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Research Article

On the Exact Solvability of Polynomial Families: Quintic and Sextic Cases with a Conjectural Extension to Higher Degrees

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Abstract: We develop a framework for identifying conditionally solvable families of quintic and sextic polynomials by reformulating polynomial solving with the coefficient relations as a constrained matrix decomposition problem. By enforcing rank-2 conditions on an associated symmetric matrix, the original polynomial equation can, for certain structured families, be reduced to lower-degree components. Since polynomial equations of degree at most four are solvable by radicals, such reductions provide explicit radical solutions whenever the reduction succeeds. We emphasize that the rank-2 condition is not sufficient for solvability in general; rather, it identifies special coefficient families for which the resulting reduced equations fall within solvable classes. The parameters introduced in the construction arise from solving an auxiliary algebraic system obtained via Gaussian elimination, and different parameter choices correspond to distinct solvable subfamilies. Several quintic and sextic examples are presented to illustrate the method. Finally, we propose a conjectural extension to higher-degree polynomials for degrees nine and above, formulated in a manner consistent with classical Galois theory, thus outlining conditions for exact solvability and inviting further exploration.

Keywords: quintic equations, sextic equations, exact solutions, polynomial solving

MSC: 12D05, 12F10, 15A18, 11R32, 11C08

1. Introduction

Solving polynomials in closed form, specifically by radicals, is a problem in algebra with a deep history [1] and has seen significant recent activity [2–9]. Mathematicians, from the time of the Renaissance, have been increasingly challenged to find closed-form solutions to polynomials. Solutions by radicals for quadratic, cubic, and quartic polynomials were discovered during the 16th century by mathematicians Scipione del Ferro, Niccolo Tartaglia, and Lodovico Ferrari [10]. However, the general quintic does not yield to these techniques, which led to the question of whether they are solvable by radicals in the first place or by extension, higher degree polynomials. The unsolvability of quintics was proved by Paolo Ruffini [11] partially at the start, and Niels Abel completed the proof, which concluded in the Abel-Ruffini theorem [12, 13]. Evariste Galois was credited with clarifying that general polynomials of degree-5 and above cannot be solved with radicals in the 19th century. He established this through his development of Galois theory. Galois showed that the solvability of a polynomial by radicals is linked to the structure of its associated Galois group, which is a group of symmetries over the

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roots of the polynomial in question [14]. If a polynomial can be solved by radicals, its Galois group must be a solvable group, but with polynomials of degree-5 or greater, the Galois group is usually not solvable; thus, there are no solutions by radicals [15, 16]. This groundbreaking result established the unsolvability of the quintic polynomial with radicals, thus creating a foundation for modern algebra.

Despite this result, particular cases of quintic and high-degree polynomials are shown to have algebraic solutions, although the general form does not. These cases go back to the fact that specific symmetries within the polynomial lead to solvable Galois groups. For example, one can construct quintics with cyclic or abelian Galois groups. Klein's study on the Icosahedral group shows this link of symmetry and solvability, where he found that restricting symmetry led to solvable cases [17] with future mathematicians building upon this work to demonstrate particular cases of high degree polynomials that are solvable [18].

1.1 *Applications of polynomial solving and its relation to splines*

Both exact and numerical solutions to polynomial equations have widespread applications across diverse scientific and engineering disciplines. In robotics, closed-form solutions to inverse kinematics problems are essential for real-time motion planning and control of manipulator systems [19]. For robots with six degrees of freedom and spherical wrists, the inverse kinematics can be reduced to solving systems of polynomial equations. In numerical approaches, solutions to a system of polynomials [20] can achieve higher precision by combining exact solutions for low-degree polynomials with numerical approximations for more complex cases. Both polynomial and non-polynomial spline methods are powerful numerical tools for solving differential equations and boundary value problems in science and engineering. Classical polynomial splines such as cubic, quadratic, and quartic splines provide stable and accurate solutions to higher order boundary value problems in applications like heat transfer and cable deflection through piecewise approximations [21, 22].

For greater flexibility, non-polynomial splines have gained prominence, including logarithmic and fractional splines, as they can achieve high-order convergence and stability for solving nonlinear inhomogeneous equations [23], and they can also serve as tools in multiresolution analysis [24]. Rational non-polynomial splines, notably NURBS, are used in computer-aided design for advanced manufacturing applications [25]. Variants of non-polynomial splines have been extensively used to solve various differential equations. Rational non-polynomial splines to solve second-order boundary value problems [26], non-polynomial cubic splines for hyperbolic differential equations with variable coefficients [27], conformable-Caputo non-polynomial splines [28] and conformable non-polynomial splines [29] for time-fractional differential equations.

1.2 *Motivations and novel contributions*

This paper pursues two major objectives. The first is to identify specific families of quintic and sextic polynomials that have algebraic solutions despite unsolvability for their general cases. Our approach leverages linear algebra methods to reduce the polynomial so that it is equivalent to solving a lower degree polynomial that has half of its degree.

The second objective of this paper is to propose a conjecture regarding polynomials of degree 9 and higher, where conditions are hypothesised that indicate the solvability of these high-degree polynomials. Our conjecture states that structured subfamilies of degree 9 and above may admit algebraic solutions under specific constraints, provided they can be reduced to polynomials of lower degree. This proposition conforms to the limitations set by Galois theory as reduction into lower degree polynomials is not guaranteed. And it provides a landscape for the solvability of higher degrees with potential consequences for algebraic theory [30] and number theory [31].

We emphasise that our method does not contradict Galois theory, as it applies only to restricted coefficient families. Contrary to traditional approaches that rely on either Galois-theoretic techniques or numerical methods, our framework offers a constructive approach through matrix decomposition. Our innovation lies in translating the polynomial coefficient constraints into rank-2 conditions on their associated symmetric matrices. While classical methods, such as Klein's icosahedral symmetries, Bring-Jerrard transforms, and Tschirnhaus transforms, focus on specific polynomial forms, our approach offers a unified matrix-based perspective applicable across multiple polynomial degrees. Our method shares conceptual similarities with Klein's work in utilising symmetry constraints, but differs in two key aspects: we employ eigenstructure analysis rather than group-theoretic techniques, and we provide a systematic framework rather than a case-by-case analysis. This work contributes to the ongoing exploration of polynomial solvability by:

1. Identifying specific families of solvable quintic and sextic polynomials showcasing where algebraic solutions exist despite limitations on the general form. After determining the solvability conditions of each family, their matrix decompositions are retrieved and included in the appendix.

2. Formulating a conjecture for degree-9 and higher polynomials, which expands the boundaries of current solvability theories and provides new directions for research regarding algebraic solutions for polynomials.

3. Offering a matrix-based framework to solve quintic and sextic polynomials where possible, which have rational roots with potential applications to higher degree polynomials, backed by numerical examples. This highlights the potential for implementing software based on the framework. Unlike numerical methods, the resulting expressions are exact and symbolic.

The rest of this paper is organised as follows: Section 2 provides related work on high-degree polynomials such as quintics, sextics, and those with higher degrees. Section 3 outlines the methodology used to identify algebraically solvable cases for both quintics and sextics, as well as formalising the general case. Sections 4 and 5 present our proofs for specific cases, while Section 6 includes computational examples for completeness, and Section 7 introduces our conjecture for higher degrees. The paper concludes with a summary of the work and possible future research directions.

2. Related work

2.1 *Solvability and Galois theory*

In algebra, a polynomial is solvable by radicals if its roots can be expressed using a finite number of arithmetic operations such as additions, subtractions, multiplications, divisions and roots on the coefficients of the polynomial. Galois theory establishes the required conditions for the existence of algebraic solutions by analysing symmetries of the roots of the polynomial. Galois established the criterion of solvability by stating that a polynomial is solvable by radicals if and only if its associated Galois group is also a solvable group [15, 16]. This links solvability to the group structure of permutations of the polynomial's roots: the Galois group. For polynomials of degree four or lower, their Galois groups are always solvable, which explains why they have solutions in terms of radicals. However, quinic polynomials or those of a higher degree are usually non-solvable. Formally, the symmetric group S_n for $n \geq 5$ is non-solvable and the general polynomial of degree n has S_n as its Galois group, thus it is unsolvable by radicals in the general case. This result is the Abel-Ruffini theorem, which shows that there is no general formula in terms of radicals solving polynomials of degree-5 or higher [18, 32]. While the general cases for polynomials of degree-5 and above lack closed-form solutions, certain particular cases have solutions. These cases of polynomials demonstrate special structural properties of their Galois group, which arise from the fact that their symmetries reduce the complexity of their root arrangements. An example is a quintic polynomial whose Galois groups are abelian or cyclic, which, in consequence, can be solved by radicals. An abelian group is one where all elements commute, while a cyclic group is a group generated by a single element. Both are examples of solvable groups, making it possible for their associated polynomials to be solvable in terms of radicals [17].

An example of one of these solvable groups is the Icosahedral symmetry group, which is usually non-solvable, but the specific configurations of the group lead to solvable cases for some quintics. Felix Klein explored this in his lectures on the Icosahedron [17]. Subsequent research has focused on finding explicit criteria for said specific cases. This can be done by examining particular types of polynomials where the Galois group does not show the full complexity usually associated with its degree. This approach has led to numerous studies on specific classes of polynomials and their algebraic solutions [30, 33].

2.2 *Results for degrees 5 and 6*

For quintics, one line of research is finding structured forms of quintic polynomials which have solvable Galois groups. In these cases, the polynomial demonstrates special symmetry or factorisation properties that reduce the associated Galois group to a cyclic or abelian form. Some forms commonly studied include cyclic quintic polynomials, where the Galois group is cyclic of order 5, and abelian quintic polynomials, which have a Galois group that is a direct product of cyclic groups. The order of a cyclic group equals the number of elements in that group. Sextic polynomials present a similar complexity but allow for structured cases due to their composite nature. Take, for example, certain sextic polynomials, which are a product of quadratics and cubics or other specific forms that lead to their associated Galois group being solvable. The analysis of sextics often involves techniques from Galois theory, like descent, where one reduces the problem to simpler, lower degree polynomial factors. Studies have shown that sextic polynomials with solvable Galois groups can often be transformed through factorisation or symmetrisation [32]. In contrast to the theory, the scientific community is interested in retrieving the actual roots of the quintics and high-degree polynomials, albeit with the theoretical limitations [20, 34]. Nowadays, matrix methods are popular for calculating roots of arbitrary polynomials by approximation, which involves retrieving the eigenvalues of the associated companion and generalised companion matrices. An example of this is the QR algorithm for retrieving eigenvalues and eigenvectors. Weierstrass's polynomial solving algorithm makes use of the QR algorithm along with some modifications; it is noted that the results are obtained by approximation. Further treatments of applying the QR algorithm in the context of polynomials or in general should be referenced in [35–40].

As solving quintic polynomials is a special case of solving polynomials of arbitrary degree, it can be seen that the standard approach for quintic polynomials is through solving with the same numerical methods discussed, providing approximate solutions. A current well-known algorithm for computing solutions to the quintic polynomials is that of King and Canfield [41], which is a modification of Kiepert's algorithm [42]. Their approach involves Tschirnhaus transformations [43] which were first demonstrated by Kiepert to transform the quintic polynomial into the the Jacobi sextic form of $s^6+\frac{10}{\Delta}s^3-\frac{12g_2}{\Delta^2}s+\frac{5}{\Delta^2}=0$ which are then solved using techniques from elliptic curves which involve a pair of infinite series. It is interesting to note that the Bring-Jerrard form x^5+ax+b is commonly used when approaching quintic polynomials, as it can be shown that using the resultant, any quintic polynomial can be reduced to its Bring-Jerrard form [44–46]. For the case of the sextic polynomial, there are schemes to solve them approximately, but they make use of advanced techniques such as complex geometry [47] and Kampé de Fériet functions defined by infinite series [48, 49]. These methods leave room for improvement. The mathematical community would benefit from a framework that provides exact solutions while remaining simple to implement.

2.3 *Works on degrees 7, 8, 9 and higher*

Similarly to quintics and sextics, there are indications that specific subfamilies of higher degree polynomials have algebraic solutions for specific cases. Recent work in algebraic theory suggests that increasing degrees of polynomials also lead to increasing opportunities to classify solvable polynomials based on a structured Galois group with additional symmetry or decomposition properties [18, 30]. There has been work on solving solvable sextics through the use of decomposition methods into lower degree polynomials, as in [50] or by relating the coefficients of the sextic to its Galois group structure [51]. While for the case of octic polynomials, there is work presenting parameterisations with quintic and sextic polynomials, thus making them easier to solve through employing six arbitrary parameters, this also implies that solvable cases can be found through explicit radicals [52]. Decomposition techniques, similar to the sextic case, are used to solve the octics [53], and interestingly, symmetry-breaking methods are explored in six-dimensional projective spaces, hinting at generalisations for even higher degrees [54]. This paper proposes a conjecture that polynomials of degree 9 and higher have specific cases which are solvable. This conjecture builds on patterns observed with lower degree polynomials and leverages the ideas developed in the paper, which may provide a new frontier for solvability in higher-dimensional representations.

3. Methodology

The central concept in this work involves utilising the matrix representation of a polynomial to facilitate its decomposition into a pair of lower degree polynomials, thereby simplifying its solution process. This decomposition requires that the polynomial's matrix representation have rank-2, a property which will be formalised later as a conjecture. To establish the framework, we first show the generalised theory from [55], as it provides the theoretical foundation upon which this paper builds. The discussion begins with the general case, followed by detailed analyses of the quintic and sextic cases, which lead to two supporting lemmas.

3.1 *General case*

An n-degree polynomial $P_n(x)$ with real coefficients and $x \in \mathbb{R}$ can be expressed in terms of a matrix product, a formulation that provides a powerful algebraic tool for analysing its structure and properties. This representation generalises techniques from the previous subsection and is expressed as follows:

$$P_n(x) = \sum_{i=0}^{n} b_n x^n = \begin{bmatrix} x^{\lceil \frac{n}{2} \rceil} \\ x^{\lceil \frac{n}{2} \rceil - 1} \\ \vdots \\ x \\ 1 \end{bmatrix}^T \mathbf{A}_n \begin{bmatrix} x^{\lceil \frac{n}{2} \rceil} \\ x^{\lceil \frac{n}{2} \rceil - 1} \\ \vdots \\ x \\ 1 \end{bmatrix}.$$

Where $\mathbf{A}_n$ is a symmetric matrix with real terms and the sum of anti-diagonals equals the coefficient of the polynomial $P_n(x)$, i.e., $b_{n+2-k} = \sum_{i+j=k} a_{i,\,j}$ for $k \in \{0,\, 1,\, \ldots,\, n\}$. It is also noted that the leading coefficient satisfies $b_n = 1$. We now demonstrate the following Lemma,

Lemma 1 The matrix $\mathbf{A}_n$ is not unique, different ways of distributing the values of each coefficient on its corresponding anti-diagonal are possible.

The zero polynomial can be expressed as a matrix equation, which ensures that the sum of elements along each anti-diagonal vanishes, maintaining the characteristic nullification of all its coefficients. The equation is as follows:

$$0 = \begin{bmatrix} x^{\lceil \frac{n}{2} \rceil} \\ x^{\lceil \frac{n}{2} \rceil - 1} \\ \vdots \\ x \\ 1 \end{bmatrix}^T \mathbf{A}_n^0 \begin{bmatrix} x^{\lceil \frac{n}{2} \rceil} \\ x^{\lceil \frac{n}{2} \rceil - 1} \\ \vdots \\ x \\ 1 \end{bmatrix}.$$

Where the sum of anti-diagonals is null, i.e., $\forall k \in \{0,\, 1,\, \ldots,\, n\}$: $\sum_{i+j=k} a_{i,\,j} = 0$. There are many ways to define explicitly the matrix $\mathbf{A}_n^0$, one of them is as follows:

$$a^0_{i,\,j} = \begin{cases} 0 & \text{if } i+j \le 3 \text{ or } i+j \ge 2\left\lceil \frac{n}{2} \right\rceil + 1 \\ t_{i+j-3} & \text{if } i+j \text{ is even and } i \ne j \\ -2\min\left(\left\lceil \frac{n}{2} \right\rceil - i,\, i-1\right) t_{i+j-3} & \text{if } i+j \text{ is even and } i = j \\ (-1)^{i+1} t_{i+j-3} & \text{if } i+j \text{ is odd and } \frac{i+j-1}{2} \text{ is even, } i \le \left\lceil \frac{n}{2} \right\rceil + 1 \\ \left(-\frac{i+j-3}{2}\right)^{\delta(i-j+1)} t_{i+j-3} & \text{if } i+j \text{ is odd and } \frac{i+j-1}{2} \text{ are odd, } i \le \left\lceil \frac{n}{2} \right\rceil + 1 \\ (-1)^{i+1} t_{i+j-3} & \text{if } i+j \text{ is odd and } \frac{2\left\lceil \frac{n}{2} \right\rceil - i - j - 1}{2} \text{ is even, } i > \left\lceil \frac{n}{2} \right\rceil + 1 \\ \left(-\frac{i+j-3}{2}\right)^{\delta(i-j+1)} t_{i+j-3} & \text{if } i+j \text{ and } \frac{2\left\lceil \frac{n}{2} \right\rceil - i - j - 1}{2} \text{ are odd, } i > \left\lceil \frac{n}{2} \right\rceil + 1 \end{cases} \quad , \forall i > j: \quad a^0_{i,\,j} = a^0_{j,\,i}.$$

Thus it is observed the symmetric matrix $\mathbf{A}^0_n$ has $2\left\lceil \frac{n}{2} \right\rceil - 3$ parameters $t_1, t_2, \dots t_{2\left\lceil \frac{n}{2} \right\rceil - 3} \in \mathbb{R}$. The number of parameters can be increased or decreased, so this decomposition is not unique. Below are two examples of the zero matrix $\mathbf{A}^0_5$,

Example 1

$$\mathbf{A}^0_7 = \begin{bmatrix} 0 & 0 & t_1 & t_2 & t_3 \\ 0 & -2t_1 & -t_2 & -t_3 & t_4 \\ t_1 & -t_2 & -4t_3 & -t_4 & t_5 \\ t_2 & t_3 & -t_4 & -2t_5 & 0 \\ t_3 & t_4 & t_5 & 0 & 0 \end{bmatrix}.$$

Example 2

$$\mathbf{A}^0_7 = \begin{bmatrix} 0 & 0 & t_1 & t_2 & t_3 \\ 0 & -2t_1 & -t_2 & -t_3 & t_4 \\ t_1 & -t_2 & 0 & -t_4 & t_5 \\ t_2 & -t_3 & -t_4 & -2t_5 & 0 \\ t_3 & t_4 & t_5 & 0 & 0 \end{bmatrix}.$$

One can verify these examples by checking if the sum in each anti-diagonal goes to zero. Therefore, each polynomial $P_n(x)$ of degree n can be written as a matrix product:

$$P_n(x) = \sum_{i=0}^{n} b_n x^n = \begin{bmatrix} x^{\left\lceil \frac{n}{2} \right\rceil} \\ x^{\left\lceil \frac{n}{2} \right\rceil - 1} \\ \vdots \\ x \\ 1 \end{bmatrix}^T \left(\mathbf{A}_n + \mathbf{A}^0_n\right) \begin{bmatrix} x^{\left\lceil \frac{n}{2} \right\rceil} \\ x^{\left\lceil \frac{n}{2} \right\rceil - 1} \\ \vdots \\ x \\ 1 \end{bmatrix}.$$

The notation simplifies to:

$$X = \begin{bmatrix} x^{\lceil \frac{n}{2} \rceil} & x^{\lceil \frac{n}{2} \rceil - 1} & \dots & x & 1 \end{bmatrix}^T.$$

Which results in,

$$P_n(x) = \mathbf{X}^T (\mathbf{A}_n + \mathbf{A}_n^0)\mathbf{X}.$$

By forcing the rank of the matrix $\mathbf{A}_n + \mathbf{A}_n^0$ to be equal to two: for polynomials of degree 3 and 4, this can always be forced. However, for higher dimensions, we will encounter a hard system of nonlinear equations that need to be solved. Extra assumptions about the coefficients of the polynomial are needed to solve the system, and, as a consequence, the rank of the matrix will be equal to two. If the rank of $\mathbf{A}_n + \mathbf{A}_n^0$ is two, at most two non-zero eigenvalues can be found in the orthogonal diagonalisation of $\mathbf{A}_n + \mathbf{A}_n^0 = \mathbf{V}\mathbf{D}\mathbf{V}^T$,

$$P_n(x) = \mathbf{X}^T \mathbf{V}\mathbf{D}\mathbf{V}^T \mathbf{X}.$$

We note $\mathbf{Y} = \mathbf{V}^T \mathbf{X}$, so:

$$P_n(x) = \mathbf{Y}^T \mathbf{D}\mathbf{Y} = \mathbf{Y}^T \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \mathbf{Y}.$$

Resulting in,

$$\begin{aligned} P_n(x) = &\lambda_1 \left(\mathbf{V}_{1,\,1} x^{\lceil \frac{n}{2} \rceil} + \mathbf{V}_{1,\,2} x^{\lceil \frac{n}{2} \rceil - 1} + \cdots + \mathbf{V}_{1,\,\lceil \frac{n}{2} \rceil + 1} \right)^2 \\ &+ \lambda_2 \left(\mathbf{V}_{2,\,1} x^{\lceil \frac{n}{2} \rceil} + \mathbf{V}_{2,\,2} x^{\lceil \frac{n}{2} \rceil - 1} + \cdots + \mathbf{V}_{2,\,\lceil \frac{n}{2} \rceil + 1} \right)^2. \end{aligned}$$

Where λ_1 and λ_2 are the eigenvalues of $\mathbf{A}_n + \mathbf{A}_n^0$, and $(\mathbf{V}_{1,\,1}, \mathbf{V}_{1,\,2}, \dots, \mathbf{V}_{1,\,\lceil \frac{n}{2} \rceil + 1})$ and $(\mathbf{V}_{2,\,1}, \mathbf{V}_{2,\,2}, \dots, \mathbf{V}_{2,\,\lceil \frac{n}{2} \rceil + 1})$ are the corresponding orthonormal eigenvectors. Thus the solution of $P_n(x) = 0$ can be found easily by solving polynomials of degrees $\lceil \frac{n}{2} \rceil$ as follows:

$$\begin{aligned} &\lambda_1 \left(\mathbf{V}_{1,\,1} x^{\lceil \frac{n}{2} \rceil} + \mathbf{V}_{1,\,2} x^{\lceil \frac{n}{2} \rceil - 1} + \cdots + \mathbf{V}_{1,\,\lceil \frac{n}{2} \rceil + 1} \right)^2 \\ = &-\lambda_2 \left(\mathbf{V}_{2,\,1} x^{\lceil \frac{n}{2} \rceil} + \mathbf{V}_{2,\,2} x^{\lceil \frac{n}{2} \rceil - 1} + \cdots + \mathbf{V}_{2,\,\lceil \frac{n}{2} \rceil + 1} \right)^2. \end{aligned}$$

Therefore, the preceding analysis leads us to the following key relation,

$$\mathbf{V}_{1,\,1}x^{\lceil \frac{n}{2} \rceil} + \mathbf{V}_{1,\,2}x^{\lceil \frac{n}{2} \rceil - 1} + \cdots + \mathbf{V}_{1,\,\lceil \frac{n}{2} \rceil + 1}$$

$$= \pm \sqrt{\frac{-\lambda_2}{\lambda_1}} \left(\mathbf{V}_{2,\,1}x^{\lceil \frac{n}{2} \rceil} + \mathbf{V}_{2,\,2}x^{\lceil \frac{n}{2} \rceil - 1} + \cdots + \mathbf{V}_{2,\,\lceil \frac{n}{2} \rceil + 1} \right).$$

This equation succinctly captures the relationship between the two orthonormal eigenvectors, $\mathbf{V}_1$ and $\mathbf{V}_2$, corresponding to the eigenvalues λ_1 and λ_2 of the matrix $\mathbf{A}_n + \mathbf{A}_n^0$. The presence of the square root term $\sqrt{\frac{-\lambda_2}{\lambda_1}}$ signifies the relative scaling between these two eigenvectors, which provides a critical step in the decomposition of the original polynomial into simpler, solvable components. The resulting reduced cubic and quartic equations obtained in this framework are algebraically equivalent to the original polynomial under constraints which is discussed later on in detail. Specifically, substituting the solutions of the reduced equations back into the original polynomial verifies that all roots are preserved. The reduction therefore constitutes an exact algebraic decomposition rather than an approximation. This is demonstrated in the Computational Examples section for both quintics and sectics.

3.2 *Quintic and sextic case*

A general monic quintic polynomial is of the form,

$$P_5(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e.$$

Where a, b, c, d, e are the coefficients that define the polynomial. This form highlights the primary structure of the quintic polynomial, with the leading coefficient normalised to 1 for simplicity. Proceeding with the analysis, the polynomial can be represented as follows, where $\mathbf{X} = \begin{bmatrix} x^3 & x^2 & x & 1 \end{bmatrix}^T$,

$$P_5(x) = \begin{bmatrix} x^3 & x^2 & x & 1 \end{bmatrix} \mathbf{A} \begin{bmatrix} x^3 \\ x^2 \\ x \\ 1 \end{bmatrix} = \mathbf{X}^T \mathbf{A} \mathbf{X}.$$

Where $\mathbf{A}$ is of the form,

$$\mathbf{A} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & a & \frac{b}{2} & 0 \\ 0 & \frac{b}{2} & c & \frac{d}{2} \\ 0 & 0 & \frac{d}{2} & e \end{bmatrix}.$$

Evaluating the matrix product shows that it is equivalent to the polynomial as formulated in the previous subsection. Since the matrix $\mathbf{A}$ is real and symmetric, it indeed can be orthogonally diagonalised,

$$\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^T.$$

The matrix $\mathbf{V}$ consists of the orthonormal eigenvectors of $\mathbf{A}$ as its columns, e.g. $\mathbf{V} = \begin{bmatrix} \mathbf{V}_1 & \mathbf{V}_2 & \mathbf{V}_3 & \mathbf{V}_4 \end{bmatrix}$ and $\mathbf{D}$ is a diagonal matrix where the diagonal entries are the eigenvalues λ_i of $\mathbf{A}$. Now use the following substitution,

$$\mathbf{V}^T\mathbf{X} = \begin{bmatrix} \mathbf{V}_1 & \mathbf{V}_2 & \mathbf{V}_3 & \mathbf{V}_4 \end{bmatrix}^T \begin{bmatrix} x^3 \\ x^2 \\ x \\ 1 \end{bmatrix} = \mathbf{Y} = \begin{bmatrix} y_3 \\ y_2 \\ y_1 \\ y_0 \end{bmatrix}.$$

With y_i as polynomials of degree 3 in terms of x, resulting in the form,

$$cP_5(x) = \mathbf{Y}^T\mathbf{D}\mathbf{Y} = \begin{bmatrix} y_3 & y_2 & y_1 & y_0 \end{bmatrix} \begin{bmatrix} \lambda_4 & 0 & 0 & 0 \\ 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{bmatrix} \begin{bmatrix} y_3 \\ y_2 \\ y_1 \\ y_0 \end{bmatrix} = \sum_{i=1}^{4} \lambda_i y_i^2 = 0.$$

Consider when the matrix $\mathbf{A}$ is of rank-2; this means we have only two nonzero eigenvalues, which significantly simplifies the polynomial representation and solution process. A rank-2 matrix implies that the polynomial $P_n(x)$ can be reduced to a system involving just two independent components, corresponding to the orthonormal eigenvectors associated with these nonzero eigenvalues,

$$\sum_{i=1}^{2} \lambda_i y_i^2 = 0 \Rightarrow \lambda_3 y_3^2 = \lambda_4 y_4^2 \Rightarrow |y_3| = |y_4|\sqrt{\frac{-\lambda_4}{\lambda_3}}.$$

The imposition of a rank-2 condition on the associated matrix does *not* by itself imply solvability by radicals. Its role is to produce algebraic relations among the coefficients that may, in favorable cases, reduce the original polynomial to lower-degree equations. Since equations of degree at most four are solvable by radicals, solvability follows only when such a reduction is achieved. The present approach is therefore orthogonal to classical Galois-theoretic criteria: we do not attempt to determine the Galois group of a generic polynomial; instead, we identify explicit coefficient families for which solvability is guaranteed by construction. Gaussian elimination is used solely to derive the algebraic constraint system governing these families and is not, by itself, a proof of solvability. There is also another perspective in terms of the polynomial basis. This is so since we can think of it as expanding the basis of the polynomials as in below Figure 1 where we consider the basis elements (eigenvectors) corresponding to nonzero eigenvalues.

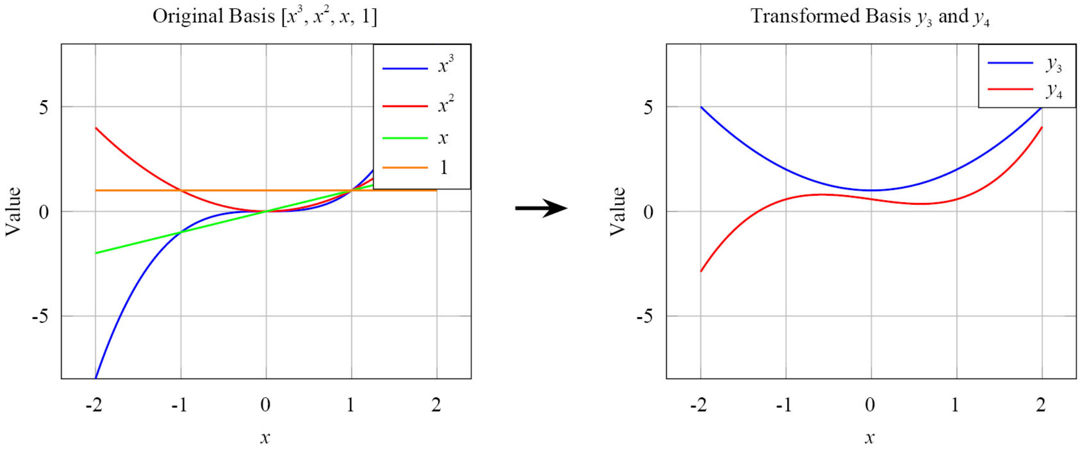

Figure 1. Original polynomial basis transformed into a new basis corresponding to nonzero eigenvalues

This implies that the system can be solved as both y_1 and y_4 are at most degree 3 polynomials. Expanding the above leads to:

$$\lambda_3(y_3)^2+\lambda_4(y_4)^2=0,$$

$$\Rightarrow \lambda_3(\mathbf{V}_3\mathbf{X})^2+\lambda_4(\mathbf{V}_4\mathbf{X})^2=0.$$

Upon simplification,

$$\mathbf{V}_3\mathbf{X}=\pm\sqrt{\frac{-\lambda_4}{\lambda_3}}\mathbf{V}_4\mathbf{X},$$

$$\Rightarrow \mathbf{V}_{31}x^3+\mathbf{V}_{32}x^2+\mathbf{V}_{33}x+\mathbf{V}_{34}=\pm\sqrt{\frac{-\lambda_4}{\lambda_3}}(\mathbf{V}_{41}x^3+\mathbf{V}_{42}x^2+\mathbf{V}_{43}x+\mathbf{V}_{44}).$$

Thus, one ends up with the following closed-form solutions, where solving the quartics provides at most six solutions for the quintic polynomial in this example, where one of the solutions is erroneous if so. This means that solving Equation (1) provides three solutions, and Equation (2) provides the other three.

$$(\mathbf{V}_{31}+\sqrt{\frac{-\lambda_4}{\lambda_3}}\mathbf{V}_{41})x^3+(\mathbf{V}_{32}+\sqrt{\frac{-\lambda_4}{\lambda_3}}\mathbf{V}_{42})x^2+(\mathbf{V}_{33}+\sqrt{\frac{-\lambda_4}{\lambda_3}}\mathbf{V}_{43})x+(\mathbf{V}_{34}+\sqrt{\frac{-\lambda_4}{\lambda_3}}\mathbf{V}_{44})=0, \quad (1)$$

$$(\mathbf{V}_{31}-\sqrt{\frac{-\lambda_4}{\lambda_3}}\mathbf{V}_{41})x^3+(\mathbf{V}_{32}-\sqrt{\frac{-\lambda_4}{\lambda_3}}\mathbf{V}_{42})x^2+(\mathbf{V}_{33}-\sqrt{\frac{-\lambda_4}{\lambda_3}}\mathbf{V}_{43})x+(\mathbf{V}_{34}-\sqrt{\frac{-\lambda_4}{\lambda_3}}\mathbf{V}_{44})=0. \quad (2)$$

Solving the two cases in the previous work provided the full solution set, but it will be seen that there is no need for this when solving quintics. The matrix **A** can be reduced to rank-2 by introducing the construction colorbluewith r, s, and $t \in \mathbb{R}$, which does not change the quintic equation:

$$P_5(x) = \mathbf{A}_{s,\,t,\,r} = \begin{bmatrix} 0 & \frac{1}{2} & s & t \\ \frac{1}{2} & a-2s & \frac{b}{2}-t & -r \\ s & \frac{b}{2}-t & c+2r & \frac{d}{2} \\ t & -r & \frac{d}{2} & e \end{bmatrix}.$$

The parameters s, t, and r arise as auxiliary variables introduced to parametrise solutions of the algebraic constraint system obtained from the rank condition. They are not intrinsic invariants of the polynomial, nor are they chosen arbitrarily; rather, each admissible choice corresponds to a specific solvable subfamily. Different parameter selections generate distinct exact solutions, and exploring alternative parameter regimes remains an open problem.

Evaluating the matrix product for $P_5(x)$ shows that it is equivalent even if we introduce the construction for $\mathbf{A}_{s,\,t,\,r}$. Thus, the goal is to find values of r, s, $t \in \mathbb{R}$ that force the matrix to be rank-2, then use this new, modified matrix to solve the polynomial. A similar matrix can be used for the sextic defined by the monic sextic polynomial of the form:

$$P_6(x) = x^6 + fx^5 + ax^4 + bx^3 + cx^2 + dx + e.$$

This is true since they use the same steps with an adjustment to the matrix **A**. Specifically, the methodology used for the quintic case can be seamlessly extended to handle the sextic polynomial by incorporating one additional degree term. This extension naturally follows the symmetry and structure of the matrix formulation. As in the quintic case, the goal is to decompose the polynomial into simpler components by representing it in matrix form, but here the matrix must account for the higher degree:

$$P_6(x) = \mathbf{A}_{s,\,t,\,r} = \begin{bmatrix} 1 & \frac{f}{2} & s & t \\ \frac{f}{2} & a-2s & \frac{b}{2}-t & -r \\ s & \frac{b}{2}-t & c+2r & \frac{d}{2} \\ t & -r & \frac{d}{2} & e \end{bmatrix}.$$

This matrix encapsulates the coefficients of the sextic polynomial in a manner similar to the quintic case, but with slight modifications to reflect the additional terms. Now that we have built up the necessary theory, we state the following lemmas and prove them in the following section.

4. Main results

For both lemmas, it is noted for notation that with an example subcondition such as $b = 2b'$ for a quintic polynomial, say, then b' is considered the coefficient of x^3, e.g. $p(x) = fx^5 + ax^4 + b'x^3 + cx^2 + dx + e$.

Lemma 2 Let $p(x) = fx^5 + ax^4 + bx^3 + cx^2 + dx + e$ be a quintic equation with real coefficients and $x \in \mathbb{R}$. If the coefficients f, a, b, c, d, e satisfy any of the following conditions, then the roots of the quintic equation can be expressed in radicals:

1. Quintics with $f = 1$ and $a = b = 0$:
• Condition 1: $c^2 d + e^2 = 0$.
• Condition 2: $c = 0$ and $d^5 = e^4$.
• Condition 3: $c = 0$ and $e^4 + 6^4 d^5 = 0$.
2. Quintics with $f = 1$ and $a = 0$:
• Condition 4: $c^2 d + e^2 - bec = 0$.
• Condition 5: $e = c = -2\sqrt{\frac{-b}{4}}$, and $d + 1 = b$.
• Condition 6: $c = e + be^{\frac{1}{3}} + e^{\frac{2}{3}}$, and $d = -e^{\frac{2}{3}} - be^{\frac{1}{3}} - 2e$.
3. Quintics with $f = 2$, $a' = 2a$, $b' = 2b$, and $d' = 2d$:
• Condition 7:

$$c - 2(\sqrt{-b})^3 + 2(\frac{d}{\sqrt{-b}}) = 0, \quad e + 2(\sqrt{-b})^5 + 2d'\sqrt{-b} = 0.$$

• Condition 8:

$$c + 2(\frac{e}{2d})^3 + 2b(\frac{e}{2d}) + 2(\frac{-e}{\sqrt{d}}) - 2(\frac{-e}{\sqrt{d}})(\sqrt{d}) = 0.$$

Lemma 3 Let $p(x) = gx^6 + fx^5 + ax^4 + bx^3 + cx^2 + dx + e = gx^6 + fx^5 + ax^4 + b'x^3 + cx^2 + d'x + e$ be a sextic equation with real coefficients and $x \in \mathbb{R}$. If the coefficients g, f, a, b, b', c, d, d', e satisfy one of the following conditions, then the roots of the sextic equation can be expressed in radicals:
1. Sextics with $g = 1$ and $f = a = b = c = 0$:
• Condition 1: $d = -2\left(\frac{-e}{2}\right)^{1/3}\sqrt{e}$.
• Condition 2: $e^5 2^8 = 5^5 d^6$.
2. Sextics with $g = 1$, $f = 0$, $a \neq 0$:
• Condition 3: $b' = 2b$, $d' = 2d$, $b = \sqrt{e}$, and $d^2 = ce$.

4.1 *Proof of Lemma 2, condition one*

Proof. A matrix of the form below is considered

$$\mathbf{A}_{t,\,r} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & t \\ \frac{1}{2} & 0 & -t & -r \\ 0 & -t & c+2r & \frac{d}{2} \\ t & -r & \frac{d}{2} & e \end{bmatrix}.$$

Applying Gaussian elimination,

$$\begin{bmatrix} \frac{1}{2} & 0 & -t & -r \\ 0 & \frac{1}{2} & 0 & t \\ 0 & 0 & c+2r & \frac{d+4t^2}{2} \\ 0 & 0 & \frac{d+4t^2}{2} & 4rt+e \end{bmatrix}.$$

To zero out the bottom two rows, it is required to solve the following system of equations,

$$c+2r=0, \tag{3}$$

$$d+4t^2=0, \tag{4}$$

$$4rt+e=0. \tag{5}$$

Solving Equation (3) and plugging it to Equation (5),

$$r=-\frac{c}{2}, \quad t=\frac{e}{2c}. \tag{6}$$

Plugging these results into Equation (4):

$$d+4(\frac{e}{2c})^2=0,$$

$$\Rightarrow d+\frac{e^2}{c^2}=0. \tag{7}$$

Multiplying both sides by c,

$$\Rightarrow c^2d+e^2=0.$$

Thus, the polynomial in this example is solvable given that the above constraint on its coefficients is satisfied. As in the previous example, the nonzero eigenvalues are considered, which are Equation (A3) and Equation (A4), with the normalised versions of the corresponding eigenvectors in Equation (A7) and Equation (A9). Now these values can be plugged into Equation (1) and Equation (2), where the only difference is the formulas defining the eigenvalues and eigenvectors. □

4.2 *Proof of Lemma 2, condition two*

Proof. A general matrix of the form below is used,

$$\mathbf{A}_{s,\,t,\,r} = \begin{bmatrix} 0 & \frac{1}{2} & -s & t \\ \frac{1}{2} & 2s & \frac{b}{2}-t & -r \\ -s & \frac{b}{2}-t & c+2r & \frac{d}{2} \\ t & -r & \frac{d}{2} & e \end{bmatrix}.$$

An arbitrary assumption $r = 0$ is made.

$$\mathbf{A}_{s,\,t} = \begin{bmatrix} 0 & \frac{1}{2} & -s & t \\ \frac{1}{2} & 2s & \frac{b}{2}-t & 0 \\ -s & \frac{b}{2}-t & c & \frac{d}{2} \\ t & 0 & \frac{d}{2} & e \end{bmatrix}.$$

Upon applying Gaussian elimination, the following matrix is the result,

$$\begin{bmatrix} \frac{1}{2} & 2s & \frac{b-2t}{2} & 0 \\ 0 & \frac{1}{2} & -s & t \\ 0 & 0 & c+8s^3+2bs-4st & \frac{d+4t^2-2bt-16s^2t}{2} \\ 0 & 0 & \frac{d+4t^2-2bt-16s^2t}{2} & 8st^2+e \end{bmatrix}.$$

Indicating that the following system has to be solved in order to force rank-2.

$$c+8s^3+2bs-4st = 0,$$

$$d+4t^2-2bt-16s^2t = 0,$$

$$8st^2+e = 0.$$

This system can be simplified further as it is recalled that the quintic considered in this subsection has coefficients satisfying $b = 0$, $c = 0$, and $d \neq 0$.

$$8s^3-4st = 0, \tag{8}$$

$$d+4t^2-16s^2t = 0, \tag{9}$$

$$8st^2+e=0. \tag{10}$$

Solving Equation (8) one gets the following,

$$t=2s^2. \tag{11}$$

And applying Equation (11) to Equation (9), this results in

$$4t^2-8t(t)+d=0 \Rightarrow d=4t^2. \tag{12}$$

Substituting into Equation (10),

$$2ds+e=0 \Rightarrow s=\frac{-e}{2d}. \tag{13}$$

Plugging in the values of s and t from Equation (13) and Equation (12) into Equation (11),

$$\pm\sqrt{\frac{d}{4}}=\frac{e^2}{2d^2} \Rightarrow \frac{d}{4}=\frac{e^4}{4d^4} \Rightarrow d^5=e^4. \tag{14}$$

Thus, the quintic is only solvable when the coefficients satisfy Equation (14). Plugging in Equation (12) and Equation (13) into $\mathbf{A}_{s,\,t}$, one retrieves nonzero eigenvalues defined in Equation (A13) and Equation (A14) along with the normalised versions of the their corresponding eigenvectors in Equation (A17) and Equation (A19). These equations can be plugged into Equation (1) and Equation (2) as before, deriving a closed-form solution. □

4.3 *Proof of Lemma 2, condition three*

Proof. To solve the quintic, a matrix of the form below is considered,

$$\mathbf{A}_{s,\,t,\,r}=\begin{bmatrix} 0 & \frac{1}{2} & -s & t \\ \frac{1}{2} & 2s & -t & -r \\ -s & -t & 2r & \frac{d}{2} \\ t & -r & \frac{d}{2} & e \end{bmatrix}.$$

Applying Gaussian elimination,

$$\begin{bmatrix} \frac{1}{4} & 0 & \frac{4s^2-t}{2} & \frac{-r-4st}{2} \\ 0 & \frac{1}{4} & -\frac{s}{2} & \frac{t}{2} \\ 0 & 0 & \frac{r+4s^3-2st}{2} & \frac{d-4rs+4t^2-16s^2t}{8} \\ 0 & 0 & \frac{d-4rs+4t^2-16s^2t}{8} & \frac{8st^2+4rt+e}{4} \end{bmatrix}.$$

Thus, the following system has to be solved,

$$r+4s^3-2st=0, \tag{15}$$

$$d-4rs+4t^2-16s^2t=0, \tag{16}$$

$$e+8st^2+4rt=0. \tag{17}$$

A condition is assumed to simplify solving the above system,

$$d=4rs. \tag{18}$$

Plugging this into Equation (16), simplifies to

$$4t^2-16s^2t=0 \Rightarrow t=4s^2. \tag{19}$$

Inserting Equation (19) into Equation (15),

$$4s^3+r-2s(4s^2)=0 \Rightarrow r=4s^3. \tag{20}$$

Using both Equation (19) and Equation (20) into Equation (17),

$$e+8s(4s^2)^2+4(4s^3)(4s^2)=0 \Rightarrow e+192s^5=0. \tag{21}$$

Lastly, Equation (20) has to be inserted into Equation (18) to represent it completely in terms of s

$$d=4s(4s^3)=16s^4. \tag{22}$$

These set of equations imply that plugging Equation (22) into Equation (21) leads to the condition,

$$e^4 + 6^4 d^5 = 0. \tag{23}$$

Assuming the coefficients of the quintic satisfy Equation (23), plugging in Equations (19)–(22) into the matrix $\mathbf{A}_{s,\,t,\,r}$ results in the nonzero Eigenvalues defined in Equation (A23) and Equation (A24) with the normalised versions of the Eigenvectors defined in Equations (A27)–(A30). Plugging them into Equation (1) and Equation (1) yields the roots of the quintic. □

4.4 *Proof of Lemma 2, condition four*

Proof. The matrix to apply Gaussian elimination on is of the form,

$$\mathbf{A}_{t,\,r} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & t \\ \frac{1}{2} & 0 & \frac{b}{2}-t & -r \\ 0 & \frac{b}{2}-t & c+2r & \frac{d}{2} \\ t & -r & \frac{d}{2} & e \end{bmatrix}.$$

Using Gaussian elimination,

$$\begin{bmatrix} \dfrac{1}{2} & 0 & \dfrac{b-2t}{2} & -r \\ 0 & \dfrac{1}{2} & 0 & t \\ 0 & 0 & c+2r & \dfrac{d+4t^2-2bt}{2} \\ 0 & 0 & \dfrac{d+4t^2-2bt}{2} & 4rt+e \end{bmatrix}.$$

As in the previous section, to zero out the bottom two rows to achieve rank-2, the following system of equations must be solved.

$$c + 2r = 0, \tag{24}$$

$$d + 4t^2 - 2bt = 0, \tag{25}$$

$$4rt + e = 0. \tag{26}$$

It can be seen that by solving Equation (24) and plugging it into Equation (26), one gets the following solutions for r and t.

$$r = \frac{-c}{2}, \quad t = \frac{e}{2c}. \tag{27}$$

Plugging this into Equation (25),

$$d + 4(\frac{e}{2c})^2 - 2b\frac{e}{2c} = 0,$$

$$\Rightarrow d + \frac{e^2}{c^2} - \frac{be}{c} = 0. \tag{28}$$

Multiplying both sides by c^2,

$$\Rightarrow c^2 d + e^2 - bec = 0. \tag{29}$$

Thus, if the coefficients of the polynomial $x^5 + bx^3 + cx^2 + dx + e = 0$ satisfy Equation (29), one can solve the quintic. In the appendix, the corresponding eigenvalues and eigenvectors are attained. Thus, the solutions to the quintic can be extracted using the nonzero eigenvalues in Equation (A33) and Equation (A34) with their corresponding normalised versions of the eigenvectors defined in Equation (A37) and Equation (A39). This is done by plugging them into Equation (1) and Equation (2), then solving them. □

4.5 *Proof of Lemma 2, condition five*

Proof. A matrix of the form below is considered,

$$\mathbf{A}_{s,\,t} = \begin{bmatrix} 0 & \frac{1}{2} & -s & t \\ \frac{1}{2} & 2s & \frac{b}{2} - t & 0 \\ -s & \frac{b}{2} - t & c & \frac{d}{2} \\ t & 0 & \frac{d}{2} & e \end{bmatrix}.$$

With Gaussian elimination resulting in the matrix,

$$\begin{bmatrix} \frac{1}{4} & 0 & \frac{b+8s^2-2t}{4} & -2st \\ 0 & \frac{1}{4} & \frac{-s}{2} & \frac{t}{2} \\ 0 & 0 & \frac{c+8s^3+2bs-4st}{4} & \frac{d+4t^2-2bt-16s^2t}{8} \\ 0 & 0 & \frac{d+4t^2-2bt-16s^2t}{8} & \frac{8st^2+e}{4} \end{bmatrix}.$$

The system to solve is thus,

$$c + 8s^3 + 2bs - 4st = 0, \tag{30}$$

$$d + 4t^2 - 2bt - 16s^2t = 0, \tag{31}$$

$$8st^2 + e = 0. \quad (32)$$

Approaching Equation (32),

$$s = \frac{-e}{8t^2}. \quad (33)$$

Plugging this result into Equation (30),

$$c - \frac{e^3}{64t^6} - \frac{be}{4t^2} + \frac{e}{2t} = 0 \Rightarrow 64t^6c - e^3 - 16bet^4 + 32et^5 = 0. \quad (34)$$

Equation (34) shows that one has to solve a sextic in order to find the required conditions to determine the solvability and solutions of the polynomial in this subsection. But what if Equation (33) is plugged into Equation (31)?

$$d + 4t^2 - 2bt - 16(\frac{-e}{8t^2})^2t = 0 \Rightarrow d + 4t^2 - 2bt - \frac{e^2}{4t^3} = 0. \quad (35)$$

Multiplying both sides of Equation (35) by t^3,

$$t^3d + 4t^5 - 2bt^4 - \frac{e^2}{4} = 0. \quad (36)$$

Thus, a quintic has to be evaluated to proceed further. Proceeding further for either case when encountering a quintic or sextic requires that they are solvable in the first place, which is another issue altogether. Thus, it is best to make some assumptions for s and t to proceed further without complication. Before proceeding with any steps, it is observed that the substitution $b = -4s^2$ simplifies Equation (30) and Equation (31). Applying this substitution to Equation (31),

$$d + 4t^2 - 2bt + 4(-4s^2)t = 0 \Rightarrow d + 4t^2 + 2bt = 0. \quad (37)$$

Now using this substitution on Equation (30),

$$c - 2(-4s^2) + 2bs - 4st = 0 \Rightarrow c - 4st = 0. \quad (38)$$

Another substitution is now made $t = \frac{-1}{2}$ to simplify Equation (38)

$$c - 4s(\frac{-1}{2}) = 0 \Rightarrow c + 2s = 0. \quad (39)$$

Also simplifying Equation (32)

$$8s(\frac{-1}{2})^2 + e = 0 \Rightarrow e + 2s = 0. \tag{40}$$

Equation (39) and Equation (40) imply the requirement that $e = c = -2s$. Plugging the substitution for t into Equation (41),

$$d + 4(\frac{-1}{2})^2 + 2b\frac{-1}{2} = 0 \Rightarrow d + 1 = b. \tag{41}$$

Thus when the polynomial's coefficients satisfy $e = c = -2\sqrt{\frac{-b}{4}}$ and $d + 1 = b$, one can substitute $s = \sqrt{\frac{-b}{4}}$ and $t = \frac{-1}{2}$ into the matrix to retrieve the eigenvalues defined in Equation (A43) and Equation (A44) along with their corresponding normalised versions of the eigenvectors in Equation (A48) and Equation (A50). Substituting the eigenvalues and eigenvectors into Equation (1) and Equation (2). The implication of making use of assumptions through substitutions shows that one can have different configurations for s and t along with their corresponding conditions. Meaning that it is possible to solve the polynomial with different solutions for s and t, given that the polynomial satisfies the respective conditions. □

4.6 *Proof of Lemma 2, condition six*

Proof. Another assumption, $s = t$ is considered to solve Equations (30), (31), and (32). This results in,

$$c + 8t^3 + 2bt - 4t^2 = 0, \tag{42}$$

$$d + 4t^2 - 2bt - 16t^3 = 0, \tag{43}$$

$$8t^3 + e = 0. \tag{44}$$

Solving Equation (44),

$$t = (\frac{-e}{8})^{\frac{1}{3}}. \tag{45}$$

Plugging this into Equation (42) and Equation (43),

$$c = -8((\frac{-e}{8})^{\frac{1}{3}})^3 - 2b(\frac{-e}{8})^{\frac{1}{3}} + 4((\frac{-e}{8})^{\frac{1}{3}})^2 \Rightarrow c = e + be^{\frac{1}{3}} + e^{\frac{2}{3}}, \tag{46}$$

$$d = -4((\frac{-e}{8})^{\frac{1}{3}})^2 + 2b(\frac{-e}{8})^{\frac{1}{3}} + 16((\frac{-e}{8})^{\frac{1}{3}})^3 \Rightarrow d = -e^{\frac{2}{3}} - be^{\frac{1}{3}} - 2e. \tag{47}$$

Thus when the quintic's coefficients satisfies Equation (46) and Equation (47), setting $s = t = (\frac{-e}{8})^{\frac{1}{3}}$ reduces the matrix $\mathbf{A}_{s,t}$ to rank-2 and thus solvable. Thus the eigenvalues defined by Equation (A53) and Equation (A54) with their corresponding normalised versions of the eigenvectors defined by Equations (A57)–(A60) provide the roots of the quintic when plugged into Equation (1) and Equation (2). □

4.7 *Proof of Lemma 2, condition seven*

Proof. The quintic is presented using the following matrix,

$$A_{s,t,r} = \begin{bmatrix} 0 & 1 & -s & t \\ 1 & 2s & b-t & -r \\ -s & b-t & c+2r & d \\ t & -r & d & e \end{bmatrix}.$$

Applying Gaussian elimination,

$$\begin{bmatrix} 1 & 0 & b+2s^2-t & -r-2st \\ 0 & 1 & -s & t \\ 0 & 0 & c+2s^3+2bs+2r-2st & d-rs+t^2-bt-2s^2t \\ 0 & 0 & d-rs+t^2-bt-2s^2t & e+2st^2+2rt \end{bmatrix}.$$

Thus, we have to solve the following system to achieve rank-2,

$$c+2s^3+2bs+2r-2st = 0, \tag{48}$$

$$d-rs+t^2-bt-2s^2t = 0, \tag{49}$$

$$e+2st^2+2rt = 0. \tag{50}$$

Assume the following conditions,

$$d = rs, \tag{51}$$

$$b = -s^2. \tag{52}$$

Using the above conditions, Equation (49) evaluates to the following,

$$t^2 - s^2t = 0 \Rightarrow t = s^2. \tag{53}$$

Plugging Equation (53) and Equation (52) into Equation (48) and Equation (50) as needed,

$$c+2s^3+2bs+2r-2s^3=0 \Rightarrow c-2s^3+2r=0, \tag{54}$$

$$e+2s^5+2rs^2=0 \Rightarrow e+2s^5-2br=0. \tag{55}$$

To simplify further, Equation (51) and Equation (52) are equivalent to the below respectively,

$$r=\frac{d}{\sqrt{-b}}, \tag{56}$$

$$s=\sqrt{-b}. \tag{57}$$

Applying this back to Equation (54) and Equation (55).

$$c-2(\sqrt{-b})^3+2(\frac{d}{\sqrt{-b}})=0, \tag{58}$$

$$e+2(\sqrt{-b})^5+2(\frac{d}{\sqrt{-b}})(\sqrt{-b})^2=0 \Rightarrow e+2(\sqrt{-b})^5+2d\sqrt{-b}=0. \tag{59}$$

Given that the coefficients of this polynomial satisfy Equation (58) and Equation (59), then substituting Equations (51)–(57) into the matrix $\mathbf{A}_{s,t,r}$, the quintic can be solved by plugging in the computed nonzero eigenvalues in Equation (A63) and Equation (A64) with the normalised versions of the corresponding eigenvectors defined by Equations (A67)–(A70) into Equation (1) and Equation (2). □

4.8 *Proof of Lemma 2, condition eight*

Proof. Now consider the following alternate conditions,

$$e=-rt, \tag{60}$$

$$b=2t. \tag{61}$$

Plugging the first condition into Equation (50),

$$2st^2+rt=0 \Rightarrow r=-2st. \tag{62}$$

Substituting the above into Equation (49) along with Equation (61),

$$d-t^2=0 \Rightarrow d=t^2. \tag{63}$$

Putting the above into Equation (60),

$$r = \frac{-e}{\sqrt{d}}. \tag{64}$$

Making use of both Equation (63) and Equation (64) into Equation (62),

$$s = \frac{r}{-2t} = \frac{\frac{-e}{\sqrt{d}}}{-2(\sqrt{d})} \Rightarrow s = \frac{e}{2d}. \tag{65}$$

Plugging in Equations (63)–(65) into Equation (48),

$$c + 2(\frac{e}{2d})^3 + 2b(\frac{e}{2d}) + 2(\frac{-e}{\sqrt{d}}) - 2(\frac{-e}{\sqrt{d}})(\sqrt{d}) = 0. \tag{66}$$

Given that the coefficients of the polynomial for the complex condition in Equation (66), the solution is defined by the Eigenvalues in Equation (A73) and Equation (A74) with their corresponding normalised versions of the Eigenvectors defined in Equations (A77)–(A80) being plugged into Equation (1) and Equation (2). □

4.9 *Proof of Lemma 3, condition one*

Proof. A matrix of the form below is used, where $r = 0$,

$$\mathbf{A}_{s,t} = \begin{bmatrix} 1 & 0 & -s & t \\ 0 & 2s & -t & 0 \\ -s & -t & 0 & \frac{d}{2} \\ t & 0 & \frac{d}{2} & e \end{bmatrix}.$$

Applying Gaussian elimination and assuming $s \neq 0$,

$$\begin{bmatrix} 1 & 0 & -s & t \\ 0 & 2s & -t & 0 \\ 0 & 0 & -2s^3 - t^2 & ds + 2s^2t \\ 0 & 0 & ds + 2s^2t & -2st^2 + 2se \end{bmatrix}.$$

This leads to the system being solved

$$-2s^3 - t^2 = 0, \tag{67}$$

$$ds + 2s^2t = 0, \tag{68}$$

$$-2st^2 + 2se = 0. \tag{69}$$

From Equation (69), it can be seen that,

$$t^2 = e \Rightarrow t = \pm\sqrt{e}. \tag{70}$$

Plugging Equation (70) into Equation (67),

$$-2s^3 - e = 0 \Rightarrow s = (\frac{-e}{2})^{\frac{1}{3}}. \tag{71}$$

Plugging Equation (70) and Equation (71) into Equation (68),

$$d + 2(\frac{-e}{2})^{\frac{1}{3}}\sqrt{e} = 0 \Rightarrow d = -2(\frac{-e}{2})^{\frac{1}{3}}\sqrt{e}. \tag{72}$$

Therefore when the sextic of form $x^6 + dx + e = 0$ satisfies the above Equation (72), it is solvable by plugging in the nonzero eigenvalues defined in Equation (A83) and Equation (A84) with the normalised versions of the corresponding eigenvectors in Equation (A88) and Equation (A90). Plugging them into Equation (1) and Equation (2) provides the closed-form solution. □

4.10 *Proof of Lemma 3, condition two*

Proof. A similar matrix is used as in the previous subsection, but $r \neq 0$.

$$\mathbf{A}_{s,\,t,\,r} = \begin{bmatrix} 1 & 0 & -s & t \\ 0 & 2s & -t & -r \\ -s & -t & 2r & \frac{d}{2} \\ t & -r & \frac{d}{2} & e \end{bmatrix}.$$

By Gaussian elimination with $s \neq 0$,

$$\begin{bmatrix} 2s & 0 & -2s^2 & 2st \\ 0 & 2s & -t & -r \\ 0 & 0 & -2s^3 + 4rs - t^2 & ds - rt + 2s^2t \\ 0 & 0 & ds - rt + 2s^2t & -r^2 - 2st^2 + 2se \end{bmatrix}.$$

Thus, the system below is considered,

$$-2s^3 + 4rs - t^2 = 0, \tag{73}$$

$$ds - rt + 2s^2t = 0, \tag{74}$$

$$-r^2 + 2es - 2st^2 = 0. \tag{75}$$

The difference compared to the previous subsection's general case is that it assumes a different condition to simplify solving the system.

$$d = -st. \tag{76}$$

Substituting into Equation (74),

$$(-st)s - rt + 2s^2t = 0 \Rightarrow r = s^2. \tag{77}$$

Applying Equation (77) into Equation (73),

$$-2s^3 + 4s(s^2) - t^2 = 0 \Rightarrow \sqrt{2s^3} = t. \tag{78}$$

Using both Equation (77) and Equation (78) into Equation (75),

$$-(s^2)^2 + 2es - 2s(2s^3) = 0 \Rightarrow e = \frac{5s^3}{2}. \tag{79}$$

Finally, to represent Equation (76) in terms of s,

$$d = -s(\sqrt{2s^3}). \tag{80}$$

Making use of both Equation (79) and Equation (80) implies that the coefficients of the sextic have to satisfy the condition,

$$e^5 = (\frac{5}{2})^5 \frac{d^6}{2^3} \Rightarrow e^5 = \frac{5^5 d^6}{2^8}. \tag{81}$$

Thus given that the above condition defined by Equation (81) is satisfied, plugging Equations (77)–(81) into $\mathbf{A}_{s,t,r}$ yields the eigenvalues in Equation (A93) and Equation (A94) with their respective normalised versions of the eigenvectors defined by Equations (A97)–(A100). And plugging these into Equation (1) and Equation (2) yields the roots to the sextic. □

4.11 *Proof of Lemma 3, condition three*

Proof. The sextic can be represented by the matrix,

$$\mathbf{A}_{s,\,t,\,r} = \begin{bmatrix} 1 & 0 & -s & t \\ 0 & a+2s & b-t & -r \\ -s & b-t & c+2r & d \\ t & -r & d & e \end{bmatrix}.$$

Whereby Gaussian elimination with the added requirement $a \neq -2s$,

$$\begin{bmatrix} a+2s & 0 & -2s^2-as & at+2st \\ 0 & a+2s & b-t & -r \\ 0 & 0 & -b^2+ac-2s^3-as^2+2cs+2ar+4rs-t^2+2bt & ad+2ds+br+2s^2t+ast-rt \\ 0 & 0 & ad+2ds+br+2s^2t+ast-rt & ae+2fs-r^2-at^2-2st^2 \end{bmatrix}.$$

Thus, achieving rank-2 requires solving the following system,

$$-b^2+ac-2s^3-as^2+2cs+2ar+4rs-t^2+2bt=0,$$

$$ad+2ds+br+2s^2t+ast-rt=0,$$

$$ae+2es-r^2-at^2-2st^2=0.$$

It is noted that this system can be expressed in terms of factors,

$$(a+2s)(e-t^2)-r^2=0, \tag{82}$$

$$(a+2s)(d+st)+r(b-t)=0, \tag{83}$$

$$(a+2s)(c-s^2+2r)-(b-t)^2=0. \tag{84}$$

Assume the conditions,

$$b=t, \tag{85}$$

$$r=0. \tag{86}$$

Resulting in the system becoming,

$$(a+2s)(e-t^2)=0, \tag{87}$$

$$(a+2s)(d+st)=0, \tag{88}$$

$$(a+2s)(c-s^2)=0. \tag{89}$$

Since $a \neq -2s$, Equations (87)–(89) become,

$$e=t^2 \Rightarrow t=\sqrt{e} \Rightarrow b=\sqrt{e}, \tag{90}$$

$$d=-st, \tag{91}$$

$$c=s^2 \Rightarrow s=\sqrt{c}. \tag{92}$$

This also implies $a \neq -2\sqrt{c}$. Plugging Equation (90) and Equation (92) into Equation (91),

$$d=-\sqrt{ce} \Rightarrow d^2=ce. \tag{93}$$

Thus when the sextic's coefficients satisfy Equation (93) and Equation (90) along with $a \neq -2\sqrt{c}$, the sextic can be solved using Equation (1) and Equation (2) by plugging in the eigenvalues in Equations (A103) and (A104) along with their normalised versions of the eigenvectors in Equations (A107) and (A108) which are found by plugging Equations (90)–(92) into $\mathbf{A}_{s,\,t,\,r}$. □

We now demonstrate using the techniques in the previous section to solve polynomials with integer coefficients.

5. Computational examples

The examples here were chosen arbitrarily and to the best of our knowledge there is no discussion of solving these specific polynomials.

Example 3 Find the roots of the following quintic polynomial, $x^5-3x^4-23x^3+51x^2+94x-120$.

Plugging in values for the coefficients, the following matrix is formed, where values of $s,\,t,\,r$ have to be found to force rank-2,

$$\mathbf{A}_{s,\,t,\,r}=\begin{bmatrix} 0 & \frac{1}{2} & s & t \\ \frac{1}{2} & -3-2\cdot s & \frac{-23}{2}-t & -r \\ s & \frac{-23}{2}-t & 51+2r & \frac{94}{2} \\ t & -r & \frac{94}{2} & -120 \end{bmatrix}.$$

To figure out how to set r, s, and t so that the rank of the matrix is two, Gaussian elimination will be performed on the symbolic matrix. Upon applying Gaussian elimination, one gets the row reduced echelon form,

$$\begin{bmatrix} \frac{1}{2} & -2s-3 & \frac{-2t-23}{2} & -r \\ 0 & \frac{1}{2} & s & t \\ 0 & 0 & 2r-8s^3-12s^2+46s+4st+51 & 2rs+2t^2-8s^2t-12st+23t+47 \\ 0 & 0 & 2rs+2t^2-8s^2t-12st+23t+47 & -8st^2-12t^2+4rt-120 \end{bmatrix}.$$

Assuming the following equation is true

$$2r-8s^3-12s^2+46s+4st+51=0, \tag{94}$$

$$\begin{bmatrix} \frac{1}{2} & -2s-3 & \frac{-2t-23}{2} & -r \\ 0 & \frac{1}{2} & s & t \\ 0 & 0 & 0 & 2rs+2t^2-8s^2t-12st+23t+47 \\ 0 & 0 & 2rs+2t^2-8s^2t-12st+23t+47 & -8st^2-12t^2+4rt-120 \end{bmatrix}.$$

Now, add the following equation as an extra constraint

$$2rs+2t^2-8s^2t-12st+23t+47=0, \tag{95}$$

$$\begin{bmatrix} \frac{1}{2} & -2s-3 & \frac{-2t-23}{2} & -r \\ 0 & \frac{1}{2} & s & t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -8st^2-12t^2+4rt-120 \end{bmatrix}.$$

Adding the final constraint, one achieves,

$$-8st^2-12t^2+4rt-120=0, \tag{96}$$

$$\begin{bmatrix} \frac{1}{2} & -2s-3 & \frac{-2t-23}{2} & -r \\ 0 & \frac{1}{2} & s & t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\operatorname{rank}\left(\begin{bmatrix} 0 & \frac{1}{2} & s & t \\ \frac{1}{2} & -2s-3 & \frac{-2t-23}{2} & -r \\ s & \frac{-2t-23}{2} & 2r+51 & 47 \\ t & -r & 47 & -120 \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} \frac{1}{2} & -2s-3 & \frac{-2t-23}{2} & -r \\ 0 & \frac{1}{2} & s & t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) = 2$$

The system of three equations being satisfied (Equations (1)–(3)) is noted as it will be the key condition for r, s, and t to set the matrix to rank-2. This system has three equations and three variables; thus, it is solvable.

$$2r - 8s^3 - 12s^2 + 46s + 4st + 51 = 0,$$

$$2rs + 2t^2 - 8s^2t - 12st + 23t + 47 = 0,$$

$$-8st^2 - 12t^2 + 4rt - 120 = 0.$$

Inputting the above system into an algebraic solver provides 10 sets of possible solutions for r, s, and t. Plugging these values into $\mathbf{A}_{s,\,t,\,r}$, the appropriate eigenvalues and eigenvectors of the nonzero eigenvalues are retrieved and plugged into the equation $y_1 = y_2\sqrt{\frac{-\lambda_2}{\lambda_1}}$. Where it is recalled that y_i is the product of the matrix consisting of the eigenvectors as its columns and $\mathbf{X}$ the vector containing powers of x, thus, for each r, s, t, a solution is derived for the quintic, meaning that there are 10 sets of solutions. Recall that it is also required to solve the case when signs are different, as shown in $y_1 = -y_2\sqrt{\frac{-\lambda_2}{\lambda_1}}$. However, this is not required due to having multiple solutions, which implies that solution values that appear more than once belong to the proper solution set, which is the roots of the polynomial. Thus, the roots are $\{-4, -2, 1, 3, 5\}$, the polynomial formed from these roots is equal to the polynomial that had to be solved in the example, thus confirming that the roots are correct.

Example 4 Find the roots of the following quintic polynomial, $x^5 - 57x^4 + 1{,}227x^3 - 12{,}547x^2 + 6{,}1236x - 114{,}660$.

Proceeding as before,

$$\mathbf{A} = \begin{bmatrix} 0 & \frac{1}{2} & s & t \\ \frac{1}{2} & -57-2\cdot s & \frac{1{,}227}{2}-t & -r \\ s & \frac{1{,}227}{2}-t & -12{,}547+2r & \frac{61{,}236}{2} \\ t & -r & \frac{61{,}236}{2} & -114{,}660 \end{bmatrix}.$$

With Gaussian elimination resulting in,

$$\begin{bmatrix} \frac{1}{2} & -2s-57 & \frac{-2t+1{,}227}{2} & -r \\ 0 & \frac{1}{2} & s & t \\ 0 & 0 & 2r-8s^3-228s^2-2{,}454s+4st-12{,}547 & 2rs+2t^2-8s^2t-228st-1{,}227t+30{,}618 \\ 0 & 0 & 2rs+2t^2-8s^2t-228st-1{,}227t+30{,}618 & -8st^2-228t^2+4rt-114{,}660 \end{bmatrix}.$$

Assuming the following equation is true

$$2r-8s^3-228s^2-2,454s+4st-12,547=0, \tag{97}$$

$$\begin{bmatrix} \frac{1}{2} & -2s-57 & \frac{-2t+1,227}{2} & -r \\ 0 & \frac{1}{2} & s & t \\ 0 & 0 & 0 & 2rs+2t^2-8s^2t-228st-1,227t+30,618 \\ 0 & 0 & 2rs+2t^2-8s^2t-228st-1,227t+30,618 & -8st^2-228t^2+4rt-114,660 \end{bmatrix}.$$

Now adding the below equation as an extra constraint

$$2rs+2t^2-8s^2t-228st-1,227t+30,618=0, \tag{98}$$

$$\begin{bmatrix} \frac{1}{2} & -2s-57 & \frac{-2t+1,227}{2} & -r \\ 0 & \frac{1}{2} & s & t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -8st^2-228t^2+4rt-114,660 \end{bmatrix}.$$

Adding the final constraint, one achieves,

$$-8st^2-228t^2+4rt-114,660=0, \tag{99}$$

$$\begin{bmatrix} \frac{1}{2} & -2s-57 & \frac{-2t+1,227}{2} & -r \\ 0 & \frac{1}{2} & s & t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$\operatorname{rank}\left(\begin{bmatrix} 0 & \frac{1}{2} & s & t \\ \frac{1}{2} & -2s-57 & \frac{-2t+1,227}{2} & -r \\ s & \frac{-2t+1,227}{2} & 2r-12,547 & 30,618 \\ t & -r & 30,618 & -114,660 \end{bmatrix}\right)=\operatorname{rank}\left(\begin{bmatrix} \frac{1}{2} & -2s-57 & \frac{-2t+1,227}{2} & -r \\ 0 & \frac{1}{2} & s & t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right)=2.$$

Thus, the system to solve from Equations (4)–(6) is

$$2r-8s^3-228s^2-2,454s+4st-12,547=0,$$

$$2rs+2t^2-8s^2t-228st-1,227t+30,618=0,$$

$$-8st^2-228t^2+4rt-114,660=0.$$

The solutions for r, s, t are again multiple sets, and this leads to multiple solutions as in the previous example. It is noted that some solutions don't correctly lead to rank-2, hence there may be fewer possible solutions to the polynomial compared to the previous case. Thus, it can be calculated that the roots of the quintic in this example are $\{6, 7, 10, 13, 21\}$. Again, the polynomial formed from these roots is equal to the polynomial that had to be solved in the example, thus confirming that the roots are correct. For completeness, the solutions for when $y_1=-y_2\sqrt{\frac{-\lambda_2}{\lambda_1}}$ can be used for each corresponding solution set, leading to calculating a single set of roots in one iteration.

It can be seen from the previous examples that there is a relationship between the system of equations to solve and the corresponding polynomial. For a quintic polynomial $P_5(x)$, the following matrix can be forced to rank-2:

$$\mathbf{A}=\begin{bmatrix} 0 & \frac{1}{2} & r & s \\ \frac{1}{2} & a-2r & \frac{b}{2}-s & -t \\ r & \frac{b}{2}-s & c+2t & \frac{d}{2} \\ s & -t & \frac{d}{2} & e \end{bmatrix}.$$

Given that the system of equations below is satisfied, they are expressed in terms of r, s, t, where all the coefficients of the quintic are nonzero.

$$2r-8s^3+4as^2-2bs+4st+c=0,$$

$$2rs+2t^2-8s^2t+4ast-bt+\frac{d}{2}=0,$$

$$-8st^2+4at^2+4rt+e=0.$$

Example 5 Decompose the following sextic polynomial so that it's easier to solve, $x^6-11x^4+2x^3+x^2-2x+1$. It is observed that the sextic polynomial satisfies Lemma 3, Condition Three, which satisfies the conditions:

$$d^2=ce,$$

$$b=\sqrt{e},$$

$$a\neq-2\sqrt{c}.$$

Since plugging in the coefficients of the sextic,

$$d^2 = ce \Rightarrow (1)^2 = 1(1),$$

$$b = \sqrt{e} \Rightarrow 1 = \sqrt{1},$$

$$a \neq -2\sqrt{c} \Rightarrow -11 \neq -2\sqrt{1}.$$

Thus the eigenvalues in Equations (A103) and (A104) along with their normalised versions of the eigenvectors in Equations (A107) and (A108) can be used. This results in the eigenvalues,

$$\lambda_3 = a + 2\sqrt{c} = -11 + 2 = -9,$$

$$\lambda_4 = 1 + c + b^2 = 1 + 1 + 1 = 3.$$

With the respective eigenvectors,

$$\mathbf{V}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$\mathbf{V}_4 = \begin{bmatrix} \frac{1}{b} \\ 0 \\ -\frac{\sqrt{c}}{b} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

Where normalised,

$$\mathbf{V}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$\mathbf{V}_4 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}.$$

Then Equation (1) and Equation (2) are used to decompose the sextic as follows:

$$0x^3 + (1)x^2 + 0x + 0 = \pm\sqrt{\frac{1}{3}}(\frac{1}{\sqrt{3}}x^3 + 0x^2 - \frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}).$$

Thus, a pair of cubics is acquired,

$$3x^2 = (x^3 - x + 1) \Rightarrow x^3 - 3x^2 - x + 1 = 0,$$

$$3^2 = -(x^3 - x + 1) \Rightarrow x^3 + 3x^2 - x + 1 = 0.$$

Solving each cubic gives the roots of the sextic to confirm equivalence. Since they have the same roots as the sextic, their product should be equivalent to the original sextic.

$$(x^3 - 3x^2 - x + 1)(x^3 + 3x^2 - x + 1) = x^6 - 11x^4 + 2x^3 + x^2 - 2x + 1.$$

Example 6 Decompose the following sextic polynomial so that it is easier to solve:

$$p(x) = x^6 - 2x^4 + 3x^3 + \frac{4}{9}x^2 - 2x + \frac{9}{4}.$$

It is observed that the sextic polynomial satisfies Lemma 3, Condition Three with rational (fractional) coefficients:

$$d^2 = ce,$$

$$b = \sqrt{e},$$

$$a \neq 0.$$

Plugging in the coefficients:

$$b = \frac{3}{2} = \sqrt{\frac{9}{4}} = \sqrt{e},$$

$$d^2 = (-1)^2 = 1 = \frac{4}{9} \cdot \frac{9}{4} = ce,$$

$$a = -2 \neq 0.$$

The eigenvalues in Equations (A103) and (A104) and their normalised eigenvectors in Equations (A107) and (A108) can be used, with $b = b'/2$ and $d = d'/2$:

$$\lambda_3 = a + 2\sqrt{c} = -2 + 2\sqrt{\frac{4}{9}} = -2 + \frac{4}{3} = -\frac{2}{3},$$

$$\lambda_4 = 1 + c + b^2 = 1 + \frac{4}{9} + \left(\frac{3}{2}\right)^2 = 1 + \frac{4}{9} + \frac{9}{4} = \frac{133}{36}.$$

The corresponding eigenvectors are:

$$\mathbf{V}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$\mathbf{V}_4 = \begin{bmatrix} \frac{2}{3} \\ 0 \\ -\frac{2/3}{3/2} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ 0 \\ -\frac{4}{9} \\ 1 \end{bmatrix}.$$

Normalised:

$$\mathbf{V}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$\mathbf{V}_4 = \frac{1}{\sqrt{(2/3)^2 + (4/9)^2 + 1^2}} \begin{bmatrix} \frac{2}{3} \\ 0 \\ -\frac{4}{9} \\ 1 \end{bmatrix}.$$

Then Equation (1) and Equation (2) are used to decompose the sextic as follows

$$0x^3 + (1)x^2 + 0x + 0 = \pm\sqrt{\frac{27}{8}} \left(\frac{2}{3}x^3 + 0x^2 - \frac{4}{9}x + 1\right).$$

Thus, a pair of cubics is acquired,

$$\sqrt{\frac{8}{27}}x^2 = \left(\frac{2}{3}x^3 - \frac{4}{9}x + 1\right) \Rightarrow \frac{2}{3}x^3 - \sqrt{\frac{8}{27}}x^2 - \frac{4}{9}x + 1 = 0,$$

$$\sqrt{\frac{8}{27}}x^2 = -\left(\frac{2}{3}x^3 - \frac{4}{9}x + 1\right) \Rightarrow \frac{2}{3}x^3 + \sqrt{\frac{8}{27}}x^2 - \frac{4}{9}x + 1 = 0.$$

Observe that to ensure the term with the highest degree has a unit coefficient, we multiply the product below by $\frac{9}{4}$. Solving each cubic gives the roots of the sextic to confirm equivalence. Since they have the same roots as the sextic, their product should be equivalent to the original sextic.

$$\frac{9}{4}\left(\frac{2}{3}x^3 - \sqrt{\frac{8}{27}}x^2 - \frac{4}{9}x + 1\right)\left(\frac{2}{3}x^3 + \sqrt{\frac{8}{27}}x^2 - \frac{4}{9}x + 1\right) = x^6 - 2x^4 + 3x^3 + \frac{4}{9}x^2 - 2x + \frac{9}{4} = 0.$$

Example 7 Decompose the following sextic polynomial so that it is easier to solve:

$$p(x) = x^6 - x^4 + 2\pi x^3 + \frac{1}{\pi^2}x^2 - 2x + \pi^2.$$

It is observed that the sextic polynomial satisfies Lemma 3, Condition Three with mixed rational and irrational coefficients:

$$d^2 = ce,$$

$$b = \sqrt{e},$$

$$a \neq 0.$$

Plugging in the coefficients:

$$b = \pi = \sqrt{\pi^2} = \sqrt{e},$$

$$d^2 = (-1)^2 = 1 = \frac{1}{\pi^2} \cdot \pi^2 = ce,$$

$$a = -1 \neq 0.$$

The eigenvalues in Equations (A103) and (A104) and their normalised eigenvectors in Equations (A107) and (A108) can be used, with $b = b'/2$ and $d = d'/2$:

$$\lambda_3 = a + 2\sqrt{c} = -1 + 2\sqrt{\frac{1}{\pi^2}} = -1 + \frac{2}{\pi},$$

$$\lambda_4 = 1 + c + b^2 = 1 + \frac{1}{\pi^2} + \pi^2.$$

The corresponding eigenvectors are:

$$\mathbf{V}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$\mathbf{V}_4 = \begin{bmatrix} \frac{1}{b} \\ 0 \\ -\frac{\sqrt{c}}{b} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\pi} \\ 0 \\ -\frac{1}{\pi^2} \\ 1 \end{bmatrix}.$$

Normalised:

$$\mathbf{V}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$\mathbf{V}_4 = \frac{1}{\sqrt{\frac{1}{\pi^2}+\frac{1}{\pi^4}+1}} \begin{bmatrix} \frac{1}{\pi} \\ 0 \\ -\frac{1}{\pi^2} \\ 1 \end{bmatrix}.$$

Then Equations (1) and (2) are used to decompose the sextic as follows:

$$0x^3+(1)x^2+0x+0 = \pm\sqrt{\frac{1}{\left(-\frac{2}{\pi^3}+\frac{1}{\pi^2}\right)}}\left(\frac{1}{\pi}x^3+0x^2-\frac{1}{\pi^2}x+1\right).$$

Thus, a pair of cubics is obtained:

$$\frac{x^3}{\pi} = -\left(\sqrt{\left(-\frac{2}{\pi^3}+\frac{1}{\pi^2}\right)}x^2-\frac{x}{\pi^2}+1\right) \Rightarrow \frac{x^3}{\pi}+\sqrt{\left(-\frac{2}{\pi^3}+\frac{1}{\pi^2}\right)}x^2-\frac{x}{\pi^2}+1=0,$$

$$\frac{x^3}{\pi} = \left(\sqrt{\left(-\frac{2}{\pi^3}+\frac{1}{\pi^2}\right)}x^2-\frac{x}{\pi^2}+1\right) \Rightarrow \frac{x^3}{\pi}-\sqrt{\left(-\frac{2}{\pi^3}+\frac{1}{\pi^2}\right)}x^2-\frac{x}{\pi^2}+1=0.$$

Observe that to ensure the term with the highest degree has a unit coefficient, we multiply the product below by π^2. Multiplying the two cubics reproduces the original sextic:

$$\left(\frac{x^3}{\pi}+\sqrt{\left(-\frac{2}{\pi^3}+\frac{1}{\pi^2}\right)}x^2-\frac{x}{\pi^2}+1\right)\left(\frac{x^3}{\pi}-\sqrt{\left(-\frac{2}{\pi^3}+\frac{1}{\pi^2}\right)}x^2-\frac{x}{\pi^2}+1\right) = x^6-x^4+2\pi x^3+\frac{1}{\pi^2}x^2-2x+\pi^2.$$

6. Conjecture on higher degrees

From the discussions in the previous sections, the following can be conjectured, noting that solving a polynomial $P_n(x)$ of degree n is equivalent in difficulty to solving the closest even degree above it, since we are solving reduced polynomials of order $\lceil \frac{n}{2} \rceil$.

Conjecture 1 For any polynomial $P_n(x)$ of degree $n \geq 9$, by choosing the right two matrices $\mathbf{A}_n$ and $\mathbf{A}_n^0$, it can be expressed as a linear combination of the squares of two polynomials of degree $\lceil \frac{n}{2} \rceil$, such that

$$P_n(x) = \lambda_1\left(\sum_{i=0}^{\lceil \frac{n}{2} \rceil} a_i x^i\right)^2 + \lambda_2\left(\sum_{i=0}^{\lceil \frac{n}{2} \rceil} b_i x^i\right)^2.$$

Where λ_1 and λ_2 are constants, and a_i and b_i are coefficients for the polynomials of degree $\lceil \frac{n}{2} \rceil$. Additionally, if $P_n(x)$ is monic (i.e., the leading coefficient of $P_n(x)$ is 1) and n is odd, then it must satisfy the condition:

$$\lambda_1 a^2_{\frac{n+1}{2}} + \lambda_2 b^2_{\frac{n+1}{2}} = 0.$$

This conjecture draws inspiration from observations of the solvability of lower degree polynomials and naturally extends the framework to higher degrees. By carefully analysing the rank of the matrix representations $\mathbf{A}_n$ and $\mathbf{A}_n^0$, solvable cases can be identified within the generally unsolvable domain of polynomials of degree $n \geq 9$. It is noted that the additional condition for the odd case, as mentioned in the conjecture, arises from the fact that the odd-degree polynomial without setting the extra condition would have an extra degree term. For example, considering a quintic, without satisfying the condition, a term with factor x^6 would be included, which is not desired, hence the importance of the odd condition. The implication of this conjecture is that there are possible applications to symbolic computation, algebraic geometry, and developing efficient algorithms to solve high-degree polynomials given that they can be reduced into lower degree polynomials. The conjectural extension to higher-degree polynomials is exploratory and does not claim general solvability. In particular, decompositions of degree nine or ten polynomials into lower-degree components do not circumvent Galois-theoretic obstructions, since solvability ultimately depends on the solvability of the resulting factors. The conjecture is consistent with known impossibility results and merely suggests that additional structured families may exist.

7. Discussion

The lemmas presented in this paper significantly advance the understanding of algebraic solutions of higher-degree polynomials. By identifying specific cases of quintic and sextic polynomials with solvable structures, it is demonstrated that even within the generally unsolvable realm of higher-degree polynomials, there exist algebraically solvable subfamilies. These cases offer concrete examples of how certain structural properties influence solvability. Furthermore, the derived conjecture for polynomials of degree 9 and higher extends these insights by suggesting a broader framework for solvability based on matrix decompositions. This framework is not only a bridge between theoretical algebra and computational approaches but also introduces a new definition of solvability within polynomial theory. This work builds on foundational concepts in algebra by exploring new methods for identifying solvable cases of quintic and sextic polynomials. While Galois theory demonstrates that general polynomials of degree-5 or higher lack algebraic solutions due to their non-solvable Galois groups, we introduce a novel framework based on matrix representations and rank constraints to systematically identify solvable subfamilies. Unlike traditional approaches such as Bring-Jerrard forms or Klein's work on the Icosahedral group, the framework demonstrated in the paper leverages the eigenstructure of symmetric matrices associated with the polynomial. By reducing the matrix rank and expressing polynomials as a sum of squares, a direct solution is provided for specific cases. This approach not only clarifies the structural properties that enable solvability but also extends these principles to higher-degree polynomials through the conjecture, offering a new avenue for research in algebraic solvability and symbolic computation. The presented examples focus on exact symbolic solutions, in contrast with numerical methods that provide approximations. While the framework offers exactness, it applies only to polynomials whose coefficients satisfy nontrivial algebraic constraints. As the degree increases, the resulting constraint systems grow rapidly and become difficult to solve, limiting practical applicability. Extending the examples to non-integer coefficients further illustrates the universality of the method within its restricted scope. Despite its contributions, this study has several limitations. First, the identified solvable cases rely on specific assumptions about the coefficients of the polynomials, which may limit their applicability to broader classes of polynomials. Additionally, the approach requires the polynomial's matrix representation to achieve a rank-2 form, which can lead to a complex system of nonlinear equations that may not be easily solvable. Additionally, computing eigenstructures for higher-degree polynomials would be demanding on resources. The proposed conjecture, while promising, remains unproven in its entirety and is based on observed patterns rather than a formal, generalised proof. Finally, what has been explored in this paper has been constrained to degrees 5, 6, and 9 or higher, leaving an opportunity for further investigation into degrees 7 and 8.

Future research could focus on several promising directions. One possibility is the rigorous proof of the conjecture for polynomials of degree 9 and higher. This could involve exploring additional matrix decomposition techniques or developing new methods for identifying solvable subfamilies. Another natural direction is to make use of the framework to analyse solvability for degrees 7 and 8, which may reveal additional insights into the boundaries of algebraic solvability. Finally, exploring the interplay between matrix-based approaches and advanced computational algebra systems could lead to more efficient algorithms for identifying and solving higher-degree polynomials.

8. Conclusion

In this study, we have explored conditions under which specific classes of quintic and sextic polynomials admit algebraic solutions, despite the general impossibility as shown by Galois theory. Using matrix representations and polynomial-solving techniques, we identified particular forms of quintics and sextics with solvable structures and provided explicit solutions for these cases. This analysis highlights the unique structural properties within these polynomial families that enable algebraic solvability.

Furthermore, we propose a conjecture for polynomials of degree 9 and higher, suggesting that carefully structured subfamilies may also have algebraic solutions. This conjecture, grounded in observed patterns of symmetry and decomposability, extends current solvability theories and opens pathways for further investigation into polynomial structures at higher degrees.

Our findings contribute to the broader understanding of polynomial theory by challenging the boundaries of algebraic solvability and offering a systematic approach to identify solvable forms as well as provide their solutions. This work not only enhances theoretical insights into polynomial solvability but also provides an essential framework for future research into high-degree polynomial solutions, with implications for algebraic theory. Future research could aim to test and refine our conjecture, exploring applications in higher-dimensional representations and alternative polynomial forms. Due to the generality of this framework, future research would also involve extracting more particular cases, as the paper has only gone through a portion of them due to assumptions made when solving the polynomial system.

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Conflict of interest

The authors declare no competing financial interest.

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Appendix A
A.1 *Closed form equations for Lemma 2 condition one*

The Eigenvalues:

$$\lambda_1 = 0, \tag{A1}$$

$$\lambda_2 = 0, \tag{A2}$$

$$\lambda_3 = \frac{c^2 e - \sqrt{c^6 + c^4 e^2 + c^4 + 2c^2 e^2 + e^4}}{2c^2}, \tag{A3}$$

$$\lambda_4 = \frac{c^2 e + \sqrt{c^6 + c^4 e^2 + c^4 + 2c^2 e^2 + e^4}}{2c^2}. \tag{A4}$$

Corresponding Eigenvectors:

$$\mathbf{V}_1 = \begin{bmatrix} -c \\ -\frac{e}{c} \\ 0 \\ 1 \end{bmatrix}, \tag{A5}$$

$$\mathbf{V}_2 = \begin{bmatrix} \frac{e}{c} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \tag{A6}$$

$$\mathbf{V}31_{numerator} = c\left(c^4(-e) - c^2 e^3 + c^2\sqrt{c^6 + c^4 e^2 + c^4 + 2c^2 e^2 + e^4} + e^2\sqrt{c^6 + c^4 e^2 + c^4 + 2c^2 e^2 + e^4}\right),$$

$$\mathbf{V}31_{denominator} = -2c^6 e - 2c^4 e^3 - c^4 e - 2c^2 e^3 + c^4\sqrt{c^6 + c^4 e^2 + c^4 + 2c^2 e^2 + e^4} +$$

$$2c^2 e^2\sqrt{c^6 + c^4 e^2 + c^4 + 2c^2 e^2 + e^4} - e^5,$$

$$\mathbf{V}32_{numerator} = c\left(c^6 + c^4 e^2 + c^4 + 2c^2 e^2 - c^2 e\sqrt{c^6 + c^4 e^2 + c^4 + 2c^2 e^2 + e^4} + e^4\right),$$

$$\mathbf{V}32_{denominator} = 2c^6 e + 2c^4 e^3 + c^4 e + 2c^2 e^3 - c^4\sqrt{c^6 + c^4 e^2 + c^4 + 2c^2 e^2 + e^4} -$$

$$2c^2 e^2\sqrt{c^6 + c^4 e^2 + c^4 + 2c^2 e^2 + e^4} + e^5,$$

$$\mathbf{V}_{33_{numerator}} = c^4\left(-e^2\right) - c^2e^4 + c^2e\sqrt{c^6+c^4e^2+c^4+2c^2e^2+e^4} +$$

$$e^3\sqrt{c^6+c^4e^2+c^4+2c^2e^2+e^4},$$

$$\mathbf{V}_{33_{denominator}} = -2c^6e - 2c^4e^3 - c^4e - 2c^2e^3 + c^4\sqrt{c^6+c^4e^2+c^4+2c^2e^2+e^4} +$$

$$2c^2e^2\sqrt{c^6+c^4e^2+c^4+2c^2e^2+e^4} - e^5,$$

$$\mathbf{V}_{34} = 1. \tag{A7}$$

Where

$$\mathbf{V}_3 = \begin{bmatrix} \frac{\mathbf{V}_{31_{numerator}}}{\mathbf{V}_{31_{denominator}}} \\ \frac{\mathbf{V}_{32_{numerator}}}{\mathbf{V}_{32_{denominator}}} \\ \frac{\mathbf{V}_{33_{numerator}}}{\mathbf{V}_{33_{denominator}}} \\ \mathbf{V}_{34} \end{bmatrix}. \tag{A8}$$

$$\mathbf{V}_{41_{numerator}} = c\left(c^4e + c^2e^3 + c^2\sqrt{c^6+c^4e^2+c^4+2c^2e^2+e^4} + e^2\sqrt{c^6+c^4e^2+c^4+2c^2e^2+e^4}\right),$$

$$\mathbf{V}_{41_{denominator}} = 2c^6e + 2c^4e^3 + c^4e + 2c^2e^3 + c^4\sqrt{c^6+c^4e^2+c^4+2c^2e^2+e^4} +$$

$$2c^2e^2\sqrt{c^6+c^4e^2+c^4+2c^2e^2+e^4} + e^5,$$

$$\mathbf{V}_{42_{numerator}} = c\left(c^6 + c^4e^2 + c^4 + 2c^2e^2 + c^2e\sqrt{c^6+c^4e^2+c^4+2c^2e^2+e^4} + e^4\right),$$

$$\mathbf{V}_{42_{denominator}} = 2c^6e + 2c^4e^3 + c^4e + 2c^2e^3 + c^4\sqrt{c^6+c^4e^2+c^4+2c^2e^2+e^4} +$$

$$2c^2e^2\sqrt{c^6+c^4e^2+c^4+2c^2e^2+e^4} + e^5,$$

$$\mathbf{V}_{43_{numerator}} = c^4e^2 + c^2e^4 + c^2e\sqrt{c^6+c^4e^2+c^4+2c^2e^2+e^4} +$$

$$e^3\sqrt{c^6+c^4e^2+c^4+2c^2e^2+e^4},$$

$$\mathbf{V}_{43_{denominator}} = 2c^6e + 2c^4e^3 + c^4e + 2c^2e^3 + c^4\sqrt{c^6 + c^4e^2 + c^4 + 2c^2e^2 + e^4} +$$

$$2c^2e^2\sqrt{c^6 + c^4e^2 + c^4 + 2c^2e^2 + e^4} + e^5,$$

$$\mathbf{V}_{44} = 1. \tag{A9}$$

Where

$$\mathbf{V}_4 = \begin{bmatrix} \frac{\mathbf{V}_{41_{numerator}}}{\mathbf{V}_{41_{denominator}}} \\ \frac{\mathbf{V}_{42_{numerator}}}{\mathbf{V}_{42_{denominator}}} \\ \frac{\mathbf{V}_{43_{numerator}}}{\mathbf{V}_{43_{denominator}}} \\ \mathbf{V}_{44} \end{bmatrix}. \tag{A10}$$

A.2 *Closed form equations for Lemma 2 condition two*

The Eigenvalues:

$$\lambda_1 = 0, \tag{A11}$$

$$\lambda_2 = 0, \tag{A12}$$

$$\lambda_3 = \frac{1}{2}\left(-\sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1} + e - \sqrt[5]{e}\right), \tag{A13}$$

$$\lambda_4 = \frac{1}{2}\left(\sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1} + e - \sqrt[5]{e}\right). \tag{A14}$$

Corresponding Eigenvectors:

$$\mathbf{V}_1 = \begin{bmatrix} -2e^{3/5} \\ -e^{2/5} \\ 0 \\ 1 \end{bmatrix}, \tag{A15}$$

$$\mathbf{V}_2 = \begin{bmatrix} -e^{2/5} \\ -\sqrt[5]{e} \\ 1 \\ 0 \end{bmatrix}, \tag{A16}$$

$$\mathbf{V}_{31_{numerator}} = 2e^{6/5} + 3e^{4/5} + 3e^{2/5} + \sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1}\, e^{3/5} + 2\sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1}\, \sqrt[5]{e} + 1,$$

$$\mathbf{V}_{31_{denominator}} = \left(e^{4/5} - 1\right) e^{2/5} \left(\sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1} - e + \sqrt[5]{e}\right),$$

$$\mathbf{V}_{32_{numerator}} = e^{9/5} + 4e^{7/5} + 6e^{3/5} + \sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1}\, e^{4/5} + 4\sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1}\, e^{2/5} + \sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1} + 4e + 3\sqrt[5]{e},$$

$$\mathbf{V}_{32_{denominator}} = \left(e^{4/5} - 1\right) e^{2/5} \left(\sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1} - e + \sqrt[5]{e}\right),$$

$$\mathbf{V}_{33_{numerator}} = e^{9/5} + 2e^{7/5} + 3e^{3/5} + 2\sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1}\, e^{2/5} + \sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1} + e + 2\sqrt[5]{e},$$

$$\mathbf{V}_{33_{denominator}} = \left(e^{4/5} - 1\right) \sqrt[5]{e} \left(\sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1} - e + \sqrt[5]{e}\right),$$

$$\mathbf{V}_{34} = 1. \tag{A17}$$

Where

$$\mathbf{V}_3 = \begin{bmatrix} \frac{\mathbf{V}_{31_{numerator}}}{\mathbf{V}_{31_{denominator}}} \\ \frac{\mathbf{V}_{32_{numerator}}}{\mathbf{V}_{32_{denominator}}} \\ \frac{\mathbf{V}_{33_{numerator}}}{\mathbf{V}_{33_{denominator}}} \\ \mathbf{V}_{34} \end{bmatrix}. \tag{A18}$$

$$\mathbf{V}_{41_{numerator}} = -2e^{6/5} - 3e^{4/5} - 3e^{2/5} + \sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1}e^{3/5} +$$

$$2\sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1}\sqrt[5]{e} - 1,$$

$$\mathbf{V}_{41_{denominator}} = \left(e^{4/5} - 1\right) e^{2/5} \left(\sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1} + e - \sqrt[5]{e}\right),$$

$$\mathbf{V}_{42_{numerator}} = e^{9/5} + 4e^{7/5} + 6e^{3/5} - \sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1}e^{4/5} -$$

$$4\sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1}e^{2/5} -$$

$$\sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1} + 4e + 3\sqrt[5]{e},$$

$$\mathbf{V}_{42_{denominator}} = \left(e^{4/5} - 1\right) e^{2/5} \left(\sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1} + e - \sqrt[5]{e}\right),$$

$$\mathbf{V}_{43_{numerator}} = e^{9/5} + 2e^{7/5} + 3e^{3/5} - 2\sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1}e^{2/5} -$$

$$\sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1} + e + 2\sqrt[5]{e},$$

$$\mathbf{V}_{43_{denominator}} = \left(e^{4/5} - 1\right) \sqrt[5]{e} \left(\sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1} + e - \sqrt[5]{e}\right),$$

$$\mathbf{V}_{44} = 1. \tag{A19}$$

Where

$$\mathbf{V}_4 = \begin{bmatrix} \frac{\mathbf{V}_{41_{numerator}}}{\mathbf{V}_{41_{denominator}}} \\ \frac{\mathbf{V}_{42_{numerator}}}{\mathbf{V}_{42_{denominator}}} \\ \frac{\mathbf{V}_{43_{numerator}}}{\mathbf{V}_{43_{denominator}}} \\ \mathbf{V}_{44} \end{bmatrix}. \tag{A20}$$

A.3 *Closed form equations for Lemma 2 condition three*

The Eigenvalues:

$$\lambda_1 = 0, \tag{A21}$$

$$\lambda_2 = 0, \tag{A22}$$

$$\lambda_3 = \frac{1}{2}\left(-\sqrt{2,304\ 2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\ 2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1} + 48\sqrt[3]{2}e\sqrt[3]{e}^2 - 2^{2/3}\sqrt[3]{e} - 4e\right), \tag{A23}$$

$$\lambda_4 = \frac{1}{2}\left(\sqrt{2,304\ 2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\ 2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1} + 48\sqrt[3]{2}e\sqrt[3]{e}^2 - 2^{2/3}\sqrt[3]{e} - 4e\right). \tag{A24}$$

Corresponding Eigenvectors:

$$\mathbf{V}_1 = \begin{bmatrix} -20e \\ -4\sqrt[3]{2}\sqrt[3]{e}^2 \\ 0 \\ 1 \end{bmatrix}, \tag{A25}$$

$$\mathbf{V}_2 = \begin{bmatrix} 0 \\ -2^{2/3}\sqrt[3]{e} \\ 1 \\ 0 \end{bmatrix}, \tag{A26}$$

$$\mathbf{V}31_{numerator} = -7,680\sqrt[3]{e}^2e^3 - 160\ 2^{2/3}e^3 - 40\sqrt[3]{2}\sqrt[3]{e}e^2 +$$

$$80\ 2^{2/3}\sqrt{2,304\ 2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\ 2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1}e^2 +$$

$$10\sqrt[3]{2}\sqrt[3]{e}\sqrt{2,304\ 2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\ 2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1}e +$$

$$5\sqrt[3]{e}^2\sqrt{2,304\ 2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\ 2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1} + 40\sqrt[3]{e}^2e + 5\ 2^{2/3}e,$$

$$\mathbf{V}31_{denominator} = 20\sqrt[3]{e}^2(-9,216e^4 - 512\ 2^{2/3}\sqrt[3]{e}e^3 - 248\sqrt[3]{2}\sqrt[3]{e}^2e^2 - 24e^2 +$$

$$96\ 2^{2/3}\sqrt[3]{e}\sqrt{2,304\ 2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\ 2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1}e^2 +$$

$$2\sqrt[3]{2}\sqrt[3]{e}^2\sqrt{2,304\ 2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\ 2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1}e +$$

$$\sqrt{2,304\ 2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\ 2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1}e - 2^{2/3}\sqrt[3]{e}e - \sqrt[3]{2}\sqrt[3]{e}^2),$$

$$\mathbf{V}_{32_{numerator}} = -3,840\,2^{2/3}\sqrt[3]{e}^2e^4 - 1,760\sqrt[3]{2}e^4 - 1,040\sqrt[3]{e}e^3 - 160\,2^{2/3}\sqrt[3]{e}^2e^2 - 30\sqrt[3]{2}e^2 +$$

$$80\sqrt[3]{2}\sqrt{2,304\,2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\,2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1}e^3 -$$

$$40\sqrt[3]{e}\sqrt{2,304\,2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\,2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1}e^2 -$$

$$10\,2^{2/3}\sqrt[3]{e}^2\sqrt{2,304\,2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\,2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1}e - 5\sqrt[3]{e}e,$$

$$\mathbf{V}_{32_{denominator}} = 20e\sqrt[3]{e}(9,216e^4 + 512\,2^{2/3}\sqrt[3]{e}e^3 + 248\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 24e^2 -$$

$$96\,2^{2/3}\sqrt[3]{e}\sqrt{2,304\,2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\,2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1}e^2 -$$

$$2\sqrt[3]{2}\sqrt[3]{e}^2\sqrt{2,304\,2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\,2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1}e -$$

$$\sqrt{2,304\,2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\,2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1}e + 2^{2/3}\sqrt[3]{e}e + \sqrt[3]{2}\sqrt[3]{e}^2),$$

$$\mathbf{V}_{33_{numerator}} = -1,536\sqrt[3]{2}\sqrt[3]{e}^2e^3 - 704e^3 - 208\,2^{2/3}\sqrt[3]{e}e^2 +$$

$$32\sqrt{2,304\,2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\,2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1}e^2 -$$

$$8\,2^{2/3}\sqrt[3]{e}\sqrt{2,304\,2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\,2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1}e -$$

$$4\sqrt[3]{2}\sqrt[3]{e}^2\sqrt{2,304\,2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\,2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1} -$$

$$64\sqrt[3]{2}\sqrt[3]{e}^2e - 12e - 2^{2/3}\sqrt[3]{e},$$

$$\mathbf{V}_{33_{denominator}} = 4(-9,216e^4 - 512\,2^{2/3}\sqrt[3]{e}e^3 - 248\sqrt[3]{2}\sqrt[3]{e}^2e^2 - 24e^2 +$$

$$96\,2^{2/3}\sqrt[3]{e}\sqrt{2,304\,2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\,2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1}e^2 +$$

$$2\sqrt[3]{2}\sqrt[3]{e}^2\sqrt{2,304\,2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\,2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1}e +$$

$$\sqrt{2,304\,2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\,2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1}e - 2^{2/3}\sqrt[3]{e}e - \sqrt[3]{2}\sqrt[3]{e}^2),$$

$$\mathbf{V}_{34} = 1. \tag{A27}$$

Where

$$\mathbf{V}_3 = \begin{bmatrix} \frac{\mathbf{V}_{31_{numerator}}}{\mathbf{V}_{31_{denominator}}} \\ \frac{-\mathbf{V}_{32_{numerator}}}{\mathbf{V}_{32_{denominator}}} \\ \frac{\mathbf{V}_{33_{numerator}}}{\mathbf{V}_{33_{denominator}}} \\ \mathbf{V}_{34} \end{bmatrix}. \tag{A28}$$

$$\begin{aligned}
\mathbf{V}_{41_{numerator}} = & -7,680\sqrt[3]{e}^2 e^3 - 160\ 2^{2/3}e^3 - 40\sqrt[3]{2}\sqrt[3]{e}e^2 - \\
& 80\ 2^{2/3}\sqrt{2,304\ 2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\ 2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1}e^2 - \\
& 10\sqrt[3]{2}\sqrt[3]{e}\sqrt{2,304\ 2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\ 2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1}e - \\
& 5\sqrt[3]{e}^2\sqrt{2,304\ 2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\ 2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1} + 40\sqrt[3]{e}^2e + 5\ 2^{2/3}e,
\end{aligned}$$

$$\begin{aligned}
\mathbf{V}_{41_{denominator}} = & 20\sqrt[3]{e}^2(9,216e^4 + 512\ 2^{2/3}\sqrt[3]{e}e^3 + 248\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 24e^2 + \\
& 96\ 2^{2/3}\sqrt[3]{e}\sqrt{2,304\ 2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\ 2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1}e^2 + \\
& 2\sqrt[3]{2}\sqrt[3]{e}^2\sqrt{2,304\ 2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\ 2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1}e + \\
& \sqrt{2,304\ 2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\ 2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1}e + 2^{2/3}\sqrt[3]{e}e + \sqrt[3]{2}\sqrt[3]{e}^2),
\end{aligned}$$

$$\begin{aligned}
\mathbf{V}_{42_{numerator}} = & -3,840\ 2^{2/3}\sqrt[3]{e}^2e^4 - 1,760\sqrt[3]{2}e^4 - 1,040\sqrt[3]{e}e^3 - 160\ 2^{2/3}\sqrt[3]{e}^2e^2 - 30\sqrt[3]{2}e^2 - \\
& 80\sqrt[3]{2}\sqrt{2,304\ 2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\ 2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1}e^3 + \\
& 40\sqrt[3]{e}\sqrt{2,304\ 2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\ 2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1}e^2 + \\
& 10\ 2^{2/3}\sqrt[3]{e}^2\sqrt{2,304\ 2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\ 2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1}e - 5\sqrt[3]{e}e,
\end{aligned}$$

$$\mathbf{V}_{42_{denominator}} = 20e\sqrt[3]{e}(9,216e^4 + 512\ 2^{2/3}\sqrt[3]{e}e^3 + 248\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 24e^2 +$$

$$96\ 2^{2/3}\sqrt[3]{e}\sqrt{2,304\ 2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\ 2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1}e^2 +$$

$$2\sqrt[3]{2}\sqrt[3]{e}^2\sqrt{2,304\ 2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\ 2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1}e +$$

$$\sqrt{2,304\ 2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\ 2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1}e + 2^{2/3}\sqrt[3]{e}e + \sqrt[3]{2}\sqrt[3]{e}^2),$$

$$\mathbf{V}_{43_{numerator}} = -1,536\sqrt[3]{2}\sqrt[3]{e}^2e^3 - 704e^3 - 208\ 2^{2/3}\sqrt[3]{e}e^2 -$$

$$32\sqrt{2,304\ 2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\ 2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1}e^2 +$$

$$8\ 2^{2/3}\sqrt[3]{e}\sqrt{2,304\ 2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\ 2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1}e +$$

$$4\sqrt[3]{2}\sqrt[3]{e}^2\sqrt{2,304\ 2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\ 2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1} -$$

$$64\sqrt[3]{2}\sqrt[3]{e}^2e - 12e - 2^{2/3}\sqrt[3]{e},$$

$$\mathbf{V}_{43_{denominator}} = 4(9,216e^4 + 512\ 2^{2/3}\sqrt[3]{e}e^3 + 248\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 24e^2 +$$

$$96\ 2^{2/3}\sqrt[3]{e}\sqrt{2,304\ 2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\ 2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1}e^2 +$$

$$2\sqrt[3]{2}\sqrt[3]{e}^2\sqrt{2,304\ 2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\ 2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1}e +$$

$$\sqrt{2,304\ 2^{2/3}\sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e}^2e^2 + 224e^2 + 24\ 2^{2/3}\sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e}^2 + 1}e + 2^{2/3}\sqrt[3]{e}e + \sqrt[3]{2}\sqrt[3]{e}^2),$$

$$\mathbf{V}_{44} = 1. \tag{A29}$$

Where

$$\mathbf{V}_4 = \begin{bmatrix} \frac{-\mathbf{V}_{41_{numerator}}}{\mathbf{V}_{41_{denominator}}} \\ \frac{-\mathbf{V}_{42_{numerator}}}{\mathbf{V}_{42_{denominator}}} \\ \frac{-\mathbf{V}_{43_{numerator}}}{\mathbf{V}_{43_{denominator}}} \\ \mathbf{V}_{44} \end{bmatrix}. \tag{A30}$$

A.4 *Closed form equations for Lemma 2 condition four*

The Eigenvalues:

$$\lambda_1 = 0, \tag{A31}$$

$$\lambda_2 = 0, \tag{A32}$$

$$\lambda_3 = \frac{ce^{3/2} - \sqrt{(c^2+e^2)\left(2\sqrt[3]{2}c^2\sqrt[3]{e}^2 + c^2e + e\right)}}{2c\sqrt{e}}, \tag{A33}$$

$$\lambda_4 = \frac{ce^{3/2} + \sqrt{(c^2+e^2)\left(2\sqrt[3]{2}c^2\sqrt[3]{e}^2 + c^2e + e\right)}}{2c\sqrt{e}}. \tag{A34}$$

Corresponding Eigenvectors:

$$\mathbf{V}_1 = \begin{bmatrix} -c \\ -\frac{e}{c} \\ 0 \\ 1 \end{bmatrix}, \tag{A35}$$

$$\mathbf{V}_2 = \begin{bmatrix} -\frac{2^{2/3}c\sqrt[3]{e}}{\sqrt{e}} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \tag{A36}$$

$$\mathbf{V}_{31_{numerator}} = c^3e^{3/2} - e^2\sqrt{(c^2+e^2)\left(2\sqrt[3]{2}c^2\sqrt[3]{e}^2 + c^2e + e\right)} - c^2\sqrt{(c^2+e^2)\left(2\sqrt[3]{2}c^2\sqrt[3]{e}^2 + c^2e + e\right)} + ce^{7/2},$$

$$\mathbf{V}_{31_{denominator}} = -2c^4e^{3/2} - 2\sqrt[3]{2}c^4\sqrt{e}\sqrt[3]{e}^2 - 2c^2e^{7/2} - 2\sqrt[3]{2}c^2e^{5/2}\sqrt[3]{e}^2 - c^2e^{3/2} +$$

$$2ce^2\sqrt{(c^2+e^2)\left(2\sqrt[3]{2}c^2\sqrt[3]{e}^2+c^2e+e\right)} + c^3\sqrt{(c^2+e^2)\left(2\sqrt[3]{2}c^2\sqrt[3]{e}^2+c^2e+e\right)} - e^{7/2},$$

$$\mathbf{V}_{32_{numerator}} = c\left(2\sqrt[3]{2}c^4\sqrt[3]{e}^2 + c^4e + c^2e^3 + 2\sqrt[3]{2}c^2e^2\sqrt[3]{e}^2 - ce^{3/2}\sqrt{(c^2+e^2)\left(2\sqrt[3]{2}c^2\sqrt[3]{e}^2+c^2e+e\right)} + c^2e + e^3\right),$$

$$\mathbf{V}_{32_{denominator}} = \sqrt{e}(2c^4e^{3/2} + 2\sqrt[3]{2}c^4\sqrt{e}\sqrt[3]{e}^2 + 2c^2e^{7/2} + 2\sqrt[3]{2}c^2e^{5/2}\sqrt[3]{e}^2 + c^2e^{3/2} -$$

$$2ce^2\sqrt{(c^2+e^2)\left(2\sqrt[3]{2}c^2\sqrt[3]{e}^2+c^2e+e\right)} - c^3\sqrt{(c^2+e^2)\left(2\sqrt[3]{2}c^2\sqrt[3]{e}^2+c^2e+e\right)} + e^{7/2}),$$

$$\mathbf{V}_{33_{numerator}} = -2^{2/3}c^4e^2\sqrt[3]{e} + 2^{2/3}c^2e^4\sqrt[3]{e} - 2^{2/3}ce^{5/2}\sqrt[3]{e}\sqrt{(c^2+e^2)\left(2\sqrt[3]{2}c^2\sqrt[3]{e}^2+c^2e+e\right)} -$$

$$2^{2/3}c^3\sqrt{e}\sqrt[3]{e}\sqrt{(c^2+e^2)\left(2\sqrt[3]{2}c^2\sqrt[3]{e}^2+c^2e+e\right)},$$

$$\mathbf{V}_{33_{denominator}} = -c^4e^{5/2} - c^2e^{9/2} - e^{3/2}\left(c^2+e^2\right)\left(2\sqrt[3]{2}c^2\sqrt[3]{e}^2+c^2e+e\right) +$$

$$2ce^3\sqrt{(c^2+e^2)\left(2\sqrt[3]{2}c^2\sqrt[3]{e}^2+c^2e+e\right)} + c^3e\sqrt{(c^2+e^2)\left(2\sqrt[3]{2}c^2\sqrt[3]{e}^2+c^2e+e\right)},$$

$$\mathbf{V}_{34} = 1. \tag{A37}$$

Where

$$\mathbf{V}_3 = \begin{bmatrix} \frac{-\mathbf{V}_{31_{numerator}}}{\mathbf{V}_{31_{denominator}}} \\ \frac{\mathbf{V}_{32_{numerator}}}{\mathbf{V}_{32_{denominator}}} \\ \frac{-\mathbf{V}_{33_{numerator}}}{\mathbf{V}_{33_{denominator}}} \\ \mathbf{V}_{34} \end{bmatrix}. \tag{A38}$$

$$\mathbf{V}_{41_{numerator}} = -c^3e^{3/2} - e^2\sqrt{(c^2+e^2)\left(2\sqrt[3]{2}c^2\sqrt[3]{e}^2+c^2e+e\right)} - c^2\sqrt{(c^2+e^2)\left(2\sqrt[3]{2}c^2\sqrt[3]{e}^2+c^2e+e\right)} - ce^{7/2},$$

$$\mathbf{V}41_{denominator} = 2c^4e^{3/2} + 2\sqrt[3]{2}c^4\sqrt{e}\sqrt[3]{e}^2 + 2c^2e^{7/2} + 2\sqrt[3]{2}c^2e^{5/2}\sqrt[3]{e}^2 + c^2e^{3/2} +$$

$$2ce^2\sqrt{(c^2+e^2)\left(2\sqrt[3]{2}c^2\sqrt[3]{e}^2 + c^2e + e\right)} + c^3\sqrt{(c^2+e^2)\left(2\sqrt[3]{2}c^2\sqrt[3]{e}^2 + c^2e + e\right)} + e^{7/2},$$

$$\mathbf{V}42_{numerator} = c(2\sqrt[3]{2}c^4\sqrt[3]{e}^2 + c^4e + c^2e^3 + 2\sqrt[3]{2}c^2e^2\sqrt[3]{e}^2$$

$$ce^{3/2}\sqrt{(c^2+e^2)\left(2\sqrt[3]{2}c^2\sqrt[3]{e}^2 + c^2e + e\right)} + c^2e + e^3),$$

$$\mathbf{V}42_{denominator} = \sqrt{e}(2c^4e^{3/2} + 2\sqrt[3]{2}c^4\sqrt{e}\sqrt[3]{e}^2 + 2c^2e^{7/2} + 2\sqrt[3]{2}c^2e^{5/2}\sqrt[3]{e}^2 + c^2e^{3/2} +$$

$$2ce^2\sqrt{(c^2+e^2)\left(2\sqrt[3]{2}c^2\sqrt[3]{e}^2 + c^2e + e\right)} + c^3\sqrt{(c^2+e^2)\left(2\sqrt[3]{2}c^2\sqrt[3]{e}^2 + c^2e + e\right)} + e^{7/2}),$$

$$\mathbf{V}43_{numerator} = 2^{2/3}c^4e^2\sqrt[3]{e} + 2^{2/3}c^2e^4\sqrt[3]{e} + 2^{2/3}ce^{5/2}\sqrt[3]{e}\sqrt{(c^2+e^2)\left(2\sqrt[3]{2}c^2\sqrt[3]{e}^2 + c^2e + e\right)} +$$

$$2^{2/3}c^3\sqrt{e}\sqrt[3]{e}\sqrt{(c^2+e^2)\left(2\sqrt[3]{2}c^2\sqrt[3]{e}^2 + c^2e + e\right)},$$

$$\mathbf{V}43_{denominator} = -c^4e^{5/2} - c^2e^{9/2} - e^{3/2}\left(c^2+e^2\right)\left(2\sqrt[3]{2}c^2\sqrt[3]{e}^2 + c^2e + e\right) -$$

$$2ce^3\sqrt{(c^2+e^2)\left(2\sqrt[3]{2}c^2\sqrt[3]{e}^2 + c^2e + e\right)} - c^3e\sqrt{(c^2+e^2)\left(2\sqrt[3]{2}c^2\sqrt[3]{e}^2 + c^2e + e\right)},$$

$$\mathbf{V}_{44} = 1. \tag{A39}$$

Where

$$\mathbf{V}_4 = \begin{bmatrix} \frac{\mathbf{V}41_{numerator}}{\mathbf{V}41_{denominator}} \\ \frac{\mathbf{V}42_{numerator}}{\mathbf{V}42_{denominator}} \\ \frac{\mathbf{V}43_{numerator}}{\mathbf{V}43_{denominator}} \\ \mathbf{V}_{44} \end{bmatrix}. \tag{A40}$$

A.5 *Closed form equations for Lemma 2 condition five*

The Eigenvalues:

$$\lambda_1 = 0, \tag{A41}$$

$$\lambda_2 = 0, \tag{A42}$$

$$\lambda_3 = \frac{1}{2}\left(-\sqrt{2}\sqrt{b^2-3b+2}-\sqrt{-b}\right), \tag{A43}$$

$$\lambda_4 = \frac{1}{2}\left(\sqrt{2}\sqrt{b^2-3b+2}-\sqrt{-b}\right). \tag{A44}$$

Corresponding Eigenvectors:

$$\mathbf{V}_1 = \begin{bmatrix} -2\sqrt{-b} \\ 1 \\ 0 \\ 1 \end{bmatrix}, \tag{A45}$$

$$\mathbf{V}_2 = \begin{bmatrix} b-1 \\ \sqrt{-b} \\ 1 \\ 0 \end{bmatrix}, \tag{A46}$$

$$\mathbf{V}{31}_{numerator} = b^2+\sqrt{2}\sqrt{-b}\sqrt{b^2-3b+2}b-\sqrt{2}\sqrt{-b}\sqrt{b^2-3b+2}-4b+4,$$

$$\mathbf{V}{31}_{denominator} = \left(b^2-2\right)\left(\sqrt{2}\sqrt{b^2-3b+2}+\sqrt{-b}\right),$$

$$\mathbf{V}{32}_{numerator} = 3\sqrt{-b}b^2-\sqrt{2}\sqrt{b^2-3b+2}b^2+2\sqrt{2}\sqrt{b^2-3b+2}b-2\sqrt{2}\sqrt{b^2-3b+2}-8\sqrt{-b}b+6\sqrt{-b},$$

$$\mathbf{V}{32}_{denominator} = \left(b^2-2\right)\left(\sqrt{2}\sqrt{b^2-3b+2}+\sqrt{-b}\right),$$

$$\mathbf{V}{33}_{numerator} = 2b^3-3b^2+\sqrt{2}\sqrt{-b}\sqrt{b^2-3b+2}-2b+4,$$

$$\mathbf{V}{33}_{denominator} = \left(b^2-2\right)\left(\sqrt{2}\sqrt{b^2-3b+2}+\sqrt{-b}\right),$$

$$\mathbf{V}_{34} = 1. \tag{A47}$$

Where

$$\mathbf{V}_3 = \begin{bmatrix} \frac{\mathbf{V}_{31_{numerator}}}{\mathbf{V}_{31_{denominator}}} \\ \frac{\mathbf{V}_{32_{numerator}}}{\mathbf{V}_{32_{denominator}}} \\ \frac{\mathbf{V}_{33_{numerator}}}{\mathbf{V}_{33_{denominator}}} \\ \mathbf{V}_{34} \end{bmatrix}. \tag{A48}$$

$$\mathbf{V}_{41_{numerator}} = b^2 - \sqrt{2}\sqrt{-b}\sqrt{b^2-3b+2}b + \sqrt{2}\sqrt{-b}\sqrt{b^2-3b+2} - 4b + 4,$$

$$\mathbf{V}_{41_{denominator}} = \left(b^2-2\right)\left(\sqrt{-b} - \sqrt{2}\sqrt{b^2-3b+2}\right),$$

$$\mathbf{V}_{42_{numerator}} = 3\sqrt{-b}b^2 + \sqrt{2}\sqrt{b^2-3b+2}b^2 - 2\sqrt{2}\sqrt{b^2-3b+2}b + 2\sqrt{2}\sqrt{b^2-3b+2} - 8\sqrt{-b}b + 6\sqrt{-b},$$

$$\mathbf{V}_{42_{denominator}} = \left(b^2-2\right)\left(\sqrt{-b} - \sqrt{2}\sqrt{b^2-3b+2}\right),$$

$$\mathbf{V}_{43_{numerator}} = 2b^3 - 3b^2 - \sqrt{2}\sqrt{-b}\sqrt{b^2-3b+2} - 2b + 4,$$

$$\mathbf{V}_{43_{denominator}} = \left(b^2-2\right)\left(\sqrt{-b} - \sqrt{2}\sqrt{b^2-3b+2}\right),$$

$$\mathbf{V}_{44} = 1. \tag{A49}$$

Where

$$\mathbf{V}_4 = \begin{bmatrix} \frac{\mathbf{V}_{41_{numerator}}}{\mathbf{V}_{41_{denominator}}} \\ \frac{\mathbf{V}_{42_{numerator}}}{\mathbf{V}_{42_{denominator}}} \\ \frac{\mathbf{V}_{43_{numerator}}}{\mathbf{V}_{43_{denominator}}} \\ \mathbf{V}_{44} \end{bmatrix}. \tag{A50}$$

A.6 *Closed form equations for Lemma 2 condition six*

The Eigenvalues:

$$\lambda_1 = 0, \tag{A51}$$

$$\lambda_2 = 0, \tag{A52}$$

$$\lambda_3 = -\frac{\sqrt{b^2-3b+2}}{\sqrt{2}} - \frac{\sqrt{-b}}{2}, \tag{A53}$$

$$\lambda_4 = \frac{\sqrt{b^2-3b+2}}{\sqrt{2}} - \frac{\sqrt{-b}}{2}. \tag{A54}$$

Corresponding Eigenvectors:

$$\mathbf{V}_1 = \begin{bmatrix} -2\sqrt{-b} \\ 1 \\ 0 \\ 1 \end{bmatrix}, \tag{A55}$$

$$\mathbf{V}_2 = \begin{bmatrix} b-1 \\ \sqrt{-b} \\ 1 \\ 0 \end{bmatrix}, \tag{A56}$$

$$\mathbf{V}{31}_{numerator} = -b^2 + \sqrt{2}(-b)^{3/2}\sqrt{b^2-3b+2} + \sqrt{2}\sqrt{-b}\sqrt{b^2-3b+2} + 4b - 4,$$

$$\mathbf{V}{31}_{denominator} = \left(b^2-2\right)\left(\sqrt{2}\sqrt{b^2-3b+2} + \sqrt{-b}\right),$$

$$\mathbf{V}{32}_{numerator} = \sqrt{2}b^2\sqrt{b^2-3b+2} - 2\sqrt{2}b\sqrt{b^2-3b+2} + 2\sqrt{2}\sqrt{b^2-3b+2} - 3(-b)^{5/2} + 8b\sqrt{-b} - 6\sqrt{-b},$$

$$\mathbf{V}{32}_{denominator} = \left(b^2-2\right)\left(\sqrt{2}\sqrt{b^2-3b+2} + \sqrt{-b}\right),$$

$$\mathbf{V}{33}_{numerator} = -2b^3 + 3b^2 - \sqrt{2}\sqrt{-b}\sqrt{b^2-3b+2} + 2b - 4,$$

$$\mathbf{V}{33}_{denominator} = \left(b^2-2\right)\left(\sqrt{2}\sqrt{b^2-3b+2} + \sqrt{-b}\right),$$

$$\mathbf{V}_{34} = 1. \tag{A57}$$

Where

$$\mathbf{V}_3 = \begin{bmatrix} \frac{\mathbf{V}_{31_{numerator}}}{\mathbf{V}_{31_{denominator}}} \\ \frac{\mathbf{V}_{32_{numerator}}}{\mathbf{V}_{32_{denominator}}} \\ \frac{\mathbf{V}_{33_{numerator}}}{\mathbf{V}_{33_{denominator}}} \\ \mathbf{V}_{34} \end{bmatrix}, \tag{A58}$$

$$\mathbf{V}_{41_{numerator}} = b^2 + \sqrt{2}(-b)^{3/2}\sqrt{b^2-3b+2} + \sqrt{2}\sqrt{-b}\sqrt{b^2-3b+2} - 4b + 4,$$

$$\mathbf{V}_{41_{denominator}} = \left(b^2-2\right)\left(\sqrt{-b} - \sqrt{2}\sqrt{b^2-3b+2}\right),$$

$$\mathbf{V}_{42_{numerator}} = \sqrt{2}b^2\sqrt{b^2-3b+2} - 2\sqrt{2}b\sqrt{b^2-3b+2} + 2\sqrt{2}\sqrt{b^2-3b+2} + 3(-b)^{5/2} + 8(-b)^{3/2} + 6\sqrt{-b},$$

$$\mathbf{V}_{42_{denominator}} = \left(b^2-2\right)\left(\sqrt{-b} - \sqrt{2}\sqrt{b^2-3b+2}\right),$$

$$\mathbf{V}_{43_{numerator}} = -2b^3 + 3b^2 + \sqrt{2}\sqrt{-b}\sqrt{b^2-3b+2} + 2b - 4,$$

$$\mathbf{V}_{43_{denominator}} = \left(b^2-2\right)\left(\sqrt{-b} - \sqrt{2}\sqrt{b^2-3b+2}\right),$$

$$\mathbf{V}_{44} = 1. \tag{A59}$$

Where

$$\mathbf{V}_4 = \begin{bmatrix} \frac{\mathbf{V}_{41_{numerator}}}{\mathbf{V}_{41_{denominator}}} \\ \frac{\mathbf{V}_{42_{numerator}}}{\mathbf{V}_{42_{denominator}}} \\ \frac{\mathbf{V}_{43_{numerator}}}{\mathbf{V}_{43_{denominator}}} \\ \mathbf{V}_{44} \end{bmatrix}. \tag{A60}$$

A.7 *Closed form equations for Lemma 2 condition seven*

The Eigenvalues:

$$\lambda_1 = 0, \tag{A61}$$

$$\lambda_2 = 0, \tag{A62}$$

$$\lambda_3 = -\sqrt{(b^2-b+1)\left(-b^3+b^2-\frac{d^2}{b}+2bd-b+1\right)}+\frac{bd}{\sqrt{-b}}+(-b)^{5/2}-(-b)^{3/2}-\sqrt{-b}, \tag{A63}$$

$$\lambda_4 = \sqrt{(b^2-b+1)\left(-b^3+b^2-\frac{d^2}{b}+2bd-b+1\right)}+\frac{bd}{\sqrt{-b}}+(-b)^{5/2}-(-b)^{3/2}-\sqrt{-b}. \tag{A64}$$

Corresponding Eigenvectors:

$$\mathbf{V}_1 = \begin{bmatrix} \frac{d}{\sqrt{-b}}-2(-b)^{3/2} \\ b \\ 0 \\ 1 \end{bmatrix}, \tag{A65}$$

$$\mathbf{V}_2 = \begin{bmatrix} 0 \\ -\sqrt{-b} \\ 1 \\ 0 \end{bmatrix}, \tag{A66}$$

$$\mathbf{V}3_{1_{numerator}} = \left(b^2-b+1\right)\left(\sqrt{(b^2-b+1)\left(-b^3+b^2-\frac{d^2}{b}+2bd-b+1\right)}-\frac{bd}{\sqrt{-b}}-(-b)^{5/2}+(-b)^{3/2}+\sqrt{-b}\right),$$

$$\begin{aligned}\mathbf{V}3_{1_{denominator}} = {} & 2\left(b^2-b+1\right)d^2+\frac{d}{\sqrt{-b}}\Bigg(2b^2\sqrt{(b^2-b+1)\left(-b^3+b^2-\frac{d^2}{b}+2bd-b+1\right)}- \\ & b\sqrt{(b^2-b+1)\left(-b^3+b^2-\frac{d^2}{b}+2bd-b+1\right)}+ \\ & \sqrt{(b^2-b+1)\left(-b^3+b^2-\frac{d^2}{b}+2bd-b+1\right)}- \\ & 4(-b)^{9/2}-3(-b)^{7/2}-2(-b)^{5/2}+2(-b)^{3/2}+\sqrt{-b}\Bigg)- \\ & b\left(-2b^5+2b^4-2b^3+b^2-2(-b)^{5/2}\sqrt{(b^2-b+1)\left(-b^3+b^2-\frac{d^2}{b}+2bd-b+1\right)}-b+1\right),\end{aligned}$$

$$\mathbf{V}_{32_{numerator}} = 2b^4 - 4b^3 - \frac{\left(b^2-b+1\right)d^2}{b} + 5b^2 +$$

$$\frac{bd\left(-\sqrt{(b^2-b+1)\left(-b^3+b^2-\frac{d^2}{b}+2bd-b+1\right)}+(-b)^{5/2}+(-b)^{3/2}+\sqrt{-b}\right)}{\sqrt{-b}} +$$

$$2(-b)^{3/2}\sqrt{(b^2-b+1)\left(-b^3+b^2-\frac{d^2}{b}+2bd-b+1\right)} +$$

$$2\sqrt{-b}\sqrt{(b^2-b+1)\left(-b^3+b^2-\frac{d^2}{b}+2bd-b+1\right)} - 3b + 1,$$

$$\mathbf{V}_{32_{denominator}} = 2\left(b^2-b+1\right)d^2 + \frac{d}{\sqrt{-b}}\left(2b^2\sqrt{(b^2-b+1)\left(-b^3+b^2-\frac{d^2}{b}+2bd-b+1\right)} -\right.$$

$$b\sqrt{(b^2-b+1)\left(-b^3+b^2-\frac{d^2}{b}+2bd-b+1\right)} + \sqrt{(b^2-b+1)\left(-b^3+b^2-\frac{d^2}{b}+2bd-b+1\right)} -$$

$$\left. 4(-b)^{9/2} - 3(-b)^{7/2} - 2(-b)^{5/2} + 2(-b)^{3/2} + \sqrt{-b}\right) -$$

$$b\left(-2b^5+2b^4-2b^3+b^2-2(-b)^{5/2}\sqrt{(b^2-b+1)\left(-b^3+b^2-\frac{d^2}{b}+2bd-b+1\right)} - b + 1\right),$$

$$\mathbf{V}_{33_{numerator}} = \sqrt{-b}\left(2b^4 - 4b^3 - \frac{\left(b^2-b+1\right)d^2}{b} + 5b^2 +\right.$$

$$\frac{bd\left(-\sqrt{(b^2-b+1)\left(-b^3+b^2-\frac{d^2}{b}+2bd-b+1\right)}+(-b)^{5/2}+(-b)^{3/2}+\sqrt{-b}\right)}{\sqrt{-b}} +$$

$$2(-b)^{3/2}\sqrt{(b^2-b+1)\left(-b^3+b^2-\frac{d^2}{b}+2bd-b+1\right)} +$$

$$\left. 2\sqrt{-b}\sqrt{(b^2-b+1)\left(-b^3+b^2-\frac{d^2}{b}+2bd-b+1\right)} - 3b + 1\right),$$

$$\mathbf{V}_{33_{denominator}} = 2\left(b^2-b+1\right)d^2 + \frac{d}{\sqrt{-b}}\left(2b^2\sqrt{\left(b^2-b+1\right)\left(-b^3+b^2-\frac{d^2}{b}+2bd-b+1\right)}-\right.$$

$$b\sqrt{\left(b^2-b+1\right)\left(-b^3+b^2-\frac{d^2}{b}+2bd-b+1\right)}+\sqrt{\left(b^2-b+1\right)\left(-b^3+b^2-\frac{d^2}{b}+2bd-b+1\right)}-$$

$$\left.4(-b)^{9/2}-3(-b)^{7/2}-2(-b)^{5/2}+2(-b)^{3/2}+\sqrt{-b}\right)-$$

$$b\left(-2b^5+2b^4-2b^3+b^2-2(-b)^{5/2}\sqrt{\left(b^2-b+1\right)\left(-b^3+b^2-\frac{d^2}{b}+2bd-b+1\right)}-b+1\right),$$

$$\mathbf{V}_{34} = 1. \tag{A67}$$

Where

$$\mathbf{V}_3 = \begin{bmatrix} \frac{\mathbf{V}_{31_{numerator}}}{\mathbf{V}_{31_{denominator}}} \\ \frac{\mathbf{V}_{32_{numerator}}}{\mathbf{V}_{32_{denominator}}} \\ \frac{\mathbf{V}_{33_{numerator}}}{\mathbf{V}_{33_{denominator}}} \\ \mathbf{V}_{34} \end{bmatrix}. \tag{A68}$$

$$\mathbf{V}_{41_{numerator}} = \left(b^2-b+1\right)\left(\sqrt{\left(b^2-b+1\right)\left(-b^3+b^2-\frac{d^2}{b}+2bd-b+1\right)}+\frac{bd}{\sqrt{-b}}+(-b)^{5/2}-(-b)^{3/2}-\sqrt{-b}\right),$$

$$\mathbf{V}_{41_{denominator}} = 2\left(b^2-b+1\right)d^2 - \frac{d}{\sqrt{-b}}\left(2b^2\sqrt{\left(b^2-b+1\right)\left(-b^3+b^2-\frac{d^2}{b}+2bd-b+1\right)}-\right.$$

$$b\sqrt{\left(b^2-b+1\right)\left(-b^3+b^2-\frac{d^2}{b}+2bd-b+1\right)}+\sqrt{\left(b^2-b+1\right)\left(-b^3+b^2-\frac{d^2}{b}+2bd-b+1\right)}+$$

$$\left.4(-b)^{9/2}+3(-b)^{7/2}+2(-b)^{5/2}-2(-b)^{3/2}-\sqrt{-b}\right)-$$

$$b\left(-2b^5+2b^4-2b^3+b^2+2(-b)^{5/2}\sqrt{\left(b^2-b+1\right)\left(-b^3+b^2-\frac{d^2}{b}+2bd-b+1\right)}-b+1\right),$$

$$\mathbf{V}_{42_{numerator}} = 2b^4 - 4b^3 - \frac{\left(b^2 - b + 1\right)d^2}{b} +$$

$$5b^2 + \frac{bd\left(\sqrt{(b^2 - b + 1)\left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1\right)} + (-b)^{5/2} + (-b)^{3/2} + \sqrt{-b}\right)}{\sqrt{-b}} -$$

$$2(-b)^{3/2}\sqrt{(b^2 - b + 1)\left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1\right)} -$$

$$2\sqrt{-b}\sqrt{(b^2 - b + 1)\left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1\right)} - 3b + 1,$$

$$\mathbf{V}_{42_{denominator}} = 2\left(b^2 - b + 1\right)d^2 - \frac{d}{\sqrt{-b}}\left(2b^2\sqrt{(b^2 - b + 1)\left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1\right)} -\right.$$

$$b\sqrt{(b^2 - b + 1)\left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1\right)} + \sqrt{(b^2 - b + 1)\left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1\right)} +$$

$$\left.4(-b)^{9/2} + 3(-b)^{7/2} + 2(-b)^{5/2} - 2(-b)^{3/2} - \sqrt{-b}\right) -$$

$$b\left(-2b^5 + 2b^4 - 2b^3 + b^2 + 2(-b)^{5/2}\sqrt{(b^2 - b + 1)\left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1\right)} - b + 1\right),$$

$$\mathbf{V}_{43_{numerator}} = \sqrt{-b}\left(2b^4 - 4b^3 - \frac{\left(b^2 - b + 1\right)d^2}{b} + 5b^2 +\right.$$

$$\frac{bd\left(\sqrt{(b^2 - b + 1)\left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1\right)} + (-b)^{5/2} + (-b)^{3/2} + \sqrt{-b}\right)}{\sqrt{-b}} -$$

$$2(-b)^{3/2}\sqrt{(b^2 - b + 1)\left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1\right)} -$$

$$\left.2\sqrt{-b}\sqrt{(b^2 - b + 1)\left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1\right)} - 3b + 1\right),$$

$$\mathbf{V}_{43_{denominator}} = 2\left(b^2-b+1\right)d^2 - \frac{d}{\sqrt{-b}}\left(2b^2\sqrt{\left(b^2-b+1\right)\left(-b^3+b^2-\frac{d^2}{b}+2bd-b+1\right)}-\right.$$

$$b\sqrt{\left(b^2-b+1\right)\left(-b^3+b^2-\frac{d^2}{b}+2bd-b+1\right)}+\sqrt{\left(b^2-b+1\right)\left(-b^3+b^2-\frac{d^2}{b}+2bd-b+1\right)}+$$

$$\left.4(-b)^{9/2}+3(-b)^{7/2}+2(-b)^{5/2}-2(-b)^{3/2}-\sqrt{-b}\right)-$$

$$b\left(-2b^5+2b^4-2b^3+b^2+2(-b)^{5/2}\sqrt{\left(b^2-b+1\right)\left(-b^3+b^2-\frac{d^2}{b}+2bd-b+1\right)}-b+1\right),$$

$$\mathbf{V}_{44} = 1. \tag{A69}$$

Where

$$\mathbf{V}_4 = \begin{bmatrix} \frac{\mathbf{V}_{41_{numerator}}}{\mathbf{V}_{41_{denominator}}} \\ \frac{\mathbf{V}_{42_{numerator}}}{\mathbf{V}_{42_{denominator}}} \\ \frac{\mathbf{V}_{43_{numerator}}}{\mathbf{V}_{43_{denominator}}} \\ \mathbf{V}_{44} \end{bmatrix}. \tag{A70}$$

A.8 *Closed form equations for Lemma 2 condition eight*

The Eigenvalues:

$$\lambda_1 = 0, \tag{A71}$$

$$\lambda_2 = 0, \tag{A72}$$

$$\lambda_3 = \frac{-\frac{e^3}{d^{3/2}}-\sqrt{\frac{e^6}{d^3}+64d^3(d+1)^2+16d\left(2d^{3/2}+d^2+3d+2\sqrt{d}+2\right)e^2+\frac{8\left(d+\sqrt{d}+1\right)e^4}{d}}+4\sqrt{d}\left(d-\sqrt{d}+1\right)e}{8d^{3/2}}, \tag{A73}$$

$$\lambda_4 = \frac{-\frac{e^3}{d^{3/2}}+\sqrt{\frac{e^6}{d^3}+64d^3(d+1)^2+16d\left(2d^{3/2}+d^2+3d+2\sqrt{d}+2\right)e^2+\frac{8\left(d+\sqrt{d}+1\right)e^4}{d}}+4\sqrt{d}\left(d-\sqrt{d}+1\right)e}{8d^{3/2}}. \tag{A74}$$

Corresponding Eigenvectors:

$$\mathbf{V}_1 = \begin{bmatrix} 0 \\ -\sqrt{d} \\ 0 \\ 1 \end{bmatrix}, \tag{A75}$$

$$\mathbf{V}_2 = \begin{bmatrix} -\frac{e^2}{2d^2} - \sqrt{d} \\ \frac{e}{2d} \\ 1 \\ 0 \end{bmatrix}, \tag{A76}$$

$$\mathbf{V}31_{\text{numerator}} = \left(4\left(d^2+d\right)+\frac{e^2}{d}\right)\left(-\frac{e^3}{d^{3/2}}-\right.$$

$$\sqrt{\frac{e^6}{d^3}+64d^3(d+1)^2+16d\left(2d^{3/2}+d^2+3d+2\sqrt{d}+2\right)e^2+\frac{8\left(d+\sqrt{d}+1\right)e^4}{d}}+$$

$$\left.4\sqrt{d}\left(d-\sqrt{d}+1\right)e\right),$$

$$\mathbf{V}31_{\text{denominator}} = 2d\left(\frac{\left(2d+\sqrt{d}+2\right)e^4}{d^2}+16d^2(d+1)^2+4\left(d^{3/2}+2d^2+5d+\sqrt{d}+3\right)e^2-\right.$$

$$\left.\frac{\left(2d-\sqrt{d}+2\right)e\sqrt{\frac{e^6}{d^3}+64d^3(d+1)^2+16d\left(2d^{3/2}+d^2+3d+2\sqrt{d}+2\right)e^2+\frac{8\left(d+\sqrt{d}+1\right)e^4}{d}}}{\sqrt{d}}\right),$$

$$\mathbf{V}_{32} = \frac{1}{\sqrt{d}},$$

$$\mathbf{V}33_{numerator} = -\frac{e^7}{d^{7/2}}-\frac{4\left(\sqrt{d}+1\right)^2e^5}{d^{3/2}}-8\sqrt{d}\left(3d^{3/2}+2d+3\sqrt{d}+1\right)e^3+$$

$$32d^{5/2}\left(d^{5/2}+2d^{3/2}-2d^2-3d+\sqrt{d}-1\right)e-$$

$$\frac{e^4\sqrt{\frac{e^6}{d^3}+64d^3(d+1)^2+16d\left(2d^{3/2}+d^2+3d+2\sqrt{d}+2\right)e^2+\frac{8\left(d+\sqrt{d}+1\right)e^4}{d}}}{d^2}-$$

$$4\sqrt{d}e^2\sqrt{\frac{e^6}{d^3}+64d^3(d+1)^2+16d\left(2d^{3/2}+d^2+3d+2\sqrt{d}+2\right)e^2+\frac{8\left(d+\sqrt{d}+1\right)e^4}{d}}-$$

$$8d^{5/2}(d+1)\sqrt{\frac{e^6}{d^3}+64d^3(d+1)^2+16d\left(2d^{3/2}+d^2+3d+2\sqrt{d}+2\right)e^2+\frac{8\left(d+\sqrt{d}+1\right)e^4}{d}},$$

$$\mathbf{V}_{33_{denominator}}=4d^2\left(\frac{\left(2d+\sqrt{d}+2\right)e^4}{d^2}+16d^2(d+1)^2+4\left(d^{3/2}+2d^2+5d+\sqrt{d}+3\right)e^2-\right.$$

$$\left.\frac{\left(2d-\sqrt{d}+2\right)e\sqrt{\frac{e^6}{d^3}+64d^3(d+1)^2+16d\left(2d^{3/2}+d^2+3d+2\sqrt{d}+2\right)e^2+\frac{8\left(d+\sqrt{d}+1\right)e^4}{d}}}{\sqrt{d}}\right),$$

$$\mathbf{V}_{34}=1. \tag{A77}$$

Where

$$\mathbf{V}_3=\begin{bmatrix}\frac{\mathbf{V}_{31_{numerator}}}{\mathbf{V}_{31_{denominator}}}\\ \mathbf{V}_{32}\\ \frac{\mathbf{V}_{33_{numerator}}}{\mathbf{V}_{33_{denominator}}}\\ \mathbf{V}_{34}\end{bmatrix}. \tag{A78}$$

$$\mathbf{V}_{41_{\text{numerator}}}=\left(4\left(d^2+d\right)+\frac{e^2}{d}\right)\left(-\frac{e^3}{d^{3/2}}+\right.$$

$$\sqrt{\frac{e^6}{d^3}+64d^3(d+1)^2+16d\left(2d^{3/2}+d^2+3d+2\sqrt{d}+2\right)e^2+\frac{8\left(d+\sqrt{d}+1\right)e^4}{d}}+$$

$$\left.4\sqrt{d}\left(d-\sqrt{d}+1\right)e\right),$$

$$\mathbf{V}_{41_{denominator}} = 2d\left(\frac{\left(2d+\sqrt{d}+2\right)e^4}{d^2} + 16d^2(d+1)^2 + 4\left(d^{3/2}+2d^2+5d+\sqrt{d}+3\right)e^2 - \right.$$

$$\left.\frac{\left(-2d+\sqrt{d}-2\right)e\sqrt{\frac{e^6}{d^3}+64d^3(d+1)^2+16d\left(2d^{3/2}+d^2+3d+2\sqrt{d}+2\right)e^2+\frac{8\left(d+\sqrt{d}+1\right)e^4}{d}}}{\sqrt{d}}\right),$$

$$\mathbf{V}_{42} = \frac{1}{\sqrt{d}},$$

$$\mathbf{V}_{43_{numerator}} = -\frac{e^7}{d^{7/2}} - \frac{4\left(\sqrt{d}+1\right)^2 e^5}{d^{3/2}} - 8\sqrt{d}\left(3d^{3/2}+2d+3\sqrt{d}+1\right)e^3 +$$

$$32d^{5/2}\left(d^{5/2}+2d^{3/2}-2d^2-3d+\sqrt{d}-1\right)e +$$

$$\frac{e^4\sqrt{\frac{e^6}{d^3}+64d^3(d+1)^2+16d\left(2d^{3/2}+d^2+3d+2\sqrt{d}+2\right)e^2+\frac{8\left(d+\sqrt{d}+1\right)e^4}{d}}}{d^2} +$$

$$4\sqrt{d}e^2\sqrt{\frac{e^6}{d^3}+64d^3(d+1)^2+16d\left(2d^{3/2}+d^2+3d+2\sqrt{d}+2\right)e^2+\frac{8\left(d+\sqrt{d}+1\right)e^4}{d}} +$$

$$8d^{5/2}(d+1)\sqrt{\frac{e^6}{d^3}+64d^3(d+1)^2+16d\left(2d^{3/2}+d^2+3d+2\sqrt{d}+2\right)e^2+\frac{8\left(d+\sqrt{d}+1\right)e^4}{d}},$$

$$\mathbf{V}_{43_{denominator}} = 4d^2\left(\frac{\left(2d+\sqrt{d}+2\right)e^4}{d^2} + 16d^2(d+1)^2 + 4\left(d^{3/2}+2d^2+5d+\sqrt{d}+3\right)e^2 - \right.$$

$$\left.\frac{\left(-2d+\sqrt{d}-2\right)e\sqrt{\frac{e^6}{d^3}+64d^3(d+1)^2+16d\left(2d^{3/2}+d^2+3d+2\sqrt{d}+2\right)e^2+\frac{8\left(d+\sqrt{d}+1\right)e^4}{d}}}{\sqrt{d}}\right),$$

$$\mathbf{V}_{44} = 1. \tag{A79}$$

Where

$$\mathbf{V}_4 = \begin{bmatrix} \frac{\mathbf{V}_{41_{numerator}}}{\mathbf{V}_{41_{denominator}}} \\ \mathbf{V}_{42} \\ \frac{\mathbf{V}_{43_{numerator}}}{\mathbf{V}_{43_{denominator}}} \\ \mathbf{V}_{44} \end{bmatrix}. \tag{A80}$$

A.9 *Closed form equations for Lemma 3 condition one*

The Eigenvalues:

$$\lambda_1 = 0, \tag{A81}$$

$$\lambda_2 = 0, \tag{A82}$$

$$\lambda_3 = \frac{1}{2}\left(-\sqrt{e^2+6e+2\sqrt[3]{2}(e+2)\sqrt[3]{e}^2+2\,2^{2/3}(e+1)\sqrt[3]{e}+1}+e-2^{2/3}\sqrt[3]{e}+1\right), \tag{A83}$$

$$\lambda_4 = \frac{1}{2}\left(\sqrt{e^2+6e+2\sqrt[3]{2}(e+2)\sqrt[3]{e}^2+2\,2^{2/3}(e+1)\sqrt[3]{e}+1}+e-2^{2/3}\sqrt[3]{e}+1\right). \tag{A84}$$

Corresponding Eigenvectors:

$$\mathbf{V}_1 = \begin{bmatrix} -\sqrt{e} \\ 0 \\ 0 \\ 1 \end{bmatrix}, \tag{A85}$$

$$\mathbf{V}_2 = \begin{bmatrix} -\frac{\sqrt[3]{e}}{\sqrt[3]{2}} \\ -\frac{\sqrt{e}}{2^{2/3}\sqrt[3]{e}} \\ 1 \\ 0 \end{bmatrix}, \tag{A86}$$

$$\mathbf{V}_{31} = \frac{1}{\sqrt{e}},$$

$$\mathbf{V}_{32_{numerator}} = 4\left(\sqrt{e^2+6e+2\sqrt[3]{2}(e+2)\sqrt[3]{e}^2+2\,2^{2/3}(e+1)\sqrt[3]{e}+1}+e+1\right)\sqrt[3]{e}^2+$$

$$2^{2/3}e\left(\sqrt{e^2+6e+2\sqrt[3]{2}(e+2)\sqrt[3]{e}^2+2\,2^{2/3}(e+1)\sqrt[3]{e}+1}+e+5\right)+6\sqrt[3]{2}e\sqrt[3]{e},$$

$$\mathbf{V}_{32_{denominator}} = e\sqrt[3]{e}\left(\sqrt[3]{2}\left(-\sqrt{e^2+6e+2\sqrt[3]{2}(e+2)\sqrt[3]{e}^2+2\,2^{2/3}(e+1)\sqrt[3]{e}+1}+e+1\right)-2\sqrt[3]{e}\right),$$

$$\mathbf{V}_{33_{numerator}} = 2\left(\sqrt[3]{2}\left(\sqrt{e^2+6e+2\sqrt[3]{2}(e+2)\sqrt[3]{e}^2+2\,2^{2/3}(e+1)\sqrt[3]{e}+1}+e+1\right)\sqrt[3]{e}+(e+3)\sqrt[3]{e}^2+2^{2/3}e\right),$$

$$\mathbf{V}_{33_{denominator}} = \sqrt{e}\sqrt[3]{e}\left(\sqrt[3]{2}\left(-\sqrt{e^2+6e+2\sqrt[3]{2}(e+2)\sqrt[3]{e}^2+2\,2^{2/3}(e+1)\sqrt[3]{e}+1}+e+1\right)-2\sqrt[3]{e}\right),$$

$$\mathbf{V}_{34} = 1. \tag{A87}$$

Where

$$\mathbf{V}_3 = \begin{bmatrix} \mathbf{V}_{31} \\ \frac{\mathbf{V}_{32_{numerator}}}{\mathbf{V}_{32_{denominator}}} \\ \frac{\mathbf{V}_{33_{numerator}}}{\mathbf{V}_{33_{denominator}}} \\ \mathbf{V}_{34} \end{bmatrix}. \tag{A88}$$

$$\mathbf{V}_{41_{numerator}} = \frac{1}{\sqrt{e}},$$

$$\mathbf{V}_{42_{numerator}} = \left(-4\sqrt{e^2+6e+2\sqrt[3]{2}(e+2)\sqrt[3]{e}^2+2\,2^{2/3}(e+1)\sqrt[3]{e}+1}+4e+4\right)\sqrt[3]{e}^2+$$

$$2^{2/3}e\left(-\sqrt{e^2+6e+2\sqrt[3]{2}(e+2)\sqrt[3]{e}^2+2\,2^{2/3}(e+1)\sqrt[3]{e}+1}+e+5\right)+6\sqrt[3]{2}e\sqrt[3]{e},$$

$$\mathbf{V}_{42_{denominator}} = e\sqrt[3]{e}\left(\sqrt[3]{2}\left(\sqrt{e^2+6e+2\sqrt[3]{2}(e+2)\sqrt[3]{e}^2+2\,2^{2/3}(e+1)\sqrt[3]{e}+1}+e+1\right)-2\sqrt[3]{e}\right),$$

$$\mathbf{V}_{43_{numerator}} = 2\left(\sqrt[3]{2}\left(-\sqrt{e^2+6e+2\sqrt[3]{2}(e+2)\sqrt[3]{e}^2+2\,2^{2/3}(e+1)\sqrt[3]{e}+1}+e+1\right)\sqrt[3]{e}+(e+3)\sqrt[3]{e}^2+2^{2/3}e\right),$$

$$\mathbf{V}_{43_{denominator}} = \sqrt{e}\sqrt[3]{e}\left(\sqrt[3]{2}\left(\sqrt{e^2+6e+2\sqrt[3]{2}(e+2)\sqrt[3]{e}^2+2\,2^{2/3}(e+1)\sqrt[3]{e}+1}+e+1\right)-2\sqrt[3]{e}\right),$$

$$\mathbf{V}_{44} = 1. \tag{A89}$$

Where

$$\mathbf{V}_4 = \begin{bmatrix} \mathbf{V}_{41} \\ \frac{\mathbf{V}_{42_{numerator}}}{\mathbf{V}_{42_{denominator}}} \\ \frac{\mathbf{V}_{43_{numerator}}}{\mathbf{V}_{43_{denominator}}} \\ \mathbf{V}_{44} \end{bmatrix}. \tag{A90}$$

A.10 *Closed form equations for Lemma 3 condition two*

The Eigenvalues:

$$\lambda_1 = 0, \tag{A91}$$

$$\lambda_2 = 0, \tag{A92}$$

$$\lambda_3 = \left(\frac{2}{5}\right)^{2/3} e^{2/3} - \frac{1}{4}\sqrt{-\frac{64}{5}\left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3} + 16\left(\frac{2}{5}\right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16\sqrt[3]{\frac{2}{5}}\sqrt[3]{e} + 4+} \tag{A93}$$
$$\frac{e}{2} + \sqrt[3]{\frac{2}{5}}\sqrt[3]{e} + \frac{1}{2},$$

$$\lambda_4 = \frac{1}{4}\left(4\left(\frac{2}{5}\right)^{2/3} e^{2/3} + \sqrt{-\frac{64}{5}\left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3} + 16\left(\frac{2}{5}\right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16\sqrt[3]{\frac{2}{5}}\sqrt[3]{e} + 4+}\right. \tag{A94}$$
$$\left. 2e + 4\sqrt[3]{\frac{2}{5}}\sqrt[3]{e} + 2\right).$$

Corresponding Eigenvectors:

$$\mathbf{V}_1 = \begin{bmatrix} -\frac{2\sqrt{e}}{\sqrt{5}} \\ \frac{\sqrt[3]{e}}{2^{2/3}\sqrt[3]{5}} \\ 0 \\ 1 \end{bmatrix}, \tag{A95}$$

$$\mathbf{V}_2 = \begin{bmatrix} \sqrt[3]{\frac{2}{5}}\sqrt[3]{e} \\ \frac{\sqrt[6]{e}}{\sqrt[3]{2}\sqrt[6]{5}} \\ 1 \\ 0 \end{bmatrix}, \tag{A96}$$

$$\mathbf{V}_{31_{numerator}} = \sqrt{5}\Bigg(2\left(\frac{2}{5}\right)^{2/3} e^{5/3} + \frac{8}{5}\sqrt[3]{\frac{2}{5}}e^{4/3} - \frac{8e^2}{5} +$$

$$\frac{2}{5}\left(2\sqrt{-\frac{64}{5}\left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3} + 16\left(\frac{2}{5}\right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16\sqrt[3]{\frac{2}{5}}\sqrt[3]{e} + 4} - 3\right)e +$$

$$\left(\frac{2}{5}\right)^{2/3}\left(\sqrt{-\frac{64}{5}\left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3} + 16\left(\frac{2}{5}\right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16\sqrt[3]{\frac{2}{5}}\sqrt[3]{e} + 4} - 2\right)e^{2/3} +$$

$$\sqrt{-\frac{64}{5}\left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3} + 16\left(\frac{2}{5}\right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16\sqrt[3]{\frac{2}{5}}\sqrt[3]{e} + 4} + 4\sqrt[3]{\frac{2}{5}}\sqrt[3]{e} - 2\Bigg),$$

$$\mathbf{V}_{31_{denominator}} = \sqrt{e}\Bigg(-\frac{22}{5}\left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3} + 4e^2 +$$

$$\frac{2}{5}\left(4 - 5\sqrt{-\frac{64}{5}\left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3} + 16\left(\frac{2}{5}\right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16\sqrt[3]{\frac{2}{5}}\sqrt[3]{e} + 4}\right)e -$$

$$\left(\frac{2}{5}\right)^{2/3}\left(\sqrt{-\frac{64}{5}\left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3} + 16\left(\frac{2}{5}\right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16\sqrt[3]{\frac{2}{5}}\sqrt[3]{e} + 4} - 2\right)e^{2/3} -$$

$$2\sqrt{-\frac{64}{5}\left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3} + 16\left(\frac{2}{5}\right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16\sqrt[3]{\frac{2}{5}}\sqrt[3]{e} + 4} - 8\sqrt[3]{\frac{2}{5}}\sqrt[3]{e} + 4\Bigg),$$

$$\mathbf{V}_{32_{numerator}} = 2\sqrt[3]{\frac{2}{5}}\left(\sqrt[3]{\frac{2}{5}}\sqrt[3]{e} - 1\right)\sqrt[3]{e}\Bigg(4\left(\frac{2}{5}\right)^{2/3} e^{2/3} -$$

$$\sqrt{-\frac{64}{5}\left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3} + 16\left(\frac{2}{5}\right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16\sqrt[3]{\frac{2}{5}}\sqrt[3]{e} + 4} + 2e + 4\sqrt[3]{\frac{2}{5}}\sqrt[3]{e} + 2\Bigg),$$

$$
\begin{aligned}
\mathbf{V}_{32_{denominator}} = & -\frac{22}{5}\left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3} + 4e^2 + \\
& \frac{2}{5}\left(4 - 5\sqrt{-\frac{64}{5}\left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3} + 16\left(\frac{2}{5}\right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16\sqrt[3]{\frac{2}{5}}\sqrt[3]{e} + 4}\right) e - \\
& \left(\frac{2}{5}\right)^{2/3}\left(\sqrt{-\frac{64}{5}\left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3} + 16\left(\frac{2}{5}\right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16\sqrt[3]{\frac{2}{5}}\sqrt[3]{e} + 4} - 2\right) e^{2/3} - \\
& 2\sqrt{-\frac{64}{5}\left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3} + 16\left(\frac{2}{5}\right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16\sqrt[3]{\frac{2}{5}}\sqrt[3]{e} + 4} - 8\sqrt[3]{\frac{2}{5}}\sqrt[3]{e} + 4,
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{V}_{33_{numerator}} = & \sqrt[3]{2}\sqrt[6]{5}\Bigg(-\frac{8}{5}\left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{8}{5}\sqrt[3]{\frac{2}{5}}e^{4/3} + \frac{4e^2}{5} - \\
& \frac{2}{5}\left(\sqrt{-\frac{64}{5}\left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3} + 16\left(\frac{2}{5}\right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16\sqrt[3]{\frac{2}{5}}\sqrt[3]{e} + 4} - 5\right) e - \\
& 2\left(\frac{2}{5}\right)^{2/3}\left(\sqrt{-\frac{64}{5}\left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3} + 16\left(\frac{2}{5}\right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16\sqrt[3]{\frac{2}{5}}\sqrt[3]{e} + 4} - 2\right) e^{2/3} - \\
& \sqrt{-\frac{64}{5}\left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3} + 16\left(\frac{2}{5}\right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16\sqrt[3]{\frac{2}{5}}\sqrt[3]{e} + 4} - 4\sqrt[3]{\frac{2}{5}}\sqrt[3]{e} + 2\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{V}_{33_{denominator}} = & \sqrt[6]{e}\Bigg(-\frac{22}{5}\left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3} + 4e^2 + \\
& \frac{2}{5}\left(4 - 5\sqrt{-\frac{64}{5}\left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3} + 16\left(\frac{2}{5}\right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16\sqrt[3]{\frac{2}{5}}\sqrt[3]{e} + 4}\right) e - \\
& \left(\frac{2}{5}\right)^{2/3}\left(\sqrt{-\frac{64}{5}\left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3} + 16\left(\frac{2}{5}\right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16\sqrt[3]{\frac{2}{5}}\sqrt[3]{e} + 4} - 2\right) e^{2/3} - \\
& 2\sqrt{-\frac{64}{5}\left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3} + 16\left(\frac{2}{5}\right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16\sqrt[3]{\frac{2}{5}}\sqrt[3]{e} + 4} - 8\sqrt[3]{\frac{2}{5}}\sqrt[3]{e} + 4\Bigg),
\end{aligned}
$$

$$\mathbf{V}_{34} = 1. \tag{A97}$$

Where

$$\mathbf{V}_3 = \begin{bmatrix} \dfrac{-\mathbf{V}_{31_{numerator}}}{\mathbf{V}_{31_{denominator}}} \\ \dfrac{-\mathbf{V}_{32_{numerator}}}{\mathbf{V}_{32_{denominator}}} \\ \dfrac{-\mathbf{V}_{33_{numerator}}}{\mathbf{V}_{33_{denominator}}} \\ \mathbf{V}_{34} \end{bmatrix}. \tag{A98}$$

$$\mathbf{V}_{41_{numerator}} = \sqrt{5}\Bigg(-2\left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{8}{5}\sqrt[3]{\frac{2}{5}}e^{4/3} + \frac{8e^2}{5} +$$

$$\frac{2}{5}\left(2\sqrt{-\frac{64}{5}\left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3} + 16\left(\frac{2}{5}\right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16\sqrt[3]{\frac{2}{5}}\sqrt[3]{e} + 4} + 3\right)e +$$

$$\left(\frac{2}{5}\right)^{2/3}\left(\sqrt{-\frac{64}{5}\left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3} + 16\left(\frac{2}{5}\right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16\sqrt[3]{\frac{2}{5}}\sqrt[3]{e} + 4} + 2\right)e^{2/3} +$$

$$\sqrt{-\frac{64}{5}\left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3} + 16\left(\frac{2}{5}\right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16\sqrt[3]{\frac{2}{5}}\sqrt[3]{e} + 4} - 4\sqrt[3]{\frac{2}{5}}\sqrt[3]{e} + 2\Bigg),$$

$$\mathbf{V}_{41_{denominator}} = \sqrt{e}\Bigg(-\frac{22}{5}\left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3} + 4e^2 +$$

$$\frac{2}{5}\left(5\sqrt{-\frac{64}{5}\left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3} + 16\left(\frac{2}{5}\right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16\sqrt[3]{\frac{2}{5}}\sqrt[3]{e} + 4} + 4\right)e +$$

$$\left(\frac{2}{5}\right)^{2/3}\left(\sqrt{-\frac{64}{5}\left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3} + 16\left(\frac{2}{5}\right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16\sqrt[3]{\frac{2}{5}}\sqrt[3]{e} + 4} + 2\right)e^{2/3} +$$

$$2\left(\sqrt{-\frac{64}{5}\left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3} + 16\left(\frac{2}{5}\right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16\sqrt[3]{\frac{2}{5}}\sqrt[3]{e} + 4} + 2\right) - 8\sqrt[3]{\frac{2}{5}}\sqrt[3]{e}\Bigg),$$

$$\mathbf{V}_{42_{numerator}} = 2\sqrt[3]{\frac{2}{5}}\left(\sqrt[3]{\frac{2}{5}}\sqrt[3]{e}-1\right)\sqrt[3]{e}\left(4\left(\frac{2}{5}\right)^{2/3}e^{2/3}+\right.$$

$$\left.\sqrt{-\frac{64}{5}\left(\frac{2}{5}\right)^{2/3}e^{5/3}-\frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3}+16\left(\frac{2}{5}\right)^{2/3}e^{2/3}+4e^2+\frac{24e}{5}-16\sqrt[3]{\frac{2}{5}}\sqrt[3]{e}+4}+2e+4\sqrt[3]{\frac{2}{5}}\sqrt[3]{e}+2\right),$$

$$\mathbf{V}_{42_{denominator}} = -\frac{22}{5}\left(\frac{2}{5}\right)^{2/3}e^{5/3}-\frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3}+4e^2+$$

$$\frac{2}{5}\left(5\sqrt{-\frac{64}{5}\left(\frac{2}{5}\right)^{2/3}e^{5/3}-\frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3}+16\left(\frac{2}{5}\right)^{2/3}e^{2/3}+4e^2+\frac{24e}{5}-16\sqrt[3]{\frac{2}{5}}\sqrt[3]{e}+4}+4\right)e+$$

$$\left(\frac{2}{5}\right)^{2/3}\left(\sqrt{-\frac{64}{5}\left(\frac{2}{5}\right)^{2/3}e^{5/3}-\frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3}+16\left(\frac{2}{5}\right)^{2/3}e^{2/3}+4e^2+\frac{24e}{5}-16\sqrt[3]{\frac{2}{5}}\sqrt[3]{e}+4}+2\right)e^{2/3}+$$

$$2\left(\sqrt{-\frac{64}{5}\left(\frac{2}{5}\right)^{2/3}e^{5/3}-\frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3}+16\left(\frac{2}{5}\right)^{2/3}e^{2/3}+4e^2+\frac{24e}{5}-16\sqrt[3]{\frac{2}{5}}\sqrt[3]{e}+4}+2\right)-8\sqrt[3]{\frac{2}{5}}\sqrt[3]{e},$$

$$\mathbf{V}_{43_{numerator}} = \sqrt[3]{2}\sqrt[6]{5}\left(-\frac{8}{5}\left(\frac{2}{5}\right)^{2/3}e^{5/3}-\frac{8}{5}\sqrt[3]{\frac{2}{5}}e^{4/3}+\frac{4e^2}{5}+\right.$$

$$\frac{2}{5}\left(\sqrt{-\frac{64}{5}\left(\frac{2}{5}\right)^{2/3}e^{5/3}-\frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3}+16\left(\frac{2}{5}\right)^{2/3}e^{2/3}+4e^2+\frac{24e}{5}-16\sqrt[3]{\frac{2}{5}}\sqrt[3]{e}+4}+5\right)e+$$

$$2\left(\frac{2}{5}\right)^{2/3}\left(\sqrt{-\frac{64}{5}\left(\frac{2}{5}\right)^{2/3}e^{5/3}-\frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3}+16\left(\frac{2}{5}\right)^{2/3}e^{2/3}+4e^2+\frac{24e}{5}-16\sqrt[3]{\frac{2}{5}}\sqrt[3]{e}+4}+2\right)e^{2/3}+$$

$$\left.\sqrt{-\frac{64}{5}\left(\frac{2}{5}\right)^{2/3}e^{5/3}-\frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3}+16\left(\frac{2}{5}\right)^{2/3}e^{2/3}+4e^2+\frac{24e}{5}-16\sqrt[3]{\frac{2}{5}}\sqrt[3]{e}+4}-4\sqrt[3]{\frac{2}{5}}\sqrt[3]{e}+2\right),$$

$$\mathbf{V}_{43_{denominator}} = \sqrt[6]{e}\left(-\frac{22}{5}\left(\frac{2}{5}\right)^{2/3}e^{5/3}-\frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3}+4e^2+\right.$$

$$\frac{2}{5}\left(5\sqrt{-\frac{64}{5}\left(\frac{2}{5}\right)^{2/3}e^{5/3}-\frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3}+16\left(\frac{2}{5}\right)^{2/3}e^{2/3}+4e^2+\frac{24e}{5}-16\sqrt[3]{\frac{2}{5}}\sqrt[3]{e}+4}+4\right)e+$$

$$\left(\frac{2}{5}\right)^{2/3}\left(\sqrt{-\frac{64}{5}\left(\frac{2}{5}\right)^{2/3}e^{5/3}-\frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3}+16\left(\frac{2}{5}\right)^{2/3}e^{2/3}+4e^2+\frac{24e}{5}-16\sqrt[3]{\frac{2}{5}}\sqrt[3]{e}+4}+2\right)e^{2/3}+$$

$$2\left(\sqrt{-\frac{64}{5}\left(\frac{2}{5}\right)^{2/3}e^{5/3}-\frac{16}{5}\sqrt[3]{\frac{2}{5}}e^{4/3}+16\left(\frac{2}{5}\right)^{2/3}e^{2/3}+4e^2+\frac{24e}{5}-16\sqrt[3]{\frac{2}{5}}\sqrt[3]{e}+4}+2\right)-8\sqrt[3]{\frac{2}{5}}\sqrt[3]{e}\right),$$

$$\mathbf{V}_{44}=1. \tag{A99}$$

Where

$$\mathbf{V}_4=\begin{bmatrix}\frac{\mathbf{V}_{41_{numerator}}}{\mathbf{V}_{41_{denominator}}}\\ \frac{-\mathbf{V}_{42_{numerator}}}{\mathbf{V}_{42_{denominator}}}\\ \frac{-\mathbf{V}_{43_{numerator}}}{\mathbf{V}_{43_{denominator}}}\\ \mathbf{V}_{44}\end{bmatrix}. \tag{A100}$$

A.11 *Closed form equations for Lemma 3 condition three*

The Eigenvalues:

$$\lambda_1=0, \tag{A101}$$

$$\lambda_2=0, \tag{A102}$$

$$\lambda_3=a+2\sqrt{c}, \tag{A103}$$

$$\lambda_4=1+c+b^2. \tag{A104}$$

Corresponding Eigenvectors:

$$\mathbf{V}_1=\begin{bmatrix}-b\\0\\0\\1\end{bmatrix}, \tag{A105}$$

$$\mathbf{V}_2 = \begin{bmatrix} \sqrt{c} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \tag{A106}$$

$$\mathbf{V}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \tag{A107}$$

$$\mathbf{V}_4 = \begin{bmatrix} \frac{1}{b} \\ 0 \\ -\frac{\sqrt{c}}{b} \\ 1 \end{bmatrix}. \tag{A108}$$