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Research Article

Mathematical Analysis of Coupled System with Non Local Coupled Integral Boundary Conditions

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Abstract: This work is devoted to study a class of fractional order coupled system with nonlocal coupled Integral Boundary Conditions (IBCs). We use Conformable Fractional Order Derivative (CFOD) to investigate the mentioned problem for existence theory, stability and numerical analysis. The CFOD has some useful features as compared to other kinds of fractional order operators like satisfying product, quotient and chain rules which help in computational analysis. On the use of fixed point theorems combined with the tools of nonlinear functional analysis, appropriate results are deduced for existence, uniqueness and stability of solution. The concept introduced by Ulam-Hyers (U-H) is used to derive some results for stability theory. In addition, numerical tool based on Runge-Kutta method of order four (RK4) is used to compute approximate solution to the considered problem. We present two examples for verification of our results. Also, we present the graphical illustrations for various fractional orders of the two considered examples.

Keywords: conformable operator, Fractional Differential Equations (FDEs), stability, existence results, numerical tools

MSC: 26A33, 34A08, 74S40

1. Introduction

Fractional derivatives and integrals can be used to describe complicated real-world issues that exhibit time reliance on his state, a natural extension of ordinary calculus. The said area has recently attracted a lot of attention. Authors [1] have discussed various applications of the mentioned area in details. Youssef et al. [2] have explained the memory impact of time fractional order derivatives. The properties of the said area have been used to study memory problems [2], hereditary phenamenon [3], anomalous diffusion procedures [4], and in porous media flow processes [5]. The derivatives of real order are crucial for modeling complex systems because in contrast to regular calculus, they reflect long-term memory effects. The aforementioned tools have recently found numerous uses in a wide range of scientific fields, such as physics [1], signal processing [6], biology [7], engineering [8], economics [9], and modeling of viscoelastic material [10]. Specifically, it has been used to successfully study problems with non-local and irregular behavior in quantum mechanics [11]. Also, the applications of non-singular operators were studied recently [12]. Researchers have used the mentioned area to analyze the dynamics of infectious diseases in biology [13], and to examine process of chemistry [14].

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The area related to Boundary Value Problems (BVPs) have significant importance in many areas of engineering, physics, etc. They offer a framework for comprehending phenomena like fluid dynamics, heat conduction, and wave propagation by defining limitations at a system's boundaries, such as a rod with a particular temperature at each ends. In the same way coupled systems of BVPs have various applications in diverse fields of science and technology. Particularly, problems with thermal conduction, semiconductors, and hydrodynamics all include integral boundary conditions [15]. Coupled systems are differential equation problems that are modeled for issues, where one phenomenon is influenced by another. These configurations which can be either linear or non-linear in character are widely employed in the scientific domains to investigate variety of problems. Numerous applications exist for these differential equation systems, such as the predator-prey model in ecology [16], neural networks [17], and population dynamics modeling in biology [16]. Some significant issues in physics include modeling coupled systems, such as chemical reactions [16], electric circuits [16], and mechanical oscillators [18]. Likewise, the aforementioned idea has been applied to engineering related problems [19], and market related problems in economics [20] have been studied. The processes which depend on time in past and future due to hereditary are essential in conductivity of heat related phenomenon are modeled by coupled systems under various fractional differential operators. For instance, authors studied existence theory of coupled systems of Fractional Differential Equations (FDEs) [21]. Authors [22] studied a coupled system of reaction-diffusion problems. Authors [23] investigated solvability for a couple system of nonlinear FDEs. These applications demonstrate how crucial the pair system is for resolving a range of applied field issues.

To verify the existence and validity of a physical phenomenon, qualitative and quantitative research are crucial components of mathematics, particularly of FDEs. Existence theory is necessary to determine whether a functional equation has a solution under specific conditions or not. This idea has already been investigated for a number of dynamical problems. For instance, researchers [24] studied the existence theory of fuzzy fractional order Goursat partial differential equations. Authors [25] have applied topological degree theory to deduce existence theory for evolution problem under fractional order derivative. Authors [26] demonstrated controllability results for a class of FDEs. Stability analysis is an important aspects for dynamical problems [27]. Various tools for stability analysis have been introduced in literature. The important and easily concept was given by Ulam [28] in 1940. The mentioned idea was further clarified by Hyers in 1940, we refer to [29]. Rassias [30] further generalized the concepts of Ulam-Hyers (U-H) in 1978. Additionally, it can be challenging to evaluate the exact solution of applied problems; therefore, we require reliable semi-analytical or numerical tools to compute numerical solution as closely as possible to the actual solution. Coupled systems under fractional order operators have numerous applications in modeling various real world phenomenon [31]. The derivation of analytical solution is a challenging task in many cases and hence we need sophisticated numerical techniques to compute approximate solution [32]. Mishra et al. [33] introduced numerical methods for variable order problems with Conformable Fractional Order Derivative (CFOD). Additionally, the systems with integro-differential behaviour adds memory effects make further complexity in nonlinear multi-dimensional cases. Hence, extending Runge-Kutta method of order four (RK4) with fractional order to compute the approximate solution to coupled system with coupled integral Boundary Conditions (BCs) is the main goal of this study.

To investigate a problem's complete history over an interval rather than at a single point, nonlocal initial conditions can be used [34]. When compared to local problems, the prolonged behavior of nonlocal problems makes them crucial for accurately studying complicated problems [35]. Such issues are now being analyzed qualitatively and numerically, but further research is required [36]. Since traditional fractional differential operators do not satisfy the famous product, quotient and change roles. Often in complex numerical analysis, researchers face problems due to lack of mentioned roles. The CFOD has the ability to obey the mentioned roles which make computational analysis easy to implement. Khalil et al. [37] introduced CFOD in 2015. Authors [38] have used the said derivative to study tumor mathematical model. Authors [39] have studied controllability, observability for fractional linear-quadratic problem by using CFOD. Ghoreishi et al. [40] established fractional order RK4 method for FDEs. Here, we remark that CFOD has been recently used in studying various dynamical problems. For instance, Lavín-Delgado et al. [41] used CFOD to study a class of FDEs. Hamali et al. [42] used CFOD to study a time-fractional Noyes-field model. Iqbal et al. [43] studied soliton solutions for nonlinear problem by using CFOD. Motivated by the discussion above, this manuscript examines a couple systems of FDEs for existence theory, stability and numerical solution by utilizing the RK4 method in the context of CFOD. The suggested problem with coupled integral BCs and fractional order $\alpha \in (0, 1]$ is given as follows:

$$\begin{cases} \mathbf{J}_0^{\alpha}\mathbf{x}(\mathrm{t}) - \mathbf{f_1}(\mathrm{t}, \mathbf{x}(\mathrm{t}), \mathbf{y}(\mathrm{t})) = 0, & \mathrm{t} \in \mathbb{V}, \\ \mathbf{J}_0^{\alpha}\mathbf{y}(\mathrm{t}) - \mathbf{f_2}(\mathrm{t}, \mathbf{x}(\mathrm{t}), \mathbf{y}(\mathrm{t})) = 0, & \mathrm{t} \in \mathbb{V}, \\ \mathbf{x}(0) = \mathbf{x}_0 + \int_0^{\tau} \mathbf{g}_1(\mathbf{y}(\eta))\,\mathrm{d}\eta, \\ \mathbf{y}(0) = \mathbf{y}_0 + \int_0^{\tau} \mathbf{g}_2(\mathbf{x}(\eta))\,\mathrm{d}\eta, \end{cases} \tag{1}$$

where $\mathbb{V} = [0, \tau]$, $0 < \tau < \infty$ and $\mathbf{f}_i$ $(i = 1, 2) : \mathbb{V} \times \mathbb{R}^2 \to \mathbb{R}$ and $\mathbf{g}_i$ $(i = 1, 2) : \mathbb{R} \to \mathbb{R}$ are continuous functions.

The manuscript is organized as: Section 1 contains introduction. Section 2 contains basic results. Section 3 contains existence theory and stability analysis. Section 4 is related to numerical algorithm and examples. Section 5 contains conclusion.

2. Background results

From the references [32, 37–39], we have taken the basic materials given bellow.

Definition 1 For $\mathbf{x} : [0, \infty) \to \mathbb{R}$ and $1 \geq \alpha > 0$, the CFOD is given by

$$\mathbf{J}_0^{\alpha}\mathbf{x}(\mathrm{t}) = \lim_{\mathrm{h}\to 0} \frac{\mathbf{x}(\mathrm{t} + \mathrm{h}\mathrm{t}^{1-\alpha}) - \mathbf{x}(\mathrm{t})}{\mathrm{h}}.$$

Definition 2 For $\mathbf{x} : [0, \infty) \to \mathbb{R}$ with $\alpha > 0$, the integral is given by

$$\mathbf{I}_0^{\alpha}(\mathbf{x})(\mathrm{t}) = \int_0^{\mathrm{t}} \zeta^{\alpha-1}\mathbf{x}(\zeta)\mathrm{d}\zeta.$$

Lemma 1 Let $\mathbf{x} : [0, \infty) \to \mathbb{R}$, then we have

$$\mathbf{J}_0^{\alpha}[\mathbf{I}_0^{\alpha}(\mathbf{x})(\mathrm{t})] = \mathbf{x}(\mathrm{t}).$$

Lemma 2 The result given bellow holds:

$$\mathbf{I}_0^{\alpha}[\mathbf{J}_0^{\alpha}(\mathbf{x}(\mathrm{t}))] = \mathbf{x}(\mathrm{t}) - \mathbf{x}(0).$$

The Banach space is denoted by $\mathbb{H} = \mathrm{C}(\mathbb{V})$ with norm $\|\mathbf{x}\| = \sup_{\mathrm{t}\in\mathbb{V}}\{|\mathbf{x}| : \mathbf{x} \in \mathbb{H}\}$. Additionally, $\mathbb{H} \times \mathbb{H}$ with norm $\|(\mathbf{x}, \mathbf{y})\| = \|\mathbf{x}\| + \|\mathbf{y}\|$ is also the Banach space.

Theorem 1 [35] If $\mathbf{P} : \mathbb{H} \to \mathbb{H}$ is a contraction operator, then it has a unique fixed point.

Theorem 2 [35] If $\mathbb{D} \subset \mathbb{H}$ be closed bounded, convex subset and $\mathbf{P}_1$ and $\mathbf{P}_2$ be two operators satisfy the given conditions:

(1) $\mathbf{P}_1$ is a contraction operator from $\mathbb{H}$ to $\mathbb{H}$;

(2) $\mathbf{P}_2$ is relatively compact and continuous operator.

Then $\mathbf{P}_1(\mathbf{x}) + \mathbf{P}_2(\mathbf{x}) = \mathbf{x}$ has at least one fixed point, if $\mathbf{P}_1(\mathbf{x}) + \mathbf{P}_2(\mathbf{x}) \in \mathbb{H}$ for all $\mathbf{x} \in \mathbb{H}$.

3. Qualitative study

Here, we apply Theorems 1 and 2 to investigate the existence theory of solution for the considered problem. The corresponding theory is based on some assumptions stated bellow:

($\mathbf{A}_1$) For constant $\ell_1, \flat_1 > 0$, we describe

$$\begin{cases} |\mathbf{g}_2(\mathbf{x}) - \mathbf{g}_2(\bar{\mathbf{x}})| \leq \ell_1 |\mathbf{x} - \bar{\mathbf{x}}|, \\ |\mathbf{g}_1(\mathbf{y}) - \mathbf{g}_1(\bar{\mathbf{y}})| \leq \flat_1 |\mathbf{y} - \bar{\mathbf{y}}|. \end{cases} \tag{2}$$

($\mathbf{A}_2$) Let there $\ell_2, \ell_3, \flat_2, \flat_3 > 0$ exist, corresponding to which we have

$$\begin{cases} |\mathbf{f_1}(\mathrm{t}, \mathbf{x}, \mathbf{x_1}) - \mathbf{f_1}(\mathrm{t}, \bar{\mathbf{x}}, \bar{\mathbf{x_1}})| \leq \ell_2 |\mathbf{x} - \bar{\mathbf{x}}| + \ell_3 |\mathbf{x_1} - \bar{\mathbf{x_1}}|, \\ |\mathbf{f_2}(\mathrm{t}, \mathbf{y}, \mathbf{y_1}) - \mathbf{f_2}(\mathrm{t}, \bar{\mathbf{y}}, \bar{\mathbf{y_1}})| \leq \flat_2 |\mathbf{y} - \bar{\mathbf{y}}| + \flat_3 |\mathbf{y_1} - \bar{\mathbf{y_1}}|. \end{cases} \tag{3}$$

($\mathbf{A}_3$) Corresponding to real values $\ell_4, \ell_5, \flat_4, \flat_5 > 0$, we describe the given growth conditions:

$$\begin{cases} |\mathbf{f_1}(\mathrm{t}, \mathbf{x}, \mathbf{x_1})| \leq \ell_4 |\mathbf{x}| + \ell_5 |\mathbf{x_1}|, \\ |\mathbf{f_2}(\mathrm{t}, \mathbf{y}, \mathbf{y_1})| \leq \flat_4 |\mathbf{y}| + \flat_5 |\mathbf{y_1}|. \end{cases} \tag{4}$$

Lemma 3 Solution of given linear system

$$\begin{cases} \mathbf{J}_0^{\alpha} \mathbf{x}(\mathrm{t}) = \mathbf{h_1}(\mathrm{t}), \quad \mathbf{x}(0) = \mathbf{x}_0, \\ \mathbf{J}_0^{\alpha} \mathbf{y}(\mathrm{t}) = \mathbf{h_2}(\mathrm{t}), \quad \mathbf{x}(0) = \mathbf{x}_0, \end{cases} \tag{5}$$

is equivalent to

$$\begin{cases} \mathbf{x}(\mathrm{t}) = \mathbf{x}_0 + \int_0^{\mathrm{t}} \zeta^{\alpha-1} \mathbf{h_1}(\zeta) d\zeta, \\ \mathbf{y}(\mathrm{t}) = \mathbf{y}_0 + \int_0^{\mathrm{t}} \zeta^{\alpha-1} \mathbf{h_2}(\zeta) d\zeta. \end{cases} \tag{6}$$

The proof is given in Appendix A.

Thank to Lemma 3, Eq. (1) is equivalent to

$$\begin{cases} \mathbf{x}(\mathrm{t}) = \mathbf{x}_0 + \int_0^\tau \mathbf{g}_1(\mathbf{y}(\zeta))\, d\zeta + \int_0^{\mathrm{t}} \zeta^{\alpha-1}\mathbf{f_1}(\zeta, \mathbf{x}(\zeta), \mathbf{y}(\zeta)) d\zeta, \\ \mathbf{y}(\mathrm{t}) = \mathbf{y}_0 + \int_0^\tau \mathbf{g}_2(\mathbf{x}(\zeta))\, d\zeta + \int_0^{\mathrm{t}} \zeta^{\alpha-1}\mathbf{f_2}(\zeta, \mathbf{x}(\zeta), \mathbf{y}(\zeta)) d\zeta. \end{cases} \tag{7}$$

Define $\Omega : \mathbb{H} \times \mathbb{H} \to \mathbb{H} \times \mathbb{H}$ by

$$(\mathbf{x}, \mathbf{y}) = \Omega(\mathbf{x}, \mathbf{y}) = \begin{pmatrix} \mathbf{x}_0 + \int_0^\tau \mathbf{g}_1(\mathbf{y}(\zeta))\, d\zeta + \int_0^{\mathrm{t}} \zeta^{\alpha-1}\mathbf{f_1}(\zeta, \mathbf{x}(\zeta), \mathbf{y}(\zeta)) d\zeta \\ \mathbf{y}_0 + \int_0^\tau \mathbf{g}_2(\mathbf{x}(\zeta))\, d\zeta + \int_0^{\mathrm{t}} \zeta^{\alpha-1}\mathbf{f_2}(\zeta, \mathbf{x}(\zeta), \mathbf{y}(\zeta)) d\zeta \end{pmatrix}. \tag{8}$$

Let define $\Phi : \mathbb{H} \times \mathbb{H} \to \mathbb{H} \times \mathbb{H}$ by

$$\Phi(\mathbf{x}, \mathbf{y}) = \begin{pmatrix} \mathbf{x}_0 + \int_0^\tau \mathbf{g}_1(\mathbf{y}(\zeta))\, d\zeta \\ \mathbf{y}_0 + \int_0^\tau \mathbf{g}_2(\mathbf{x}(\zeta))\, d\zeta \end{pmatrix}, \tag{9}$$

and $\Psi : \mathbb{D} \to \mathbb{D}$ by

$$\Psi(\mathbf{x}, \mathbf{y}) = \begin{pmatrix} \int_0^{\mathrm{t}} \zeta^{\alpha-1}\mathbf{f_1}(\zeta, \mathbf{x}(\zeta), \mathbf{y}(\zeta))\, d\zeta \\ \int_0^{\mathrm{t}} \zeta^{\alpha-1}\mathbf{f_2}(\zeta, \mathbf{x}(\zeta), \mathbf{y}(\zeta))\, d\zeta \end{pmatrix}. \tag{10}$$

In addition, $\Phi = (\Phi_1, \Phi_2)$, where

$$\begin{cases} \Phi_1(\mathbf{y}) = \mathbf{x}_0 + \int_0^\tau \mathbf{g}_1(\mathbf{y}(\zeta))\, d\zeta \\ \Phi_2(\mathbf{x}) = \mathbf{y}_0 + \int_0^\tau \mathbf{g}_2(\mathbf{x}(\zeta))\, d\zeta \end{cases} \tag{11}$$

and $\Psi(\mathbf{x}, \mathbf{y}) = (\Psi_1, \Psi_2)(\mathbf{x}, \mathbf{y})$, where

$$\begin{cases} \Psi_1(\mathbf{x}, \mathbf{y}) = \int_0^t \zeta^{\alpha-1} \mathbf{f_1}(\zeta, \mathbf{x}(\zeta), \mathbf{y}(\zeta))\, d\zeta \\ \Psi_2(\mathbf{x}, \mathbf{y}) = \int_0^t \zeta^{\alpha-1} \mathbf{f_2}(\zeta, \mathbf{x}(\zeta), \mathbf{y}(\zeta))\, d\zeta. \end{cases} \tag{12}$$

On the basis of Eq. (9) and Eq. (10), we have

$$(\mathbf{x}, \mathbf{y}) = \Omega(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x}, \mathbf{y}) + \Psi(\mathbf{x}, \mathbf{y}). \tag{13}$$

Theorem 3 For problem (1) with condition $\mathbb{L} = \ell_1 \tau + \flat_1 \tau < 1$, there exists at least one solution.
Proof. Consider

$$\mathbb{D} = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{H} \times \mathbb{H} : \|(\mathbf{x}, \mathbf{y})\| \leq \rho,\ \rho > 0\} \subset \mathbb{H} \times \mathbb{H},$$

such that $(\mathbf{x}, \mathbf{y})$, $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \mathbb{D}$, then using assumptions $(\mathbf{A}_1, \mathbf{A}_2)$

$$\begin{aligned} \sup_{t\in\mathbb{V}} |\Phi(\mathbf{x}, \mathbf{y}) - \Phi(\bar{\mathbf{x}}, \bar{\mathbf{y}})| = \sup_{t\in\mathbb{V}} \Bigg\{ \Bigg| & \mathbf{x}_0 + \int_0^\tau \mathbf{g}_1(\mathbf{y}(\zeta))\, d\zeta + \mathbf{y}_0 + \int_0^\tau \mathbf{g}_2(\mathbf{x}(\zeta))\, d\zeta \\ & - \left(\bar{\mathbf{x}}_0 + \int_0^\tau \mathbf{g}_1(\bar{\mathbf{y}}(\zeta))\, d\zeta + \bar{\mathbf{y}}_0 + \int_0^\tau \mathbf{g}_1\left(\mathbf{x}(\bar{\zeta})\right) d\zeta \right) \Bigg| \Bigg\}, \\ & \leq \sup_{t\in\mathbb{V}} \left\{ \int_0^\tau |\mathbf{g}_2(\mathbf{x}(\zeta)) - \mathbf{g}_2(\bar{\mathbf{x}}(\zeta))|\, d\zeta + \int_0^\tau |\mathbf{g}_1(\mathbf{y}(\zeta)) - \mathbf{g}_1(\bar{\mathbf{y}}(\zeta))|\, d\zeta \right\}, \\ & \leq \ell_1 \tau \|\mathbf{x} - \bar{\mathbf{x}}\| + \flat_1 \tau \|\mathbf{y} - \bar{\mathbf{y}}\|, \\ & \leq \mathbb{L}[\|(\mathbf{x}, \mathbf{y}) - (\bar{\mathbf{x}}, \bar{\mathbf{y}})\|], \end{aligned}$$

which gives

$$\|\Phi(\mathbf{x}, \mathbf{y}) - \Phi(\bar{\mathbf{x}}, \bar{\mathbf{y}})\| \leq \mathbb{L}[\|(\mathbf{x}, \mathbf{y}) - (\bar{\mathbf{x}}, \bar{\mathbf{y}})\|].$$

Therefore, Φ is a contraction operator.
To deduce that Ψ is completely continuous, take $(\mathbf{x}, \mathbf{y}) \in \mathbb{D}$, one has by using the assumption $(\mathbf{A}_3)$

$$\sup_{t\in\mathbb{V}}|\Psi(\mathbf{x},\mathbf{y})| = \sup_{t\in\mathbb{V}}\left\{\left|\int_0^t \zeta^{\alpha-1}\mathbf{f_1}(\zeta,\mathbf{x}(\zeta),\mathbf{y}(\zeta))\,d\zeta + \int_0^t \zeta^{\alpha-1}\mathbf{f_2}(\zeta,\mathbf{x}(\zeta),\mathbf{y}(\zeta))\,d\zeta\right|\right\},$$

$$\leq \sup_{t\in\mathbb{V}}\left\{\int_0^t \zeta^{\alpha-1}(\ell_4|\mathbf{x}(\zeta)|+\ell_5|\mathbf{y}(\zeta)|)\,d\zeta + \int_0^t \zeta^{\alpha-1}(\flat_4|\mathbf{x}(\zeta)|+\flat_5|\mathbf{y}(\zeta)|)\,d\zeta\right\},$$

$$\leq [\ell_4+\ell_5]\frac{\rho\tau^\alpha}{\alpha} + [\flat_4+\flat_5]\frac{\rho\tau^\alpha}{\alpha},$$

$$<\infty,$$

that is

$$\|\Psi(\mathbf{x},\mathbf{y})\| < \infty.$$

Thus Ψ is bounded. Due to the continuity of $\mathbf{f_1}$ and $\mathbf{f_2}$, the mentioned operator Ψ is continuous. In addition, to obtain uniform continuity, take $t_1 < t_2$, one has

$$|\Psi(\mathbf{x}(t_2),\mathbf{y}(t_2)) - \Psi(\mathbf{x}(t_1),\mathbf{y}(t_1))|$$

$$= \left|\int_0^{t_2} \zeta^{\alpha-1}\mathbf{f_1}(\zeta,\mathbf{x}(\zeta),\mathbf{y}(\zeta))\,d\zeta + \int_0^{t_2} \zeta^{\alpha-1}\mathbf{f_2}(\zeta,\mathbf{x}(\zeta),\mathbf{y}(\zeta))\,d\zeta\right|$$

$$-\left(\left|\int_0^{t_1} \zeta^{\alpha-1}\mathbf{f_1}(\zeta,\mathbf{x}(\zeta),\mathbf{y}(\zeta))\,d\zeta + \int_0^{t_1} \zeta^{\alpha-1}\mathbf{f_2}(\zeta,\mathbf{x}(\zeta),\mathbf{y}(\zeta))\,d\zeta\right|\right)$$

$$\leq [\ell_4+\ell_5]\frac{\rho}{\alpha}\times[t_2^\alpha - t_1^\alpha] + [\flat_4+\flat_5]\frac{\rho}{\alpha}\times[t_2^\alpha - t_1^\alpha] \to 0 \quad \text{as} \quad t_1 \to t_2.$$

Thus Ψ is equi-continuous and relatively compact by using Arzelà-Ascoli theorem, therefore the problem (1) has at least one solution in view of Theorem 2. □

Theorem 4 Thank to assumptions ($\mathbf{A}_1$, $\mathbf{A}_2$), and if $\mathbb{C} = (\ell_1+\flat_1)\tau + (\ell_2+\ell_3+\flat_2+\flat_3)\frac{\tau^\alpha}{\alpha} < 1$, then system (1) has a unique solution.

Proof. Considering $(\mathbf{x},\mathbf{y}), (\bar{\mathbf{x}},\bar{\mathbf{y}}) \in \mathbb{H}\times\mathbb{H}$, we proceed as follows:

$$|\Omega(\mathbf{x},\mathbf{y}) - \Omega(\bar{\mathbf{x}},\bar{\mathbf{y}})|$$

$$= \left|\mathbf{x}_0 + \int_0^\tau \mathbf{g}_1(\mathbf{y}(\zeta))\,d\zeta + \mathbf{y}_0 + \int_0^\tau \mathbf{g}_2(\mathbf{x}(\zeta))\,d\zeta + \int_0^t \zeta^{\alpha-1}\mathbf{f_1}(\zeta,\mathbf{x}(\zeta),\mathbf{y}(\zeta))\,d\zeta\right.$$

$$+\int_0^{t}\zeta^{\alpha-1}\mathbf{f_2}(\zeta,\mathbf{x}(\zeta),\mathbf{y}(\zeta))\,\mathrm{d}\zeta-\left(\left|\bar{\mathbf{x}}_0+\int_0^{\tau}\mathbf{g}_1(\bar{\mathbf{y}}(\zeta))\,\mathrm{d}\zeta+\bar{\mathbf{y}}_0+\int_0^{\tau}\mathbf{g}_2(\bar{\mathbf{x}}(\zeta))\,\mathrm{d}\zeta\right.\right.$$

$$\left.\left.+\int_0^{t}\zeta^{\alpha-1}\mathbf{f_1}(\zeta,\bar{\mathbf{x}}(\zeta),\bar{\mathbf{y}}(\zeta))\,\mathrm{d}\zeta+\int_0^{t}\zeta^{\alpha-1}\mathbf{f_2}(\zeta,\bar{\mathbf{x}}(\zeta),\bar{\mathbf{y}}(\zeta))\,\mathrm{d}\zeta\right|\right)\right|,$$

$$\leq\int_0^{\tau}|\mathbf{g}_2(\mathbf{x}(\zeta))-\mathbf{g}_2(\bar{\mathbf{x}}(\zeta))|\mathrm{d}\zeta+\int_0^{\tau}|\mathbf{g}_1(\mathbf{y}(\zeta))-\mathbf{g}_1(\bar{\mathbf{y}}(\zeta))|\mathrm{d}\zeta$$

$$+\int_0^{t}\zeta^{\alpha-1}|\mathbf{f_1}(\zeta,\mathbf{x}(\zeta),\mathbf{y}(\zeta))\ \mathbf{f_1}(\zeta,\bar{\mathbf{x}}(\zeta),\bar{\mathbf{y}}(\zeta))|\,\mathrm{d}\zeta$$

$$+\int_0^{t}\zeta^{\alpha-1}|\mathbf{f_2}(\zeta,\mathbf{x}(\zeta),\mathbf{y}(\zeta))\ \mathbf{f_2}(\zeta,\bar{\mathbf{x}}(\zeta),\bar{\mathbf{y}}(\zeta))|\,\mathrm{d}\zeta,$$

$$\leq(\ell_1\tau)\|\mathbf{x}-\bar{\mathbf{x}}\|+(\flat_1\tau)\|\mathbf{y}-\bar{\mathbf{y}}\|+[\ell_2\|\mathbf{x}-\bar{\mathbf{x}}\|+\ell_3\|\mathbf{y}-\bar{\mathbf{y}}\|]\,\frac{\tau^{\alpha}}{\alpha}$$

$$+[\flat_2\|\mathbf{x}-\bar{\mathbf{x}}\|+\flat_3\|\mathbf{y}-\bar{\mathbf{y}}\|]\,\frac{\tau^{\alpha}}{\alpha},$$

$$=\left[\ell_1\tau+\ell_2\frac{\tau^{\alpha}}{\alpha}+\flat_2\frac{\tau^{\alpha}}{\alpha}\right]\|\mathbf{x}-\bar{\mathbf{x}}\|+\left[\flat_1\tau+\ell_3\frac{\tau^{\alpha}}{\alpha}+\flat_3\frac{\tau^{\alpha}}{\alpha}\right]\|\mathbf{y}-\bar{\mathbf{y}}\|,$$

$$\leq\mathbb{C}\|(\mathbf{x},\mathbf{y})-(\bar{\mathbf{x}},\bar{\mathbf{y}})\|.$$

We see that Ω is a contraction operator and hence by using Theorem 1, problem (1) has a unique solution. □

3.1 *U-H stability*

After the existence theory, the important aspect of any mathematical problems is its stability. To insure that a dynamical system is stable, by incorporating a small change in the system it does retain to its original state by removing the disturbance from the system. Stability, in mathematical language concerns with the dynamics and response of the trajectories of the solution of a mathematical problem with small change in its initial state. The solution of a dynamical system is stable if the solution trajectory do not change too much from initial position after small perturbation in initial state. To answer the query, “Under what circumstances the approximate solution must be nearly equal to exact solution of a functional equation?” was answered by Ulam’s [28] in 1940 and further explored and generalized by Hayer’s and Rassias [29, 30]. Hence, before going to a stable numerical technique it is crucial to check the stability, particularly the U-H stability of the solution of the problem under consideration.

To deduce stability results, let we have two mappings θ, ψ independent of $\mathbf{x}$, $\mathbf{y}$, such that for any $\varepsilon>0$, we have

$$|\theta(\mathrm{t})|<\frac{\varepsilon}{2},\ \ |\psi(\mathrm{t})|<\frac{\varepsilon}{2}.$$

We reconsider Eq. (1) as follows:

$$\begin{cases} \mathbf{J}_0^{\alpha}\mathbf{x}(\mathrm{t}) = \mathbf{f_1}(\mathrm{t}, \mathbf{x}(\mathrm{t}), \mathbf{y}(\mathrm{t})) + \theta(\mathrm{t}), \\ \mathbf{J}_0^{\alpha}\mathbf{y}(\mathrm{t}) = \mathbf{f_2}(\mathrm{t}, \mathbf{y}(\mathrm{t}), \mathbf{x}(\mathrm{t})) + \psi(\mathrm{t}), \\ \mathbf{x}(0) = \mathbf{x}_0 + \int_0^{\tau} \mathbf{g}_1(\mathbf{y}(\zeta))\, \mathrm{d}\zeta, \\ \mathbf{y}(0) = \mathbf{y}_0 + \int_0^{\tau} \mathbf{g}_2(\mathbf{x}(\zeta))\, \mathrm{d}\zeta. \end{cases} \tag{14}$$

Lemma 4 The solution of Eq. (14) satisfies the given relation:

$$\|\Omega(\mathbf{x}, \mathbf{y}) - (\mathbf{x}, \mathbf{y})\| \leq \frac{\tau^{\alpha}\varepsilon}{\alpha}. \tag{15}$$

Proof. Thank to Lemma 2, from Eq. (14), we have after simplification by using Theorem 4

$$(\mathbf{x}(\mathrm{t}), \mathbf{y}(\mathrm{t})) = \begin{cases} \mathbf{x}_0 + \int_0^{\tau} \mathbf{g}_1(\mathbf{y}(\zeta))\, \mathrm{d}\zeta + \int_0^{\mathrm{t}} \zeta^{\alpha-1}\mathbf{f_1}(\zeta, \mathbf{x}(\zeta), \mathbf{y}(\zeta))\, \mathrm{d}\zeta \\ + \int_0^{\mathrm{t}} \zeta^{\alpha-1}\theta(\zeta)\mathrm{d}\zeta + \mathbf{y}_0 + \int_0^{\tau} \mathbf{g}_2(\mathbf{x}(\zeta))\, \mathrm{d}\zeta \\ + \int_0^{\mathrm{t}} \zeta^{\alpha-1}\mathbf{f_2}(\zeta, \mathbf{x}(\zeta), \mathbf{y}(\zeta))\, \mathrm{d}\zeta + \int_0^{\mathrm{t}} \zeta^{\alpha-1}\psi(\zeta)\mathrm{d}\zeta, \end{cases}$$

$$= \Omega(\mathbf{x}(\mathrm{t}), \mathbf{y}(\mathrm{t})) + \int_0^{\mathrm{t}} \zeta^{\alpha-1}\theta(\zeta)\mathrm{d}\zeta + \int_0^{\mathrm{t}} \zeta^{\alpha-1}\psi(\zeta)\mathrm{d}\zeta,$$

which means that

$$(\mathbf{x}(\mathrm{t}), \mathbf{y}(\mathrm{t})) = \Omega(\mathbf{x}(\mathrm{t}), \mathbf{y}(\mathrm{t})) + \int_0^{\mathrm{t}} \zeta^{\alpha-1}\theta(\zeta)\mathrm{d}\zeta + \int_0^{\mathrm{t}} \zeta^{\alpha-1}\psi(\zeta)\mathrm{d}\zeta. \tag{16}$$

From Eq. (16), we obtain

$$\|(\mathbf{x}, \mathbf{y}) - \Omega(\mathbf{x}, \mathbf{y})\| \leq \max_{\mathrm{t}\in\mathbb{V}} \left[\int_0^{\mathrm{t}} \zeta^{\alpha-1}|\theta(\zeta)|\mathrm{d}\zeta + \int_0^{\mathrm{t}} \zeta^{\alpha-1}|\psi(\zeta)|\mathrm{d}\zeta\right] \leq \left[\frac{\varepsilon}{2} + \frac{\varepsilon}{2}\right]\frac{\tau^{\alpha}}{\alpha} = \frac{\tau^{\alpha}\varepsilon}{\alpha}.$$

□

Theorem 5 The results of Eq. (1) is U-H stable with the condition $\mathbb{C} < 1$.

Proof. If $(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \in \mathbf{X} \times \mathbf{X}$ is any solution of system (1) and take unique solution $(\mathbf{x}, \mathbf{y}) \in \mathbf{X} \times \mathbf{X}$ of system (1), then applying Lemma 4, we obtain

$$
\begin{aligned}
|(\mathbf{x}(\mathrm{t}), \mathbf{y}(\mathrm{t})) - (\bar{\mathbf{x}}(\mathrm{t}), \bar{\mathbf{y}}(\mathrm{t}))| &= |(\mathbf{x}(\mathrm{t}), \mathbf{y}(\mathrm{t})) - \Omega(\bar{\mathbf{x}}(\mathrm{t}), \bar{\mathbf{y}}(\mathrm{t}))| \\
&= |(\mathbf{x}(\mathrm{t}), \mathbf{y}(\mathrm{t})) - \Omega(\mathbf{x}(\mathrm{t}), \mathbf{y}(\mathrm{t})) \\
&\quad + \Omega(\mathbf{x}(\mathrm{t}), \mathbf{y}(\mathrm{t})) - \Omega(\bar{\mathbf{x}}(\mathrm{t}), \bar{\mathbf{y}}(\mathrm{t}))| \\
&\leq |(\mathbf{x}(\mathrm{t}), \mathbf{y}(\mathrm{t})) - \Omega(\mathbf{x}(\mathrm{t}), \mathbf{y}(\mathrm{t}))| \\
&\quad + |\Omega(\mathbf{x}(\mathrm{t}), \mathbf{y}(\mathrm{t})) - \Omega(\bar{\mathbf{x}}(\mathrm{t}), \bar{\mathbf{y}}(\mathrm{t}))| \\
&\leq \frac{\tau^{\alpha}\varepsilon}{\alpha} + \mathbb{C}|(\mathbf{x}(\mathrm{t}), \mathbf{y}(\mathrm{t})) - (\bar{\mathbf{x}}(\mathrm{t}), \bar{\mathbf{y}}(\mathrm{t}))| \\
&\leq \frac{\tau^{\alpha}}{(1-\mathbb{C})\alpha}\varepsilon,
\end{aligned}
$$

hence, we get

$$
\|(\mathbf{x}, \mathbf{y}) - (\bar{\mathbf{x}}, \bar{\mathbf{y}})\| \leq \frac{\tau^{\alpha}}{(1-\mathbb{C})\alpha}\varepsilon. \tag{17}
$$

Thus, the solution of Eq. (1) is U-H stable. Further, let there exists a non decreasing mapping $\theta : [0, \tau] \to (0, \infty)$, such that $\theta(\varepsilon) = \varepsilon$ and $\theta(0) = 0$, then we have from Eq. (17)

$$
\|(\mathbf{x}, \mathbf{y}) - (\bar{\mathbf{x}}, \bar{\mathbf{y}})\| \leq \frac{\tau^{\alpha}}{(1-\mathbb{C})\alpha}\theta(\varepsilon). \tag{18}
$$

Consequently, the solution of Eq. (1) is generalized U-H stable. □

4. Numerical investigation

To deal the proposed system (1) numerically, we follow RK4 method [33], for which set $t_{n+1} = t_n + h$ and $h = \frac{\tau}{n}$ is step size. Split the interval $\mathbb{V}$ into sub intervals as $[\mathrm{t}_k, \mathrm{t}_{k+1}]$. Consider the given class with $\mathbf{u} = (\mathbf{x}, \mathbf{y})$

$$
\begin{cases}
\mathbf{J}_0^{\alpha}\mathbf{u}(\mathrm{t}) = \mathcal{F}(\mathrm{t}, \mathbf{u}(\mathrm{t})), \quad \mathrm{t} \in \mathbb{V}, \quad 0 < \alpha \leq 1, \\
\mathbf{u}(0) = \mathbf{u}_0,
\end{cases} \tag{19}
$$

which implies that

$$\mathbf{u}^{i+1} = \mathbf{u}^i + \frac{h^\alpha}{6\alpha}[\mathbb{K}_1 + 2(\mathbb{K}_2 + \mathbb{K}_3) + \mathbb{K}_4], \quad \text{for } i = 0, 1, 2, 3, \ldots, \tag{20}$$

while

$$\mathbb{K}_1 = \digamma\left(\mathrm{t}^i, \mathbf{u}^i\right),$$

$$\mathbb{K}_2 = \digamma\left(\mathrm{t}^i + \frac{h^\alpha}{2\alpha}, \mathbf{u}^i + \frac{h^\alpha}{2\alpha}\mathbb{K}_1\right),$$

$$\mathbb{K}_3 = \digamma\left(\mathrm{t}^i + \frac{h^\alpha}{2\alpha}, \mathbf{u}^i + \frac{h^\alpha}{2\alpha}\mathbb{K}_2\right),$$

$$\mathbb{K}_4 = \digamma\left(\mathrm{t}^i + \frac{h^\alpha}{\alpha}, \mathbf{u}^i + \frac{h^\alpha}{\alpha}\mathbb{K}_3\right).$$

Thank to Eq. (20), we approximate the system (1) for $i = 0, 1, 2, 3, \ldots$ as follows:

$$\begin{cases} \mathbf{x}^{i+1} = \mathbf{x}^i + \frac{1}{h}\sum_{k=0}^{n}\left[\mathbf{g}_1(\mathbf{y}^{k+1}) - \mathbf{g}_1(\mathbf{y}^k)\right] + \frac{h^\alpha}{6\alpha}[\mathbb{K}_1 + 2(\mathbb{K}_2 + \mathbb{K}_3) + \mathbb{K}_4], \\ \mathbf{y}^{i+1} = \mathbf{y}^i + \frac{1}{h}\sum_{k=0}^{n}\left[\mathbf{g}_2(\mathbf{x}^{k+1}) - \mathbf{g}_2(\mathbf{x}^k)\right] + \frac{h^\alpha}{6\alpha}[M_1 + 2(M_2 + M_3) + M_4], \end{cases} \tag{21}$$

such that

$$\mathbb{K}_1 = \mathbf{f_1}\left(\mathrm{t}^i, \mathbf{x}^i, \mathbf{y}^i\right),$$

$$\mathbb{K}_2 = \mathbf{f_1}\left(\mathrm{t}^i + \frac{h^\alpha}{2\alpha}, \mathbf{x}^i + \frac{h^\alpha}{2\alpha}\mathbb{K}_1, \mathbf{y}^i + \frac{h^\alpha}{2\alpha}\mathbb{K}_1\right),$$

$$\mathbb{K}_3 = \mathbf{f_1}\left(\mathrm{t}^i + \frac{h^\alpha}{2\alpha}, \mathbf{x}^i + \frac{h^\alpha}{2\alpha}\mathbb{K}_2, \mathbf{y}^i + \frac{h^\alpha}{2\alpha}\mathbb{K}_2\right),$$

$$\mathbb{K}_4 = \mathbf{f_1}\left(\mathrm{t}^i + \frac{h^\alpha}{\alpha}, \mathbf{x}^i + \frac{h^\alpha}{\alpha}\mathbb{K}_3, \mathbf{y}^i + h^\alpha\mathbb{K}_3\right),$$

and

$$M_1 = \mathbf{f_2}\left(\mathrm{t}^i, \mathbf{x}^i, \mathbf{y}^i\right),$$

$$M_2 = \mathbf{f_2}\left(\mathrm{t}^i + \frac{h^\alpha}{2\alpha}, \mathbf{x}^i + \frac{h^\alpha}{2\alpha}M_1, \mathbf{y}^i + \frac{h^\alpha}{2\alpha}M_1\right),$$

$$M_3 = \mathbf{f_2}\left(\mathrm{t}^i + \frac{h^\alpha}{2\alpha}, \mathbf{x}^i + \frac{h^\alpha}{2\alpha}M_2, \mathbf{y}^i + \frac{h^\alpha}{2\alpha}M_2\right),$$

$$M_4 = \mathbf{f_2}\left(\mathrm{t}^i + \frac{h^\alpha}{\alpha}, \mathbf{x}^i + \frac{h^\alpha}{\alpha}M_3, \mathbf{y}^i + h^\alpha M_3\right).$$

Remark 1 The presented method is convergent with convergence rate $O(h^{4\alpha})$, where the rate of convergence depends on the fractional order. In case of classical derivative, the mentioned convergence rate is $O(h^4)$ which depends on step size only. For detail, about the convergence, we refer [40].

4.1 *Numerical examples*

To demonstrate the validity of our results, we present two examples here.

Example 1 Consider the following conformable coupled system

$$\begin{cases} \mathbf{J}_0^\alpha \mathbf{x}(\mathrm{t}) = \dfrac{\sin|\mathbf{x}(\mathrm{t})| + \tan|\mathbf{y}(\mathrm{t})|}{1{,}000 + \cos(\mathrm{t})}, \quad \mathrm{t} \in [0, 1]. \\ \mathbf{J}_0^\alpha \mathbf{y}(\mathrm{t}) = \dfrac{\sin|\mathbf{y}(\mathrm{t})| + \tan|\mathbf{x}(\mathrm{t})|}{1{,}000 + \cos(\mathrm{t})}, \quad \mathrm{t} \in [0, 1]. \\ \mathbf{x}(0) = 1 + \displaystyle\int_0^1 \frac{\mathbf{y}(\zeta)}{10} d\zeta. \\ \mathbf{y}(0) = 1 + \displaystyle\int_0^1 \frac{\mathbf{x}(\zeta)}{10} d\zeta. \end{cases} \tag{22}$$

Here,

$$\mathbf{f_1}(\mathrm{t}, \mathbf{x}(\mathrm{t}), \mathbf{y}(\mathrm{t})) = \frac{\sin|\mathbf{x}(\mathrm{t})| + \tan|\mathbf{y}(\mathrm{t})|}{1{,}000 + \cos(\mathrm{t})}.$$

$$\mathbf{f_2}(\mathrm{t}, \mathbf{x}(\mathrm{t}), \mathbf{y}(\mathrm{t})) = \frac{\sin|\mathbf{y}(\mathrm{t})| + \tan|\mathbf{x}(\mathrm{t})|}{1{,}000 + \cos(\mathrm{t})}.$$

$$\mathbf{g}_1(\mathbf{y}(\zeta)) = \frac{\mathbf{y}(\zeta)}{10}, \quad \mathbf{g}_2(\mathbf{x}(\zeta)) = \frac{\mathbf{x}(\zeta)}{10}.$$

Hence, we have

$$|\mathbf{g}_1(\mathbf{y}(\zeta)) - \mathbf{g}_1(\bar{\mathbf{y}}(\zeta))| \leq \frac{1}{10}|\mathbf{y} - \bar{\mathbf{y}}|.$$

$$|\mathbf{g}_2(\mathbf{x}(\zeta)) - \mathbf{g}_2(\bar{\mathbf{x}}(\zeta))| \leq \frac{1}{10}|\mathbf{x} - \bar{\mathbf{x}}|.$$

$$|\mathbf{f_1}(\mathrm{t}, \mathbf{x}(\mathrm{t}), \mathbf{y}(\mathrm{t})) - \mathbf{f_1}(\mathrm{t}, \bar{\mathbf{x}}(\mathrm{t}), \bar{\mathbf{y}}(\mathrm{t}))| \leq \frac{1}{1{,}000}|\mathbf{x} - \bar{\mathbf{x}}| + \frac{1}{1{,}000}|\mathbf{y} - \bar{\mathbf{y}}|.$$

$$|\mathbf{f_2}(\mathrm{t}, \mathbf{x}(\mathrm{t}), \mathbf{y}(\mathrm{t})) - \mathbf{f_2}(\mathrm{t}, \bar{\mathbf{x}}(\mathrm{t}), \bar{\mathbf{y}}(\mathrm{t}))| \leq \frac{1}{1{,}000}|\mathbf{x} - \bar{\mathbf{x}}| + \frac{1}{1{,}000}|\mathbf{y} - \bar{\mathbf{y}}|.$$

$$|\mathbf{f_1}(\mathrm{t}, \mathbf{x}(\mathrm{t}), \mathbf{y}(\mathrm{t}))| \leq \frac{1}{1{,}000}|\mathbf{x}| + \frac{1}{1{,}000}|\mathbf{y}|.$$

and

$$|\mathbf{f_2}(\mathrm{t}, \mathbf{x}(\mathrm{t}), \mathbf{y}(\mathrm{t}))| \leq \frac{1}{1{,}000}|\mathbf{x}| + \frac{1}{1{,}000}|\mathbf{y}|.$$

The coefficients are computed as $\ell_1 = \flat_1 = \frac{1}{10}$, $\ell_2 = \flat_2 = \ell_3 = \flat_3 = \ell_4 = \flat_4 = \ell_5 = \flat_5 = \frac{1}{1{,}000}$. Further, let $\alpha = 0.5$ then, we have

$$\ell = \max\{\ell_1\tau, \flat_1\tau\} = 0.1 < 1,$$

and

$$\mathbb{C} = \max\left\{\left[\ell_1\tau + \ell_2\frac{\tau^\alpha}{\alpha} + \flat_2\frac{\tau^\alpha}{\alpha}\right], \left[\flat_1\tau + \ell_3\frac{\tau^\alpha}{\alpha} + \flat_3\frac{\tau^\alpha}{\alpha}\right]\right\} = 0.104 < 1.$$

All the requirements of Theorems 3, 4, and 5 are verified and thus Example 1 has at least one unique solution which U-H stable. Additionally, some plots against different fractional orders are given in Figures 1 and 2.

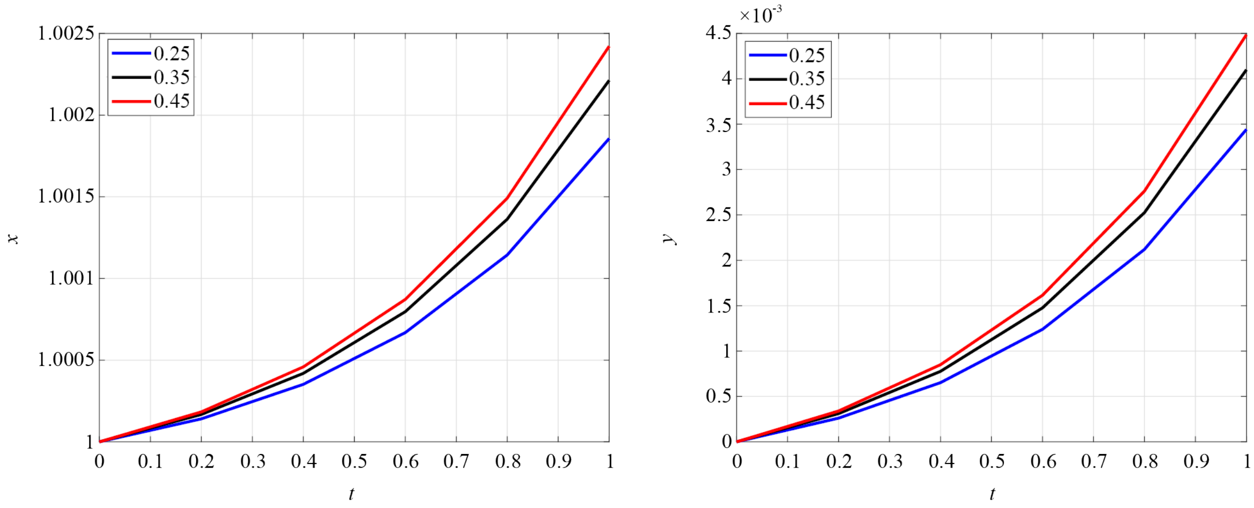

Figure 1. Numerical results of Example 1 for fractional orders in (0, 0.50)

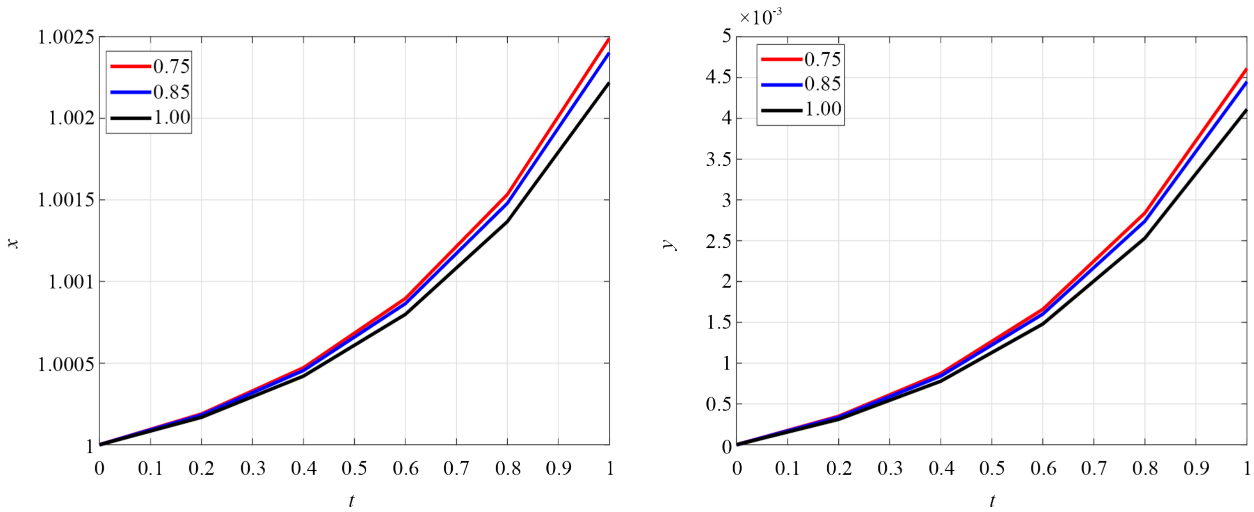

Figure 2. Numerical results of Example 1 for fractional orders in (0.60, 1.00]

For graphical illustration, we have taken $h = 0.05$. From Figures 1 and 2, we see that solutions of both equations show growth. The growth process of both equations in Figure 1 is slower at smaller fractional order $\alpha = 0.25$ as compared to larger fractional order $\alpha = 0.45$. But in Figure 2, the growth rate is slower at higher fractional order $\alpha = 1.0$. We see that as the fractional orders are smaller then the convergence process is slower as compared to larger fractional orders. In Figure 2, we see that as $\alpha \to 1$, the curves tend closer to the solution at integer order of the problem.

Example 2 Here, we apply our analysis to another coupled system, that is defined as follows:

$$\begin{cases} \mathbf{J}_0^{\alpha}\mathbf{x}(\mathrm{t}) = \dfrac{\exp(-\pi \mathrm{t}) + |\mathbf{x}(\mathrm{t})| + |\mathbf{y}(\mathrm{t})|}{100 + \sin|\mathbf{x}(\mathrm{t})| + \sin|\mathbf{y}(\mathrm{t})|}, \quad \mathrm{t} \in [0, 2]. \\[2ex] \mathbf{J}_0^{\alpha}\mathbf{y}(\mathrm{t}) = \dfrac{\exp(-\sin(\mathrm{t})) + |\mathbf{x}(\mathrm{t})| + |\mathbf{y}(\mathrm{t})|}{100 + \sin|\mathbf{x}(\mathrm{t})| + \sin|\mathbf{y}(\mathrm{t})|}, \quad \mathrm{t} \in [0, 2]. \\[2ex] \mathbf{x}(0) = 4 + \displaystyle\int_0^2 \frac{\mathbf{y}(\zeta)}{20} \mathrm{d}\zeta. \\[2ex] \mathbf{y}(0) = 4 + \displaystyle\int_0^2 \frac{\mathbf{x}(\zeta)}{20} \mathrm{d}\zeta. \end{cases} \tag{23}$$

Here,

$$\mathbf{f_1}(\mathrm{t}, \mathbf{x}(\mathrm{t}), \mathbf{y}(\mathrm{t})) = \frac{\exp(-\pi \mathrm{t}) + |\mathbf{x}(\mathrm{t})| + |\mathbf{y}(\mathrm{t})|}{100 + \sin|\mathbf{x}(\mathrm{t})| + \sin|\mathbf{y}(\mathrm{t})|},$$

$$\mathbf{f_2}(\mathrm{t}, \mathbf{x}(\mathrm{t}), \mathbf{y}(\mathrm{t})) = \frac{\exp(-\sin(\mathrm{t})) + |\mathbf{x}(\mathrm{t})| + |\mathbf{y}(\mathrm{t})|}{100 + \sin|\mathbf{x}(\mathrm{t})| + \sin|\mathbf{y}(\mathrm{t})|},$$

$$\mathbf{g}_1(\mathbf{y}(\zeta)) = \frac{\mathbf{y}(\zeta)}{20}, \quad \mathbf{g}_2(\mathbf{x}(\zeta)) = \frac{\mathbf{x}(\zeta)}{20}.$$

Hence, we have

$$|\mathbf{g}_1(\mathbf{y}(\zeta)) - \mathbf{g}_1(\bar{\mathbf{y}}(\zeta))| \le \frac{1}{20}|\mathbf{y} - \bar{\mathbf{y}}|,$$

$$|\mathbf{g}_2(\mathbf{x}(\zeta)) - \mathbf{g}_2(\bar{\mathbf{x}}(\zeta))| \le \frac{1}{20}|\mathbf{x} - \bar{\mathbf{x}}|,$$

$$|\mathbf{f_1}(\mathrm{t}, \mathbf{x}(\mathrm{t}), \mathbf{y}(\mathrm{t})) - \mathbf{f_1}(\mathrm{t}, \bar{\mathbf{x}}(\mathrm{t}), \bar{\mathbf{y}}(\mathrm{t}))| \le \frac{1}{100}|\mathbf{x} - \bar{\mathbf{x}}| + \frac{1}{100}|\mathbf{y} - \bar{\mathbf{y}}|,$$

$$|\mathbf{f_2}(\mathrm{t}, \mathbf{x}(\mathrm{t}), \mathbf{y}(\mathrm{t})) - \mathbf{f_2}(\mathrm{t}, \bar{\mathbf{x}}(\mathrm{t}), \bar{\mathbf{y}}(\mathrm{t}))| \le \frac{1}{100}|\mathbf{x} - \bar{\mathbf{x}}| + \frac{1}{100}|\mathbf{y} - \bar{\mathbf{y}}|,$$

$$|\mathbf{f_1}(\mathrm{t}, \mathbf{x}(\mathrm{t}), \mathbf{y}(\mathrm{t}))| \le \frac{1}{100}|\mathbf{x}| + \frac{1}{100}|\mathbf{y}|,$$

and

$$|\mathbf{f_2}(\mathrm{t}, \mathbf{x}(\mathrm{t}), \mathbf{y}(\mathrm{t}))| \le \frac{1}{100}|\mathbf{x}| + \frac{1}{100}|\mathbf{y}|.$$

The axillary constants can be computed as: $\ell_1 = \flat_1 = \dfrac{1}{20}$, $\ell_2 = \flat_2 = \ell_3 = \flat_3 = \ell_4 = \flat_4 = \ell_5 = \flat_5 = \dfrac{1}{100}$. Further, let $\alpha = 0.8$ then, we have

$$\mathbb{L} = \max\{\ell_1 \tau, \flat_1 \tau\} = 0.2 < 1,$$

and

$$\mathbb{C} = \max\left\{\left[\ell_1 \tau + \ell_2 \frac{\tau^\alpha}{\alpha} + \flat_2 \frac{\tau^\alpha}{\alpha}\right], \left[\flat_1 \tau + \ell_3 \frac{\tau^\alpha}{\alpha} + \flat_3 \frac{\tau^\alpha}{\alpha}\right]\right\} = 0.143527 < 1.$$

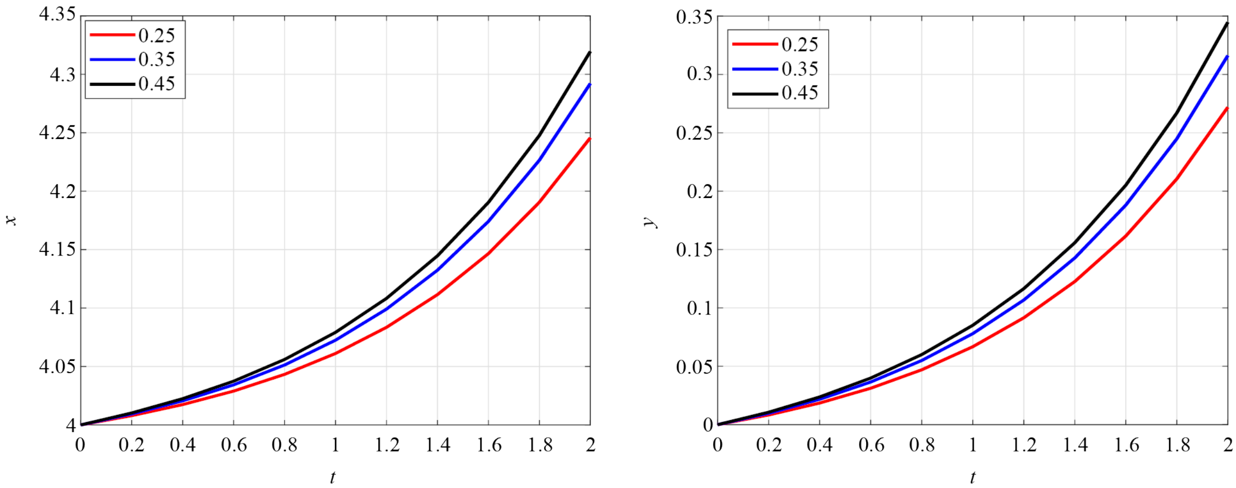

Figure 3. Numerical results of Example 2 for fractional orders in (0, 0.50)

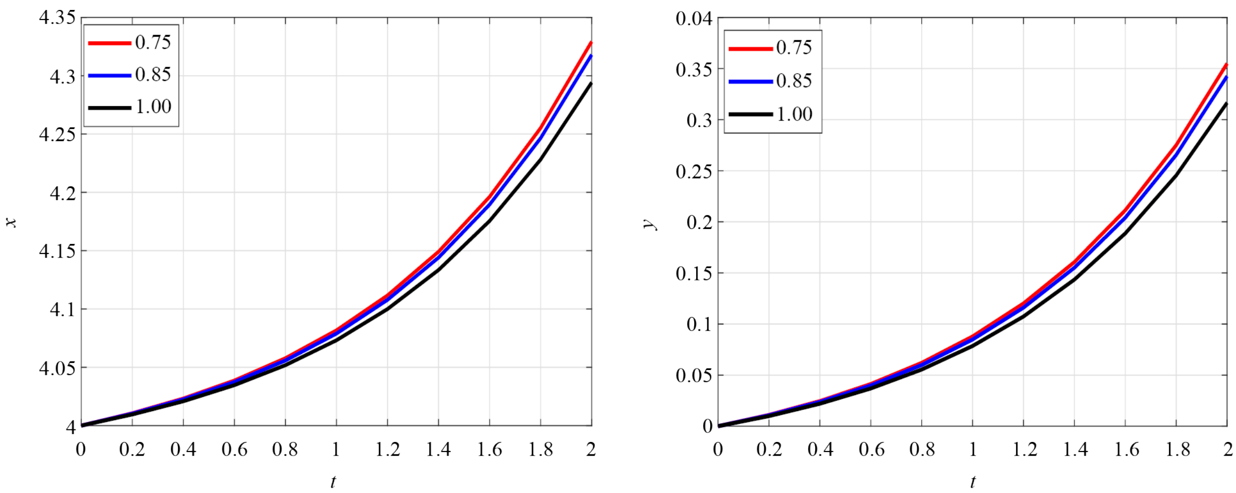

Figure 4. Numerical results of Example 2 for fractional orders in (0.60, 1.00]

The requirements of Theorems 3, 4, and 5 are met. As a result, our considered system (1) has a U-H stable solution and at least one unique solution. We provide the numerical interpretation of Example 2 using various fractional order values in Figures 3 and 4.

To provide a visual representation, we have used $h = 0.05$. Figures 3 and 4 demonstrate that both equations' solutions exhibit growth. At lesser fractional order $\alpha = 0.25$, the growth process of both equations in Figure 3 is slower than at bigger fractional order $\alpha = 0.45$. However, with greater fractional order $\alpha = 1.0$, the growth rate is slower in Figure 4. We see that the convergence process is slower for smaller fractional orders than for bigger fractional orders. As $\alpha \to 1$, the curves in Figure 4 trend toward the solution at the problem's integer order.

5. Concluding remarks

The differentials and integral operators of fractional orders in recent times have been found valuable tools to investigate various real world problems that display inherited or memory-dependent characteristics. Since researchers have produced numerous differential operators of fractional order. They have their own advantages and disadvantages. For instance, some fundamental roles including product, quotient and chain roles are not obeyed by traditional fractional differential operators. Therefore, CFOD was introduced in 2015. This research work has been devoted to study the existence theory and stability analysis of coupled system with nonlocal integral coupled BCs under CFOD. Further, RK4 method was utilized to deduce some numerical investigations. Successfully, we have implemented the mentioned numerical tools for the considered problem and two pertinent examples were testified. Due to simplicity of CFOD, easily we have extended the traditional fixed point theory to obtain sufficient conditions for existence theory of solution. Also, criteria for stability was implemented by using U-H concepts. The numerical tool based on RK4 was extended to the proposed problem. Several graphical illustrations were given. In the future, the mentioned methodology will be exercised to study tumor growth model by using CFOD.

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Conflict of interest

The authors declare no conflict of interests.

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Appendix A

The proof of Theorem 3 is given bellow:
Proof. Applying Lemma 2 to (5), one has

$$\begin{cases} \mathbf{x}(\mathrm{t}) - \mathbf{x}(0) = \mathbf{I}_0^{\alpha} \mathbf{h_1}(\mathrm{t}), \\ \mathbf{y}(\mathrm{t}) - \mathbf{y}(0) = \mathbf{I}_0^{\alpha} \mathbf{h_2}(\mathrm{t}). \end{cases} \tag{24}$$

Inserting conditions Eq. (24) implies

$$\begin{cases} \mathbf{x}(\mathrm{t}) = \mathbf{x}_0 + \int_0^{\mathrm{t}} \zeta^{\alpha-1} \mathbf{h_1}(\zeta) \mathrm{d}\zeta, \\ \mathbf{y}(\mathrm{t}) = \mathbf{y}_0 + \int_0^{\mathrm{t}} \zeta^{\alpha-1} \mathbf{h_2}(\zeta) \mathrm{d}\zeta. \end{cases} \tag{25}$$

□