Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

Research Article

Approximation of Fixed Points for *E*-Type Enriched Nonexpansive Mappings with Applications

Mujahid Abbas[1,2*], Ahad Hamoud Alotaibi[3], Muhammad Waseem Asghar[1]

[1] Department of Mechanical Engineering Science, Faculty of Engineering and the Built Environment, University of Johannesburg, Johannesburg 2092, South Africa
[2] Department of Medical Research, China Medical University, Taichung 40402, Taiwan
[3] Department of Mathematics, Faculty of Science and Arts, King Abdulaziz University, Rabigh 21911, Saudi Arabia
E-mail: mujahida@uj.ac.za

Received: 6 November 2025; **Revised:** 22 December 2025; **Accepted:** 6 February 2026

Abstract: In this research, we propose an inertial-type iterative scheme constructed by an *E*-type enriched nonexpansive mappings in the framework of uniformly convex Banach spaces. Strong and weak convergence results along with stability result for the proposed algorithm are presented. Numerical examples and comparisons with well known existing iterative schemes with the help of graphs are presented to demonstrate the efficiency of our proposed scheme. It is shown that the algorithm presented herein converges more faster to fixed point and competitive on the examples considered for this class of mappings. As applications, we apply the scheme to solve variational inequality problems, constrained optimization problems, and split feasibility problems. Moreover, we use the proposed approach to solve a functional differential equation.

***Keywords*:** variational inequality problem, fixed point problem, split feasibility problems, minimization problem

MSC: 47H09, 47H10, 47J25, 65J15

1. Introduction and preliminaries

Many problems in applied sciences can be modeled in functional inclusions and equations which often can be reformulated as fixed point problems. Classical examples include the minimization of loss functions in artificial neural networks, the modeling of infectious disease dynamics, the evolution of solutions to differential equations, the construction of stationary distributions in stochastic processes, finding the optimal control solutions and the formulation of Nash equilibrium problems. These diverse applications signifies a vital role of fixed point theory in analyzing and solving complex systems. Many problems arising in applied sciences can be reformulated as fixed point problems. In recent years, fixed point methods have been successfully employed in the analytical and numerical investigation of memory dependent fractional models describing behavioral learning dynamics, where nonlocal operators introduce strong nonlinear effects. Similarly, fixed point techniques provide a rigorous framework to establish convergence of numerical algorithms and stability of solutions under parameter variations. Moreover, fractional differential equations appearing in the theory of chemical graphs often lead to complex nonlinear systems, for which fixed point theory offers a robust and systematic approach to ensure solvability and to design efficient iterative procedures, several recent developments in fixed point

Copyright ©2026 Mujahid Abbasi, et al.
DOI: https://doi.org/10.37256/cm.7320269093
This is an open-access article distributed under a CC BY license
(Creative Commons Attribution 4.0 International License)
https://creativecommons.org/licenses/by/4.0/

theory can be seen in [1–4]. In practice, finding an exact solution is often impossible, so an efficient iterative schemes that approximate solutions with few iterations and low computational cost are highly desirable. The main features of Iterative algorithms are stability, dependence on the data, convergence rate, computational cost and well-posedness. For these reasons, the development of faster and more reliable iterative methods have attracted the attention of several mathematicians and computer scientists.

A mapping F is called Lipschitzian mapping if it satisfy the following inequality:

$$\|Fr - Fs\| \leq L\|r - s\|,$$

for all $r, s \in \Gamma$, where $L > 0$ is a constant. For $L \in (0, 1)$, F is said contraction and F is nonexpansive if we take $L = 1$ in the above inequality. The mapping F is known as quasi-nonexpansive if $\|Fr - r^*\| \leq \|r - r^*\|$, for all $r \in \Gamma$, $r^* \in \mathrm{Fix}(F)$, here $\mathrm{Fix}(F)$ is fixed points set of F.

The Banach contraction result provides one of the most simplest algorithms known as Picard iterative scheme, in approximating the solution of fixed point equations and is based on the idea of iterative application of contraction mapping on the initial guess of the solution. Mathematically, if r_0 is an initial guess of the fixed point equation given by $F(r) = r$, then other approximations of the solution are obtained by $r_{n+1} = Fr_n$ ([5]). However, contraction result is not applicable in approximating the solution of fixed point equations of certain classes of mappings which are very useful in convex optimization and in many real world problems of practical interest. In particular, the solution of fixed point equations involving nonexpansive mappings (which are natural extensions of contractions) need not be approximated by Picard iterative scheme. Indeed, the Picard iteration scheme constructed with the help of a nonexpansive mapping given by $Fr = 1 - r$ on $[0, 1]$, fail to converge despite the fact that fixed point set of F is nonempty. Note that, $\frac{1}{2}$ is a fixed point of F and if we choose any initial guess in $[0, 1]$ other than $\frac{1}{2}$, then the Picard sequence obtained by F does not converge. However, if we use a weighted average of an identity mapping I and F given by $F_\lambda = (1 - \lambda)I + \lambda F$, where $\lambda \in (0, 1)$, then it converges to an element in $\mathrm{Fix}(F)$, if we use any initial guess, and $\mathrm{Fix}(F_\lambda) = \mathrm{Fix}(F)$. This shows that in certain cases, developing an iterative sequence by an averaged mapping is useful than using a mapping F itself.

Throughout we denote, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}^+$ the set of natural, real numbers and positive real numbers, respectively. Thy symbol Γ stands for a nonempty closed convex closed convex subset of Banach space $\mathscr{B}$ which is also uniformly convex and F for a self mapping on Γ.

Suzuki [6] proposed an important class of mappings called it the mappings that satisfy the condition (C).

Definition 1 [6] Assume F is mapping on Γ, we say F has the condition (C), if

$$\frac{1}{2}\|r - Fr\| \leq \|r - s\| \quad \text{implies that} \quad \|Fr - Fs\| \leq \|r - s\|,$$

for all $r, s \in \Gamma$.

Suzuki [6] proved several fixed point theorems and its convergence results for such class mappings. It is important to note: any nonexpansive mapping satisfies the above condition (C); however, there exist some mappings which are discontinuous (means not expansive) that fulfill the condition (C). Later, Garcia-Falset [7] introduced a broader class of mappings that satisfy condition (E), which generalized the condition (C) and given by:

Definition 2 A mapping F on the set Γ have the property (E) or (E_ξ) on Γ, if there is a constant $\xi \geq 1$ with

$$\|r - Fs\| \leq \xi\|r - Fr\| + \|r - s\|,$$

for all $r, s \in \Gamma$.

Further note that, nonexpansivness implies the condition (E_1) fulfills, but the converse is not always true. Moreover, if F have the condition (C), then it fulfills the condition (E_3), however, the converse does not generally hold. There also exist some mappings that satisfy the condition (E) but fail to fulfil the condition (C) (see, for example, [7]).

Shukla and Panicher [8] proposed a new more general mapping which is given by:

Definition 3 The mapping $F: \Gamma \to \Gamma$ is called an E-type enriched nonexpansive mapping if

$$\|b(r-s)+r-Fs\| \leq \xi\|r-Fr\|+(b+1)\|r-s\|, \tag{1}$$

holds for all $r, s \in \mathscr{B}$, $b \in [0, \infty)$ and $\xi \in [1, \infty)$.

Several authors used multi-step iterative methods to study existence and convergence for these generalized mappings. Let us recall the following iterative schemes.

Choose an initial guess r_1 in Γ. The sequence r_n is given by

$$r_{n+1} = (1-\theta_n)r_n + \theta_n F r_n, \quad n \in \mathbb{N} \tag{2}$$

is known as Mann iteration sequence [9], where $\{\theta_n\}$ is an appropriate sequence of parameters in $(0, 1)$.

The sequence $\{r_n\}$ defined by

$$\begin{cases} r_1 \in \Gamma \\ r_{n+1} = (1-\theta_n)\, r_n + \theta_n F s_n \\ s_n = (1-\eta_n)\, r_n + \eta_n F r_n \end{cases} \tag{3}$$

is called Ishikawa scheme [10], where $\{\eta_n\}, \{\theta_n\} \subset (0, 1)$ are sequence of parameters.

After that Noor [11] introduced a scheme which is given as follows:

$$\begin{cases} r_1 \in \Gamma \\ r_{n+1} = (1-\theta_n)r_n + \theta_n F s_n \\ s_n = (1-\eta_n)r_n + \eta_n F t_n \\ t_n = (1-\sigma_n)r_n + \sigma_n F r_n, \end{cases} \tag{4}$$

where $\{\sigma_n\}, \{\eta_n\}, \{\theta_n\} \subset (0, 1)$.

An S-iteration scheme was proposed by Agarwal et al. [12] and give by:

$$\begin{cases} r_1 \in \Gamma \\ r_{n+1} = (1-\theta_n)Fr_n + \theta_n Fs_n \\ s_n = (1-\eta_n)r_n + \eta_n Fr_n, \end{cases} \tag{5}$$

where $\{\eta_n\}$, $\{\theta_n\}$ are sequences in $(0, 1)$.

An iterative scheme $\{r_n\}$ introduced in [13] is given by:

$$\begin{cases} r_1 \in \Gamma \\ r_{n+1} = (1-\theta_n)Fs_n + \theta_n Ft_n \\ s_n = (1-\eta_n)Fr_n + \eta_n Ft_n \\ t_n = (1-\sigma_n)r_n + \sigma_n Fr_n, \end{cases} \tag{6}$$

where $\{\sigma_n\}$, $\{\eta_n\}$ and $\{\theta_n\}$ in $(0, 1)$ satisfy certain conditions. Note that, it has a better convergence than than $S-$ iteration in term of convergence rate.

Thakur iterative sequence $\{r_n\}$ proposed by Thakur et al. in [14] is given by:

$$\begin{cases} r_1 \in \Gamma \\ r_{n+1} = (1-\theta_n)Ft_n + \theta_n Fs_n \\ s_n = (1-\eta_n)t_n + \eta_n Ft_n \\ t_n = (1-\sigma_n)r_n + \sigma_n Fr_n. \end{cases} \tag{7}$$

It is was observed that the above sequence convergence faster than iterative sequence in [13].

The M-iteration scheme [15] is given as follows:

$$\begin{cases} r_1 \in \Gamma \\ r_{n+1} = Fs_n \\ s_n = Ft_n \\ t_n = (1-\sigma_n)r_n + \sigma_n Fr_n. \end{cases} \tag{8}$$

Recently, Khan et al. [16] proposed an iterative scheme to approximate the fixed points for mappings with property (E) and is given as follows

$$\begin{cases} r_1 \in \Gamma \\ r_{n+1} = Fs_n \\ s_n = F((1-\sigma_n)Fr_n + \sigma_n F^2 r_n). \end{cases} \tag{9}$$

Recent research work has focused on developing iterative algorithms with higher convergence rate, better stability and clearer data-dependency properties (see for example [17–20]). Hacıoğlu [21] proposed an algorithm and studied its convergence results for almost contraction mappings. Houmani and Turcanu [22] applied an iteration to solve fixed points equation of mappings with property (E). Moreover, the inertial-type iterative methods have received considerable attention in recent years for solving fixed point problems due to their ability to accelerate convergence. Notable contributions in this direction include the inertial schemes in primal dual techniques and the forward backward inertial approaches. These methods introduce momentum like terms into classical fixed point iterations. In convergence analysis of iterative algorithms, specifically for a more such general class of mappings than the mapping satisfy the condition (E) have received less attention along with the inertial technique. Motivated by the need to address these gaps, this paper proposes a new inertial AA-type iterative process that is independent of, and in many cases faster than the several well-known schemes for average mapping to fixed points problem for E-type enriched nonexpansive mappings and is defined as follows:

$$\begin{cases} r_1 \in \Gamma \\ r_{n+1} = F_\lambda s_n \\ s_n = F_\lambda((1-\theta_n)F_\lambda v_n + \theta_n F_\lambda t_n) \\ t_n = F_\lambda((1-\eta_n)u_n + \eta_n F_\lambda v_n) \\ v_n = (1-\sigma_n)u_n + \sigma_n F_\lambda u_n, \\ u_n = r_n + o_n(r_n - r_{n-1}), \quad \forall\, n \in \mathbb{N} \end{cases} \tag{10}$$

where $\{\sigma_n\}$, $\{\eta_n\}$ and $\{\theta_n\}$ are the parameters lie in $(0,\, 1)$ and o_n with $0 \leq o_n \leq \hat{o}_n$ and $\hat{o}_n$ given as

$$\hat{o}_n = \begin{cases} \min\{o,\, \frac{\vartheta_n}{\|r_n - r_{n-1}\|}\}, & \text{if} \quad r_n \neq r_{n-1}, \\ o, & \text{otherwise.} \end{cases} \tag{11}$$

Also $o > 0$ and $\{\vartheta_n\} \subset \mathbb{R}^+$ with $\lim_{n\to\infty} \frac{\vartheta_n}{\beta_n} = 0$. We proved the convergence results for the proposed method when applied to generalized nonexpansive mappings. We also investigate stability and presented numerical experiments for the

validity of the theoretical results. Finally, we illustrate the applicability of our approach in solving the the of variational inequalities, constrained minimization and split feasibility problems. After that ,we solve some delay functional equation.

The following concepts are used in the convergence results.

Definition 4 [23] Assume that $\{p_n\}$ and $\{q_n\}$ are two sequences which converge to r^* with the following error estimates

$$\|p_n - r^*\| \leq \alpha_n,$$

$$\|q_n - r^*\| \leq \beta_n.$$

If $\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = 0$, then $\{p_n\}$ converges faster than $\{q_n\}$.

Definition 5 [24] A Banach space $\mathscr{B}$ fulfills the Opial's property, if for any sequence $\{r_n\} \rightharpoonup r^* \in \mathscr{B}$, also for every $a \in \mathscr{B}$ with $r^* \neq a$, we have the following:

$$\liminf_{n\to\infty} \|r_n - r^*\| < \liminf_{n\to\infty} \|r_n - a\|.$$

Ostrowski [25] proposed the idea of stability for an iterative scheme, which is given as follows:

Definition 6 Suppose that $\{r_n\}$ is an approximate sequence of the sequence $\{a_n\}$ in $\Gamma \subset \mathscr{B}$. The iterative sequence $a_{n+1} = T(F_\lambda, a_n)$, where T defines a sequence which converges to $a^* \in \mathrm{Fix}(F)$, is said to be F-stable provided that $\lim_{n\to\infty} e_n = 0$ iff $\lim_{n\to\infty} r_n = a^*$, where $\{e_n\}$ is defined as follows:

$$e_n = \|r_{n+1} - T(F, r_n)\|, \ \forall\, n \in \mathbb{N}.$$

Note that, a mapping $F\colon \Gamma \to \mathscr{B}$ is called demiclosed on Γ, if the graph of $I - F$ is sequentially closed under weak-strong convergence.

Definition 7 [26] A mapping $F\colon \Gamma \to \mathscr{B}$ is demiclosed on 0 if, for each $\{a_n\}$ in Γ with $a_n \rightharpoonup a$ and $(I-F)a_n \to 0$, we have $(I-F)a = 0$.

Lemma 1 [6] Assume F is nonexpansive and self map on Γ. Then $I - F$ is know as demiclosed at zero.

Lemma 2 [23] Assume that $\{r_n\}$ and $\{b_n\} \subset \mathbb{R}+$ with:

$$r_{n+1} \leq (1 - \varepsilon_n) r_n + b_n$$

where $\varepsilon_n \in (0, 1)$ for all $n \in \mathbb{N}$ with $\Sigma_{n=1}^{\infty} \varepsilon_n = \infty$. If $\lim_{n\to\infty} \frac{b_n}{\varepsilon_n} = 0$, then $\lim_{n\to\infty} r_n = 0$.

Proposition 1 [7] If $F\colon \Gamma \to \mathscr{B}$ satisfies the property (E) on Γ, with $\mathrm{Fix}(F) \neq \emptyset$ then F is quasi-nonexpansive mapping.

Lemma 3 [27] Suppose that $0 < \alpha \leq \theta_n \leq \beta < 1$, $\forall n \in \mathbb{N}$. If $\{a_n\}, \{q_n\} \subset \mathscr{B}$ with:

$$\limsup_{n\to\infty} \|a_n\| \ \leq \ a, \ \limsup_{n\to\infty} \|b_n\| \leq a \text{ and}$$

$$\limsup_{n\to\infty} \|(1 - \theta_n)a_n + \theta_n b_n\| \ = \ a,$$

for $a \geq 0$, then we have $\lim_{n\to\infty} \|a_n - b_n\| = 0$.

Lemma 4 [28] If $0 \leq q < 1$, $r_n \geq 0$, and $s_n \in (0, 1)$ with $s_n \to 0$, and

$$r_{n+1} \leq qr_n + s_n \quad \text{for all } n,$$

then $r_n \to 0$.

Remark 1 We assume the following conditions: for some $\beta_n \subset (0, 1)$ such that $\sum_{n=1}^{\infty} \beta_n < \infty$ and

$$\lim_{n\to\infty} \frac{o_n}{\beta_n} \|r_n - r_{n-1}\| = 0.$$

The remaining part of the manuscript is organized in the following ways: In Section 2, we proved the convergence properties of the proposed algorithm. Section 3 contains numerical comparisons with existing iterative processes and Section 4 discusses different applications.

2. Convergence analysis

Theorem 1 If F: $\Gamma \to \Gamma$ a E-type enriched nonexpansive mapping, then $\text{Fix}(F) \neq \emptyset$.

Proof. As F is E-type enriched nonexpansive mapping, by the definition we have:

$$\|b(r-s) + r - Fs\| \leq \xi \|r - Fr\| + (b+1)\|r - s\|,$$

for all $r, s \in \mathscr{B}$. If we set, $b = \dfrac{1-\lambda}{\lambda}$, then $\lambda \in (0, 1)$ and by the above inequality, we have

$$\|(1-\lambda)(r-s) + \lambda(r - Fs)\| \leq \xi\lambda \|r - Fr\| + \|r - s\|,$$

and

$$\|r - ((1-\lambda)y + \lambda Fs)\| \leq \xi\lambda \|r - Fr\| + \|r - s\|. \tag{12}$$

Let F_λ be given by

$$F_\lambda r = (1-\lambda)r + \lambda Fr \quad \text{for all } r \in \mathscr{B}.$$

Clearly,

$$\|F_\lambda r - r\| = \lambda \|Fr - r\|. \tag{13}$$

From (12), we obtain that

$$\|r - F_\lambda s\| \leq \xi \|r - F_\lambda r\| + \|r - s\|. \tag{14}$$

Thus, the operator F_λ satisfies the condition (E), by Theorem 2 of [7] we get that F_λ has at the least one fixed point and we have $\mathrm{Fix}(F_\lambda) = \mathrm{Fix}(F) \neq \emptyset$. □

Lemma 5 If $F\colon \Gamma \to \Gamma$ is a E-type enriched nonexpansive mapping, sequence $\{r_n\}$ is defined in (10) and $\mathrm{Fix}(F) \neq \emptyset$, then $\lim_{n\to\infty} \|r_n - r^*\|$ exists for all $r^* \in \mathrm{Fix}(F)$.

Proof. Suppose that $r^* \in \mathrm{Fix}(F_\lambda)$. By the Theorem (1), the mapping F_λ satisfies the condition (E) and by the Proposition 1, it is quasi-nonexpansive. It follows from (10) that

$$\begin{aligned} \|u_n - r^*\| &= \|r_n + o_n(r_n - r_{n-1}) - r^*\| \\ &\leq \|r_n - r^*\| + o_n\|r_n - r_{n-1}\| \\ &= \|r_n - r^*\| + \beta_n \frac{o_n}{\beta_n}\|r_n - r_{n-1}\|. \end{aligned} \tag{15}$$

By *Remark* 1, $\lim_{n\to\infty} \frac{o_n}{\beta_n}\|r_n - r_{n-1}\| = 0$. Thus, there exists positive real constant $M > 0$ with

$$\frac{o_n}{\beta_n}\|r_n - r_{n-1}\| \leq M, \qquad \text{for all } \ n \geq 1.$$

By Equation (15) we obtain that

$$\|u_n - r^*\| \leq \|r_n - r^*\| + \beta_n M. \tag{16}$$

Also,

$$\begin{aligned} \|v_n - r^*\| &= \|(1-\sigma_n)u_n + \sigma_n F_\lambda u_n - r^*\| \\ &\leq (1-\sigma_n)\|u_n - r^*\| + \sigma_n\|F_\lambda u_n - r^*\| \\ &\leq (1-\sigma_n)\|u_n - r^*\| + \sigma_n\|u_n - r^*\| \\ &= \|u_n - r^*\|. \end{aligned} \tag{17}$$

From (10) and (17), we have

$$\begin{aligned}\|t_n - r^*\| &= \|F_\lambda((1-\eta_n)u_n + \eta_n F_\lambda v_n) - r^*\| \\ &\leq \|(1-\eta_n)u_n + \eta_n F_\lambda v_n - r^*\| \\ &\leq (1-\eta_n)\|u_n - r^*\| + \eta_n\|F_\lambda v_n - r^*\| \\ &\leq (1-\eta_n)\|u_n - r^*\| + \eta_n\|v_n - r^*\| \\ &\leq \|u_n - r^*\|. \end{aligned} \tag{18}$$

Using (18), we get that

$$\begin{aligned}\|s_n - r^*\| &= \|F_\lambda((1-\theta_n)F_\lambda v_n + \theta_n F_\lambda t_n) - r^*\| \\ &\leq \|(1-\theta_n)F_\lambda v_n + \theta_n F_\lambda t_n - r^*\| \\ &\leq (1-\theta_n)\|F_\lambda v_n - r^*\| + \theta_n\|F_\lambda t_n - r^*\| \\ &\leq (1-\theta_n)\|v_n - r^*\| + \theta_n\|t_n - r^*\| \\ &\leq (1-\theta_n)\|u_n - r^*\| + \theta_n\|u_n - r^*\| \\ &= \|u_n - r^*\|. \end{aligned} \tag{19}$$

Finally, by (10), (16) and (19), we have

$$\begin{aligned}\|r_{n+1} - r^*\| &= \|F_\lambda s_n - r^*\| \\ &\leq \|s_n - r^*\| \\ &\leq \|u_n - r^*\| \\ &\leq \|r_n - r^*\| + \beta_n M. \end{aligned} \tag{20}$$

Note that, $\beta_n \to 0$ as $n \to \infty$, there exists a $K \in \mathbb{N}$ such that the tail of the sequence $\{\|r_n - r^*\|\}$ is monotone decreasing and bounded below for $n > K$. Then, by the monotone convergence theorem, $\lim_{n\to\infty}\|r_n - r^*\|$ exists for all $r^* \in \mathrm{Fix}(F_\lambda) = \mathrm{Fix}(F)$. □

Lemma 6 If $F: \Gamma \to \Gamma$ and sequence $\{r_n\}$ as given in Lemma 5 and $\mathrm{Fix}(F) \neq \emptyset$, then we have $\lim_{n\to\infty} \|r_n - F_\lambda r_n\| = 0$, where $\lambda = \frac{1}{b+1}$.

Proof. Suppose that $r^* \in \mathrm{Fix}(F)$. By Lemma 5, $\lim_{n\to\infty} \|r_n - r^*\|$ exists, therefore it is bounded. Further, assume that

$$\lim_{n\to\infty} \|r_n - r^*\| = l. \tag{21}$$

Take the limit sup as $n \to \infty$ in (16), (17), (18) and (19), we obtain

$$\limsup_{n\to\infty} \|u_n - r^*\| \leq l, \limsup_{n\to\infty} \|v_n - r^*\| \leq l, \text{ and}$$

$$\limsup_{n\to\infty} \|t_n - r^*\| \leq l, \limsup_{n\to\infty} \|s_n - r^*\| \leq l. \tag{22}$$

As, F_λ is quasi-nonexpansive mapping, we obtain that

$$\|F_\lambda u_n - r^*\| \leq \|u_n - r^*\|, \|F_\lambda v_n - r^*\| \leq \|v_n - r^*\|,$$

$$\|F_\lambda r_n - r^*\| \leq \|r_n - r^*\|, \|F_\lambda t_n - r^*\| \leq \|t_n - r^*\|,$$

$$\|F_\lambda s_n - r^*\| \leq \|s_n - r^*\|.$$

Again by above inequalities, we get

$$\limsup_{n\to\infty} \|F_\lambda r_n - r^*\| \leq l, \quad \text{and} \quad \limsup_{n\to\infty} \|F_\lambda u_n - r^*\| \leq l,$$

$$\limsup_{n\to\infty} \|F_\lambda v_n - r^*\| \leq l, \quad \text{and} \quad \limsup_{n\to\infty} \|F_\lambda t_n - r^*\| \leq l, \quad \text{and}$$

$$\limsup_{n\to\infty} \|F_\lambda s_n - r^*\| \leq l. \tag{23}$$

Note that,

$$l = \liminf_{n\to\infty} \|r_{n+1} - r^*\| = \liminf_{n\to\infty} \|F_\lambda s_n - r^*\| \leq \liminf_{n\to\infty} \|s_n - r^*\|. \tag{24}$$

From (22) and (24) we get

$$\lim_{n\to\infty} \|s_n - r^*\| = l.$$

Thus,

$$\|s_n - r^*\| = \|F_\lambda((1-\theta_n)F_\lambda v_n + \theta_n F_\lambda t_n) - r^*\|$$

$$\leq \|(1-\theta_n)F_\lambda v_n + \theta_n F_\lambda t_n - r^*\|$$

$$\leq (1-\theta_n)\|v_n - r^*\| + \theta_n\|t_n - r^*\|$$

$$\leq \|v_n - r^*\|. \tag{25}$$

If we take lim inf in (25), then we have that

$$l \leq \liminf_{n\to\infty} \|v_n - r^*\|,$$

which implies that

$$\lim_{n\to\infty} \|v_n - r^*\| = l.$$

If we set $r_0 = r_1$, then we have

$$l = \lim_{n\to\infty} \|(1-\sigma_n)r_n + \sigma_n F_\lambda r_n - r^*\|$$

$$= \lim_{n\to\infty} \|(1-\sigma_n)(r_n - r^*) + \sigma_n(F_\lambda r_n - r^*)\|. \tag{26}$$

From (21), (23) and (26) and by Lemma 3, we have

$$\lim_{n\to\infty} \|r_n - F_\lambda r_n\| = 0.$$

□

Theorem 2 If $F\colon \Gamma \to \Gamma$ is a E-type enriched nonexpansive mapping, $\mathscr{B}$ satisfies the Opial's condition, $\{r_n\}$ which is given by (10) with $\mathrm{Fix}(F) \neq \emptyset$, implies the $\{r_n\}$ gives the weak converges a fixed point of F.

Proof. If $r^* \in \mathrm{Fix}(F)$, then we have $\lim_{n\to\infty}\|r_n - r^*\|$ exists. Here we need to prove that $\{r_n\}$ has only one subsequential limit in $\mathrm{Fix}(F)$. Suppose that $r^{'}$ and $r^{''}$ are two distinct weak cluster points of $\{r_{n_i}\}$ and $\{r_{n_j}\}$ of $\{r_n\}$. Since, $\lim_{n\to\infty}\|r_n - F_\lambda r_{n_k}\| = 0$ also $I - F_\lambda$ is demiclosed, by Lemma 1 we get $F_\lambda r^{'} = r^{'}$. Similarly, we have $r^{''} \in \mathrm{Fix}(F_\lambda)$. From Lemma 5, $\lim_{n\to\infty}\|r_n - r^{''}\|$ exists. To prove the uniqueness, suppose that $r^{'} \neq r^{''}$. If we use the Opial's property, then we obtain that

$$\lim_{n\to\infty} \|r_n - r^{'}\| \leq \lim_{n_i\to\infty} \|r_{n_i} - r^{'}\|$$

$$< \lim_{n_i \to \infty} \|r_{n_i} - r''\|$$

$$= \lim_{n \to \infty} \|r_n - r''\|$$

$$= \lim_{n_j \to \infty} \|r_{n_j} - r''\|$$

$$< \lim_{n_j \to \infty} \|r_{n_j} - r'\|$$

$$= \lim_{n \to \infty} \|r_n - r'\|, \tag{27}$$

a contradiction and this implies $\{r_n\}$ converges to a point weakly in $\mathrm{Fix}(F_\lambda) = \mathrm{Fix}(F)$. □

Theorem 3 Suppose that F and $\{r_n\}$ are the mapping and the sequence and $\mathrm{Fix}(F) \neq \emptyset$. The sequence $\{r_n\}$ converges to fixed point of F iff $\liminf_{n\to\infty} d(r_n, \mathrm{Fix}(F)) = 0$ or $\limsup_{n\to\infty} d(r_n, \mathrm{Fix}(F)) = 0$.

Proof. Necessity can be seen is very obvious.

Assume that $\liminf_{n\to\infty} d(r_n, \mathrm{Fix}(F_\lambda)) = 0$. From the Lemma 6, $\lim_{n\to\infty} \|r_n - r^*\|$ exists for all $r^* \in \mathrm{Fix}(F_\lambda)$. As $\liminf_{n\to\infty} d(r_n, \mathrm{Fix}(F_\lambda)) = 0$, $\lim_{n\to\infty} d(r_n, \mathrm{Fix}(F_\lambda)) = 0$. We now show $\{r_n\}$ is Cauchy in Γ'. For $\varepsilon > 0$. By the given assumption, there exists $m_0 \in \mathbb{N}$ with

$$d(r_n, \mathrm{Fix}(F_\lambda)) < \frac{\varepsilon}{2},$$

for all $n \geq m_0$, which gives that

$$\inf\{\|r_n - r^*\| : r^* \in \mathrm{Fix}(F_\lambda)\} < \frac{\varepsilon}{2}.$$

Thus, $\inf\{\|r_{m_0} - r^*\| : r^* \in \mathrm{Fix}F(F_\lambda)\} < \frac{\varepsilon}{2}$ and hence there exists $r^* \in \mathrm{Fix}(F_\lambda)$ such that

$$\|r_{m_0} - r^*\| < \frac{\varepsilon}{2}.$$

If $m, n \geq m_0$, then we get that

$$\|r_{m+n} - r_n\| \leq \|r_{m+n} - r^*\| + \|r_n - r^*\|$$

$$\leq \|r_{m_0} - r^*\| + \|r_{m_0} - r^*\|$$

$$\leq 2\|r_{m_0} - r^*\|$$

$$\leq \varepsilon, \tag{28}$$

which gives that $\{r_n\}$ is Cauchy in the set Γ. Since by the closed and boundedness of the set Γ implies that there exists a point $a^* \in \Gamma$ and $\lim_{n\to\infty} r_n = a^*$. Here, $\lim_{n\to\infty} d(r_n, \mathrm{Fix}(F_\lambda)) = 0$ gives

$$d(a^*, \mathrm{Fix}(F_\lambda)) = 0, \quad \text{which implies that} \quad a^* \in \mathrm{Fix}(F_\lambda) = \mathrm{Fix}(F).$$

□

Theorem 4 If $\Gamma^{'}$ is compact and convex subset of $\mathscr{B}$, $F\colon \Gamma^{'} \to \Gamma^{'}$, $\{r_n\}$ are mapping and sequence as above, respectively and $\mathrm{Fix}(F) \neq \emptyset$, then $r_n \to r^* \in \mathrm{Fix}(F)$.

Proof. Assume that $\mathrm{Fix}(F) \neq \emptyset$. Using the Lemma 6, we have that $\lim_{n\to\infty} \|r_n - F_\lambda r_n\| = 0$. As $\Gamma^{'}$ is compact, there exist a subsequence $\{r_{n_k}\}$ of $\{r_n\}$ such that $r_{n_k} \to r^* \in \Gamma^{'}$, Note that,

$$\|r_{n_k} - F_\lambda r^*\| \leq \xi \|r_{n_k} - F_\lambda r_{n_k}\| + \|r_{n_k} - r^*\|.$$

On taking limit as $k \to \infty$, we have $r_{n_k} \to F_\lambda r^*$. Thus, $F_\lambda r^* = r^*$. Lemma 5 gives that $\lim_{n\to\infty} \|r_n - r^*\|$ exists. Hence, $r_n \to r^* \in \mathrm{Fix}(F_\lambda) = \mathrm{Fix}(F)$. □

The condition (I) on a mapping proposed by Sentor and Dotson [29] is very important and for proving the strong convergence of a sequence constructed with such mapping.

Definition 8 A mapping $F\colon \Gamma^{'} \to \Gamma^{'}$ fulfils the condition (I), if there mapping $\phi\colon [0, \infty) \to [0, \infty)$ which is a nondecreasing with $\phi(a) > 0$ and $\phi(0) = 0 \; \forall \; a > 0$ which satisfy

$$d(r^{'}, Fr^{'}) \geq \phi(d(r^{'}, \mathrm{Fix}(F))),$$

for all $r^{'} \in \Gamma^{'}$, where $d(r^{'}, \mathrm{Fix}(F)) = \inf\{d(r^{'}, r^*)\colon r^* \in \mathrm{Fix}(F)\}$.

Employing the condition (I), the strong convergence result can be prove as follows:

Theorem 5 If $F\colon \Gamma^{'} \to \Gamma^{'}$ is a E-type enriched nonexpansive mapping with $\lambda = \frac{1}{b+1}$, and fulfills the condition (I). If $\{r_n\}$ is a sequence given in (10) and $\mathrm{Fix}(F) \neq \emptyset$, then $r_n \to r^* \in \mathrm{Fix}(F)$.

Proof. Now using the Lemma 6, we get that

$$\lim_{n\to\infty} \|r_n - F_\lambda r_n\| = 0.$$

If we use the condition (I) then we have

$$\lim_{n\to\infty} \phi(d(r_n, \mathrm{Fix}(F_\lambda))) \leq \lim_{n\to\infty} \|r_n - F_\lambda r_n\| = 0,$$

which implies that $\lim_{n\to\infty} \phi(d(r_n, \mathrm{Fix}(F_\lambda))) = 0$. As $\phi\colon [0, \infty) \to [0, \infty)$ is nondecreasing and $\phi(0) = 0$ and $\phi(a) > 0$, $\forall a \in (0, \infty)$, $\lim_{n\to\infty} d(r_n, \mathrm{Fix}(F_\lambda)) = 0$. By Theorem 3, we have $\{r_n\} \to r^* \in \mathrm{Fix}(F_\lambda) = \mathrm{Fix}(F)$. □

Let $F\colon \Gamma^{'} \to \Gamma^{'}$. Assume $\psi\colon [0, \infty) \to [0, \infty)$ is monotone increasing and continuous with $\lambda\psi(r) = \psi(\lambda r)$ and $\psi(0) = 0$ for all $r, s \in \Gamma^{'}$ with following inequality holds:

$$\|b(r-s) + Fr - Fs\| \leq \psi(\|r - Fr\|) + c(b+1)\|r - s\|. \tag{29}$$

Theorem 6 If $F\colon \Gamma \to \Gamma$ is a mapping that satisfies (29) and $\{r_n\}$ given in (10), then $r_n \to r^* \in \mathrm{Fix}(F)$.

Proof. If we set $b = \frac{1}{\lambda} - 1$, then (29) reduce to

$$\|Fr - Fs\| \leq \psi(\|r - F_\lambda r\|) + c\|r - s\|. \tag{30}$$

From (16), we have

$$\|u_n - r^*\| \leq \|r_n - r^*\| + \beta_n M. \tag{31}$$

Assume that, $r^* \in \mathrm{Fix}(F)$. Then using (30), we get

$$\begin{aligned}
\|v_n - r^*\| &= \|(1-\sigma_n)u_n + \sigma_n F_\lambda u_n - r^*\| \\
&\leq (1-\sigma_n)\|u_n - r^*\| + \sigma_n \|F_\lambda u_n - r^*\| \\
&\leq (1-\sigma_n)\|u_n - r^*\| + \sigma_n [\psi(\|r^* - F_\lambda r^*\|) + c\|u_n - r^*\|] \\
&= (1-(1-c)\sigma_n)\|u_n - r^*\|.
\end{aligned}$$

From (17) and (30), we have

$$\begin{aligned}
\|t_n - r^*\| &= \|F_\lambda((1-\eta_n)u_n + \eta_n F_\lambda v_n) - r^*\| \\
&\leq 2\psi(\|r^* - F_\lambda r^*\|) + c[(1-\eta_n)\|u_n - r^*\| + \eta_n c\|v_n - r^*\|] \\
&\leq c[(1-\eta_n)\|u_n - r^*\| + \eta_n c\|u_n - r^*\|] \\
&= c[1 - \eta_n + \eta_n c]\|u_n - r^*\| \\
&\leq c[1-(1-c)\eta_n)][1-(1-c)\sigma_n)]\|u_n - r^*\|.
\end{aligned}$$

Also,

$$\begin{aligned}
\|s_n - r^*\| &= \|F_\lambda((1-\theta_n)F_\lambda v_n + \theta_n F_\lambda t_n) - r^*\| \\
&\leq \psi(\|r^* - F_\lambda r^*\|) + c\|(1-\theta_n)F_\lambda v_n + \theta_n F_\lambda t_n - r^*\| \\
&\leq (1+2c)\psi(\|r^* - F_\lambda r^*\|) + c[(1-\theta_n)c\|v_n - r^*\| + \theta_n c\|t_n - r^*\|]
\end{aligned}$$

$$= c^2[(1-\theta_n)\|v_n - r^*\| + \theta_n\|t_n - r^*\|]. \tag{32}$$

Note that,

$$(1-\theta_n)\|v_n - r^*\| \leq (1-\theta_n)(1-(1-c)\sigma_n)\|u_n - r^*\|$$

$$= [(1-(\theta_n+\sigma_n)+(\sigma_n-\theta_n\sigma_n)c]\|u_n - r^*\|,$$

and

$$\theta_n\|t_n - r^*\| \leq \theta_n[c[1-(1-c)\eta_n)][1-(1-c)\sigma_n)]\|u_n - r^*\|]$$

$$= c[\theta_n - \theta_n\eta_n - \theta_n\sigma_n + c(\theta_n\eta_n + \theta_n\sigma_n) + \theta_n\eta_n\sigma_n$$

$$- 2c\theta_n\eta_n\sigma_n + c^2\theta_n\eta_n\sigma_n]\|u_n - r^*\|.$$

By (32), we have

$$\|s_n - r^*\| \leq c^2\big(1-\theta_n-\sigma_n+\theta_n\sigma_n + c[\theta_n+\sigma_n-2\theta_n\sigma_n-\theta_n\eta_n+\theta_n\eta_n\sigma_n]$$

$$+ c^2[\theta_n\eta_n+\theta_n\sigma_n-2\theta_n\eta_n\sigma_n] + c^3[\theta_n\eta_n\sigma_n]\big)\|u_n - r^*\|$$

$$\leq c^2\big(1-\theta_n-\sigma_n+\theta_n\sigma_n + c[\theta_n+\sigma_n-2\theta_n\sigma_n-\theta_n\eta_n+\theta_n\eta_n\sigma_n]$$

$$+ c[\theta_n\eta_n+\theta_n\sigma_n-2\theta_n\eta_n\sigma_n] + c[\theta_n\eta_n\sigma_n]\big)\|u_n - r^*\|$$

$$= c^2[1-\theta_n-\sigma_n+\theta_n\sigma_n+c\theta_n+c\sigma_n-c\theta_n\sigma_n]\|u_n - r^*\|$$

$$= c^2[1-(1-c)(\theta_n+\sigma_n-\theta_n\sigma_n)]\|u_n - r^*\|.$$

By using (31), we have

$$\|r_{n+1} - r^*\| = \|F_\lambda s_n - r^*\|$$

$$\leq \psi(\|r^* - F_\lambda r^*\|) + c\|s_n - r^*\|$$

$$\leq c\big[c^2[1-(1-c)(\theta_n+\sigma_n-\theta_n\sigma_n)]\big]\|u_n - r^*\|$$

$$\leq c^3[1-(1-c)(\theta_n+\sigma_n-\theta_n\sigma_n)]\|u_n-r^*\|$$

$$\leq c^3[1-(1-c)(\theta_n+\sigma_n-\theta_n\sigma_n)](\|r_n-r^*\|+\beta_n M). \tag{33}$$

Inductively, we obtain that

$$\|r_{n+1}-r^*\| \leq c^{3n}[1-(1-c)(\theta_n+\sigma_n-\theta_n\sigma_n)]\big(\|r_1-r^*\|+M(\sum_{j=1}^{n}\beta_j)\big). \tag{34}$$

As $0 < c^{3n}[1-(1-c)(\theta_n+\sigma_n-\theta_n\sigma_n)] < 1$, $\{r_n\}$ converges to r^*. □

Theorem 7 If Γ is nonempty closed and convex subset of a Banach space $\mathscr{B}$ and $F\colon \Gamma \to \Gamma$ is a mapping which satisfies (29). Then the sequence $\{r_n\}$ defined by (10) is F_λ-stable, where $\lambda = \frac{1}{b+1}$.

Proof. Assume that $\{p_n\}$ is an approximate sequence of $\{r_n\}$ in Γ. From (10), we have $r_{n+1} = f(F_\lambda, r_n)$ and $e_n = \|p_{n+1} - f(F_\lambda, p_n)\|$, $n \in \mathbb{N}$.

We now prove $\lim_{n\to\infty} e_n = 0$ iff $\lim_{n\to\infty} p_n = r^*$.

If $\lim_{n\to\infty} e_n = 0$ then, from (10) we have

$$\|p_{n+1}-r^*\| \leq \|p_{n+1}-f(F_\lambda, p_n)\| + \|f(F_\lambda, p_n)-r^*\|$$

$$= e_n + \|r_{n+1}-r^*\|. \tag{35}$$

Using Theorem 6, we get

$$\|p_{n+1}-r^*\| \leq e_n + c^3[1-(1-c)(\theta_n+\sigma_n-\theta_n\sigma_n)](\|p_n-r^*\|+\beta_n M).$$

If we set,

$$A_n = \|p_n - r^*\|, \text{ and } B_n = (1-c)(\theta_n+\sigma_n-\theta_n\sigma_n).$$

Then, we obtain that

$$A_n \leq c^3(1-B_n)A_n + \beta_n M + e_n.$$

As, $\lim_{n\to\infty} e_n = 0$, also $\frac{e_n}{B_n} \to 0$ as $n \to \infty$.

If we use Lemma 2, then we obtain that $\lim_{n\to\infty} A_n = 0$ and further we have $\lim_{n\to\infty} p_n = r^*$.

Conversely, let $\lim_{n\to\infty} p_n = r^*$. Note that,

$$e_n = \|p_{n+1} - f(F_\lambda, p_n)\|$$

$$\leq \|p_{n+1} - r^*\| + \|f(F_\lambda, p_n) - r^*\|$$

$$\leq \|p_{n+1} - r^*\| + c^3[1-(1-c)(\theta_n + \sigma_n - \theta_n\sigma_n)]\|p_n - r^*\| + \beta_n M, \tag{36}$$

which give $\lim_{n\to\infty} e_n = 0$. □

3. Numerical experiment

Here we present some numerical examples to visualize the convergence. For all of the numerical computations the MATLAB version R2018a was used. We illustrate the convergence of a sequence obtained from two different mappings (given in Examples 1 and 2) to show the performance of our proposed algorithm in (10) with other comparable algorithms. We used the different parameters to visualize the efficiency of different iterative algorithms. We use different cases given as follows.

Case 1:

$$\sigma_n = \frac{1}{n^2+2n+1}, \quad \eta_n = \frac{3n}{4n^2+4n+5}, \quad \theta_n = \frac{n^2+1}{n^2+n+7}.$$

Case 2:

$$\sigma_n = \frac{n^2+n}{n^2+4n+5}, \quad \eta_n = \frac{n+1}{n^2+n+1}, \quad \theta_n = \frac{n^2+1}{n^2+2n+10}.$$

Case 3:

$$\sigma_n = \frac{8n}{(9n+1)^2}, \quad \eta_n = \frac{15n}{(n+7)^2}, \quad \theta_n = \frac{n^2+1}{n^2+n+7}.$$

Case 4:

$$\sigma_n = \frac{8n}{(2n+5)^2}, \quad \eta_n = \frac{n^2}{(n+3)^2}, \quad \theta_n = \frac{n^2+1}{n^2+3n+1}.$$

Example 1 Take $\Gamma = [0, 4] \subset \mathbb{R}$ and $F\colon \Gamma \to \Gamma$ by

$$F(r) = \begin{cases} 0, & \text{if} \quad r \in [0, 4), \\ \\ 3, & r = 4. \end{cases}$$

To show, F is E-type enriched nonexpansive mapping, here we have the different cases:
Case 1: For $r \leq 3$ and $s = 4$,

$$\begin{aligned}\|b(r-s)+r-Fs\| &= \|b(r-4)+(r-3)\| \\ &\leq b\|(r-4)\|+\|(r-3)\| \\ &\leq \|r\|+(b+1)\|r-4\| \\ &\leq \|r-Fr\|+(b+1)\|r-s\|.\end{aligned}$$

Case 2: If $r > 3$ and $s = 4$, then we have

$$\begin{aligned}\|b(r-s)+r-Fs\| &= \|b(r-4)+(r-3)\| \\ &\leq b\|(r-4)\|+\|(r-3)\| \\ &\leq \|r\|+(b+1)\|r-4\| \\ &\leq \|r-Fr\|+(b+1)\|r-s\|.\end{aligned}$$

Case 3: If $r = 4$ and $s \neq 4$, then we get

$$\begin{aligned}\|b(r-s)+r-Fs\| &= \|b(s-4)+4)\| \\ &\leq 4+b\|s-4\| \\ &\leq 4\|r-Fr\|+(b+1)\|r-s\|.\end{aligned}$$

For $\lambda = \frac{1}{2}$, we have

$$F_{\frac{1}{2}}(r) = \begin{cases} \frac{r}{2}, & \text{if} \quad r \in [0,4), \\ \\ \frac{7}{2}, & r = 4. \end{cases}$$

Note that, $\mathrm{Fix}(F) = \mathrm{Fix}(F_\lambda) = \{0\}$. Take the initial value $r_0 = 2$, $r_1 = 4$, and compare convergence of our proposed algorithm with the other existing some algorithms. The Figures 1, 2 and Table 1 give our proposed sequence shows the batter performance in the sense of convergence than the other existing sequences.

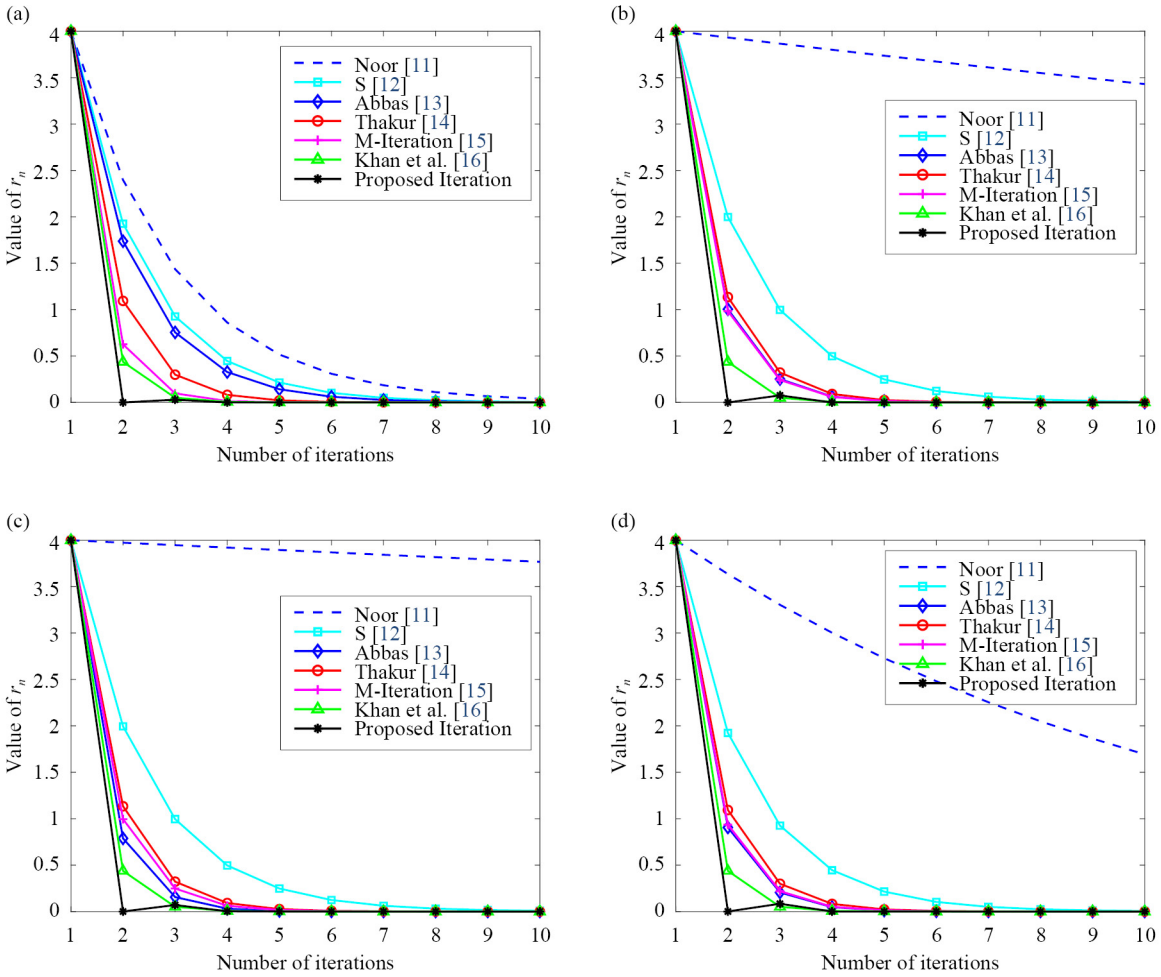


Figure 1. Convergence behavior for mapping defined in Example 1

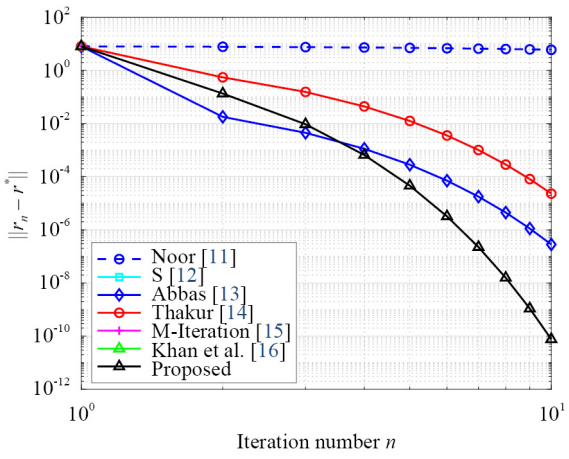


Figure 2. Log-log accuracy profile for Example 1

Table 1. Comparison of iterative methods for Example 1

Method	Iterations	Final Error	CPU Time
Noor [11]	10	5.95560704472654	0.0217918
S-Iteration [12]	10	0	0.0067203
Abbas [13]	10	$2.81329613889126 \times 10^{-7}$	0.0093002
Thakur [14]	10	$2.30456005612844 \times 10^{-5}$	0.0096969
M-Iteration [15]	10	0	0.0083084
Khan et al. [16]	10	0	0.0058506
Proposed	10	$7.63103302335805 \times 10^{-11}$	0.0021468

Example 2 Let $\Gamma = [0, 8] \subset \mathbb{R}$ be equipped with a usual norm. Define the mapping $F: \Gamma \to \Gamma$ by

$$F(r) = \cos r + \sin r.$$

Note that, F is E-type enriched nonexpansive mapping. If we set $\lambda = \frac{1}{2}$, then we get

$$F_{\frac{1}{2}}(r) = \frac{r + \cos r + \sin r}{2}.$$

Also, $\mathrm{Fix}(F) = \mathrm{Fix}(F_\lambda) = \{1.25872\}$.Take the initial value $r_2 = 2$, $r_1 = 8$, and compare the convergence behavior of our proposed algorithm with the other existing comparable algorithms. We observe, from the Figures 3, 4 and Table 2 give our proposed sequence shows the batter performance in the sense of convergence than the other existing sequences.

Table 2. Comparison of iterative methods for Example 2

Method	Iterations	Final Error	CPU Time (s)
Noor [11]	10	7.04453137676481	0.0190782
S-Iteration [12]	10	1.25872895946892	0.0062787
Abbas [13]	10	1.25872817749299	0.0082705
Thakur [14]	10	1.25872817750530	0.0078922
M-Iteration [15]	10	1.25872817749279	0.0084314
Khan et al. [16]	10	1.25872817749282	0.0046147
Proposed	10	1.25872817749268	0.0037568

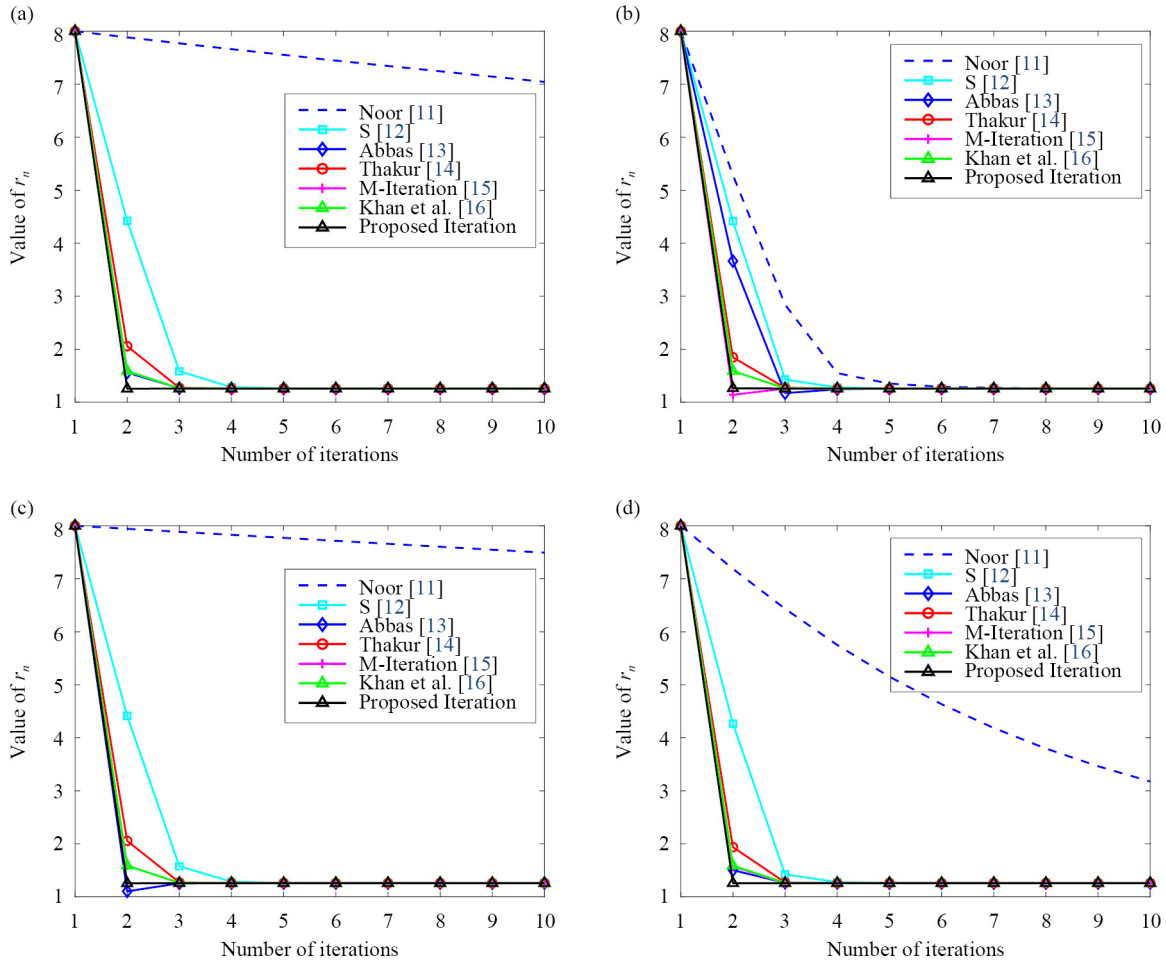


Figure 3. Convergence behavior for mapping defined in Example 2

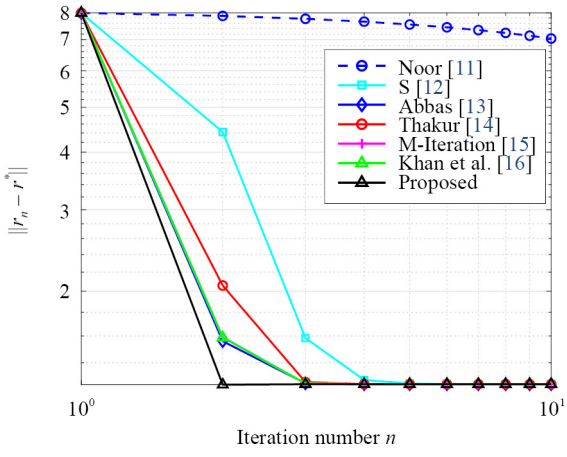


Figure 4. Log-log accuracy profile for Example 2

Example 3 Let $\Gamma = [0, 4] \subset \mathbb{R}$, define the mapping F: $\Gamma \to \Gamma$ by

$$F(r) = \frac{1-r}{2} + \frac{\cos 2r}{4},$$

F is E-type enriched nonexpansive mapping and if we set $\lambda = \frac{2}{3}$, then we have

$$F_{\frac{2}{3}}(r) = \frac{\cos 2r}{6} + \frac{1}{3}.$$

Also, $\mathrm{Fix}(F) = \mathrm{Fix}(F_\lambda) = \{0.43975\}$. We set the initial values $r_2 = 2.5$, $r_1 = 3$, and then compare the convergence behavior of different algorithms. Note that, Figures 5, 6 and Table 3 show our scheme has the batter convergence than the other comparable sequences.

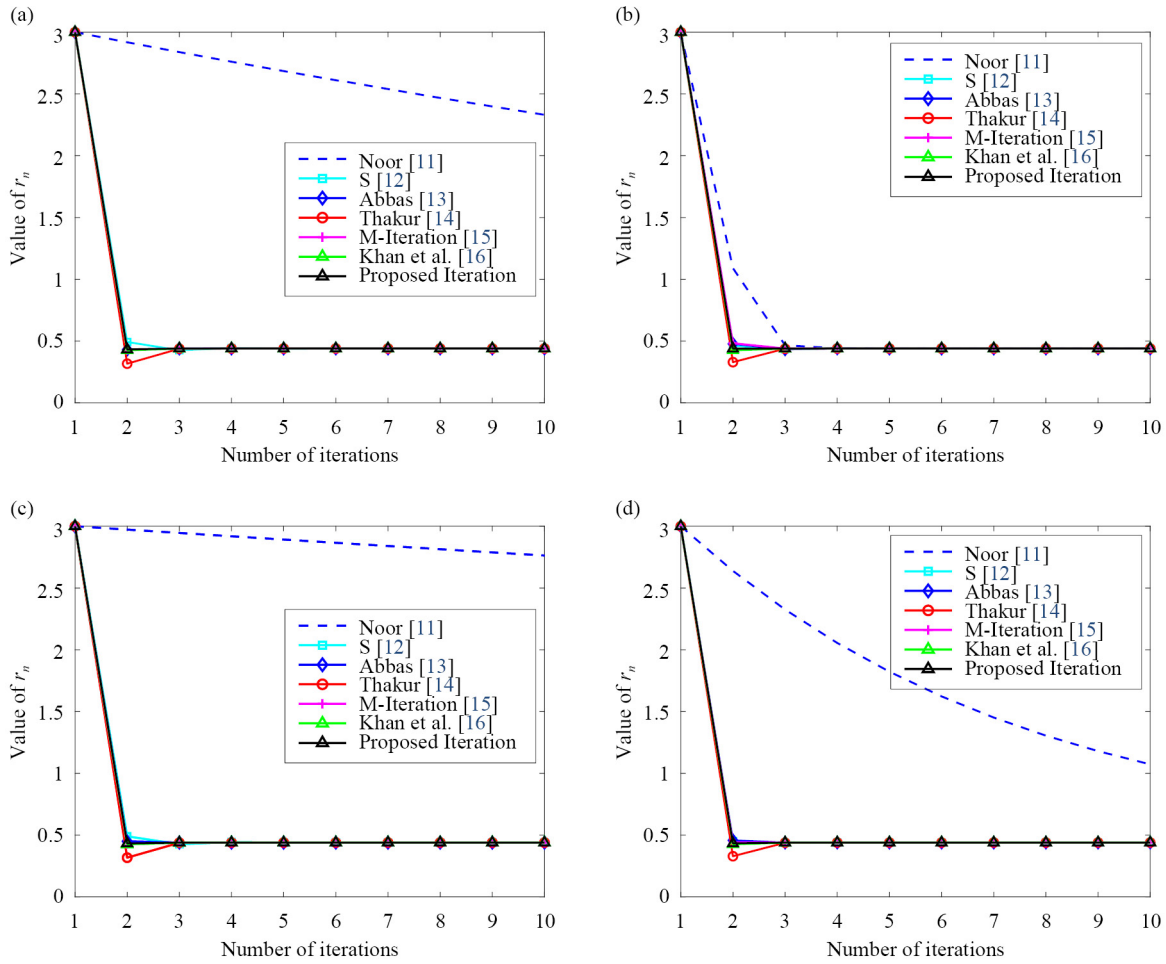


Figure 5. Convergence behavior for mapping defined in Example 3

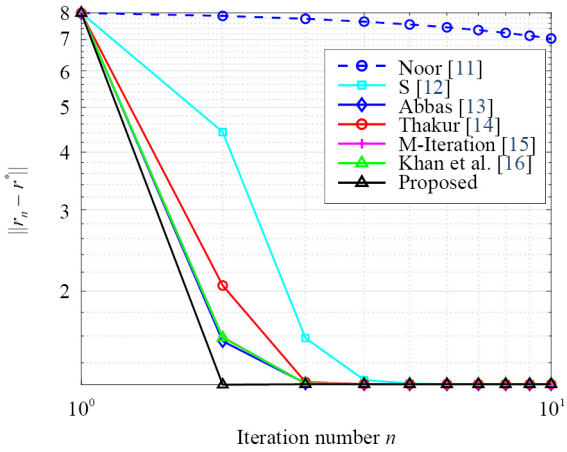


Figure 6. Log-log accuracy profile for Example 3

Table 3. Comparison of iterative methods for Example 3

Method	Iterations	Final Error	CPU Time (s)
Noor [11]	10	6.06060047978628	0.0189922
S-Iteration [12]	10	0.439618485718373	0.0068764
Abbas [13]	10	0.439622217580906	0.0085525
Thakur [14]	10	0.439622217579072	0.0081376
M-Iteration [15]	10	0.439622217592465	0.0077617
Khan et al. [16]	10	0.439622217597609	0.0050013
Proposed	10	0.439622217579027	0.0032279

4. Applications

In the subsequence section, we present an important applications of our results. First, we recall some concepts: In the sequel we suppose H is Hilbert space and Γ is its a nonempty closed convex subset. The mapping $F: \Gamma \to H$ is called:

1. *monotone* if $\langle Fr - Fs, r - s\rangle \geq 0$ for all $r, s \in \Gamma$,
2. β-*strongly monotone* if there exists a constant $\beta > 0$ such that $\langle Fr - Fs, r - s\rangle \geq \beta\|r - s\|^2$ for all $r, s \in \Gamma$,
3. γ-*inverse strongly monotone* (γ-ism) if we have a fixed $\gamma > 0$ and

$$\langle Fr - Fs, r - s\rangle \geq \gamma\|Fr - Fs\|^2, \quad \forall r, s \in \Gamma.$$

As, every Hilbert space is uniformly convex, and a nonexpansive mapping is a particular case of a mapping that satisfies condition (E), our results applicable to the following classes of problems and applications.

4.1 *Variational inequality problem*

An important nonlinear problem known as the Variational Inequality Problems (VIP) and these inequalities provide an efficient and unified framework for dealing with a wide class of nonlinear boundary value problems linked with Partial

Differential Equations (PDEs) of elliptic or parabolic type. A variational inequality generally describes a relationship between an operator acting on elements of a certain set and the vectors that belong to that same set. Over the years, The variational inequality problem has been studied by numerous researchers from various perspectives because of its broad theoretical significance and wide range of practical applications. In particular, these inequalities serve as useful mathematical tools to model the several important practical problems for example the image and signal processing, optimal control, and other applied science fields (for more details, see [30–34]).

Formally, the VIP of a nonlinear operator F defined on Γ is denoted by $VI(\Gamma, F)$ and introduced in [35]. The problem $VI(\Gamma, F)$ is to find a vector $r^* \in \Gamma$ such that

$$\langle Fr^*, r^* - s\rangle \geq 0, \quad \forall s \in \Gamma.$$

The collection of vectors which satisfy the above inequality represents by Ω_1. It is important to note that the VIP is closely connected to several fundamental mathematical problems, for instance, complementarity and convex minimization problems, the the task to $r' \in H$ that satisfies $Fr' = 0$, which is essentially called it the null point problem.

The VIP $(VI(\Gamma, F))$ can also be expressed as a fixed point equation. Specifically, it is equivalent to finding a point $r^* \in \Gamma$ satisfying

$$r^* = P_\Gamma(I - \alpha F)r^*,$$

where $\alpha > 0$ is a fixed and P_Γ denotes the metric projection from H onto Γ. If we define the operator $T_\alpha = P_\Gamma(I - \alpha F)$, in this case, the problem can be simplified to determining a fixed point of the corresponding operator T_α, that is, $T_\alpha r^* = r^*$. Moreover, if F is L-Lipschitz and β-strongly monotone, and hence T_α becomes a contraction mapping on Γ whenever $0 < \alpha < 2\beta/L^2$. Using contraction principle, it can be deduced that $\Omega_1 = \{r^*\}$, and the sequence obtained through the Picard iterative process,

$$r_{n+1} = T_\alpha r_n, \quad n \in \mathbb{N},$$

converges to a fixed point r^*.

If the mappings are nonexpansive, the related fixed point problem that provides the solution of VIP, along with its related result and proof, can be derived from Lemma 6, which is stated as follows.

Theorem 8 Let Γ be a set as describe above F: $\Gamma \to H$ be an L-Lipschitz continuous and β-strongly monotone operator, and $\{\theta_n\}$, $\{\eta_n\}$ and $\{\sigma_n\}$ as given in (10). Then the iterative sequence $\{r_n\}$ defined by

$$\begin{cases} r_0 \in \Gamma \\ r_{n+1} = P_\Gamma(1 - \alpha F)s_n \\ s_n = P_\Gamma(1 - \alpha F)\big((1 - \theta_n)P_\Gamma(1 - \alpha F)v_n + \theta_n P_\Gamma(1 - \alpha F)t_n\big) \\ t_n = P_\Gamma(1 - \alpha F)\big((1 - \eta_n)u_n + \eta_n P_\Gamma(1 - \alpha F)v_n\big) \\ v_n = (1 - \sigma_n)u_n + \sigma_n P_\Gamma(1 - \alpha F)u_n \\ u_n = r_n + o_n(r_n - r_{n-1}), \quad \forall\, n \in \mathbb{N} \end{cases} \tag{37}$$

converges weakly to $r^* \in \Omega_1$.

4.2 *Constrained optimization problem*

Constrained optimization problems are fundamental in various fields engineering and sciences, particularly in signal and image processing, where the objective is typically to minimize a convex and differentiable function F over a feasible region defined by a closed convex set Γ in H. Such problems arise naturally when certain physical, structural, or data-related restrictions must be satisfied while finding an optimal solution.

Let Γ be a nonempty, closed, and convex subset of a Hilbert space H, and let P_Γ denote the metric projection from H onto Γ. Assume that a mapping $F: \Gamma \to H$ is γ-ism for some constant $\gamma > 0$. It is well established result that the operator $P_\Gamma(I - \alpha F)$ is nonexpansive whenever $\alpha \in (0, 2\gamma)$. Moreover, recall the fact that, every Lipschitz continuous with constant L is $\frac{2}{L}$-ism. Now, we define an iterative process that produces a sequence of points in Γ, that weakly converges to the point that is minimizer of the function F over Γ.

Theorem 9 Suppose that Γ is a set as describe above and F is differentiable and convex $U \subset \Gamma$, where U is open. Also ∇F Lipschitzian map on U with constant L, set $\alpha \in (0, 2/L)$. Suppose F has the minimizers of over Γ exist. Consider a sequence $\{r_n\} \subset \Gamma$ defined by

$$\begin{cases} r_0 \in \Gamma \\ r_{n+1} = P_\Gamma(1 - \alpha\nabla F)s_n \\ s_n = P_\Gamma(1 - \alpha\nabla F)\big((1 - \theta_n)P_\Gamma(1 - \alpha\nabla F)v_n + \theta_n P_\Gamma(1 - \alpha\nabla F)t_n\big) \\ t_n = P_\Gamma(1 - \alpha\nabla F)\big((1 - \eta_n)u_n + \eta_n P_\Gamma(1 - \alpha\nabla F)v_n\big) \\ v_n = (1 - \sigma_n)u_n + \sigma_n P_\Gamma(1 - \alpha\nabla F)u_n, \\ u_n = r_n + o_n(r_n - r_{n-1}), \quad \forall\, n \in \mathbb{N} \end{cases} \tag{38}$$

where $\{\theta_n\}, \{\eta_n\}, \{\sigma_n\} \in (0, 1)$. Then $\{r_n\}$ weakly converges to a minimizer of F.

4.3 *Split Feasibility Problem (SFP)*

The SFP [36], has become a significant area of research because of its broad theoretical and practical applications. In general, the SFP seeks to find a point in a closed convex subset of a Hilbert (or normed) space so that the image of a bounded linear map lie in convex and closed subset of other space. This formulation provides a powerful mathematical framework for many real applications, including image reconstruction and recognition, signal processing, and intensity-modulated radiation therapy, among others. Due to its practical importance and mathematical depth, the SFP has drawn substantial attention from many researchers (see [37–41] for additional details and related results).

Solving the fixed point equation for nonexpansive is curial because it plays a key role for SFP, as it provides a natural link between feasibility and fixed point equation of a mapping.

Recall that the SFP associated with Γ and F, denoted by $SFP(\Gamma, F)$, involves finding a point $r \in \Gamma$ with Fr belongs to Γ', where Γ is a closed convex subset of a Hilbert space H_1, Γ' is a closed convex subset of another Hilbert space H_2, and $F: H_1 \to H_2$ is bounded linear. Equivalently, the SFP in the form of fixed point equation is as follows:

$$r = P_\Gamma(I - \gamma F^*(I - P_{\Gamma'})F)r,$$

where P_Γ and $P_{\Gamma'}$ are the projection mappings, $\gamma > 0$ is fixed, and F^* represents the adjoint operator for mapping F. Further the mapping $P_\Gamma(I - \gamma F^*(I - P_{\Gamma'})F)$ is nonexpansive.

A mapping F is called an average mapping if it could be represented by

$$F = (1-\alpha)I + \alpha T,$$

here $\alpha \in (0, 1)$ and T is nonexpansive operator.

Define the function $h\colon \Gamma \to \mathbb{R}$ by

$$h(r) := \tfrac{1}{2}\|(F - P_{\Gamma'}F)r\|, \quad r \in \Gamma.$$

We then consider the minimization problem

$$\min_{r\in\Gamma} h(r).$$

According to [42], the gradient of h is given by $\nabla h = F^*(I - P_{\Gamma'})F$, where F^* denotes the adjoint of F. Also the operator $I - P_{\Gamma'}$ is nonexpansive, so we have ∇h is L-Lipschitz continuous with $L = \|F\|^2$. Consequently, ∇h is $1/L$-ism, and for any $0 < \alpha < 2/L$, the operator $I - \alpha\nabla h$ is an averaged operator.

As a result, the composition $P_\Gamma(I - \alpha\nabla h)$ is also an averaged operator. Let us define

$$F := P_\Gamma(I - \alpha\nabla h).$$

It is easy to observe that the solution set of the split feasibility problem $SFP(\Gamma, F)$ coincides with the set of fixed points of F.

We now introduce an iterative scheme that can be employed to approximate the solutions of the problem $SFP(\Gamma, F)$.

Theorem 10 Assume that F, h are as give above and $\{\theta_n\}$, $\{\eta_n\}$ and $\{\sigma_n\}$ as given in (10). Then the iterative sequence $\{r_n\}$ in Γ defined by:

$$\begin{cases} r_0 \in \Gamma \\ r_{n+1} = P_\Gamma(1 - \alpha\nabla h)s_n \\ s_n = P_\Gamma(1 - \alpha\nabla h)((1-\theta_n)P_\Gamma(1 - \alpha\nabla h)v_n + \theta_n P_\Gamma(1 - \alpha\nabla h)t_n) \\ t_n = P_\Gamma(1 - \alpha\nabla h)((1-\eta_n)u_n + \eta_n P_\Gamma(1 - \alpha\nabla h)v_n) \\ v_n = (1-\sigma_n)u_n + \sigma_n P_\Gamma(1 - \alpha\nabla h)u_n \\ u_n = r_n + o_n(r_n - r_{n-1}), \quad \forall\, n \in \mathbb{N} \end{cases} \tag{39}$$

weakly converges to solution of $SFP(\Gamma, F)$.

4.4 *Application delay functional differential equation*

El and Omar [43] proved the existence for Volterra–Stieltjes functional equation involving derivation. Similar equations have also been studied by other authors (see [44, 45]).

Here, we provide the weak solution by approximating it with the sequence defined in (10).

Assume that the Banach space $\mathbb{B}$ is reflexive having norm $\|\cdot\|_{\mathbb{B}}$ with its dual $\mathbb{B}^*$. Let $L > 0$ and $I = [0, L]$. The set of continuous functions is denoted by $\mathscr{C}[0, L]$ with the norm given by:

$$\|r\|_{\mathscr{C}} = \sup_{j\in I} \|r(j)\|_{\mathbb{B}}, \ \ r \in \mathscr{C}[0, L].$$

A second-order initial value problem is given by:

$$\frac{d}{dj} r(j) = f_1\left(j, \int_0^{\phi(j)} f_2(j, s, r(s)) d_s g(j, s)\right), \ \ j \in I \tag{40}$$

with the initial condition

$$r(0) = r_0. \tag{41}$$

Assume that,

i. The ϕ is a mapping on I is which is not increasing and continuous with $\phi(j) \leq j$.

ii. Take $q = 1, 2$, every map f_q form $I \times \mathbb{B}$ to $\mathbb{B}$ is continuous weakly and satisfies

$$|\phi(f_q(j, r) - f_q(j, s))| \leq c_q |\phi(r - s)|, \ c_q > 0, \ \forall (j, r), \ (j, s) \in I \times \mathbb{B}, \ \phi \in \mathbb{B}^*$$

where, each c_q is a Lipschitz constant.

iii. The g is continuous function from $I \times \mathbb{R}$ to $\mathbb{R}$ with

$$K = \max\{\sup|g(j, \phi(j))| + \sup|g(j, 0)|\} \text{ is finite.}$$

iv. $c_1 c_2 K j < 1$.

Lemma 7 [43] Solving an initial value problem (40) and (41) is same as to solve the integral equation given by:

$$r(j) = r_0 + \int_0^j f_1\left(\tau, \int_0^{\phi(j)} f_2(\tau, \alpha, r(\alpha))\right) d_\alpha g(\tau, \alpha) d\tau.$$

Let F_1 be an operator on $\mathscr{C}[0, L]$ given by

$$F_1(r(j)) = r_0 + \int_0^j f_1\left(\tau, \int_0^{\phi(j)} f_2(\tau, \alpha, r(\alpha))\right) d_\alpha g(\tau, \alpha) d\tau.$$

We now approximate the solution of (40) and (41) by using the iterative scheme given in (10).

Theorem 11 Suppose that (i)-(iv) hold and $\{r_n\}$ is a sequence in (10) obtained by an operator F_1. Then $\{r_n\}$ converges to r^*, which is indeed a weak solution of initial value problem given by (40) and (41).

Proof. Note that,

$$\begin{aligned}
\|F_1u_n - r^*\|_{\mathscr{C}} &= \|F_1u_n - F_1r^*\| \\
&\leq \left\| r_0 + \int_0^j f_1\left(\tau, \int_0^{\phi(\tau)} f_2(\tau, \alpha, u_n(\alpha))\right) d_\alpha g(\tau, \alpha) d\tau - \right. \\
&\qquad \left. r_0 - \int_0^j f_1\left(\tau, \int_0^{\phi(\tau)} f_2(\tau, \alpha, r^*(\alpha))\right) d_\alpha g(\tau, \alpha) d\tau \right\|_{\mathscr{C}} \\
&= \left| F_1\left(\int_0^j f_1\left(\tau, \int_0^{\phi(\tau)} f_2(\tau, \alpha, u_n(\alpha))\right) d_\alpha g(\tau, \alpha) d\tau - \right. \right. \\
&\qquad \left. \left. \int_0^j f_1\left(\tau, \int_0^{\phi(\tau)} f_2(\tau, \alpha, r^*(\alpha))\right) d_\alpha g(\tau, \alpha)\right) d\tau \right|.
\end{aligned}$$

Using assumption (ii), we have

$$\begin{aligned}
\|F_1r_n - r^*\|_{\mathscr{C}} &\leq \int_0^j c_1 \left| F_1\left(\int_0^{\phi(\tau)} f_2(\tau, \alpha, u_n(\alpha))) d_\alpha g(\tau, \alpha) - \right. \right. \\
&\qquad \left. \left. \int_0^{\phi(\tau)} f_2(\tau, \alpha, r^*(\alpha)) d_\alpha g(\tau, \alpha)\right) \right| d\tau \\
&\leq c_1 \int_0^j \int_0^{\phi(\tau)} \left| F_1\left(f_2(\tau, \alpha, u_n(\alpha))) - \right. \right. \\
&\qquad \left. \left. f_2(\tau, \alpha, r^*(\alpha))\right) d_\alpha g(\tau, \alpha) \right| d\tau,
\end{aligned}$$

and

$$\begin{aligned}
\|F_1u_n - r^*\|_{\mathscr{C}} &\leq c_1 \int_0^j \int_0^{\phi(\tau)} c_2 \left| F_1\left(u_n(\alpha) - r^*(\alpha)\right) d_\alpha g(\tau, \alpha) \right| d\tau \\
&\leq c_1c_2 \|u_n - r^*\|_{\mathscr{C}} \int_0^j \int_0^{\phi(\tau)} d_\alpha g(\tau, \alpha) d\tau
\end{aligned}$$

$$= c_1c_2\|u_n - r^*\|_{\mathscr{C}} \int_0^j (g(\tau, \phi(\tau)) - g(\tau, 0))d\tau$$

$$\leq c_1c_2K\|u_n - r^*\|_{\mathscr{C}} \int_0^j d\tau$$

$$= c_1c_2Kj\|u_n - r^*\|_{\mathscr{C}}. \tag{42}$$

As $c_1c_2Kj < 1$, by (42) we have

$$\|F_1u_n - r^*\|_{\mathscr{C}} \leq \|u_n - r^*\|_{\mathscr{C}}. \tag{43}$$

Now, let $u_n^{'} = (1-\eta_n)u_n + \eta_nF_1v_n$. Following arguments similar to those used in obtaining (42) and by (43), we get that

$$\|u_n^{'} - r^*\|_{\mathscr{C}} = \|(1-\eta_n)u_n + \eta_nF_1v_n - r^*\|_{\mathscr{C}}$$

$$\leq (1-\eta_n)\|u_n - r^*\|_{\mathscr{C}} + \eta_n\|F_1v_n - r^*\|_{\mathscr{C}}$$

$$\leq \|u_n - r^*\|_{\mathscr{C}}. \tag{44}$$

Also,

$$\|t_n - r^*\|_{\mathscr{C}} \leq \|F_1u_n^{'} - r^*\|_{\mathscr{C}} \leq \|u_n^{'} - r^*\|_{\mathscr{C}} \leq \|u_n - r^*\|_{\mathscr{C}}. \tag{45}$$

Similarly, if we set $s_n^{'} = (1-\theta_n)F_1v_n + \theta_nF_1t_n$, then we have

$$\|s_n^{'} - r^*\|_{\mathscr{C}} = \|(1-\theta_n)F_1v_n + \theta_nF_1t_n - r^*\|_{\mathscr{C}}$$

$$\leq (1-\theta_n)\|F_1v_n - r^*\|_{\mathscr{C}} + \theta_n\|F_1t_n - r^*\|_{\mathscr{C}}$$

$$\leq \|u_n - r^*\|_{\mathscr{C}}, \tag{46}$$

and

$$\|s_n - r^*\|_{\mathscr{C}} \leq \|F_1s_n^{'} - r^*\|_{\mathscr{C}} \leq \|u_n - r^*\|_{\mathscr{C}}. \tag{47}$$

Similar to what obtained in (42), we get that

$$\|r_{n+1} - r^*\|_{\mathscr{C}} \leq \|F_1 s_n - r^*\|_{\mathscr{C}} \leq c_1 c_2 K j \|u_n - r^*\|_{\mathscr{C}}. \tag{48}$$

If we set $\omega = c_1 c_2 K j \in (0, 1)$, then by (16) and (42), we obtain that

$$\|r_{n+1} - r^*\|_{\mathscr{C}} \leq \omega[\|r_n - r^*\|_{\mathscr{C}} + \beta_n M]. \tag{49}$$

From Lemma 4, we have $r_n \to r^*$. □

5. Conclusions

The purpose of the given work in this manuscript is to introduce the inertial-type algorithm for fixed point of E-type enriched nonexpansive mappings in the setup of uniformly convex Banach spaces. The proposed algorithm was analyzed theoretically and numerically. The strong and weak convergence results are obtained, and stability properties of the proposed iterative scheme were studied. Numerical experiments and graphical comparisons demonstrated that the proposed iterative sequence has higher convergence rate over some known existing approaches. Furthermore, efficiency and applicability of the proposed scheme were employed in approximating the solution of variational inequality problems, constrained optimization problems, and split feasibility problems. In addition, we applied the algorithm to solve a particular functional equation. Overall, the presented results confirm that the proposed inertial iterative scheme is a simple, efficient, and reliable tool to solve the fixed points equation other nonlinear operator equations in applied mathematics.

Author's contribution

All authors were involved in the conception, design, and computational aspects of the study. Asghar prepared the initial version of the manuscript, while the remaining authors provided feedback, revisions, and then approved final submission.

Funding

There is no funding for the preparation of this manuscript.

Conflict of interest

The authors declare no conflict of interest associated with this work.

References

[1] Tas N. An overview of triple-composed S-metric spaces with few fixed-point theorems. *Fundamental Journal of Mathematics and Applications*. 2025; 8(3): 169–179. Available from: https://doi.org/10.33401/fujma.1623200.

[2] Atalan Y. A new iterative process based on almost contraction operators. *Fundamental Journal of Mathematics and Applications*. 2025; 8(3): 161–168. Available from: https://doi.org/10.33401/fujma.1478028.

[3] Acar C, Ozturk V. Fixed point theorems for almost α–ψ-contractive mappings in F-metric spaces. *Fundamental Journal of Mathematics and Applications*. 2024; 7(4): 203–211. Available from: https://doi.org/10.33401/fujma.1400093.

[4] Sahin A, Demir B. Some fixed point theorems for the new generalizations of *P*-contractive maps. *Fundamental Journal of Mathematics and Applications*. 2024; 7(4): 218–225. Available from: https://doi.org/10.33401/fujma.1466353.

[5] Banach S. Sur les operations dans les ensembles abstraits et leurs applications [On operations in abstract sets and their applications]. *Fundamenta Mathematicae*. 1922; 3: 133–187.

[6] Suzuki T. Fixed point theorems and convergence theorems for some generalized nonexpansive mappings. *Journal of Mathematical Analysis and Applications*. 2008; 340: 1088–1095. Available from: https://doi.org/10.1016/j.jmaa.2007.09.023.

[7] García-Falset J, Llorens-Fuster E, Suzuki T. Fixed point theory for a class of generalized nonexpansive mappings. *Journal of Mathematical Analysis and Applications*. 2011; 375(1): 185–195. Available from: https://doi.org/10.1016/j.jmaa.2010.08.069.

[8] Shukla R, Panicker R. Generalized enriched nonexpansive mappings and their fixed point theorems. *Abstract and Applied Analysis*. 2023; 2023(1): 5572893. Available from: https://doi.org/10.1155/2023/5572893.

[9] Mann WR. Mean value methods in iteration. *Proceedings of the American Mathematical Society*. 1953; 4: 506–510. Available from: https://doi.org/10.1090/S0002-9939-1953-0054846-3.

[10] Ishikawa S. Fixed points by a new iteration method. *Proceedings of the American Mathematical Society*. 1974; 44: 147–150. Available from: https://www.jstor.org/stable/2039245.

[11] Noor MA. New approximation schemes for general variational inequalities. *Journal of Mathematical Analysis and Applications*. 2000; 251: 217–229. Available from: https://doi.org/10.1006/jmaa.2000.7042.

[12] Agarwal RP, O'Regan D, Sahu DR. Iterative construction of fixed points of nearly asymptotically nonexpansive mappings. *Journal of Nonlinear and Convex Analysis*. 2007; 8(1): 61–79. Available from: http://yokohamapublishers.jp/online2/opjnca/vol8/p61.html.

[13] Abbas M, Nazir T. A new faster iteration process applied to constrained minimization and feasibility problems. *Matematicki Vesnik*. 2014; 66: 223–234. Available from: https://vesnik.matf.bg.ac.rs/vol/mv14211.pdf.

[14] Thakur BS, Thakur D, Postolache M. A new iteration scheme for approximating fixed points of nonexpansive mappings. *Filomat*. 2016; 30: 2711–2720. Available from: http://doiserbia.nb.rs/img/doi/0354-5180/2016/0354-51801610711T.pdf.

[15] Ullah K, Arshad M. Numerical reckoning of fixed points for Suzuki's generalized nonexpansive mappings via new iteration process. *Filomat*. 2018; 32(1): 187–196. Available from: https://doi.org/10.2298/FIL1801187U.

[16] Khan SH, Dilawer H, Iqbal H, Abbas M. Convergence and ω^2 stability analysis of a hybrid-type iterative scheme with application to functional delay differential equations. *Axioms*. 2025; 14(6): 475. Available from: https://doi.org/10.3390/axioms14060475.

[17] Gürsoy F, Khan AR, Ertürk M, Karakaya V. Weak w^2 and data dependence of Mann iteration method in Hilbert spaces. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*. 2019; 113: 11–20. Available from: https://doi.org/10.1007/s13398-017-0447-y.

[18] Hacıoğlu E, Gürsoy F, Maldar S, Atalan Y, Milovanović GV. Iterative approximation of fixed points and applications to two-point second-order boundary value problems and to machine learning. *Applied Numerical Mathematics*. 2021; 167: 143–172. Available from: https://doi.org/10.1016/j.apnum.2021.04.020.

[19] Khatoon S, Uddin I, Ali J, George R. Common fixed points of two nonexpansive mappings via a faster iteration procedure. *Journal of Function Spaces*. 2021; 2021: 9913540. Available from: https://doi.org/10.1155/2021/9913540.

[20] Maniu G. On a three-step iteration process for Suzuki mappings with a qualitative study. *Numerical Functional Analysis and Optimization*. 2020; 41: 929–949. Available from: https://doi.org/10.1080/01630563.2020.1719415.

[21] Hacıoğlu E. A comparative study on iterative algorithms of almost contractions in the context of convergence, stability, and data dependency. *Computational and Applied Mathematics*. 2021; 40: 282. Available from: https://link.springer.com/article/10.1007/s40314-021-01671-8.

[22] Houmani H, Turcanu T. CQ-type algorithm for reckoning best proximity points of EP-operators. *Symmetry*. 2019; 12: 4. Available from: https://doi.org/10.3390/sym12010004.

[23] Berinde V. Picard iteration converges faster than Mann iteration for a class of quasi-contractive operators. *Fixed Point Theory and Applications*. 2004; 2004: 1–9. Available from: https://doi.org/10.1155/S1687182004311058.

[24] Opial Z. Weak convergence of the sequence of successive approximations for nonexpansive mappings. *Bulletin of the American Mathematical Society*. 1967; 73: 591–597.

[25] Ostrowski AM. The round-off stability of iterations. *Zeitschrift für Angewandte Mathematik und Mechanik*. 1967; 47: 77–81. Available from: https://doi.org/10.1002/ZAMM.19670470202.

[26] Gallagher TM. The demiclosedness principle for mean nonexpansive mappings. *Journal of Mathematical Analysis and Applications*. 2016; 439: 832–842. Available from: https://doi.org/10.1016/j.jmaa.2016.03.029.

[27] Schu J. Weak and strong convergence to fixed points of asymptotically nonexpansive mappings. *Bulletin of the Australian Mathematical Society*. 1991; 43: 153–159. Available from: https://doi.org/10.1017/S0004972700028884.

[28] Henry D. *Geometric Theory of Semilinear Parabolic Equations*. Vol. 840. Springer; 2006. Available from: https://doi.org/10.1112/BLMS/14.3.270.

[29] Senter HF, Dotson WG. Approximating fixed points of nonexpansive mappings. *Proceedings of the American Mathematical Society*. 1974; 44: 375–380. Available from: https://www.jstor.org/stable/2040440.

[30] Ceng LC. A modified inertial subgradient extragradient method for solving pseudomonotone variational inequalities and common fixed point problems. *Fixed Point Theory*. 2020; 21: 93–108.

[31] Ceng LC. A subgradient-extragradient method for bilevel equilibrium problems with the constraints of variational inclusion systems and fixed point problems. *Communications in Optimization Theory*. 2021; 2021(4): 1–16. Available from: https://cot.mathres.org/issues/COT20214.pdf.

[32] Liu L, Cho SY, Yao JC. Convergence analysis of an inertial Tseng's extragradient algorithm for solving pseudomonotone variational inequalities and applications. *Journal of Nonlinear and Variational Analysis*. 2021; 5: 627–644. Available from: https://doi.org/10.23952/jnva.5.2021.4.09.

[33] Liu L, Qin X. Strong convergence theorems for solving pseudo-monotone variational inequality problems and applications. *Optimization*. 2022; 71: 3603–3626. Available from: https://doi.org/10.1080/02331934.2021.1905641.

[34] Liu L, Yao JC. Iterative methods for solving variational inequality problems with a double hierarchical structure in Hilbert spaces. *Optimization*. 2023; 72(10): 2195–2216. Available from: https://doi.org/10.1080/02331934.2022.2074297.

[35] Kinderlehrer D, Stampacchia G. *An Introduction to Variational Inequalities and Their Applications*. SIAM; 2000. Available from: https://epubs.siam.org/doi/book/10.1137/1.9780898719451.

[36] Censor Y, Elfving T. A multiprojection algorithm using Bregman projections in a product space. *Numerical Algorithms*. 1994; 8(2): 221–239. Available from: https://doi.org/10.1007/BF02142692.

[37] Bin Dehaish BA, Latif A, Bakodah HO, Qin X. A regularization projection algorithm for various problems with nonlinear mappings in Hilbert spaces. *Journal of Inequalities and Applications*. 2015; 2015(1): 51. Available from: https://doi.org/10.1186/s13660-014-0541-z.

[38] Lu T, Zhao L, He H. A multi-view on the CQ algorithm for split feasibility problems: From optimization lens. *Journal of Applied and Numerical Optimization*. 2020; 2: 387-399.

[39] Moudafi A. Byrne's extended CQ-algorithms in the light of Moreau Yosida regularization. *Applied Set-Valued Analysis and Optimization*. 2021; 3: 21–26. Available from: https://amu.hal.science/hal-02531553.

[40] Osilike MO, Chima EE. Mixed equilibrium and multiple-set split feasibility problems for asymptotically *k*-strictly pseudononspreading mappings. *Communications in Optimization Theory*. 2020; 2020: 1–13. Available from: https://cot.mathres.org/issues/COT202014.pdf.

[41] Xu T, Zhang Z, Shi L. Strong convergence of an explicit extragradient-like method for a split feasibility problem. *Journal of Nonlinear Functional Analysis*. 2022; 2022(16). Available from: https://doi.org/10.1155/2012/12503.

[42] Aubin JP, Cellina A. Differential inclusions with maximal monotone maps. In: *Differential Inclusions: Set-Valued Maps and Viability Theory*. Springer; 1984. p.139–171. Available from: https://doi.org/10.1007/978-3-642-69512-4_5.

[43] El-Sayed AMA, Omar YMY. On the weak solutions of a delay composite functional integral equation of Volterra-Stieltjes type in reflexive Banach space. *Mathematics*. 2022; 10(2): 245. Available from: https://doi.org/10.3390/math10020245.

[44] Gripenberg G, Londen S-O, Staffans O. *Volterra Integral and Functional Equations*. No. 34. Cambridge University Press; 1990. Available from: https://doi.org/10.1017/CBO9780511662805.

[45] Malique MA. *Numerical treatment of oscillatory delay and mixed functional differential equations arising in modelling*. Thesis. University of Liverpool; 2012. Available from: https://livrepository.liverpool.ac.uk/3062915/1/10041808.pdf.