{\mathrm{s}}\right|^{\mathfrak{M}-1} \left|\log \frac{\mathfrak{x}_2}{\varsigma}\right|^{\mathfrak{R}_j-1} \mathfrak{q}_{\mathrm{i}}(\mathrm{s}, \varsigma) \mathrm{d}\varsigma \, \mathrm{ds} \\
& + \frac{\mathfrak{p}_{\mathrm{i}}(\mathfrak{s}_1, \mathfrak{x}_1)}{\Gamma(\mathfrak{M})\Gamma(\mathfrak{R}_j)} \int_1^{\gamma(\mathfrak{s}_2)} \int_{\mathfrak{K}_{j-1}}^{\mathfrak{x}_2} \left| \left(\log \frac{\gamma(\mathfrak{s}_2)}{\varsigma}\right)^{\mathfrak{M}-1} \left(\log \frac{\mathfrak{x}_2}{\mathrm{s}}\right)^{\mathfrak{R}_j-1} \right. \\
& \left. - \left(\log \frac{\gamma(\mathfrak{s}_1)}{\varsigma}\right)^{\mathfrak{M}-1} \left(\log \frac{\mathfrak{x}_2}{\mathrm{s}}\right)^{\mathfrak{R}_j-1} \right| \mathfrak{q}_{\mathrm{i}}(\mathrm{s}, \varsigma) \mathrm{d}\varsigma \, \mathrm{ds}.
\end{aligned}$$

From the continuity of λ, γ, $\mathfrak{f}$, $\mathfrak{h}$ and as $\mathfrak{s}_1 \to \mathfrak{s}_2$ and $\mathfrak{x}_1 \to \mathfrak{x}_2$, the right-hand side of the inequality vanishes.
Step 4: $\mathfrak{N}(\mathfrak{B}_{\rho_{\mathrm{i}}})$ is equiconvergent.
Let $(\mathfrak{s}, \mathfrak{x}) \in [1, +\infty) \times \mathfrak{V}_{\mathrm{i}}$ and $\mathfrak{u} \in \mathfrak{B}_{\rho_{\mathrm{i}}}$, then we have:

$$\begin{aligned}
|(\mathfrak{N}\mathfrak{u})(\mathfrak{s}, \mathfrak{x})| \leq & |\mathfrak{f}(\mathfrak{s}, \mathfrak{x}, \mathfrak{u}(\mathfrak{s}, \mathfrak{x}), \mathfrak{u}(\lambda(\mathfrak{s}), \mathfrak{x})) - \mathfrak{f}(\mathfrak{s}, \mathfrak{x}, 0, 0)| + |\mathfrak{f}(\mathfrak{s}, \mathfrak{x}, 0, 0)| \\
& + \frac{1}{\Gamma(\mathfrak{M})\Gamma(\mathfrak{R}_j)} \int_1^{\gamma(\mathfrak{s})} \int_{\mathfrak{K}_{\mathrm{j}-1}}^{\mathfrak{x}} \left|\log \frac{\gamma(\mathfrak{s})}{\mathrm{s}}\right|^{\mathfrak{M}-1} \left|\log \frac{\mathfrak{x}}{\varsigma}\right|^{\mathfrak{R}_{\mathrm{j}}-1} \\
& \times |\mathfrak{h}(\mathfrak{s}, \mathfrak{x}, \mathrm{s}, \varsigma, \mathfrak{u}(\mathrm{s}, \varsigma), \mathfrak{u}(\eta(\mathrm{s}), \varsigma))| \frac{\mathrm{d}\varsigma \, \mathrm{ds}}{\mathrm{s}\varsigma}
\end{aligned}$$

$$\leq \frac{\mathscr{M}_{\mathfrak{i}}(|u(\mathfrak{s},\mathfrak{x})|+|u(\lambda(\mathfrak{s}),\mathfrak{x})|)}{(1+\lambda(\mathfrak{s}))(\mathfrak{L}_{\mathfrak{i}}+|u(\mathfrak{s},\mathfrak{x})|+|u(\lambda(\mathfrak{s}),\mathfrak{x})|)}+|\mathfrak{f}(\mathfrak{s},\mathfrak{x},0,0)|$$

$$+\frac{\mathfrak{p}_{\mathfrak{i}}(\mathfrak{s},\mathfrak{x})}{\Gamma(\mathfrak{M})\Gamma(\mathfrak{R}_j)}\int_1^{\gamma(\mathfrak{s})}\int_{\mathfrak{K}_{j-1}}^{\mathfrak{x}}\left|\log\frac{\gamma(\mathfrak{s})}{\mathrm{s}}\right|^{\mathfrak{M}-1}\left|\log\frac{\mathfrak{x}}{\varsigma}\right|^{\mathfrak{R}_j-1}$$

$$\times\frac{\mathfrak{q}_{\mathfrak{i}}(\mathfrak{s},\varsigma)}{1+\lambda(\mathfrak{s})+|u(\mathrm{s},\varsigma)|+|u(\eta(\mathrm{s}),\varsigma)|}\mathrm{d}\varsigma\,\mathrm{ds}$$

$$\leq \frac{\mathscr{M}_{\mathfrak{i}}}{1+\lambda(\mathfrak{s})}+|\mathfrak{f}(\mathfrak{s},\mathfrak{x},0,0)|$$

$$+\frac{\mathfrak{p}_{\mathfrak{i}}(\mathfrak{s},\mathfrak{x})}{\Gamma(\mathfrak{M})\Gamma(\mathfrak{R}_j)(1+\lambda(\mathfrak{s}))}\int_1^{\gamma(\mathfrak{s})}\int_{\mathfrak{K}_{j-1}}^{\mathfrak{x}}\left|\log\frac{\gamma(\mathfrak{s})}{\varsigma}\right|^{\mathfrak{M}-1}\left|\log\frac{\mathfrak{x}}{s}\right|^{\mathfrak{R}_j-1}d\varsigma ds$$

$$\leq \frac{\mathscr{M}_{\mathfrak{i}}}{1+\lambda(\mathfrak{s})}+|\mathfrak{f}(\mathfrak{s},\mathfrak{x},0,0)|+\frac{\mathfrak{d}_i^*}{1+\lambda(\mathfrak{s})}.$$

Hence, the following holds for all $\mathfrak{x}\in\mathfrak{V}_{\mathfrak{i}}$:

$$|u(\mathfrak{s},\mathfrak{x})|\to 0 \text{ as } \mathfrak{s}\to+\infty.$$

Hence,

$$|u(\mathfrak{s},\mathfrak{x})-u(+\infty,\mathfrak{x})|\to 0 \text{ as } \mathfrak{s}\to+\infty.$$

Our arguments in Steps (1)-(4), combined with Lemma 1, prove the continuity and compactness of $\mathfrak{N}$ on $\mathfrak{B}_{\rho_{\mathfrak{i}}}$. Through Schauder's fixed point theorem 1, this establishes the existence of a solution $u\in\mathfrak{B}_{\rho_{\mathfrak{i}}}$ to the Hadamard Equation (2), given by:

$$u(\mathfrak{s},\mathfrak{x})=\begin{cases}u_1(\mathfrak{s},\mathfrak{x}), & \mathfrak{x}\in\mathfrak{V}_1,\\ u_2(\mathfrak{s},\mathfrak{x})=\begin{cases}0, & \mathfrak{x}\in\mathfrak{V}_1,\\ u_2, & \mathfrak{x}\in\mathfrak{V}_2,\end{cases}\\ \vdots\\ u_{\mathfrak{n}}(\mathfrak{s},\mathfrak{x})=\begin{cases}0, & \mathfrak{x}\in\mathfrak{V}_{\mathfrak{n}-1},\\ u_{\mathfrak{n}}, & \mathfrak{x}\in\mathfrak{V}_{\mathfrak{n}},\end{cases}\end{cases}$$

Step 5: We now establish uniform global attractivity.

Let $\mathfrak{u}_{\mathfrak{o}}$ be a solution to (2) under the given theorem conditions, and consider the ball $\mathfrak{B}(\mathfrak{u}_{\mathfrak{o}}, \rho_{\mathfrak{i}}^{*})$ with $\rho_{\mathfrak{i}}^{*} = \frac{\mathfrak{L}_{\mathfrak{i}}\mathscr{M}_{\mathfrak{i}}^{*}}{\mathfrak{L}_{\mathfrak{i}} - 2\mathscr{M}_{\mathfrak{i}}}$, where

$$\mathscr{M}_{\mathfrak{i}}^{*} := \frac{1}{\Gamma(\mathfrak{M})\Gamma(\mathfrak{R}_j)} \sup_{(\mathfrak{s},\mathfrak{x})\in[1,+\infty)\times\mathfrak{V}_{\mathfrak{i}}} \left\{ \int_1^{\gamma(\mathfrak{s})} \int_{\mathfrak{K}_{j-1}} \left| \log \frac{\gamma(\mathfrak{s})}{\varsigma} \right|^{\mathfrak{M}-1} \left| \log \frac{\mathfrak{x}}{s} \right|^{\mathfrak{R}_j - 1} \right.$$

$$\left. \times \left| \mathfrak{h}(\mathfrak{s}, \mathfrak{x}, \varsigma, \mathrm{s}, \mathfrak{u}(\varsigma, \mathrm{s}), \mathfrak{u}(\eta(\mathrm{s}), \varsigma)) - \mathfrak{h}(\mathfrak{s}, \mathfrak{x}, \varsigma, \mathrm{s}, \mathfrak{u}_{\mathfrak{o}}(\varsigma, \mathrm{s}), \mathfrak{u}_{\mathfrak{o}}(\eta(\mathrm{s}), \varsigma)) \right| \mathrm{d}\mathrm{s}\,\mathrm{d}\varsigma;\ \mathfrak{u} \in \mathfrak{B}\mathfrak{C}_{\mathfrak{i}} \right\}.$$

Taking $\mathfrak{u} \in \mathfrak{B}(\mathfrak{u}_{\mathfrak{o}}, \rho_{\mathfrak{i}}^{*})$. Then, we have

$$\begin{aligned}
|(\mathfrak{N}\mathfrak{u})(\mathfrak{s},\mathfrak{x}) - \mathfrak{u}_{\mathfrak{o}}(\mathfrak{s},\mathfrak{x})| &= |(\mathfrak{N}\mathfrak{u})(\mathfrak{s},\mathfrak{x}) - (\mathfrak{N}\mathfrak{u}_{\mathfrak{o}})(\mathfrak{s},\mathfrak{x})| \\
&\leq |\mathfrak{f}(\mathfrak{s},\mathfrak{x},\mathfrak{u}(\mathfrak{s},\mathfrak{x}),\mathfrak{u}(\lambda(\mathfrak{s}),\mathfrak{x})) - \mathfrak{f}(\mathfrak{s},\mathfrak{x},\mathfrak{u}_{\mathfrak{o}}(\mathfrak{s},\mathfrak{x}),\mathfrak{u}_{\mathfrak{o}}(\lambda(\mathfrak{s}),\mathfrak{x}))| \\
&\quad + \frac{1}{\Gamma(\mathfrak{M})\Gamma(\mathfrak{R}_j)} \int_1^{\gamma(\mathfrak{s})} \int_{\mathfrak{K}_{j-1}} \left| \log \frac{\gamma(\mathfrak{s})}{\mathrm{s}} \right|^{\mathfrak{M}-1} \left| \log \frac{\mathfrak{x}}{\varsigma} \right|^{\mathfrak{R}_j - 1} \\
&\quad \times \left| \mathfrak{h}(\mathfrak{s},\mathfrak{x},\varsigma,\mathrm{s},\mathfrak{u}(\varsigma,\mathrm{s}),\mathfrak{u}(\eta(\mathrm{s}),\varsigma)) - \mathfrak{h}(\mathfrak{s},\mathfrak{x},\varsigma,\mathrm{s},\mathfrak{u}_{\mathfrak{o}}(\varsigma,\mathrm{s}),\mathfrak{u}_{\mathfrak{o}}(\eta(\mathrm{s}),\varsigma)) \right| \frac{\mathrm{d}\mathrm{s}\mathrm{d}\varsigma}{s\varsigma} \\
&\leq \frac{2\mathscr{M}_{\mathfrak{i}}}{\mathfrak{L}_{\mathfrak{i}}} \|\mathfrak{u} - \mathfrak{u}_{\mathfrak{o}}\|_{\mathfrak{B}\mathfrak{C}_{\mathfrak{i}}} + \mathscr{M}_{\mathfrak{i}}^{*} \\
&\leq \frac{2\mathscr{M}_{\mathfrak{i}}}{\mathfrak{L}_{\mathfrak{i}}} \rho_{\mathfrak{i}}^{*} + \mathscr{M}_{\mathfrak{i}}^{*} = \rho_{\mathfrak{i}}^{*}.
\end{aligned}$$

Hence, we see that $\mathfrak{N}$ is a continuous function satisfying $\mathfrak{N}(\mathfrak{B}(\mathfrak{u}_{\mathfrak{o}}, \rho_{\mathfrak{i}}^{*})) \subset \mathfrak{B}(\mathfrak{u}_{\mathfrak{o}}, \rho_{\mathfrak{i}}^{*})$. Additionally, for every solution $\mathfrak{u}$ of (2), we have:

$$\begin{aligned}
|(\mathfrak{N}\mathfrak{u})(\mathfrak{s},\mathfrak{x}) - \mathfrak{u}_{\mathfrak{o}}(\mathfrak{s},\mathfrak{x})| &= |(\mathfrak{N}\mathfrak{u})(\mathfrak{s},\mathfrak{x}) - (\mathfrak{N}\mathfrak{u}_{\mathfrak{o}})(\mathfrak{s},\mathfrak{x})| \\
&\leq |\mathfrak{f}(\mathfrak{s},\mathfrak{x},\mathfrak{u}(\mathfrak{s},\mathfrak{x}),\mathfrak{u}(\lambda(\mathfrak{s}),\mathfrak{x})) - \mathfrak{f}(\mathfrak{s},\mathfrak{x},\mathfrak{u}_{\mathfrak{o}}(\mathfrak{s},\mathfrak{x}),\mathfrak{u}_{\mathfrak{o}}(\lambda(\mathfrak{s}),\mathfrak{x}))| \\
&\quad + \frac{1}{\Gamma(\mathfrak{M})\Gamma(\mathfrak{R}_j)} \int_1^{\gamma(\mathfrak{s})} \int_{\mathfrak{K}_{j-1}} \left| \log \frac{\gamma(\mathfrak{s})}{\mathrm{s}} \right|^{\mathfrak{M}-1} \left| \log \frac{\mathfrak{x}}{\varsigma} \right|^{\mathfrak{R}_j - 1} \\
&\quad \times \left| \mathfrak{h}(\mathfrak{s},\mathfrak{x},\varsigma,\mathrm{s},\mathfrak{u}(\varsigma,\mathrm{s}),\mathfrak{u}(\eta(\mathrm{s}),\varsigma)) - \mathfrak{h}(\mathfrak{s},\mathfrak{x},\varsigma,\mathrm{s},\mathfrak{u}_{\mathfrak{o}}(\varsigma,\mathrm{s}),\mathfrak{u}_{\mathfrak{o}}(\eta(\mathrm{s}),\varsigma)) \right| \mathrm{d}\mathrm{s}\mathrm{d}\varsigma
\end{aligned}$$

Thus

$$|(\mathfrak{N}\mathfrak{u})(\mathfrak{s},\mathfrak{x})-\mathfrak{u}_{\mathfrak{o}}(\mathfrak{s},\mathfrak{x})| = \frac{\mathscr{M}_{\mathfrak{i}}}{\mathfrak{L}_{\mathfrak{i}}}(|\mathfrak{u}(\mathfrak{s},\mathfrak{x})-\mathfrak{u}_{\mathfrak{o}}(\mathfrak{s},\mathfrak{x})|+|\mathfrak{u}(\lambda(\mathfrak{s}),\mathfrak{x})-\mathfrak{u}_{\mathfrak{o}}(\lambda(\mathfrak{s}),\mathfrak{x})|)$$
$$+\frac{\mathfrak{p}_{\mathfrak{i}}(\mathfrak{s},\mathfrak{x})}{\Gamma(\mathfrak{M})\Gamma(\mathfrak{R}_j)}\int_1^{\gamma(\mathfrak{s})}\int_{\mathfrak{K}_{j-1}}^{\mathfrak{x}}\left|\log\frac{\gamma(\mathfrak{s})}{\mathfrak{s}}\right|^{\mathfrak{M}-1}\left|\log\frac{\mathfrak{x}}{\varsigma}\right|^{\mathfrak{R}_j-1}\mathfrak{q}_{\mathfrak{i}}(\mathfrak{s},\mathfrak{x})d\mathfrak{s}d\varsigma \tag{12}$$

Where $\mathscr{M}_i$ satisfies Assumption 2 (see page 6).
By using (12) and the fact that $\gamma(\mathfrak{s})\to+\infty$ as $\mathfrak{s}\to+\infty$, we get

$$\lim_{\mathfrak{s}\to+\infty}|(\mathfrak{N}\mathfrak{u})(\mathfrak{s},\mathfrak{x})-\mathfrak{u}_{\mathfrak{o}}(\mathfrak{s},\mathfrak{x})|$$
$$\leq\lim_{\mathfrak{s}\to\infty}\frac{\mathfrak{L}_{\mathfrak{i}}\mathfrak{p}_{\mathfrak{i}}(\mathfrak{s},\mathfrak{x})}{\Gamma(\mathfrak{M})\Gamma(\mathfrak{R}_j)}\int_1^{\gamma(\mathfrak{s})}\int_{\mathfrak{K}_{j-1}}^{\mathfrak{x}}\left|\log\frac{\gamma(\mathfrak{s})}{\mathfrak{s}}\right|^{\mathfrak{M}-1}\left|\log\frac{\mathfrak{x}}{\varsigma}\right|^{\mathfrak{R}_j-1}\mathfrak{q}_{\mathfrak{i}}(\mathfrak{s},\mathfrak{x})d\mathfrak{s}d\varsigma=0 \tag{13}$$

The result implies that the integral Equation (2) admits globally asymptotically stable solutions. □

3.1 *A case study*

We proceed to examine a Hadamard-type integral equation with partial derivatives and variable fractional order:

$$\mathfrak{u}(s,x)=\frac{sx\left(1+2\sin(u(s,x))\right)}{100\left(1+s+s^2+s^3\right)}+\frac{1}{\Gamma\left(\frac{1}{3}\right)\Gamma(\mathfrak{R}(s))}\int_1^s\int_1^x\left(\log\frac{s}{\tau}\right)^{-\frac{2}{3}}\left(\log\frac{\mathfrak{x}}{\eta}\right)^{\mathfrak{R}(s)-1}$$
$$\times\frac{\ln\left(1+2\mathfrak{x}(s\eta)^{-1}|\mathfrak{u}(\tau,\eta)|\right)}{\left(1+\tau+2|\mathfrak{u}(\tau,\eta)|\right)^2\left(1+\mathfrak{x}^2+\tau^4\right)}d\eta\,d\tau,\quad(s,x)\in[1,+\infty)\times[1,e], \tag{14}$$

where

$$\mathfrak{R}(\mathfrak{s})=\begin{cases}\frac{3}{2}, & \mathfrak{s}\in\mathfrak{V}_1:=[1,2],\\ \\ \frac{6}{5}, & \mathfrak{s}\in\mathfrak{V}_2:=]2,e],\end{cases}$$

$$\begin{cases} \text{for } i=1 \\ \mathfrak{u}_1(\tau,x)=\dfrac{\tau x\big(1+2\sin(u(\tau,x))\big)}{100(1+\tau+\tau^2+\tau^3)}+\dfrac{1}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{3}{2}\right)}\displaystyle\int_1^{\tau}\int_1^{x}\left(\log\frac{\tau}{\varsigma}\right)^{-\frac{2}{3}}\left(\log\frac{x}{\eta}\right)^{\frac{1}{2}} \\ \qquad\times\dfrac{\ln\left(1+2x(\tau\eta)^{-1}|\mathfrak{u}(\tau,\eta)|\right)}{\left(1+\tau+2|\mathfrak{u}(\tau,\eta)|\right)^2\left(1+x^2+\tau^4\right)}\,d\eta\,d\varsigma, \qquad (\tau,x)\in[1,+\infty)\times[1,2], \\ \text{for } i=2 \\ \mathfrak{u}_2(\tau,x)=\dfrac{\tau x\big(1+2\sin(u(\tau,x))\big)}{100(1+\tau+\tau^2+\tau^3)}+\dfrac{1}{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{6}{5}\right)}\displaystyle\int_1^{\tau}\int_1^{x}\left(\log\frac{\tau}{\varsigma}\right)^{-\frac{2}{3}}\left(\log\frac{x}{\eta}\right)^{\frac{1}{5}} \\ \qquad\times\dfrac{\ln\left(1+x(\tau\eta)^{-1}|\mathfrak{u}(\tau,\eta)|\right)}{\left(1+\tau+2|\mathfrak{u}(\tau,\eta)|\right)^2\left(1+x^2+\tau^4\right)}\,d\eta\,d\varsigma, \qquad (\tau,x)\in[1,+\infty)\times[2,e], \end{cases}$$

for $(\mathfrak{s},\mathfrak{x})$, $(\mathrm{s},\eta)\in[\mathfrak{1},\infty)\times\mathfrak{V}_1$, $[1,+\infty)\times\mathfrak{V}_2$, and $\mathfrak{u}$, $\mathfrak{v}$ reals.

We can easily verify that the fourth are met, since $\mathfrak{f}$ is continuous and satisfies assumption (2). In particular $\mathscr{M}_1=\mathscr{M}_2=\frac{1}{10}$, $\mathfrak{L}_1=\mathfrak{L}_2=1$. Furthermore, $\mathfrak{f}$ satisfies assumption (3) with $\mathfrak{f}^*=\frac{e}{10}$. Let us point out that $\mathfrak{h}$ meets requirement (4), wherein $\mathfrak{p}_1(\mathfrak{s},\mathfrak{x})=\mathfrak{p}_2(\mathfrak{s},\mathfrak{x})=\frac{1}{1+\mathfrak{x}^2+\mathfrak{s}^4}$ and $\mathfrak{q}_1(s,\eta)=\mathfrak{q}_2(\mathrm{s},\eta)=(s\eta)^{-1}$. Also,

$$\begin{cases} \displaystyle\lim_{\tau\to\infty}\mathfrak{p}_1(\tau,x)\int_1^{\tau}\int_1^{x}\left|\log\frac{\tau}{\sigma}\right|^{-\frac{2}{3}}\left|\log\frac{x}{\eta}\right|^{\frac{1}{2}}\mathfrak{q}_1(\sigma,\eta)\,d\eta\,d\sigma \\ =\lim_{\tau\to\infty}\frac{x}{1+x^2+\tau^4}\int_1^{\tau}\int_1^{x}\left|\log\frac{\tau}{\sigma}\right|^{-\frac{2}{3}}\left|\log\frac{x}{\eta}\right|^{\frac{1}{2}}\frac{d\eta\,d\sigma}{\sigma\eta} \\ =\lim_{\tau\to\infty}\frac{2(\log x)^{3/2}(\log\tau)^{1/3}}{3(1+x^2+\tau^4)}=0, \\ \text{and} \\ \displaystyle\lim_{\tau\to\infty}\mathfrak{p}_2(\tau,x)\int_1^{\tau}\int_1^{x}\left|\log\frac{\tau}{\sigma}\right|^{-\frac{2}{3}}\left|\log\frac{x}{\eta}\right|^{\frac{1}{5}}\mathfrak{q}_2(\sigma,\eta)\,d\eta\,d\sigma \\ =\lim_{\tau\to\infty}\frac{x}{1+x^2+\tau^4}\int_1^{\tau}\int_1^{x}\left|\log\frac{\tau}{\sigma}\right|^{-\frac{2}{3}}\left|\log\frac{x}{\eta}\right|^{\frac{1}{5}}\frac{d\eta\,d\sigma}{\sigma\eta} \\ =\lim_{\tau\to\infty}\frac{5(\log x)^{6/5}(\log\tau)^{1/3}}{2(1+x^2+\tau^4)}=0. \end{cases}$$

The following results hold for Equation (14):

• A solution exists over the product space $[1,+\infty)\times[1,e]$;

• Every solution is globally asymptotically stable.
• Figure 1 presents the numerical solution obtained for the integral equation.

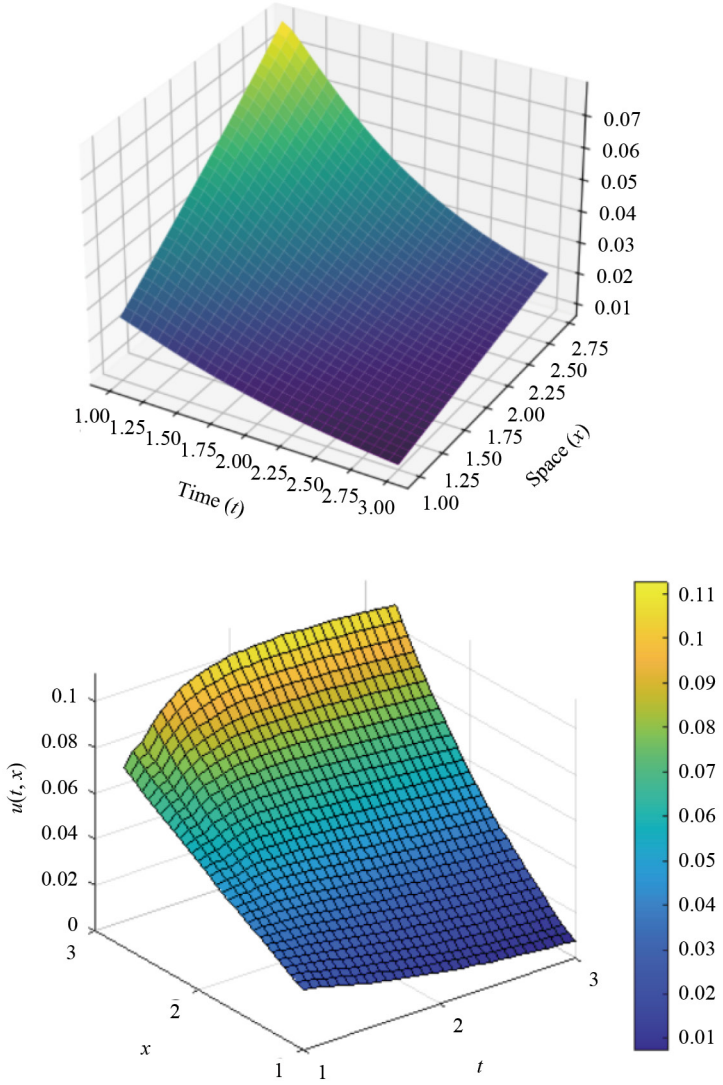


Figure 1. Numerical solution of the integral equation

$$\mathfrak{u}(\mathfrak{s},\mathfrak{x}) = \begin{cases} \mathfrak{u}_1(\mathfrak{s},\mathfrak{x}), & \mathfrak{s} \in \mathfrak{V}_1, \\ \mathfrak{u}_2(\mathfrak{s},\mathfrak{x}), & \mathfrak{s} \in \mathfrak{V}_2, \end{cases}$$

where

$$u_2(\mathfrak{s},\mathfrak{x}) = \begin{cases} 0, & \mathfrak{s} \in [1, \mathfrak{V}_1], \\ \\ u_2, & \mathfrak{s} \in \mathfrak{V}_2. \end{cases}$$

4. Conclusion

In this work, we investigate the existence, asymptotic behavior, and global stability of solutions for a class of partial integral equations involving the Hadamard fractional integral of variable order. By employing fixed–point theorems, we establish rigorous theoretical foundations, supported by detailed proofs and propositions. Our analysis identifies sufficient conditions for solution stability and illustrates these results through practical examples.

This study also opens several promising avenues for further research. A key direction is the extension of the current framework to more complex differential systems, particularly the analysis of Volterra–type partial integral equations with Hadamard–Stieltjes fractional variable order.

Acknowledgement

The authors extend their appreciation to the Deanship of Research and Graduate Studies at King Khalid University for funding this work through Large Ṙsearch Project under grant number RGP2/458/46.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this research.

Conflict of interest

The authors declare no competing financial interest.

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