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Research Article

Common Fixed Point Results of Asymptotically Regular Mappings in Generalized Metric Spaces with Application

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Abstract: In this paper, we establish common fixed point results for two regular asymptotic mappings in b-metric spaces and strong b-metric spaces. Our main result deals with regular asymptotic contracting mappings of the Hardy-Rogers-Proinov-Górnicki type. We apply one of our new results to the solution of the Fredholm linear integral equation. We also obtain a common fixed point result for multivalued regular asymptotic mappings in a strong b-metric space, which is a modification and extension of the work of previous authors. Examples are given in support of the new concepts and results.

Keywords: asymptotically regular mapping, b-metric space, common fixed point, multi-valued mapping, Fredholm linear integral equation

MSC: 47H09, 47H10, 54E35

1. Introduction

The study of fixed point theory began with Banach's Contraction Principle in 1922, which established the existence of unique fixed points for contraction mappings within complete metric spaces. Later Browder and Petryshyn [1] defined Asymptotically Regular (AR) mappings. Building on this, Krüppel [2] demonstrated that if C is a convex, closed and bounded subset of E which is uniformly convex Banach space, then any AR mapping $K\colon C \to C$ with Lipschitz norm satisfying $\liminf_{n\to\infty} \|K^n\| \le 1$ possesses a fixed point in C. This result was further generalized by Górnicki [3] in 1993, who provided conditions, for fixed points, which are sufficient for their existence. Subsequently, Ćirić [4] extended and broadened the findings of Sharma and Yuel [5] and Guay and Singh [6], establishing fixed point theorems in metric spaces without requiring completeness. More recently, Khan and Jhade [7] developed fixed point results for AR sequences and AR mappings within complete b-Metric Spaces ($bMSs$), thereby extending previous work in this area. Their results extended and generalized fixed point results of Hardy and Rogers [8] and Reich [9]. Khan and Oyetunbi [10] improved the results of Górnicki [11] and Bisht [12] for a pair of AR mappings.

In 2022, Bisht et al. [13] introduced novel classes of generalized contractive maps and established Common Fixed Point (CFP) theorems for pairs of AR mappings. Their findings expanded and refined several classical results, including those by Kannan [14], Reich [9], Hardy and Rogers [8], Ćirić [4], Jungck [15], Górnicki [3], among others. For results in the framework of $bMSs$, the reader is referred [16–18].

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Additionally, Khan et al. [19] derived *CFP* results for AR self-mappings and their averaged counterparts in convex metric spaces, employing various contractive conditions inspired by Górnicki's work on continuous AR self-mappings. For applications, we refer the interested reader to [20, 21]. Recently Alsulami and Alarfaj [22] established fixed point theorems in a complete strong b-metric space under the $\alpha-\psi$-contractive condition imposed on single-valued and multi-valued mappings.

In the present paper, we focus on establishing unique *CFP* theorems for AR mappings in *bMSs*, considering certain classes of mappings that satisfy newly introduced contractive conditions by Bisht et al. [13]. Section 4 presents an application of these results, while Section 5 extends the discussion to *CFP* theorems for single-valued and multi-valued mappings in the setting of strong *bMSs*. We also provide examples and applications to illustrate the effectiveness of our results.

2. Preliminaries

Definition 2.1 Let $\mho \neq \phi$ be a set and the mapping Φ: $\mho \times \mho \to \mathbb{R}^+$ (non-negative reals) satisfies:

(S_1) $\Phi(\nu, \rho) = 0$ iff $\nu = \rho$ for all $\nu, \rho \in \mho$;

(S_2) $\Phi(\nu, \rho) = \Phi(\rho, \nu)$ for all $\nu, \rho \in \mho$;

(S_3) $\Phi(\nu, \rho) \leq s[\Phi(\nu, \varpi) + \Phi(\varpi, \rho)]$;

for all $\nu, \rho, \varpi \in \mho$, for a constant $s \geq 1$.

Then Φ is a b-metric on $\mho$ and the pair $(\mho, \Phi)$ is known as *bMS* with coefficient s.

Remark 2.2 The collection of *bMSs* properly contains the family of metric spaces, as every metric space can be viewed as a *bMS* with coefficient $s = 1$.

To illustrate this, we provide an example of a *bMS* that does not satisfy the conditions of a metric space.

Example 2.3 [7] Let $\mho = [0, \infty)$ and Φ: $\mho \times \mho \to \mathbb{R}^+$ be defined by $\Phi(\nu, \rho) = |\nu - \rho|^2$. Let $\nu, \rho, \varpi \in \mho$, for $h = \nu - \varpi$ and $g = \varpi - \rho$, we have

$$\begin{aligned}
|\nu - \rho|^2 &= |h + g|^2 \\
&\leq (|h| + |g|)^2 \\
&\leq |h|^2 + 2|h||g| + |g|^2 \\
&\leq |h|^2 + (|h|^2 + |g|^2) + |g|^2 \quad (\text{Since } 2|h||g| \leq |h|^2 + |g|^2) \\
&= 2|h|^2 + 2|g|^2 \\
&= 2(|\nu - \varpi|^2 + |\varpi - j|^2) \\
&= 2[\Phi(\nu, \varpi) + \Phi(\varpi, j)]
\end{aligned}$$

which implies that $\Phi(\nu, y) \leq 2[\Phi(\nu, \varpi) + \Phi(\varpi, \rho)]$.

Therefore $(\mho, \Phi)$ is a *bMS* with $s = 2$.

Also, for $\nu > \rho > \varpi$, we have

$$|\nu-\rho|^2 = |h+g|^2 = (h+g)^2 > h^2+g^2 = (\nu-\varpi)^2+(\varpi-\rho)^2 = |\nu-\varpi|^2+|\varpi-\rho|^2,$$

which implies that $\Phi(\nu, \rho) > \Phi(\nu, \varpi)+\Phi(\varpi, \rho)$. Therefore $(\mho, \Phi)$ is not a metric space.

Definition 2.4 Let $(\mho, \Phi)$ be a bMS. Then $\{\nu_n\}$ in $\mho$ is called:

1. Cauchy sequence iff for all $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that for each $n, m \geq N(\varepsilon)$, we have $\Phi(\nu_n, \nu_m) < \varepsilon$.
2. Convergent to $\nu \in \mho$ iff, for all $\varepsilon > 0$, there exist $N(\varepsilon) \in \mathbb{N}$ such that for every $n \geq N(\varepsilon)$, we have $\Phi(\nu_n, \nu) < \varepsilon$.

Definition 2.5 In a bMS $(\mho, \Phi)$, a mapping $K\colon \mho \to \mho$ is said to be AR at $\nu \in \mho$ if $\lim_{n\to\infty} \Phi(K^n\nu, K^{n+1}\nu) = 0$. If K is AR at each $\nu \in \mho$, then K is said to be AR on $\mho$.

Example 2.6 [7] Let $(\mho, \Phi)$ be the bMS as in Example 2.3. Let $K\colon \mho \to \mho$ be defined as

$$K\nu = \frac{\nu}{4} \quad \text{where} \quad \nu \in \mho.$$

Then, we have

$$\lim_{n\to\infty} \Phi(K^n\nu, K^{n+1}\nu) = \lim_{n\to\infty} \left|K^n\nu - K^{n+1}\nu\right|^2 = \lim_{n\to\infty} \left|\frac{\nu}{4^n} - \frac{\nu}{4^{n+1}}\right|^2$$

$$= \lim_{n\to\infty} \left|\frac{3\nu}{4^{n+1}}\right|^2 = 0.$$

Hence K is an AR map on $\mho$.

Definition 2.7 A point $\nu \in \mho$ is called CFP of L and K if

$$L\nu = K\nu = \nu.$$

Definition 2.8 Let $\mho \neq \phi$ be a set with $s \geq 1$. Strong b-metric on $\mho$ is a function $\Phi\colon \mho\times\mho \to \mathbb{R}^+$ satisfying the following axioms for all $\nu, \rho, \varpi \in \mho$.

(Sb_1) $\Phi(\nu, \rho) = 0$ iff $\nu = \rho$ for all $\nu, \rho \in \mho$;

(Sb_2) $\Phi(\nu, \rho) = \Phi(\rho, \nu)$ for all $\nu, \rho \in \mho$;

(Sb_3) $\Phi(\nu, \rho) \leq \Phi(\nu, \varpi)+s\Phi(\varpi, \rho)$ for all $\nu, \rho, \varpi \in \mho$.

Then $(\mho, \Phi)$ is called a strong bMS.

Let us consider the class $\mathscr{F}$ of all functions $F\colon [0, \infty)\times[0, \infty) \to [0, \infty)$ satisfying $F(0, 0) = 0$ and F is continuous at $(0, 0)$.

Definition 2.9 [13] In a metric space $(\mho, \Phi)$, mappings $K, L\colon \mho \to \mho$ are called *Reich-Proinov-Górnicki* (RPG) type mappings if there exist $\eta \in [0, 1)$ s.t.

$$\Phi(K\nu, L\rho) \leq \eta\Phi(\nu, \rho)+F(\Phi(\nu, K\nu), \Phi(\rho, K\rho)) \tag{1}$$

for all $\nu, \rho \in \mho$ and some $F \in \mathscr{F}$.

Definition 2.10 [13] In a metric space $(\mho, \Phi)$, mappings $K, L\colon \mho \to \mho$ are called *Hardy-Rogers-Proinov-Górnicki* $(HRPG)$ type mappings if there exist $\eta, \theta, \vartheta \in [0, 1)$ with $\eta+\theta+\vartheta < 1$ s.t.

$$\Phi(K\nu, L\rho) \le \eta\Phi(\nu, \rho) + \theta\Phi(\nu, L\rho) + \vartheta\Phi(\rho, K\nu) + F(\Phi(\nu, K\nu), \Phi(\rho, K\rho)) \tag{2}$$

for all $\nu, \rho \in \mho$ and some $F \in \mathscr{F}$.

Definition 2.11 [13] In a metric space $(\mho, \Phi)$, mappings K, L: $\mho \to \mho$ are said to be *Ćirić-Proinov-Górnicki* (*CPG*) type mappings if there exist $\mu \in [0, 1)$ such that

$$\Phi(K\nu, L\rho) \le \mu \max\{\Phi(\nu, \rho), \Phi(\nu, L\rho), \Phi(\rho, K\nu)\} + F(\Phi(\nu, K\nu), \Phi(\rho, K\rho)) \tag{3}$$

for all $\nu, \rho \in \mho$ and some $F \in \mathscr{F}$.

Definition 2.12 Let ξ be a self-mapping on a metric space $(\mho, \Phi')$, and denote by $O_\nu(\xi) = \{\xi^n\nu: n = 1, 2, 3, \ldots\}$, the orbit of ξ at the point $\nu \in \mho$. The map ξ is called Orbitally Continuous (*OC*) at ν if, for every sequence $\{\rho_n\} \subset O_\nu(\xi)$ converging to some $\varpi \in \mho$, the sequence $\{\xi\rho_n\}$ converges to $\xi\varpi$. When ξ is *OC* at every point in $\mho$, it is called *OC*.

It is known that every continuous self-mapping on a metric space is *OC*; however, the converse does not necessarily hold, as there exist *OC* mappings that are not continuous (see [23], Examples 4 and 5).

Definition 2.13 [15] Self-mappings L and K of a metric space $(\mho, \Phi)$ are compatible iff

$$\lim_{n\to\infty} \Phi(LK\nu_n, KL\nu_n) = 0 \quad \text{whenever} \quad \{\nu_n\} \text{ is a sequence in } \mho \text{ such that}$$

$$\lim_{n\to\infty} L\nu_n = \lim_{n\to\infty} K\nu_n = t$$

for some $t \in \mho$. Thus, if $\Phi(LK\nu, KL\nu) \to 0$ as $\Phi(L\nu, K\nu) \to 0$, L and K are compatible.

Definition 2.14 [24] Let $(\mho, \Phi)$ be a metric space and K, L: $\mho \to \mho$ be two mappings. *K is AR with respect to L* at $\nu_0 \in \mho$, if there exist a sequence $\{\nu_n\}$ in $\mho$ such that $L\nu_{n+1} = K\nu_n$, $n = 1, 2, 3, \ldots$, and $\lim_{n\to\infty} \Phi(L\nu_{n+1}, L\nu_{n+2}) = 0$.

Bisht and Singh [25] have studied *CFP* of two self-mappings:

Theorem 2.15 ([25], Theorem 2.1). Let $(\mho, \Phi)$ be a complete metric space and K, L: $\mho \to \mho$ be a pair of mappings. Assume that K is *AR* w.r.t. L and satisfy the following

$$\Phi(K\nu, K\rho) \le \eta\Phi(L\nu, L\rho) + K\{\Phi(K\nu, L\nu) + \Phi(K\rho, L\rho)\} \tag{4}$$

for all $\nu, \rho \in \mho$, for some $\eta \in [0, 1)$ and for some $k \ge 0$. Then K and L have a unique *CFP* in $\mho$, given that K and L are compatible and *OC*.

Definition 2.16 In a metric space $(\mho, \Phi)$, mappings K, L: $\mho \to \mho$ are said to be contraction mappings if there exist $\eta \in [0, 1)$ such that

$$\Phi(K\nu, L\rho) \le \eta\Phi(\nu, \rho) \tag{5}$$

for all $\nu, \rho \in \mho$.

Pant and Pant [26] gave the idea of k-continuity as follows:

Definition 2.17 A self-mapping ξ of a metric space $\mho$ is said to be k-Continuous (kC), $k = 1, 2, 3, \ldots$, if $\xi^k\nu_n \to \xi t$ while $\{\nu_n\}$ is a sequence in $\mho$ such that $\xi^{k-1}\nu_n \to t$.

Example 2.18 Let $\mho = [0, 2]$ provided with the usual metric and let $\xi \colon \mho \to \mho$ be established by

$$\xi(\nu) = \begin{cases} 1, & 0 \leq \nu \leq 1, \\ 0, & \nu > 1. \end{cases}$$

Then $\xi \nu_n \to t \Rightarrow \xi^2 \nu_n \to t$ since $\xi \nu_n \to t$ implies $t = 0$ or $t = 1$ and $\xi^2 \nu_n = 1$ for all n; that is, $\xi^2 \nu_n \to 1 = \xi t$. Hence ξ is 2-continuous but discontinuous at $\nu = 1$.

Example 2.19 Let $\mho = [0, 4]$ endowed with the usual metric. Define $\xi \colon \mho \to \mho$ by

$$\xi(\nu) = \begin{cases} 1, & 0 \leq \nu \leq 1, \\ 0, & 1 < \nu \leq 3, \\ \dfrac{\nu}{3}, & 3 < \nu \leq 4. \end{cases}$$

Then $\xi^2 \nu_n \to t \Rightarrow \xi^3 \nu_n \to \xi t$ since $\xi^2 \nu_n \to t$ implies $t = 0$ or $t = 1$ and $\xi^3 \nu_n = 1 = \xi t$ for each n. Hence ξ is 3-continuous. However, $\xi \nu_n \to t$ does not imply $\xi^2 \nu_n \to \xi t$, that is, ξ is not 2-continuous.

The cited examples illustrate that the continuity of ξ^k and the k-continuity of ξ are separate requirements when $k > 1$. It is straightforward to observe that 1-continuity coincides with ordinary continuity, and

$$\text{continuity} \Rightarrow \text{2-continuity} \Rightarrow \text{3-continuity} \Rightarrow \cdots;$$

though the reverse implication does not generally hold.

3. Common fixed point results

Here is our main result. It extends Corollary 2.4 of Bisht et al. [13] in the context of a b-metric space.

Theorem 3.1 Let $(\mho, \Phi)$ be a complete *bMS* and $K, L \colon \mho \to \mho$ be *AR* mappings satisfying (2) for all $x, \rho \in \mho$. Then K and L have a unique *CFP* where K and L are *kC* for some $k \geq 1$ or *OC*.

Proof. First we show that $\Phi(K^n \nu, L^n \nu) \to 0$ as $n \to \infty$ for all $\nu \in \mho$. If $L = K$, then there is nothing to prove. So suppose that $L \neq K$. Then using (2), we obtain

$$\begin{aligned} \Phi(K^n\nu, L^n\nu) &= \Phi(KK^{n-1}\nu, LL^{n-1}\nu) \\ &\leq \eta\Phi(K^{n-1}\nu, L^{n-1}\nu) + \theta\Phi(K^{n-1}\nu, L^n\nu) + \vartheta\Phi(L^{n-1}\nu, K^n\nu) + F(\Phi(K^{n-1}\nu, K^n\nu), \Phi(L^{n-1}\nu, L^n\nu)) \\ &\leq s\eta\Phi(K^{n-1}\nu, K^n\nu) + s^2\eta\Phi(K^n\nu, L^n\nu) + s^2\eta\Phi(L^n\nu, L^{n-1}\nu) + s\theta\Phi(K^{n-1}\nu, K^n\nu) + s\theta\Phi(K^n\nu, L^n\nu) \\ &\quad + s\vartheta\Phi(L^{n-1}\nu, L^n\nu) + s\vartheta\Phi(L^n\nu, K^n\nu) + F(\Phi(K^{n-1}\nu, K^n\nu), \Phi(L^{n-1}\nu, L^n\nu)) \end{aligned}$$

$$(1-s^2\eta-s\theta-s\vartheta)\Phi(K^n\nu,\, L^n\nu)\leq s\eta\Phi(K^{n-1}\nu,\, K^n\nu)+s^2\eta\Phi(L^n\nu,\, L^{n-1}\nu)+s\theta\Phi(K^{n-1}\nu,\, K^n\nu)$$

$$+s\vartheta\Phi(L^{n-1}\nu,\, L^n\nu)+F(\Phi(K^{n-1}\nu,\, K^n\nu),\, \Phi(L^{n-1}\nu,\, L^n\nu))$$

$$\Phi(K^n\nu,\, L^n\nu)\leq\frac{\eta s}{\lambda}\Phi(K^{n-1}\nu,\, K^n\nu)+\frac{as^2}{\lambda}\Phi(L^n\nu,\, L^{n-1}\nu)+\frac{s\theta}{\lambda}\Phi(K^{n-1}\nu,\, K^n\nu)+\frac{s\vartheta}{\lambda}\Phi(L^{n-1}\nu,\, L^n\nu)$$

$$+\frac{1}{\lambda}F(\Phi(K^{n-1}\nu,\, K^n\nu),\, \Phi(L^{n-1}\nu,\, L^n\nu))$$

where $1-s^2\eta-s(\theta+\vartheta)=\lambda$.

This follows from the asymptotic regularity of L and K, along with the characteristics of the function F,

$$\Phi(K^n\nu,\, L^n\nu)\to 0 \quad \text{as} \quad n\to\infty \tag{6}$$

Now, let $\nu_0\in\mho$ be arbitrary. Consider the sequence $\nu_n=K^n\nu_0$ for $n=0,\, 1,\, 2,\, \ldots$. Then for $n,\, m\in N$ and $m>n$, we have

$$\Phi(\nu_n,\, \nu_m)=\Phi(K^n\nu_0,\, K^m\nu_0)$$

$$\leq s\Phi(K^n\nu_0,\, L^n\nu_0)+s\Phi(L^n\nu_0,\, K^m\nu_0)$$

$$\leq s\Phi(K^n\nu_0,\, L^n\nu_0)+s\Phi(K(K^{m-1}\nu_0),\, L(L^{n-1}\nu_0))$$

$$\leq s\Phi(K^n\nu_0,\, L^n\nu_0)+s\eta\Phi(K^{m-1}\nu_0,\, L^{n-1}\nu_0)+s\theta\Phi(K^{m-1}\nu_0,\, L^n\nu_0)+s\vartheta\Phi(L^{n-1}\nu_0,\, K^m\nu_0)$$

$$+sF(\Phi(K^{m-1}\nu_0,\, K^m\nu_0),\, \Phi(L^{n-1}\nu_0,\, L^n\nu_0))$$

$$\leq s\Phi(K^n\nu_0,\, L^n\nu_0)+s^2\eta\Phi(K^{m-1}\nu_0,\, K^m\nu_0)+s^3\eta\Phi(K^m\nu_0,\, K^n\nu_0)+s^4\eta\Phi(K^n\nu_0,\, L^n\nu_0)$$

$$+s^4\eta\Phi(L^n\nu_0,\, L^{n-1}\nu_0)+s^2\theta\Phi(K^{m-1}\nu_0,\, K^m\nu_0)+s^3\theta(K^m\nu_0,\, K^n\nu_0)+s^3\theta\Phi(K^n\nu_0,\, L^n\nu_0)$$

$$+s^2\vartheta\Phi(L^{n-1}\nu_0,\, L^n\nu_0)+s^3\vartheta\Phi(L^n\nu_0,\, K^n\nu_0)+s^3\vartheta\Phi(K^n\nu_0,\, K^m\nu_0)$$

$$+sF(\Phi(K^{m-1}\nu_0,\, K^m\nu_0),\, \Phi(L^{n-1}\nu_0,\, L^n\nu_0))$$

$$\leq s\Phi(K^n\nu_0,\, L^n\nu_0)+(s^3\eta+s^3\theta+s^3\vartheta)\Phi(K^n\nu_0,\, K^m\nu_0)+s^2\eta\Phi(K^{m-1}\nu_0,\, K^m\nu_0)+s^4\eta\Phi(K^n\nu_0,\, L^n\nu_0)$$

$$+s^4\eta\Phi(L^n v_0, L^{n-1}v_0)+s^2\theta\Phi(K^{m-1}v_0, K^m v_0)+s^3\theta\Phi(K^n v_0, L^n v_0)+s^2\vartheta\Phi(L^{n-1}v_0, L^n v_0)$$

$$+s^3\vartheta\Phi(L^n v_0, K^n v_0)+sF(\Phi(K^{m-1}v_0, K^m v_0),\ \Phi(L^{n-1}v_0, L^n v_0))$$

$$\leq s\Phi(K^n v_0, L^n v_0)+(s^3\eta+s^3\theta+s^3\vartheta)\Phi(v_n, v_m)+s^2\eta\Phi(K^{m-1}v_0, K^m v_0)+s^4\eta\Phi(K^n v_0, L^n v_0)$$

$$+s^4\eta\Phi(L^n v_0, L^{n-1}x_0)+s^2\theta\Phi(K^{m-1}x_0, K^m x_0)+s^3\theta\Phi(K^n v_0, L^n v_0)+s^2\vartheta\Phi(L^{n-1}v_0, L^n v_0)$$

$$+s^3\vartheta\Phi(L^n v_0, K^n v_0)+sF(\Phi(K^{m-1}v_0, K^m v_0),\ \Phi(L^{n-1}v_0, L^n v_0))$$

$$(1-s^3(\eta+\theta+\vartheta))\Phi(v_n, v_m)\leq s\Phi(K^n v_0, L^n v_0)+s^2\eta\Phi(K^{m-1}v_0, K^m v_0)+s^4\eta\Phi(K^n v_0, L^n v_0)+s^4\eta\Phi(L^n v_0, L^{n-1}v_0)$$

$$+s^2\theta\Phi(K^{m-1}v_0, K^m v_0)+s^3\theta\Phi(K^n v_0, L^n v_0)+s^2\vartheta\Phi(L^{n-1}v_0, L^n v_0)$$

$$+s^3\vartheta\Phi(L^n v_0, K^n v_0)+sF(\Phi(K^{m-1}v_0, K^m v_0),\ \Phi(L^{n-1}v_0, L^n v_0))$$

$$\Phi(v_n, v_m)\leq\frac{s}{1-s^3(\alpha)}s\Phi(K^n v_0, L^n v_0)+\frac{s^2\eta}{1-s^3(\alpha)}\Phi(K^{m-1}v_0, K^m v_0)+\frac{s^4\eta}{1-s^3(\alpha)}\Phi(K^n v_0, L^n v_0)$$

$$+\frac{s^4\eta}{1-s^3(\alpha)}\Phi(L^n v_0, L^{n-1}v_0)+\frac{s^2\theta}{1-s^3(\alpha)}\Phi(K^{m-1}v_0, K^m v_0)+\frac{s^3\theta}{1-s^3(\alpha)}\Phi(K^n v_0, L^n v_0)$$

$$+\frac{s^2\vartheta}{1-s^3(\alpha)}\Phi(L^{n-1}v_0, L^n v_0)+\frac{s^3\vartheta}{1-s^3(\alpha)}\Phi(L^n v_0, K^n v_0)$$

$$+\frac{s}{1-s^3(\alpha)}F(\Phi(K^{m-1}v_0, K^m v_0),\ \Phi(L^{n-1}v_0, L^n v_0))$$

where $\eta+\theta+\vartheta=\alpha$.

Applying limit, asymptotic regularity of L and K and (6), we get

$$\Phi(v_n, v_m)\to 0 \quad \text{as} \quad m, n\to\infty,$$

since F is continuous at (0, 0) and $F(0, 0)=0$, which shows that $\{v_n\}$ is a b-Cauchy sequence in $\mho$. By completeness property of $\mho$, let $v_n\to h\in\mho$ as $n\to\infty$ i.e. $K^n v_0\longrightarrow h$ as $n\longrightarrow\infty$. So by,

$$\Phi(L^n v_0, h)\leq\Phi(L^n v_0, K^n v_0)+\Phi(K^n v_0, h),$$

$L^n v_0\longrightarrow h$ as $n\longrightarrow\infty$.

Suppose that L and K are kC. Since $lim_{n\to\infty}\nu_{n+1} = h$, so $lim_{n\to\infty}K\nu_n = h$. Moreover, for each $k \geq 1$, we have $lim_{n\to\infty}K^k\nu_n = h$. Since $lim_{n\to\infty}K^{k-1}\nu_n = h$ and so by k-continuity of K, we get $lim_{n\to\infty}K^k\nu_n = Kh$. Thus $Kh = h$ i.e $h \in \mho$ is a fixed point of K. In a similar way, by k-continuity of L, we have $Lu = h$. Hence h is a CFP of L and K.

Next consider that L and K are OC: we have $\lim_{n\to\infty} \nu_{n+1} = \lim_{n\to\infty} \nu_n = h$. By orbital continuity of K, $\lim_{n\to\infty} \nu_n = h$ implies $\lim_{n\to\infty} K\nu_n = Kh$, i.e., $h \in \mho$ is a fixed point of K. In the same way, by orbital continuity of L, we have $Lh = h$. That is, $h \in \mho$ is a CFP of L and K.

For uniqueness of CFP, let $h \neq g$ where $g \in \mho$ is another fixed point of L and K.

$$\begin{aligned}
\Phi(h,\, g) &= \Phi(Kh,\, Kg)\\
&\leq \eta\Phi(h,\, g) + \theta\Phi(h,\, Kg) + \vartheta\Phi(g,\, Kh) + F(\Phi(h,\, h) + \Phi(g,\, g))\\
&\leq \eta\Phi(h,\, g) + \theta\Phi(h,\, g) + \vartheta\Phi(g,\, h) + F(\Phi(h,\, h) + \Phi(g,\, g))
\end{aligned}$$

$$(1 - \eta - \theta - \vartheta)\Phi(h,\, g) \leq F(0,\, 0)$$

$$(1 - \eta - \theta - \vartheta)\Phi(h,\, g) \leq 0$$

which yields $\Phi(h,\, g) = 0$, a contradiction to our supposition that $h \neq g$.

Hence the CFP of L and K is unique. □

We provide an example in support of Theorem 3.1.

Example 3.2 Let $\mho = \mathbb{R}$. Define Φ: $\mho \times \mho \to \mathbb{R}$ by $\Phi(\nu,\, \rho) = |\nu - \rho|^2$. Then $(\mho,\, \Phi)$ is a complete bMS with constant $s = 2$, since for all $\nu,\, \rho,\, \varpi \in \mho$,

$$\Phi(\nu,\, \varpi) = |\nu - \varpi|^2 \leq 2(|\nu - \rho|^2 + |\rho - \varpi|^2) = s(\Phi(\nu,\, \rho) + \Phi(\rho,\, \varpi)).$$

Consider the mappings K, L: $\mho \to \mho$ defined by

$$K(\nu) = \frac{\nu}{4}, \quad L(\nu) = \frac{\nu}{8}.$$

Both L and K are AR because for any $\nu \in \mho$,

$$\Phi(K^n\nu,\, K^{n+1}\nu) = \left|\frac{\nu}{4^n} - \frac{\nu}{4^{n+1}}\right|^2 = \left|\frac{3\nu}{4^{n+1}}\right|^2 \to 0,$$

and,

$$\Phi(L^n\nu,\, L^{n+1}\nu) = \left|\frac{\nu}{8^n} - \frac{\nu}{8^{n+1}}\right|^2 = \left|\frac{7\nu}{8^{n+1}}\right|^2 \to 0,$$

as $n \to \infty$. Next, choose constants $\eta = \theta = \vartheta = \frac{1}{4}$, so that $\eta + \theta + \vartheta = \frac{3}{4} < 1$, and define

$$F(\nu, \rho) = \frac{1}{8}(\nu + \rho).$$

For any $\nu, \rho \in \mho$, the Hardy-Rogers-Proinov-Górnicki type contractive condition (2), namely,

$$\Phi(K\nu, L\rho) \leq \eta\Phi(\nu, \rho) + \theta\Phi(\nu, L\rho) + \vartheta\Phi(\rho, K\nu) + F(\Phi(\nu, K\nu), \Phi(\rho, K\rho))$$

holds. Explicitly,

$$\left|\frac{\nu}{4} - \frac{\rho}{8}\right|^2 \leq \frac{1}{4}|\nu - \rho|^2 + \frac{1}{4}\left|\nu - \frac{\rho}{8}\right|^2 + \frac{1}{4}\left|\rho - \frac{\nu}{4}\right|^2 + \frac{1}{8}\left(\left|\nu - \frac{\nu}{4}\right|^2 + \left|\rho - \frac{\rho}{4}\right|^2\right).$$

Since L and K are linear and kC mappings on $\mathbb{R}$, they are kC with $k = 2$. Therefore, by Theorem 3.1, L and K have a unique CFP in $\mho$. Solution of

$$\nu = K(\nu) = \frac{\nu}{4} \quad \text{and} \quad \nu = L(\nu) = \frac{\nu}{8}$$

yields the unique CFP of ν is $\{0\}$. □

By taking $\theta = \vartheta = 0$ in (2), we get (1) and hence the following:

Corollary 3.3 Let $(\mho, \Phi)$ be a complete bMS and K, L: $\mho \to \mho$ be two AR mappings satisfying (1). Then K and L have a unique CFP given K and L are either OC or kC for some $k \geq 1$.

Also taking $\theta = \vartheta = 0$ and $F = 0$ in (2), we get (5) and hence the following:

Corollary 3.4 Let $(\mho, \Phi)$ be a complete bMS and K, L: $\mho \to \mho$ be two AR mappings satisfying (5). Then K and L have a unique CFP given K and L are either OC or kC for some $k \geq 1$.

Theorem 3.5 Let $(\mho, \Phi')$ be a complete b-metric space, and let K, L: $\mho \to \mho$ be two AR mappings that satisfy condition (3). If both K and L are k-continuous for some $k \geq 1$, then they possess a unique CFP.

Proof. First we show that $\Phi(K^n\nu, L^n\nu) \to 0$ as $n \longrightarrow \infty$ for all $\nu \in \mho$. Using (3), we have

$$\Phi(K^n\nu, L^n\nu) = \Phi(K(K^{n-1}\nu), L(L^{n-1}\nu))$$

$$\leq \mu \max\{\Phi(K^{n-1}\nu, L^{n-1}\nu), \Phi(K^{n-1}\nu, S^n\nu), \Phi(L^{n-1}\nu, K^n\nu)\}$$

$$+ F(\Phi(K^{n-1}\nu, K^n\nu), \Phi(L^{n-1}\nu, L^n\nu))$$

$$\Phi(K^n\nu, L^n\nu) = \mu\Gamma n + F(\Phi(K^{n-1}\nu, K^n\nu), \Phi(L^{n-1}\nu, L^n\nu)) \tag{7}$$

where $\Gamma n = \max\{\Phi(K^{n-1}\nu, L^{n-1}\nu), \Phi(K^{n-1}\nu, L^n\nu), \Phi(L^{n-1}\nu, K^n\nu)\}$.

Now if $\Gamma n = \Phi(K^{n-1}\nu, L^{n-1}\nu)$, then using S_3, we get

$$\Gamma n \leq s\Phi(K^{n-1}\nu,\, K^n\nu) + s^2\Phi(K^n\nu,\, L^n\nu) + s^2\Phi(S^n\nu,\, S^{n-1}\nu)$$

and hence from (7), we have

$$\Phi(K^n\nu,\, L^n\nu) \leq \frac{s\mu}{1-s^2\mu}\,\Phi(K^{n-1}\nu,\, K^n\nu) + \frac{s^2\mu}{1-s^2\mu}\,\Phi(L^{n-1}\nu,\, L^n\nu) \tag{8}$$

$$+\frac{1}{1-s^2\mu}\,F\big(\Phi(K^{n-1}\nu,\, K^n\nu),\, \Phi(L^{n-1}\nu,\, L^n\nu)\big) \tag{9}$$

Next if $\Gamma n = \Phi(K^{n-1}\nu,\, L^n\nu)$ or $\Gamma n = \Phi(L^{n-1}\nu,\, K^n\nu)$, then in the same way, we get the following inequalities:

$$\Phi(K^n\nu,\, L^n\nu) \leq \frac{s\mu}{1-s\mu}\,\Phi(K^{n-1}\nu,\, K^n\nu) + \frac{1}{1-s\mu}F\big(\Phi(K^{n-1}\nu,\, K^n\nu),\, \Phi(L^{n-1}\nu,\, L^n\nu)\big) \tag{10}$$

$$\Phi(K^n\nu,\, L^n\nu) \leq \frac{s\mu}{1-s\mu}\,\Phi(K^{n-1}\nu,\, K^n\nu) + \frac{1}{1-s\mu}F\big(\Phi(K^{n-1}\nu,\, K^n\nu),\, \Phi(L^{n-1}\nu,\, L^n\nu)\big) \tag{11}$$

Combining (8), (9) and (10) we obtain,

$$\Phi(K^n\nu,\, L^n\nu) \leq (\frac{s\mu}{1-s^2\mu} + \frac{s\mu}{1-s\mu})\Phi(K^{n-1}\nu,\, K^n\nu) + (\frac{s^2\mu}{1-s^2\mu} + \frac{s\mu}{1-s\mu})\Phi(L^{n-1},\, L^n\nu)$$

$$+(\frac{1}{1-s^2\mu} + \frac{2}{1-s\mu})F(\Phi(K^{n-1}\nu,\, K^n\nu),\, \Phi(L^{n-1}\nu,\, L^n\nu))$$

Now by using the properties of the function F and asymptotic regularity of L and K, we get, $\Phi(K^n\nu,\, L^n\nu) \longrightarrow 0$ as $n \longrightarrow \infty$ for all $\nu \in \mho$.

Next let $\nu_0 \in \mho$ be arbitrary and consider the sequence $\nu_n = K^n\nu_0$ for all $n = 0,\, 1,\, 2....$ Then for $n,\, m \in N$ and $m > n$ we have,

$$\begin{aligned}
\Phi(\nu_n,\, \nu_m) &= \Phi(K^n\nu_0,\, K^m\nu_0)\\
&\leq s\Phi(K^n\nu_0,\, L^n\nu_0) + s\Phi(L^n\nu_0,\, K^m\nu_0)\\
&= s\Phi(K^n\nu_0,\, L^n\nu_0) + s\Phi(K^m\nu_0,\, L^n\nu_0)\\
&\leq s\Phi(K^n\nu_0,\, L^n\nu_0) + s\Phi(K(K^{m-1}\nu_0),\, L(L^{n-1}\nu_0))\\
&\leq s\Phi(K^n\nu_0,\, L^n\nu_0) + s\mu max\{\Phi(K^{m-1}\nu_0,\, L^{n-1}\nu_0),\, \Phi(K^{m-1}\nu_0,\, L^n\nu_0),\, \Phi(L^{n-1}\nu_0,\, K^m\nu_0)\}
\end{aligned}$$

$$+sF(\Phi(K^{m-1}v_0, K^m v_0), \Phi(L^{n-1}v_0, L^n v_0))$$

$$\leq s\Phi(K^n v_0, L^n v_0) + s\mu max\{s\Phi(K^{m-1}v_0, K^m v_0) + s^2\Phi(K^m v_0, K^n v_0) + s^3\Phi(K^n v_0, L^n v_0)$$

$$+s^3\Phi(L^n v_0, L^{n-1}v_0), s\Phi(K^{m-1}v_0, K^m v_0) + s^2\Phi(K^m v_0, K^n v_0) + s^2\Phi(K^n v_0, L^n v_0), s\Phi(L^{n-1}v_0, L^n v_0)$$

$$+s^2\Phi(L^n v_0, K^n v_0) + s^2\Phi(K^n v_0, K^m v_0)\} + sF(\Phi(K^{m-1}v_0, K^m v_0), \Phi(L^{n-1}v_0, L^n v_0))$$

By taking limit and using AR property of L and K, we have

$$\Phi(v_n, v_m) \leq s\Phi(K^n v_0, L^n v_0) + s\mu max\{s^2\Phi(K^m v_0, K^n v_0)\}$$

$$\leq s\Phi(K^n v_0, L^n v_0) + s^3\mu\Phi(v_n, v_m)$$

$$(1-s^3\mu)\Phi(v_n, v_m) \leq s\Phi(K^n v_0, L^n v_0)$$

$$\Phi(v_n, v_m) \leq \frac{s}{1-s^3\mu}\Phi(K^n v_0, L^n v_0)$$

By taking limit and using (6) in the above inequality, we get

$$\Phi(v_n, v_m) \to 0 \quad \text{as} \quad m, n \to \infty,$$

since F is continuous at (0, 0) and $F(0, 0) = 0$, which shows that $\{v_n\}$ is a b-Cauchy sequence in $\mho$. As $\mho$ is complete, let $v_n \to h \in \mho$ as $n \to \infty$ i.e. $K^n v_0 \longrightarrow h$ as $n \longrightarrow \infty$. Again since, $\Phi(L^n v_0, h) \leq \Phi(L^n v_0, K^n v_0) + \Phi(K^n v_0, h)$, therefore $L^n v_0 \longrightarrow h$ as $n \longrightarrow \infty$.

Suppose that L and K are kC. Since $lim_{n\to\infty} v_{n+1} = h$, so $lim_{n\to\infty} K v_n = h$. Moreover, for each $k \geq 1$, we have $lim_{n\to\infty} K^k v_n = h$. Since $lim_{n\to\infty} K^{k-1} v_n = h$ and so by k-continuity of K, we get $lim_{n\to\infty} K^k v_n = Kh$, thus $Kh = h$ i.e. $h \in \mho$ is a fixed point of K. In a similar way, by k-continuity of L, we have $Lu = h$.

Hence h is a CFP of L and K.

To show uniqueness of CFP, let $h \neq g$ where $g \in \mho$ is another fixed point of L and K.

$$\Phi(h, g) = \Phi(Kh, Kg)$$

$$\leq \mu \max\{\Phi(h, g), \Phi(h, Kg), \Phi(Ku, g)\} + F(\Phi(h, Kh) + \Phi(g, Kg))$$

$$\leq \mu \max\{\Phi(h, g), \Phi(h, g), \Phi(h, g)\} + F(\Phi(h, h) + \Phi(g, g))$$

$$(1-\mu)\Phi(h, g) \leq 0$$

$$\Phi(h, g) \leq 0$$

a contradiction to $h \neq g$. Hence the CFP of L and K is unique. $\square$

The following example demonstrates Theorem 3.5.

Example 3.6 Let $\mho = \mathbb{R}$ and define the b-metric Φ: $\mho \times \mho \to \mathbb{R}$ by

$$\Phi(\nu, \rho) = |\nu - \rho|^2.$$

Consider the mappings K, L: $\mho \to \mho$ defined by

$$K(\nu) = \frac{\nu}{3}, \quad L(\nu) = \frac{\nu}{6}.$$

Both K and L are AR on $\mho$.

Choose $\mu = \frac{1}{2}$ and define

$$F(h, g) = \frac{1}{4}(h + g).$$

For any $\nu, \rho \in \mho$, we have

$$\Phi(K\nu, L\rho) = \left|\frac{\nu}{3} - \frac{\rho}{6}\right|^2 = \left|\frac{2\nu - \rho}{6}\right|^2 = \frac{|2\nu - \rho|^2}{36}.$$

Also,

$$\max\{\Phi(\nu, \rho), \Phi(\nu, L\rho), \Phi(\rho, K\nu)\} = \max\left\{|\nu - \rho|^2, \left|\nu - \frac{\rho}{6}\right|^2, \left|\rho - \frac{\nu}{3}\right|^2\right\}.$$

Since

$$\Phi(\nu, K\nu) = \left|\nu - \frac{\nu}{3}\right|^2 = \left|\frac{2\nu}{3}\right|^2 = \frac{4}{9}\nu^2, \quad \Phi(\rho, K\rho) = \frac{4}{9}\rho^2.$$

Hence the Čirić-Proinov-Górnicki contractive condition

$$\Phi(K\nu, L\rho) \leq \mu \max\{\Phi(\nu, \rho), \Phi(\nu, L\rho), \Phi(\rho, K\nu)\} + F(\Phi(\nu, K\nu), \Phi(\rho, K\rho))$$

$$\frac{|2\nu - \rho|^2}{36} \leq \frac{|\nu - \rho|^2}{2} + \frac{\nu^2 + \rho^2}{9}$$

holds for all $\nu, \rho \in \mho$.

Since K and L are linear and continuous, they are k-continuous with $k = 2$. Therefore, by Theorem 3.5, K and L have a unique CFP in $\mho$. Solving

$$\nu = K(\nu) = \frac{\nu}{3} \quad \text{and} \quad \nu = L(\nu) = \frac{\nu}{6},$$

we get $\nu = 0$ as the unique CFP of K and L.

Now we present a Jungck type CFP theorem [15] for two mappings.

Theorem 3.7 Let $(\mho, \Phi)$ be a complete bMS and $K, L: \mho \to \mho$ be a pair of mappings such that K is AR mapping w.r.t L and satisfy:

$$\Phi(K\nu, K\rho) \leq \mu \max\left\{\Phi(L\nu, L\rho), \Phi(K\nu, L\rho), \Phi(L\nu, K\rho)\right\} + F\left(\Phi(K\nu, L\nu), \Phi(K\rho, L\rho)\right) \tag{12}$$

for all $\nu, \rho \in \mho$, for some $\mu \in [0, 1]$ and $F \in \mathscr{F}$. If both K and L are OC and compatible, then they possess a unique CFP.

Proof. By (11), we have unique fixed point. We will only prove the existence of CFP of K and L.

As K is AR with respect to L at $x_0 \in \mho$, so there exist a sequence $\{x_n\}$ in $\mho$ such that $Kx_n = Lx_{n+1} = \rho_n$ for all $n = 1, 2, 3, ...$ and $\Phi(Lx_{n+1}, Lx_{n+2}) \to 0$ as $n \to \infty$ i.e. $\Phi(\rho_n, \rho_{n+1}) \to 0$ as $n \to \infty$.

First, we show that $\{\rho_n\}$ is a Cauchy sequence in $\mho$. For $p = 1, 2, 3, ...$, we have

$$\Phi(\rho_{n+p}, \rho_n) \leq \Phi(\rho_{n+p}, \rho_{n+p+1}) + \Phi(\rho_{n+p+1}, \rho_{n+1}) + \Phi(\rho_{n+1}, \rho_n)$$

$$\Phi(\rho_{n+p}, \rho_n) \leq \Phi(\rho_{n+p}, \rho_{n+p+1}) + \Phi(\rho_{n+1}, \rho_n) + \mu Z_{n,p} + F\left(\Phi(\rho_{n+p+1}, \rho_{n+p}), \Phi(\rho_{n+1}, \rho_n)\right) \tag{13}$$

where $Z_{n,p} = \max\{\Phi(\rho_{n+p}, \rho_n), \Phi(\rho_{n+p+1}, \rho_n), \Phi(\rho_{n+p}, \rho_{n+1})\}$.

Case I If $Z_{n,p} = \Phi(\rho_{n+p}, \rho_n)$, then (12) becomes

$$\Phi(\rho_{n+p}, \rho_n) \leq \Phi(\rho_{n+p}, \rho_{n+p+1}) + \Phi(\rho_{n+1}, \rho_n) + \mu\, \Phi(\rho_{n+p}, \rho_n) + F\left(\Phi(\rho_{n+p+1}, \rho_{n+p}), \Phi(\rho_{n+1}, \rho_n)\right)$$

$$(1-\mu)\, \Phi(\rho_{n+p}, \rho_n) \leq \Phi(\rho_{n+p}, \rho_{n+p+1}) + \Phi(\rho_{n+1}, \rho_n) + F\left(\Phi(\rho_{n+p+1}, \rho_{n+p}), \Phi(\rho_{n+1}, \rho_n)\right) \tag{14}$$

Case II If $Z_{n,p} = \Phi(\rho_{n+p+1}, \rho_n)$, then by S_3 of bMS,

$$\Phi(\rho_{n+p+1}, \rho_n) \leq s\Phi(\rho_{n+p+1}, \rho_{n+p}) + s\Phi(\rho_{n+p}, \rho_n).$$

So (12) gives,

$$\Phi(\rho_{n+p}, \rho_n) \leq \Phi(\rho_{n+p}, \rho_{n+p+1}) + \Phi(\rho_{n+1}, \rho_n) + s\mu(\Phi(\rho_{n+p+1}, \rho_{n+p}) + \Phi(\rho_{n+p}, \rho_n))$$
$$+ F(\Phi(\rho_{n+p+1}, \rho_{n+p}), \Phi(\rho_{n+1}, \rho_n))$$

$$(1-s\mu)\,\Phi(\rho_{n+p},\,\rho_n) \leq (1+s\mu)\,\Phi(\rho_{n+p},\,\rho_{n+p+1}) + \Phi(\rho_{n+1},\,\rho_n) + F\big(\Phi(\rho_{n+p+1},\,\rho_{n+p}),\,\Phi(\rho_{n+1},\,\rho_n)\big) \tag{15}$$

Case III If $Z_{n,\,p} = \Phi(\rho_{n+p},\,\rho_{n+1})$, then by S_3 of bMS,

$$\Phi(\rho_{n+p},\,\rho_{n+1}) \leq s\Phi(\rho_{n+p},\,\rho_n) + s\Phi(\rho_n,\,\rho_{n+1}).$$

We get by (12),

$$\Phi(\rho_{n+p},\,\rho_n) \leq \Phi(\rho_{n+p},\,\rho_{n+p+1}) + \Phi(\rho_{n+1},\,\rho_n) + s\mu(\Phi(\rho_{n+p},\,\rho_n) + \Phi(\rho_n,\,\rho_{n+1}))$$
$$+F(\Phi(\rho_{n+p+1},\,\rho_{n+p}),\,\Phi(\rho_{n+1},\,\rho_n))$$

$$(1-s\,\mu)\,\Phi(\rho_{n+p},\,\rho_n) \leq \Phi(\rho_{n+p},\,\rho_{n+p+1}) + (1+s\,\mu)\,\Phi(\rho_{n+1},\,\rho_n)$$
$$+F\big(\Phi(\rho_{n+p+1},\,\rho_{n+p}),\,\Phi(\rho_{n+1},\,\rho_n)\big) \tag{16}$$

Combining (13), (14) and (15), we have

$$(3-(1+2s)\mu)\Phi(\rho_{n+p},\,\rho_n) \leq (3+s\mu)(\Phi(\rho_{n+p},\,\rho_{n+p+1}) + \Phi(\rho_{n+1},\,\rho_n))$$
$$+F(\Phi(\rho_{n+p+1},\,\rho_{n+p}),\,\Phi(\rho_{n+1},\,\rho_n))$$

$$\Phi(\rho_{n+p},\,\rho_n) \leq \frac{(3+s\mu)}{(3-(1+2s)\mu)}(\Phi(\rho_{n+p},\,\rho_{n+p+1}) + \Phi(\rho_{n+1},\,\rho_n))$$
$$+\frac{1}{(3-(1+2s)\mu)}F(\Phi(\rho_{n+p+1},\,\rho_{n+p}),\,\Phi(\rho_{n+1},\,\rho_n))$$

Taking limit, using asymptotic regularity and property of F, we have

$$\Phi(\rho_{n+p},\,\rho_n) \to 0 \quad \text{as} \quad n \to \infty \quad \text{for} \quad p = 1,\,2,\,3,\,...,$$

which shows that $\{\rho_n\}$ is b-Cauchy sequence in $\mho$ and so,

$$\lim_{n\to\infty} \rho_n = \lim_{n\to\infty} Kx_n = \lim_{n\to\infty} Lx_{n+1} = h \in \mho.$$

Since L and K are OC, therefore

$$\lim_{n\to\infty} KK\nu_n = \lim_{n\to\infty} KL\nu_n = Ku$$

and

$$\lim_{n\to\infty} LK\nu_n = \lim_{n\to\infty} LL\nu_n = Lu.$$

So by compatibility of L and K, we have

$$\Phi(KL\nu_n, LK\nu_n) \to 0 \quad \text{as} \quad n \to \infty \quad \text{and hence} \quad Ku = Lu.$$

Again by compatibility of L and K, we have

$$K(Ku) = K(Lu) = L(Ku) = L(Lu).$$

By (11), we get

$$\Phi(Ku, KKu) \leq \mu \max\{\Phi(Lu, LKu),\ \Phi(Ku, LKu),\ \Phi(Lu, KKu)\} + F\left(\Phi(Ku, Lu),\ \Phi(KKu, LKu)\right).$$

Putting $L = K$, implies

$$\Phi(Ku, KKu) \leq \mu max\{\Phi(Ku, KKu), \Phi(Ku, KKu), \Phi(Ku, KKu)\} + F(\Phi(Ku, Ku), \Phi(KKu, KKu))$$

$$\leq \mu\Phi(Ku, KKu) + F(0, 0)$$

$$(1-\mu)\Phi(Ku, KKu) \leq 0$$

$$\implies \quad \Phi(Ku, KKu) = 0$$

Therefore, $Ku = K(Ku) = L(Ku)$ i.e. $Ku \in \mho$ is a CFP of L and K. □

As a special case of Theorem 3.7, we get:

Corollary 3.8 Let $(\mho, \Phi)$ be a complete bMS and K, L: $\mho \to \mho$ be two mappings s.t K is AR w.r.t L and satisfy (4). Then K and L have a unique CFP given that K and L are OC and compatible.

A combination of Theorems 3.1 and 3.7 provides the following important result:

Corollary 3.9 Let $(\mho, \Phi)$ be a complete bMS and K: $\mho \to \mho$ be an AR map satisfying

$$\Phi(K\nu, K\rho) \leq \mu \max\{\Phi(\nu, \rho),\ \Phi(\rho, T\nu),\ \Phi(\nu, T\rho)\} + F\big(\Phi(\nu, T\nu),\ \Phi(\rho, T\rho)\big)$$

for all ν, $\rho \in \mho$, for some $\mu \in [0, 1)$ and an $F \in \mathscr{F}$. Then K has a unique fixed point given K is OC on $\mho$.

Proof. In Theorem 3.1, $L = K$ and Theorem 3.7 with $L = I$ (Identity map on $\mho$), provide the result. $\square$

An application of Theorem 3.1 is given below.

Theorem 3.10 Let $(\mho, \Phi)$ be a complete bMS and K be a self mapping of $\mho$ satisfying (1). If K is AR at some point $\nu \in \mho$, then the sequence $\{K^n \nu\}$ converges to a unique fixed point of K.

Proof. By Theorem 3.1, K has a unique fixed point h. We prove that the sequence $\{K^n \nu\}$ converges to h.

By (1), we obtain

$$\Phi(h, K^n \nu) = \Phi(Ku, K^n \nu) = \Phi(Ku, K(K^{n-1} \nu))$$

$$\leq \eta \Phi(h, K^{n-1} \nu) + F(\Phi(h, Ku), \Phi(T^{n-1} \nu, T^n \nu))$$

$$\leq \eta (s\Phi(h, K^n \nu) + s\Phi(K^n \nu, K^{n-1} \nu)) + F(\Phi(h, Ku), \Phi(K^{n-1} \nu, K^n \nu))$$

$$(1 - s\eta)\Phi(h, K^n \nu) \leq s\Phi(K^n \nu, K^{n-1} \nu)) + F(\Phi(h, Ku), \Phi(K^{n-1} \nu, K^n \nu))$$

$$\Phi(h, K^n \nu) \leq \frac{s}{1 - s\eta} \Phi(K^n \nu, K^{n-1} \nu)) + \frac{1}{1 - s\eta} F(\Phi(h, Ku), \Phi(K^{n-1} \nu, K^n \nu))$$

Now taking limit and applying asymptotic regularity of K, we have

$$\Phi(h, K^n \nu) \to 0 \quad \text{as} \quad n \to \infty,$$

which shows that $\{K^n \nu\}$ converges to h. $\square$

4. Application

In this section, we utilize Corollary 3.4 to establish the existence and uniqueness of the solution to the Fredholm linear integral equation:

$$h(t) = f(t) + \mu \int_a^b K(t, \tau) h(\tau) \, d\tau, \tag{17}$$

where $K(t, \tau)$ is the kernel of the integral equation.

Theorem 4.1 Consider the linear integral Equation (17) with the continuous function $K(t, \tau)$, where $a \leq t$, $\tau \leq b$ and $\xi \in C[a, b]$. Let $M = \max_{a \leq t, \tau \leq b} |K(t, \tau)|$ and $m > 2$ be an arbitrary real number. If $|\mu| < \frac{1}{2mM(a-\tau)}$, then the linear integral Equation (17) has a unique solution in the interval $[a, b]$. Moreover, the solution is given by

$$h(t) = \xi(t) + \mu \lim_{n \to \infty} \int_a^b K(t, \tau) u_n(\tau) \, d\tau, \tag{18}$$

where $u_0(t) = u_0$, $u_{n-1} = \frac{1}{16} - \frac{1}{4m^2}$.

Proof. Let $\mho = C[a, b]$. Define Φ: $\mho \times \mho \to [0, +\infty)$ as

$$\Phi(h, g) = \max_{a \le t \le b} |h(t) - g(t)|^2. \tag{19}$$

We can easily check that $(\mho, \Phi)$ is a complete bMS with $s = 2$.
Define self-maps L and K on $\mho$ as

$$Lu(t) = \xi(t) + \mu \int_a^b K(t, \tau) h(\tau)\, d\tau, \quad \text{for all} \quad h \in C[a, b], \tag{20}$$

and

$$Ku(t) = \xi(t) + \mu \int_a^b K(t, \tau) g(\tau)\, d\tau, \quad \text{for all} \quad g \in C[a, b]. \tag{21}$$

Now we show that L and K are asymptotically regular and kC mappings as follows:
If $h \in C[a, b]$, then for the sequence $\{L^n h\}$, we have

$$L^{n+1}h(t) - L^n h(t) = \mu \int_a^b K(t, \tau)\big(L^n h(\tau) - L^{n-1}h(\tau)\big) d\tau.$$

Applying (18), we get

$$\Phi(L^{n+1}h, L^n h) = \max_{t \in [a, b]} |L^{n+1}h(t) - L^n h(t)|^2.$$

Using the integral inequality and continuity of K,

$$|L^{n+1}h(t) - L^n h(t)| \le |\mu| \int_a^b |K(t, \tau)||L^n h(\tau) - L^{n-1}h(\tau)| d\tau$$

$$\le |\mu| M(b-a) \max_{\tau \in [a, b]} |L^n h(\tau) - L^{n-1}h(\tau)|,$$

where

$$M = \max_{t,\, \tau \in [a, b]} |K(t, \tau)|.$$

Squaring and taking maximum over t, we obtain

$$\Phi(L^{n+1}h, L^n h) \le \big(|\mu| M(b-a)\big)^2 \Phi(L^n h, L^{n-1}h).$$

If

$$q = |\mu| M(b-a) < 1,$$

then by induction,

$$\Phi(L^{n+1}h,\, L^n h) \leq q^{2n} \Phi(Lh,\, h) \to 0, \quad \text{as } n \to \infty.$$

Hence,

$$\lim_{n\to\infty} \Phi(L^{n+1}h,\, L^n h) = 0,$$

gives that L is asymptotically regular. Similarly, K is asymptotically regular.

k-Continuity: For any $h,\, g \in C[a,\, b]$,

$$|Lh(t) - Lg(t)| = \left| \mu \int_a^b K(t,\, \tau)(h(\tau) - g(\tau))d\tau \right|$$

$$\leq |\mu| M(b-a) \max_{\tau \in [a,\, b]} |h(\tau) - g(\tau)|.$$

Squaring and taking maximum over t,

$$\Phi(Lh,\, Lg) \leq \big(|\mu| M(b-a)\big)^2 \Phi(h,\, g).$$

Thus, L is Lipschitz continuous (hence k-continuous) with Lipschitz constant $k = \big(|\mu| M(b-a)\big)^2$. Similarly, K is k-continuous with the same constant.

Combining (18), (19) and (20), we get

$$\Phi(Lh,\, Kg) = \max_{a \leq t \leq b} |Lh(t) - Kg(t)|^2$$

$$= \max_{a \leq t \leq b} \left| \mu \int_a^b K(t,\, \tau)h(\tau) - K(t,\, \tau)g(\tau)\, d\tau \right|^2$$

$$\leq \max_{a \leq t \leq b} |\mu|^2 \left| \int_a^b |K(t,\, \tau)||h(\tau) - g(\tau)|\, d\tau \right|^2$$

$$\leq |M(b-a)|^2 \Phi(h,\, g) \leq |M(b-a)|^2 \Phi(h,\, g) \leq \frac{1}{4m^2} \Phi(h,\, v).$$

Hence, K satisfies (5) on $\mho$ with the constant $\eta = \frac{1}{4m^2}$. So by Corollary 3.4, L and K have a unique CFP $h(t) \in \mho$ i.e. $h(t) = Kh(t)$ and $h(t) = Lh(t)$, which means that $h(t)$ is the solution of (17).

Now we will show that

$$\int_a^b K(t, \tau)h(\tau)\,d\tau = \lim_{n\to\infty} \int_a^b K(t, \tau)h_n(\tau)\,d\tau.$$

Indeed,

$$\begin{aligned}\limsup_{n\to\infty}\left|\int_a^b K(t, \tau)(h_n(t) - h(t))\,d\tau\right| &\leq \limsup_{n\to\infty}\int_a^b |K(t, \tau)||h_n(\tau) - h(\tau)|\,d\tau \\ &\leq M(b-a)\limsup_{n\to\infty}\max_{a\leq\tau\leq b}|h_n(\tau) - h(\tau)| \\ &= 0,\end{aligned}$$

in view of

$$\lim_{n\to\infty} \Phi(h_n(t), h(t)) = \lim_{n\to\infty}\max_{a\leq\tau\leq b}|h_n(\tau) - h(\tau)|^2 = 0,$$

It is now easy to show that

$$\lim_{n\to\infty}\int_a^b K(t, \tau)(h_n(\tau) - h(\tau))\,d\tau = 0.$$

Thus, equation (17) holds. □

5. Results in strong *b*-metric space

In this section, we obtain CFP results for both single valued and multi-valued AR mappings in a strong bMS.

In the following result, we extend Corollary 3.3 im the setting of a strong bMS.

Theorem 5.1 Let $(\mho, \Phi)$ be a complete sb-metric space and K, L: $\mho \to \mho$ be two AR mappings satisfying (1) for all $\nu, \rho \in \mho$. Then K and L have a CFP given that K and L are kC for some $k \geq 1$ or OC.

Proof. As before, we show that $\Phi(K^n\nu, L^n\nu) \to 0$ as $n \to \infty$ for all $\nu \in \mho$. Now if $L = K$, then there is nothing to prove. So suppose that $L \neq K$. Then using (1),

$$\begin{aligned}\Phi(K^n\nu, L^n\nu) &= \Phi(KK^{n-1}\nu, LL^{n-1}\nu) \\ &\leq \eta\Phi(K^{n-1}\nu, L^{n-1}\nu) + F(\Phi(K^{n-1}\nu, K^n\nu), \Phi(L^{n-1}\nu, L^n\nu))\end{aligned}$$

$$\leq \eta\Phi(K^{n-1}\nu,\, K^n\nu)+s\eta\Phi(K^n\nu,\, L^n\nu)+s^2\eta\Phi(L^n\nu,\, L^{n-1}\nu)+F(\Phi(K^{n-1}\nu,\, K^n\nu),\, \Phi(L^{n-1}\nu,\, L^n\nu))$$

$$(1-s\eta)\Phi(K^n\nu,\, L^n\nu)\leq \eta\Phi(K^{n-1}\nu,\, K^n\nu)+s^2\eta\Phi(L^n\nu,\, L^{n-1}\nu)+F(\Phi(K^{n-1}\nu,\, K^n\nu),\, \Phi(L^{n-1}\nu,\, L^n\nu))$$

$$\leq \frac{\eta}{(1-s\eta)}\Phi(K^{n-1}\nu,\, K^n\nu)+\frac{s^2\eta}{(1-s\eta)}\Phi(L^n\nu,\, L^{n-1}\nu)$$

$$+\frac{F}{(1-s\eta)}(\Phi(K^{n-1}\nu,\, K^n\nu),\, \Phi(L^{n-1}\nu,\, L^n\nu))$$

By asymptotic regularity of L and K and the properties of the function F, we get from the above inequality

$$\Phi(K^n\nu,\, L^n\nu)\to 0 \quad \text{as} \quad n\to\infty \tag{22}$$

Now, let $\nu_0\in\mho$ be arbitrary and consider the sequence $\nu_n=K^n\nu_0$ for all $n=0,\, 1,\, 2,\, \ldots.$ Then for $n,\, m\in N$ and $m>n$, we have

$$\Phi(\nu_n,\, \nu_m)=\Phi(K^n\nu_0,\, K^m\nu_0)$$

$$\leq \Phi(K^n\nu_0,\, L^n\nu_0)+s\Phi(L^n\nu_0,\, K^m\nu_0)$$

$$\leq \Phi(K^n\nu_0,\, L^n\nu_0)+s\Phi(K(K^{m-1}\nu_0),\, L(L^{n-1}\nu_0))$$

$$\leq \Phi(K^n\nu_0,\, L^n\nu_0)+s\eta\Phi(K^{m-1}\nu_0,\, L^{n-1}\nu_0)+sF(\Phi(K^{m-1}\nu_0,\, K^m\nu_0),\, \Phi(L^{n-1}\nu_0,\, L^n\nu_0))$$

$$\leq \Phi(K^n\nu_0,\, L^n\nu_0)+s\eta\Phi(K^{m-1}\nu_0,\, K^m\nu_0)+s^2\eta\Phi(K^m\nu_0,\, K^n\nu_0)+s^3\eta d(K^n\nu_0,\, L^n\nu_0)$$

$$+s^4\eta\Phi(L^n\nu_0,\, L^{n-1}\nu_0)+sF(\Phi(K^{m-1}\nu_0,\, K^m\nu_0),\, d(L^{n-1}\nu_0,\, L^n\nu_0))$$

$$\leq \Phi(K^n\nu_0,\, L^n\nu_0)+s\eta\Phi(K^{m-1}\nu_0,\, K^m\nu_0)+s^2\eta\Phi(\nu_n,\, \nu_m)+s^3\eta\Phi(K^n\nu_0,\, L^n\nu_0)+s^4\eta\Phi(L^n\nu_0,\, L^{n-1}\nu_0)$$

$$+sF(\Phi(K^{m-1}\nu_0,\, K^m\nu_0),\, \Phi(L^{n-1}\nu_0,\, L^n\nu_0))$$

$$(1-s^2\eta)\Phi(\nu_n,\, \nu_m)\leq \Phi(K^n\nu_0,\, L^n\nu_0)+s\eta\Phi(K^{m-1}\nu_0,\, K^m\nu_0)+s^3\eta\Phi(K^n\nu_0,\, L^n\nu_0)+s^4\eta\Phi(L^n\nu_0,\, L^{n-1}\nu_0)$$

$$+sF(\Phi(K^{m-1}\nu_0,\, K^m\nu_0),\, \Phi(L^{n-1}\nu_0,\, L^n\nu_0))$$

$$\Phi(\nu_n,\, \nu_m)\leq \frac{1}{(1-s^2\eta)}\Phi(K^n\nu_0,\, L^n\nu_0)+\frac{s\eta}{(1-s^2\eta)}\Phi(K^{m-1}\nu_0,\, K^m\nu_0)+\frac{s^3\eta}{(1-s^2\eta)}\Phi(K^n\nu_0,\, L^n\nu_0)$$

$$+\frac{s^4\eta}{(1-s^2\eta)}\Phi(L^n\nu_0,\, L^{n-1}\nu_0)+\frac{s}{(1-s^2\eta)}F(\Phi(K^{m-1}\nu_0,\, K^m\nu_0),\, \Phi(L^{n-1}\nu_0,\, L^n\nu_0))$$

Applying asymptotic regularity, taking limit and using (21) in the above inequality, we get

$$\Phi(\nu_n,\, \nu_m)\to 0 \quad \text{as} \quad m,\, n\to\infty$$

since F is continuous at (0, 0) and $F(0,\, 0)=0$, which shows that $\{\nu_n\}$ is an *sb*-Cauchy sequence in $\mho$. By completeness of $\mho$, let $\nu_n\to h\in\mho$ as $n\to\infty$ i.e.

$$K^n\nu_0\to h \quad \text{as} \quad n\to\infty. \tag{23}$$

From

$$\Phi(L^n\nu_0,\, h)\leq\Phi(L^n\nu_0,\, K^n\nu_0)+s\Phi(K^n\nu_0,\, h),$$

and (21) and (22), we obtain

$$\lim_{n\to\infty}\Phi(L^n\nu_0,\, h)\leq 0.$$

Suppose that L and K are kC. Since $\lim_{n\to\infty}\nu_{n+1}=h$, so $\lim_{n\to\infty}K\nu_n=h$. Moreover, for each $k\geq 1$, we have $\lim_{n\to\infty}K^k\nu_n=h$. Since $\lim_{n\to\infty}K^{k-1}\nu_n=h$ and K is kC, we get $\lim_{n\to\infty}K^k\nu_n=Kh$, thus $Kh=h$. In the same way, using k-continuity of L, we have $Lh=h$.

Hence h is a CFP of L and K.

Next suppose that L and K are OC: we have $\lim_{n\to\infty}\nu_{n+1}=\lim_{n\to\infty}\nu_n=h$. By orbital continuity of K, $\lim_{n\to\infty}\nu_n=h$ implies $\lim_{n\to\infty}K\nu_n=Kh$, i.e., $h\in\mho$ is a fixed point of K. In the same manner, using orbital continuity of L, we get $Lh=h$. That is, $h\in\mho$ is a CFP of L and K. □

Theorem 5.2 Let $(\mho,\, \Phi)$ be a complete *sb*-metric space. Let $L,\, K\colon \mho\to\mho$ be AR maps satisfying

$$\Phi(L\nu,\, K\rho)\leq\eta\Phi(\nu,\, \rho)+\theta[\Phi(\nu,\, L\nu)+\Phi(\rho,\, K\rho)] \tag{24}$$

where $\eta,\, \theta\in\left[0,\, \frac{1}{3}\right]$. Then L and K have a CFP.

Proof. First we show that $\Phi(K^n\nu,\, L^n\nu)\to 0$ as $n\to\infty$ for all $\nu\in\mho$. Now if $L=K$, then there is nothing to prove. So suppose that $L\neq K$. Then using (23),

$$\Phi(L^n\nu,\, K^n\nu)=\Phi(LL^{n-1}\nu,\, KK^{n-1}\nu)$$

$$\leq\eta\Phi(L^{n-1}\nu,\, K^{n-1}\nu)+\theta[\Phi(L^{n-1}\nu,\, L^n\nu)+\Phi(K^{n-1}\nu,\, K^n\nu)]$$

$$\leq \eta \Phi(L^{n-1}\nu, L^n\nu) + s\eta \Phi(L^n\nu, K^n\nu) + s^2\eta \Phi(K^n\nu, K^{n-1}\nu) + \theta[\Phi(L^{n-1}\nu, L^n\nu) + \Phi(K^{n-1}\nu, K^n\nu)]$$

$$(1 - s\eta)\Phi(L^n\nu, K^n\nu) \leq \eta \Phi(L^{n-1}\nu, L^n\nu) + s^2\eta \Phi(K^n\nu, K^{n-1}\nu) + \theta[\Phi(L^{n-1}\nu, L^n\nu) + \Phi(K^{n-1}\nu, K^n\nu)]$$

$$\Phi(L^n\nu, K^n\nu) \leq \frac{\eta}{1 - s\eta}\Phi(L^{n-1}\nu, L^n\nu) + \frac{s^2\eta}{1 - s\eta}\Phi(K^n\nu, K^{n-1}\nu) + \frac{1}{1 - s\eta}\theta[\Phi(L^{n-1}\nu, L^n\nu) + \Phi(K^{n-1}\nu, K^n\nu)].$$

It gives by asymptotic regularity of L and K:

$$\Phi(L^n\nu, K^n\nu) \to 0 \quad \text{as} \quad n \to \infty. \tag{25}$$

Now, let $\nu_0 \in \mho$ be arbitrary and consider the sequence $\nu_n = K^n\nu_0$ for all $n = 0, 1, 2, ...$ Then for $n, m \in N$ and $m > n$, we have

$$\begin{aligned}
\Phi(\nu_n, \nu_m) &= \Phi(K^n\nu_0, K^m\nu_0) \\
&\leq \Phi(K^n\nu_0, L^n\nu_0) + s\Phi(L^n\nu_0, K^m\nu_0) \\
&\leq \Phi(K^n\nu_0, L^n\nu_0) + s\Phi(L(L^{n-1}\nu_0, K(K^{m-1}\nu_0)) \\
&\leq \Phi(K^n\nu_0, L^n\nu_0) + s\eta \Phi(L^{n-1}\nu_0, K^{m-1}\nu_0)) + s\theta[\Phi(L^{n-1}\nu_0, L^n\nu_0) + \Phi(K^{m-1}\nu_0, K^m\nu_0)] \\
&\leq \Phi(K^n\nu_0, L^n\nu_0) + s\eta \Phi(L^{n-1}\nu_0, L^n\nu_0) + s^2\eta \Phi(L^n\nu_0, K^n\nu_0) + s^3\eta \Phi(K^n\nu_0, K^m\nu_0) \\
&\quad + s^4\eta \Phi(K^m\nu_0, K^{m-1}\nu_0) + s\theta[\Phi(L^{n-1}\nu_0, L^n\nu_0) + \Phi(K^{m-1}\nu_0, K^m\nu_0)] \\
&\leq \Phi(K^n\nu_0, L^n\nu_0) + s\eta \Phi(L^{n-1}\nu_0, L^n\nu_0) + s^2\eta \Phi(L^n\nu_0, K^n\nu_0) + s^3\eta \Phi(\nu_n, \nu_m) + s^4\eta \Phi(K^m\nu_0, K^{m-1}\nu_0) \\
&\quad + s\theta[\Phi(L^{n-1}\nu_0, L^n\nu_0) + \Phi(K^{m-1}\nu_0, K^m\nu_0)]
\end{aligned}$$

$$\begin{aligned}
(1 - s^3\eta)\Phi(\nu_n, \nu_m) &\leq \Phi(K^n\nu_0, L^n\nu_0) + s\eta \Phi(L^{n-1}\nu_0, L^n\nu_0) + s^2\eta \Phi(L^n\nu_0, K^n\nu_0) + s^4\eta \Phi(K^m\nu_0, K^{m-1}\nu_0) \\
&\quad + s\theta[\Phi(L^{n-1}\nu_0, L^n\nu_0) + \Phi(K^{m-1}\nu_0, K^m\nu_0)]
\end{aligned}$$

$$\begin{aligned}
\Phi(\nu_n, \nu_m) &\leq \frac{1}{(1 - s^3\eta)}\Phi(K^n\nu_0, L^n\nu_0) + \frac{s\eta}{(1 - s^3\eta)}\Phi(L^{n-1}\nu_0, L^n\nu_0) + \frac{s^2\eta}{(1 - s^3\eta)}\Phi(L^n\nu_0, K^n\nu_0) \\
&\quad + \frac{s^4\eta}{(1 - s^3\eta)}\Phi(K^m\nu_0, K^{m-1}\nu_0) + \frac{s\theta}{(1 - s^3\eta)}[\Phi(L^{n-1}\nu_0, L^n\nu_0) + \Phi(K^{m-1}\nu_0, K^m\nu_0)]
\end{aligned}$$

In the above inequality taking limit, applying asymptotical regularity of L and K and (24), we have

$$\Phi(\nu_n, \nu_m) \to 0 \quad \text{as} \quad m, n \to \infty$$

since F is continuous at (0, 0) and $F(0, 0) = 0$, which shows that $\{\nu_n\}$ is an *sb*-Cauchy sequence in $\mho$ which converges to $\rho \in \mho$. Now

$$\Phi(\rho, K\rho) \le \Phi(\rho, L\nu_n) + s\Phi(L\nu_n, K\rho)$$

$$\le \Phi(\rho, L\nu_n) + s[\eta\Phi(\nu_n, \rho) + \theta(\Phi(\nu_n, L\nu_n) + \Phi(\rho, K\rho))]$$

$$\le \Phi(\rho, L\nu_n) + s\eta\Phi(\nu_n, \rho) + s\theta\Phi(\nu_n, L\nu_n) + s\theta\Phi(\rho, K\rho)$$

$$(1 - s\theta)\Phi(\rho, K\rho) \le \Phi(\rho, \nu_{n+1}) + s\eta\Phi(\nu_n, L\nu_n) + s\theta\Phi(\nu_n, \nu_{n+1})$$

$$\Phi(\rho, K\rho) \le \frac{1}{(1 - s\theta)}\Phi(\rho, \nu_{n+1}) + \frac{s\eta}{(1 - s\theta)}\Phi(\nu_n, L\nu_n) + \frac{s\theta}{(1 - s\theta)}\Phi(\nu_n, \nu_{n+1})$$

implies $\Phi(\rho, K\rho) \le 0$.

Hence $\Phi(\rho, K\rho) = 0$, gives $\rho = K\rho$. In the same manner, we can show that $L\rho = \rho$. Hence ρ is a *CFP* of L and K. □

We conclude this section with a multi-valued version of Theorem 5.2 as follows.

Theorem 5.3 Let $(\mho, \Phi)$ be a complete strong *bMS*. Let $L, K \colon \mho \to CB(\mho)$ be multi-valued *AR* maps satisfying

$$H(L\nu, K\rho) \le \eta\Phi(\nu, \rho) + \theta[\Phi(\nu, L\nu) + \Phi(\rho, K\rho)] \tag{26}$$

for $\eta, \theta \in \left[0, \frac{1}{3}\right]$. Then L and K have a *CFP*.

Proof. We show $H(K^n\nu, L^n\nu) \to 0$ as $n \to \infty$ for all $\nu \in \mho$. Now if $L = K$, then there is nothing to prove. So suppose that $L \ne K$. Then using (25), we obtain

$$H(L^n\nu, K^n\nu) = H(LL^{n-1}\nu, KK^{n-1}\nu)$$

$$\le \eta\Phi(L^{n-1}\nu, K^{n-1}\nu) + \theta[\Phi(L^{n-1}\nu, L^n\nu) + \Phi(K^{n-1}\nu, K^n\nu)]$$

$$\le \eta\Phi(L^{n-1}\nu, L^n\nu) + s\eta\Phi(L^n\nu, K^n\nu) + s^2\eta\Phi(K^n\nu, K^{n-1}\nu)$$

$$+ \theta[\Phi(L^{n-1}\nu, L^n\nu) + \Phi(K^{n-1}\nu, K^n\nu)]$$

$$(1 - s\eta)H(L^n\nu, K^n\nu) \le \eta\Phi(L^{n-1}\nu, L^n\nu) + s^2\eta\Phi(K^n\nu, K^{n-1}\nu)$$

$$+\theta[\Phi(L^{n-1}\nu, L^n\nu) + \Phi(K^{n-1}\nu, K^n\nu)]$$

$$H(L^n\nu, K^n\nu) \leq \frac{\eta}{1-s\eta}\Phi(L^{n-1}\nu, L^n\nu) + \frac{s^2\eta}{1-s\eta}\Phi(K^n\nu, K^{n-1}\nu)$$

$$+\frac{1}{1-s\eta}\theta[\Phi(L^{n-1}\nu, L^n\nu) + d(K^{n-1}\nu, K^n\nu)],$$

which gives by asymptotic regularity of L and K:

$$H(L^n\nu, K^n\nu) \to 0 \quad \text{as} \quad n \to \infty. \tag{27}$$

Now, let $\nu_0 \in \mho$ be arbitrary and consider the sequence $\nu_n = K^n\nu_0$ for $n = 0, 1, 2, \ldots$. Then for $n, m \in N$ and $m > n$, we have

$$\Phi(\nu_n, \nu_m) = H(K^n\nu_0, K^m\nu_0)$$

$$\leq H(K^n\nu_0, L^n\nu_0) + sH(L^n\nu_0, K^m\nu_0)$$

$$\leq H(K^n\nu_0, L^n\nu_0) + sH(L(L^{n-1}\nu_0, K(K^{m-1}\nu_0))$$

$$\leq H(K^n\nu_0, L^n\nu_0) + s\eta\Phi(L^{n-1}\nu_0, K^{m-1}\nu_0)) + s\theta[\Phi(L^{n-1}\nu_0, L^n\nu_0) + \Phi(K^{m-1}\nu_0, K^m\nu_0)]$$

$$\leq H(K^n\nu_0, L^n\nu_0) + s\eta\Phi(L^{n-1}\nu_0, L^n\nu_0) + s^2\eta\Phi(L^n\nu_0, K^n\nu_0) + s^3\eta\Phi(K^n\nu_0, K^m\nu_0)$$

$$+ s^4\eta\Phi(K^m\nu_0, K^{m-1}\nu_0) + s\theta[\Phi(L^{n-1}\nu_0, L^n\nu_0) + \Phi(K^{m-1}\nu_0, K^m\nu_0)]$$

$$\leq H(K^n\nu_0, L^n\nu_0) + s\eta\Phi(L^{n-1}\nu_0, L^n\nu_0) + s^2\eta\Phi(L^n\nu_0, K^n\nu_0) + s^3\eta\Phi(\nu_n, \nu_m)$$

$$+ s^4\eta\Phi(K^m\nu_0, K^{m-1}\nu_0) + s\theta[\Phi(L^{n-1}\nu_0, L^n\nu_0) + \Phi(K^{m-1}\nu_0, K^m\nu_0)]$$

$$(1-s^3\eta)\Phi(\nu_n, \nu_m) \leq H(K^n\nu_0, L^n\nu_0) + s\eta\Phi(L^{n-1}\nu_0, L^n\nu_0) + s^2\eta\Phi(L^n\nu_0, K^n\nu_0) + s^4\eta\Phi(K^m\nu_0, K^{m-1}\nu_0)$$

$$+ s\theta[\Phi(L^{n-1}\nu_0, L^n\nu_0) + \Phi(K^{m-1}\nu_0, K^m\nu_0)]$$

$$\Phi(\nu_n, \nu_m) \leq \frac{1}{(1-s^3\eta)}H(K^n\nu_0, L^n\nu_0) + \frac{s\eta}{(1-s^3\eta)}\Phi(L^{n-1}\nu_0, L^n\nu_0) + \frac{s^2\eta}{(1-s^3\eta)}\Phi(L^n\nu_0, K^n\nu_0)$$

$$+ \frac{s^4\eta}{(1-s^3\eta)}\Phi(K^m\nu_0, K^{m-1}\nu_0) + \frac{s\theta}{(1-s^3\eta)}[\Phi(L^{n-1}\nu_0, L^n\nu_0) + \Phi(K^{m-1}\nu_0, K^m\nu_0)]$$

Applying asymptotic regularity, taking limit and using (26) in the above inequality, we obtain

$$\Phi(\nu_n, \nu_m) \to 0 \quad \text{as} \quad m, n \to \infty.$$

As before, $\{\nu_n\}$ is an sb-Cauchy sequence in $\mho$ and so converges to $\rho \in \mho$. Now from

$$\Phi(\rho, K\rho) \leq \Phi(\rho, L\nu_n) + sH(L\nu_n, K\rho)$$

$$\leq \Phi(\rho, L\nu_n) + s[\eta \Phi(\nu_n, \rho) + \theta(\Phi(\nu_n, L\nu_n) + \Phi(\rho, K\rho))]$$

$$\leq \Phi(\rho, L\nu_n) + s\eta \Phi(\nu_n, \rho) + s\theta \Phi(\nu_n, L\nu_n) + s\theta \Phi(\rho, K\rho)$$

$$(1 - s\theta)\Phi(\rho, K\rho) \leq \Phi(\rho, \nu_{n+1}) + s\eta \Phi(\nu_n, L\nu_n) + s\theta \Phi(\nu_n, \nu_{n+1})$$

$$\Phi(\rho, K\rho) \leq \frac{1}{(1 - s\theta)} \Phi(\rho, \nu_{n+1}) + \frac{s\eta}{(1 - s\theta)} \Phi(\nu_n, L\nu_n) + \frac{s\theta}{(1 - s\theta)} \Phi(\nu_n, \nu_{n+1})$$

we obtain, $\Phi(\rho, K\rho) \leq 0$.

So $\Phi(\rho, K\rho) = 0$ i.e. $\rho \in K\rho$. In the same way we can prove that $\rho \in L\rho$.

Hence ρ is a CFP of L and K. □

Here is an example to validate Theorem 5.3.

Example 5.4 Let $\mho = \mathbb{R}$ be endowed with the strong b-metric

$$\Phi(\nu, \rho) = |\nu - \rho|^2,$$

which satisfies the strong b-metric inequality with constant $s = 2$.

Consider the multi-valued mappings K, L: $\mho \to CB(\mho)$ given by

$$K(\nu) = \left\{\frac{\nu}{4}, \frac{\nu}{8}\right\}, \quad L(\nu) = \left\{\frac{\nu}{6}, 0\right\},$$

where $CB(\mho)$ denotes the class of all non-empty closed and bounded subsets of $\mho$.

To verify asymptotic regularity, observe that for any $\nu \in \mho$, sequences $\{\nu_n\}$ and $\{\rho_n\}$ defined by

$$\nu_{n+1} \in K(\nu_n), \quad \rho_{n+1} \in L(\rho_n),$$

with initial points $\nu_0, \rho_0 \in \mho$, satisfy

$$|\nu_{n+1}| \leq \frac{|\nu_n|}{4} \quad \text{or} \quad \frac{|\nu_n|}{8}, \quad \text{and} \quad |\rho_{n+1}| \leq \frac{|\rho_n|}{6} \quad \text{or} \quad 0,$$

which imply

$$\Phi(\nu_n, \nu_{n+1}) = |\nu_n - \nu_{n+1}|^2 \to 0 \quad \text{and} \quad \Phi(\rho_n, \rho_{n+1}) = |\rho_n - \rho_{n+1}|^2 \to 0,$$

as $n \to \infty$. Hence, K and L are AR.

$$\Phi(\nu, L\nu) = \min\left\{\left|\nu - \frac{\nu}{6}\right|^2, |\nu - 0|^2\right\} = \left(\frac{5|\nu|}{6}\right)^2,$$

and

$$\Phi(\rho, K\rho) = \min\left\{\left|\rho - \frac{\rho}{4}\right|^2, \left|\rho - \frac{\rho}{8}\right|^2\right\} = \min\left\{\left(\frac{3\rho}{4}\right)^2, \left(\frac{7\rho}{8}\right)^2\right\},$$

$$\Phi(\rho, K\rho) = \left(\frac{3|\rho|}{4}\right)^2.$$

Now, choose $\eta = \theta = \frac{1}{4} \in \left[0, \frac{1}{3}\right]$. For any $\nu, \rho \in \mho$, the contractive condition

$$H(L\nu, K\rho) \leq \eta \Phi(\nu, \rho) + \theta[\Phi(\nu, L\nu) + \Phi(\rho, K\rho)]$$

holds.

So the right-hand side dominates the Hausdorff distance $H(L\nu, K\rho)$ for all ν, ρ.

Therefore, by Theorem 5.3, K and L have a CFP in $\mho$.

Solving

$$\nu \in K(\nu), \quad \nu \in L(\nu),$$

we find $\nu = 0$ as a CFP of L and K.

Remark 5.5 Our Theorem 5.3:

(i) Extends and refines Theorem 2.15 for multi-valued mappings without compatibility.

(ii) Provides a version of multi-valued non-continuous mappings for Theorem 2.2 of Khan and Oyetunbi [10].

6. Conclusion

The main contribution of this study is to establish some common fixed point theorems for single-valued asymptotically regular mappings in a b-metric space. One of our new theorems is generalized for multi-valued asymptotically regular mappings in a strong b-metric space. Our results extend and improve the work of Bisht and Singh [13] and Khan and Oyetunbi [10]. We apply Corollary 3.4 to find the existence and uniqueness of the solution of a Fredholm linear integral equation. A pertinent feature of our work is that it includes examples for the new concepts and applications of the results established herein.

Confilict of interest

The authors declare no competing financial interest.

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