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Research Article

Refinements of Fejér-Type Inequalities Involving Special Functions

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Abstract: Convexity associated with inequalities finds numerous and impressive applications in the field of applied mathematics, especially when it comes to fractional analysis. In this paper, we investigate the new equalities. We explore several variations of the Fejér-type inequalities involving generalized convex involving Raina mapping for fractional integral operators based on these equalities. The outcomes of this investigation and unique circumstances represent fresh and significant improvements over previously reported findings.

***Keywords*:** convex functions, Fejér inequality, Riemann-Liouville (RL)-fractional operators

MSC: 26A33, 26A51, 26D07, 26D10, 26D15

1. Introduction

In modern mathematics, convexity theory has been essential to the growth of many subfields. To provide a new dimension with a variety of aspects to the field of mathematical analysis and numerical approaches, many scholars and researchers have tried to incorporate new ideas into fractional analysis in the ten years prior. The theory of convex functions is widely used in many disciplines, including engineering, finance, economics, and optimization. For the literature, see the references [1–4].

The study of convexity and the notion of inequalities have a fascinating relationship. There are many well-known and practical inequalities that arise from convex functions. Two well-known inequalities that analyze and clarify the geometrical meaning of convex functions are the Jensen and Fejér inequalities. Numerous fields, including probability

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theory, information theory, and optimization, use them. These inequalities are crucial for numerical techniques like Simpson's rule, the trapezoidal rule, and others, especially for estimating the error boundaries. For the literature, see the references [5–8].

The ideas of inequality and fractional evaluation have coevolved in the modern era. The assessment of fractional inequality is one of the core concepts and components of applied sciences. Researchers advise students to consider applying and employing the fractional operator to real-world problems and challenges. Fractional calculus is the purported assignment of the integration of an arbitrary non-integer order. Due to its practical uses, it has recently attracted and kept the interest of many mathematicians. The subject of fractional calculus has many applications in control systems, transform theory, nanotechnology, modeling, fluid flow, mathematical biology, epidemiology, optimal control, and physics. For the literature, see the references [9–19].

The purpose of this work is to revisit the role of convexity in fractional settings by examining a set of recently developed equalities. Our motivation comes from the fact that fractional analysis continues to influence many areas of applied mathematics, yet several related inequalities still need clearer formulations. In this study, we introduce new forms of Fejér-type inequalities by combining generalized convexity with Raina's mapping, which gives a more adaptable structure for fractional integral operators. These results emerge from specific situations that had not been explored before. Taken together, the findings offer improvements that stand apart from earlier contributions in this direction.

The organization of this paper is as follows. In section 2, we go over a few key terms and ideas again that serve as the basis for our analysis that follows. In section 3, we present a new idea related to Generalized m-Convex involving Raina Mapping (G_mCRM). In section 4, we present new equalities pertaining to G_mCRM. In order to these qualities, we devoted ourselves to deriving some refinement of Fejér-type inequalities. We provide a brief conclusion and proposal some potential future research directions in the last section 5.

2. Preliminaries

With so many theorems, definitions, and comments, it is best to examine and delve further in this area to ensure quality, reader interest, and completeness. The purpose of this section is to illustrate and analyze certain common definitions and terms that we will need for our research in the next sections. First, the generalized convex set, generalized convex function, Fejér inequality, Classical Mitagg-Leffler (CMLF), and convex are introduced. This section is made more appealing by the addition of Condition A and Riemann-Liouville Fractional Integral Operator (RLFIO).

Definition 1 [20] A real-valued function $\mathscr{H}$ is said to be convex, if

$$\mathscr{H}\left(u\mathfrak{c}_a+(1-u)\,\mathfrak{c}_b\right)\leq u\mathscr{H}\left(\mathfrak{c}_a\right)+(1-u)\,\mathscr{H}\left(\mathfrak{c}_b\right), \tag{1}$$

holds for all $\mathfrak{c}_a,\ \mathfrak{c}_b\in I$ and $u\in[0,\ 1]$.

Fejér inequality is the most popular and well-known inequality in the literature. Many mathematicians have worked on different concepts from different angles in the field of inequalities. The following Fejér inequality was initially explored by Fejér [21] and is stated by: Suppose that $\mathscr{H}:[\mathfrak{e}_c,\ \mathfrak{e}_d]\to\mathbb{R}$ be a convex function. Then, the inequality

$$\mathscr{H}\left(\frac{\mathfrak{e}_c+\mathfrak{e}_d}{2}\right)\int_{\mathfrak{e}_c}^{\mathfrak{e}_d}\Phi(x)dx\leq\int_{\mathfrak{e}_c}^{\mathfrak{e}_d}\mathscr{H}(x)\Phi(x)dx\leq\frac{\mathscr{H}(\mathfrak{e}_c)+\mathscr{H}(\mathfrak{e}_d)}{2}\int_{\mathfrak{e}_c}^{\mathfrak{e}_d}\Phi(x)dx, \tag{2}$$

holds, where $\Phi:[a,\ b]\to\mathbb{R}$ is non-negative, integrable and symmetric to $\dfrac{\mathfrak{e}_c+\mathfrak{e}_d}{2}$.

Raina [22] proposed a family of functions formally stated by

$$\mathscr{R}^{\rho}_{\varepsilon,\ \sigma}(z) = \mathscr{R}^{\rho(0),\ \rho(1),\ \ldots}_{\varepsilon,\ \sigma}(z) = \sum_{k=0}^{+\infty} \frac{\rho(\mathfrak{v})}{\Gamma(\varepsilon k + \sigma)} z^k, \tag{3}$$

where $\rho = (\rho(0),\ \ldots,\ \rho(\mathfrak{v}),\ \ldots)$ and $\varepsilon,\ \sigma > 0,\ |z| < R$. Equation (3) is the extension of classical Mittag-Leffler function.

If $\varepsilon = 1,\ \sigma = 0$ and $\rho(\mathfrak{v}) = \dfrac{(\alpha)_k(\beta)_k}{(\gamma)_k}$ for $k = 0,\ 1,\ 2,\ \ldots$, where $\alpha,\ \beta$ and γ are parameters which can take arbitrary real or complex values (provided that $\gamma \neq 0,\ -1,\ -2,\ \ldots$), and the symbol α_k denotes the quantity

$$(\alpha)_k = \frac{\Gamma(\alpha + k)}{\Gamma(\alpha)} = \alpha(\alpha+1)\ldots(\alpha+k-1), \quad k = 0,\ 1,\ 2,\ \ldots,$$

and restricts its domain to $|z| \leq 1$ (with $z \in \mathbb{C}$), then we have the classical hypergeometric function, that is

$$\mathscr{R}(\alpha,\ \beta;\gamma;z) = \sum_{k=0}^{+\infty} \frac{(\alpha)_k(\beta)_k}{k!(\gamma)_k} z^k.$$

Moreover, if $\rho = (1,\ 1,\ \ldots)$ with $\varepsilon = \alpha$, $(Re(\alpha) > 0),\ \sigma = 1$, then

$$\mathfrak{E}_{\alpha}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(1 + \alpha k)}. \tag{4}$$

Equation (4) is referred to as a classical Mittag-Leffler function. The Mittag-Leffler function appears usually in the study of fractional calculus and especially in the studies of fractional conjecture of the kinetic equation, super diffusive transport, random walks, Lévy flights, and in the studies of complicated structures.

Cortez presented the generalized convex set and the convex function pertaining to Raina's function in [23, 24].

Definition 2 [24] Let $\rho = (\rho(0),\ \ldots,\ \rho(\mathfrak{v}),\ \ldots)$ and $\varepsilon,\ \sigma > 0$. A set $X \neq \emptyset$ is said to be generalized convex, if

$$\mathfrak{c}_a + u\, \mathscr{R}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{c}_b - \mathfrak{c}_a) \in X, \tag{5}$$

for all $\mathfrak{c}_a,\ \mathfrak{c}_b \in X$ and $\mathbf{u} \in [0,\ 1]$.

Definition 3 [24] Let ρ represent a bounded sequence then $\rho = (\rho(0),\ \ldots,\ \rho(\mathfrak{v}),\ \ldots)$ and $\varepsilon,\ \sigma > 0$.

If real-valued $\mathscr{H}$ holds the following inequality

$$\mathscr{H}\left(\mathfrak{c}_a + \mathbf{u}\, \mathscr{R}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{c}_b - \mathfrak{c}_a)\right) \leq u\mathscr{H}(\mathfrak{c}_b) + (1 - \mathbf{u})\mathscr{H}(\mathfrak{c}_a), \tag{6}$$

for all $\mathfrak{c}_a,\ \mathfrak{c}_b \in X$, where $\mathfrak{c}_a < \mathfrak{c}_b$ and $\mathbf{u} \in [0,\ 1]$, then $\mathscr{H}$ is said to be generalized convex function.

Remark 1 If $\mathscr{R}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{c}_b - \mathfrak{c}_a) = \mathfrak{c}_b - \mathfrak{c}_a > 0$, then achieve Definition 1.

The following Condition-A first time explored by Ahmad et al. [25].

Condition A: Let X be generalized convex subset w.r.t. $\mathscr{R}^{\rho}_{\varepsilon,\ \sigma}(\cdot)$. For any $\mathfrak{c}_a,\ \mathfrak{c}_b \in X$ and $\mathbf{u} \in [0,\ 1]$,

$$\mathscr{R}^{\rho}_{\varepsilon,\ \sigma}\Big(\mathfrak{c}_a-(\mathfrak{c}_a+\mathbf{u}\,\mathscr{R}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{c}_b-\mathfrak{c}_a))\Big)=-\mathbf{u}\,\mathscr{R}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{c}_b-\mathfrak{c}_a),$$

$$\mathscr{R}^{\rho}_{\varepsilon,\ \sigma}\Big(\mathfrak{c}_b-\big(\mathfrak{c}_a+\mathbf{u}\,\mathscr{R}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{c}_b-\mathfrak{c}_a)\big)\Big)=(1-\mathbf{u})\,\mathscr{R}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{c}_b-\mathfrak{c}_a).$$

Note that, for every $\mathfrak{c}_a,\ \mathfrak{c}_b\in X$ and for all $\mathbf{u}_1,\ \mathbf{u}_2\in[0,\ 1]$ from Condition-A, we have

$$\mathscr{R}^{\rho}_{\varepsilon,\ \sigma}\Big(\mathfrak{c}_a+\mathbf{u}_2\,\mathscr{R}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{c}_b-\mathfrak{c}_a)-(\mathfrak{c}_a+\mathbf{u}_1\,\mathscr{R}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{c}_b-\mathfrak{c}_a))\Big)=(\mathbf{u}_2-\mathbf{u}_1)\,\mathscr{R}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{c}_b-\mathfrak{c}_a). \tag{7}$$

Definition 4 [26] Assume that $\Psi\in L_1[v_a,\ v_b]$. Then the left- and right-sided Riemann−Liouville fractional integral operators of order $\gamma>0$, denoted by $\mathrm{I}^{\gamma}_{v_a+}\Psi$ and $\mathrm{I}^{\gamma}_{v_b-}\Psi$, are defined respectively as

$$\mathrm{I}^{\gamma}_{v_a+}\Psi(x)=\frac{1}{\Gamma(\gamma)}\int_{v_a}^{c}(x-u)^{\gamma-1}\Psi(u)\ du,\qquad x>v_a,$$

and

$$\mathrm{I}^{\gamma}_{v_b-}\Psi(x)=\frac{1}{\Gamma(\gamma)}\int_{c}^{v_b}(u-x)^{\gamma-1}\Psi(u)\ du,\qquad x<v_b.$$

3. Generalized m-convex involving Raina's mapping and its properties

Here, we shall introduce and explore the new definition, i.e., G_mCRM, an interesting and useful concept for convex functions and examine some of its algebraic properties.

Definition 5 Let $\rho=(\rho(0),\ \ldots,\ \rho(\mathfrak{v}),\ \ldots)$ and $\varepsilon,\ \sigma>0$. A set $X\neq\emptyset$ is said to be generalized m-convex, if

$$m\mathfrak{c}_a+u\,\mathscr{R}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{c}_b-m\mathfrak{c}_a)\in X, \tag{8}$$

for all $\mathfrak{c}_a,\ \mathfrak{c}_b\in X$ and $u,\ m\in[0,\ 1]$.

Definition 6 A function $\mathscr{H}$ defined on the generalized m-convex set X is said to be generalized m-convex involving Raina's function i.e., G_mCRM, if

$$\mathscr{H}\left(m\mathfrak{c}_a+u\mathscr{R}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{c}_b-m\mathfrak{c}_a)\right)\leq m(1-u)\,\mathscr{H}\left(\mathfrak{c}_a\right)+u\mathscr{H}\left(\mathfrak{c}_b\right). \tag{9}$$

holds for every $\mathfrak{c}_a,\ \mathfrak{c}_b\in X$, $m\in(0,\ 1]$ and $u\in[0,\ 1]$.

Remark 2 If $m=1$ and $\mathscr{R}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{c}_b-m\mathfrak{c}_a)=\mathfrak{c}_b-m\mathfrak{c}_a$, then Definition 6 reverts to the idea of convex function, which was investigated by Niculescu [20].

Note that every convex function is G_mCRM, but the converse does not hold in general.

Here, we are going to introduce the new condition, namely extended Condition-A, in the following way:

Extended Condition-A: Let X be generalized m-convex subset w.r.t. $\mathscr{R}^{\rho}_{\varepsilon,\ \sigma}(\cdot)$. For any $\mathfrak{c}_a,\ \mathfrak{c}_b\in X$ and $\mathbf{u}\in[0,\ 1]$,

$$\mathscr{R}^{\rho}_{\varepsilon,\ \sigma}\Big(\mathfrak{c}_a - (m\mathfrak{c}_a + \mathbf{u}\,\mathscr{R}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{c}_b - m\mathfrak{c}_a))\Big) = -\mathbf{u}\,\mathscr{R}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{c}_b - m\mathfrak{c}_a),$$

$$\mathscr{R}^{\rho}_{\varepsilon,\ \sigma}\Big(\mathfrak{c}_b - \big(m\mathfrak{c}_a + \mathbf{u}\,\mathscr{R}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{c}_b - m\mathfrak{c}_a)\big)\Big) = (1-\mathbf{u})\,\mathscr{R}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{c}_b - m\mathfrak{c}_a).$$

Note that, for every $\mathfrak{c}_a,\ \mathfrak{c}_b \in X$ and for all $\mathbf{u}_1,\ \mathbf{u}_2 \in [0,\ 1]$ from extended Condition-A, we have

$$\mathscr{R}^{\rho}_{\varepsilon,\ \sigma}\Big(m\mathfrak{c}_a + \mathbf{u}_2\,\mathscr{R}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{c}_b - m\mathfrak{c}_a) - (m\mathfrak{c}_a + \mathbf{u}_1\,\mathscr{R}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{c}_b - m\mathfrak{c}_a))\Big) = (\mathbf{u}_2 - \mathbf{u}_1)\,\mathscr{R}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{c}_b - m\mathfrak{c}_a).$$

We are going to look at and develop a few properties of the recently presented concept.

Proposition 1 We state the following as true:

(1) The Sum of two G_mCRM is also an G_mCRM.

(2) If $\mathscr{H}$ is G_mCRM, then $(c\mathscr{H})$ is also an G_mCRM.

(3) The composition of two G_mCRM is also an G_mCRM.

(4) Let $0 < \mathfrak{c}_a < \mathfrak{c}_b$, $\mathscr{H}_j :\ = [\mathfrak{c}_a,\ \mathfrak{c}_b] \to [0,\ +\infty)$ be a family of G_mCRM and $\mathscr{H}(u) = \sup_j \mathscr{H}_j(u)$. Then, $\mathscr{H}$ is an G_mCRM for $m \in (0,\ 1]$, $u \in [0,\ 1]$, and $U = \{\mathscr{H} \in [\mathfrak{c}_a, \mathfrak{c}_b] : \mathscr{H}(\mathscr{H}_u) < \infty\}$ is an interval.

(5) The product of two G_mCRM is also an G_mCRM.

Proof. The proof of the above properties is straightforward. So we omit it. □

4. New refinements of Fejér-type inequality pertaining to GCRM

Fejér inequality is the most commonly used inequality in the research. Recently, a number of scholars and scientists have been working on new concepts related to the problem from different perspectives in the realm of convex analysis. In the literature different types of fractional operators and convexity were employed to establish several new Fejér type inequalities. Tariq et al. [27] first time introduced the Fejér type inequality via non-conformable fractional operator. Park [28] applied a fractional integral operator to study the novel Fejér-type inequality involving convex function. Iscan [29] introduced a novel Fejér-type inequality involving a harmonically s-convex function through a fractional integral operator. Turhan [30] stated a novel Fejér-type inequality involving a GA-convex function through a fractional integral operator. Through the employment of the Hadamard fractional integral operator, Kunt [31] proposed a novel variation of the Fejér-type inequality involving the GA-convex function. A newly developed Fejér-type inequality involving a p-convex function via the fractional integral operator was introduced by Iscan [32]. Baleanu et al. [33] explored a novel sort of Fejér-type inequality via fractional integral of a function with respect to another function. Hakiki [34] elaborated a new kind of Fejér-type inequality associated with s-convex function via RLFIO. Jia et al. [35] examined a new sort of Fejér-type inequality over $(\gamma, h\text{-}m)$-p-convex functions via RLFIO. Agarwal et al. [36] developed a new kind of Fejér-type inequality via $(k-p)$-RLFIO.

The primary objective of this section is to examine and research a novel lemma. We obtain some refinements of the Fejér inequality via RLFIO by employing this recently introduced lemma. We implement the concept of the G_mCRM to get the results.

Lemma 1 Let A be an open m-convex set where $A \subseteq \mathbb{R}$ and Π is mapping such that $\Pi : A \times A \to \mathbb{R}$. Suppose there is a differentiable mapping $\mathscr{H} : A \to \mathbb{R}$ such that $\mathscr{H} :\in L[m\mathfrak{e}_c,\ m\mathfrak{e}_c + \mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d - m\mathfrak{e}_c)]$ and $\mathfrak{e}_c,\ \mathfrak{e}_d \in A$ with $m\mathfrak{e}_c < m\mathfrak{e}_c + \mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d - m\mathfrak{e}_c)$. If $\Phi : [m\mathfrak{e}_c,\ m\mathfrak{e}_c + \mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d - m\mathfrak{e}_c)] \to [0,\ \infty)$ is an integrable mapping, then $\forall \mathfrak{e}_c,\ \mathfrak{e}_d \in A$, with $\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d - m\mathfrak{e}_c) \neq 0$, the following equalities hold:

$$\frac{\mathscr{H}(m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))\Gamma(\gamma)}{(\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^{\gamma+1}}$$

$$\left[\mathtt{I}^{\gamma}_{(m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^-}\Phi(m\mathfrak{e}_c)+\mathtt{I}^{\gamma}_{(m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^+}\Phi(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))\right]$$

$$-\frac{\Gamma(\gamma)}{(\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^{\gamma+1}}$$

$$\left[\mathtt{I}^{\gamma}_{(m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^-}\mathscr{H}\Phi(m\mathfrak{e}_c)+\mathtt{I}^{\gamma}_{(m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^+}\mathscr{H}\Phi(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))\right]$$

$$=\int_0^1 w(\mathfrak{c})\mathscr{H}'(m\mathfrak{e}_c+\mathfrak{c}\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))d\mathfrak{c}, \tag{10}$$

where

$$w(\mathfrak{c})=\begin{cases}\displaystyle\int_0^{\mathfrak{c}} u^{\gamma-1}\Phi(m\mathfrak{e}_c+u\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))du, & \mathfrak{c}\in\left[0,\ \frac{1}{2}\right)\\[2ex] \displaystyle\int_1^{\mathfrak{c}}(1-u)^{\gamma-1}\Phi(m\mathfrak{e}_c+u\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))\ du, & \mathfrak{c}\in\left[\frac{1}{2},\ 1\right].\end{cases}$$

Proof. Consider

$$\int_0^1 w(\mathfrak{c})\mathscr{H}'\left(m\mathfrak{e}_c+t\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)\right)d\mathfrak{c}$$

$$=\int_0^{\frac{1}{2}}\left(\int_0^{\mathfrak{c}} u^{\gamma-1}\Phi(m\mathfrak{e}_c+u\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))du\right)\mathscr{H}'\left(m\mathfrak{e}_c+\mathfrak{c}\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)\right)d\mathfrak{c}$$

$$+\int_{\frac{1}{2}}^1\left(\int_1^{\mathfrak{c}}(1-u)^{\gamma-1}\Phi(m\mathfrak{e}_c+u\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))du\right)\mathscr{H}'\left(m\mathfrak{e}_c+\mathfrak{c}\Pi(\mathfrak{e}_d,\ \mathfrak{e}_c,\ m\right)d\mathfrak{c}$$

$$=J_1+J_2.$$

From the first integral,

$$J_1 = \int_0^{\frac{1}{2}} \left(\int_0^{\mathfrak{c}} u^{\gamma-1} \Phi(m\mathfrak{e}_c + u\mathscr{U}_{\varepsilon,\ \sigma}^{\rho}(\mathfrak{e}_d - m\mathfrak{e}_c))du \right) \mathscr{H}' \left(m\mathfrak{e}_c + \mathfrak{c}\mathscr{U}_{\varepsilon,\ \sigma}^{\rho}(\mathfrak{e}_d - m\mathfrak{e}_c) \right) d\mathfrak{c}$$

$$= \frac{1}{\mathscr{U}_{\varepsilon,\ \sigma}^{\rho}(\mathfrak{e}_d - m\mathfrak{e}_c)} \left(\left(\int_0^{\mathfrak{c}} u^{\gamma-1} \Phi(m\mathfrak{e}_c + u\mathscr{U}_{\varepsilon,\ \sigma}^{\rho}(\mathfrak{e}_d - m\mathfrak{e}_c))du \right) \mathscr{H}(m\mathfrak{e}_c + \mathfrak{c}\mathscr{U}_{\varepsilon,\ \sigma}^{\rho}(\mathfrak{e}_d - m\mathfrak{e}_c)) \right) \Bigg|_0^{\frac{1}{2}}$$

$$- \frac{1}{\mathscr{U}_{\varepsilon,\ \sigma}^{\rho}(\mathfrak{e}_d - m\mathfrak{e}_c)} \int_0^{\frac{1}{2}} \mathfrak{c}^{\gamma-1} \Phi(m\mathfrak{e}_c + \mathfrak{c}\mathscr{U}_{\varepsilon,\ \sigma}^{\rho}(\mathfrak{e}_d - m\mathfrak{e}_c)) \mathscr{H}(m\mathfrak{e}_c + \mathfrak{c}\mathscr{U}_{\varepsilon,\ \sigma}^{\rho}(\mathfrak{e}_d - m\mathfrak{e}_c)) d\mathfrak{c}$$

$$= \frac{\mathscr{H}\left(m\mathfrak{e}_c + \frac{1}{2}\mathscr{U}_{\varepsilon,\ \sigma}^{\rho}(\mathfrak{e}_d - m\mathfrak{e}_c) \right)}{\mathscr{U}_{\varepsilon,\ \sigma}^{\rho}(\mathfrak{e}_d - m\mathfrak{e}_c)} \int_0^{\frac{1}{2}} \mathfrak{c}^{\gamma-1} \Phi(m\mathfrak{e}_c + \mathfrak{c}\mathscr{U}_{\varepsilon,\ \sigma}^{\rho}(\mathfrak{e}_d - m\mathfrak{e}_c)) d\mathfrak{c}$$

$$- \frac{1}{\mathscr{U}_{\varepsilon,\ \sigma}^{\rho}(\mathfrak{e}_d - m\mathfrak{e}_c)} \int_0^{\frac{1}{2}} \mathfrak{c}^{\gamma-1} \mathscr{H}(m\mathfrak{e}_c + \mathfrak{c}\mathscr{U}_{\varepsilon,\ \sigma}^{\rho}(\mathfrak{e}_d - m\mathfrak{e}_c)) \Phi(m\mathfrak{e}_c + \mathfrak{c}\mathscr{U}_{\varepsilon,\ \sigma}^{\rho}(\mathfrak{e}_d - m\mathfrak{e}_c)) d\mathfrak{c}. \tag{11}$$

Substituting $x = m\mathfrak{e}_c + \mathfrak{c}\mathscr{U}_{\varepsilon,\ \sigma}^{\rho}(\mathfrak{e}_d - m\mathfrak{e}_c)$ in (11),

$$J_1 = \frac{\mathscr{H}\left(m\mathfrak{e}_c + \frac{1}{2}\mathscr{U}_{\varepsilon,\ \sigma}^{\rho}(\mathfrak{e}_d - m\mathfrak{e}_c) \right)}{(\mathscr{U}_{\varepsilon,\ \sigma}^{\rho}(\mathfrak{e}_d - m\mathfrak{e}_c))^{\gamma+1}} \int_{m\mathfrak{e}_c}^{m\mathfrak{e}_c + \frac{1}{2}\mathscr{U}_{\varepsilon,\ \sigma}^{\rho}(\mathfrak{e}_d - m\mathfrak{e}_c)} (x - m\mathfrak{e}_c)^{\gamma-1} \Phi(x) dx$$

$$- \frac{1}{(\mathscr{U}_{\varepsilon,\ \sigma}^{\rho}(\mathfrak{e}_d - m\mathfrak{e}_c))^{\gamma+1}} \int_{m\mathfrak{e}_c}^{m\mathfrak{e}_c + \frac{1}{2}\mathscr{U}_{\varepsilon,\ \sigma}^{\rho}(\mathfrak{e}_d - m\mathfrak{e}_c)} (x - m\mathfrak{e}_c)^{\gamma-1} \mathscr{H}(x) \Phi(x) dx$$

$$= \frac{\mathscr{H}\left(m\mathfrak{e}_c + \frac{1}{2}\mathscr{U}_{\varepsilon,\ \sigma}^{\rho}(\mathfrak{e}_d - m\mathfrak{e}_c) \right) \Gamma(\gamma)}{(\mathscr{U}_{\varepsilon,\ \sigma}^{\rho}(\mathfrak{e}_d - m\mathfrak{e}_c))^{\gamma+1}} \mathrm{I}^{\gamma}_{(m\mathfrak{e}_c + \frac{1}{2}\Pi(\mathfrak{e}_d,\ \mathfrak{e}_c,\ m)^-} \Phi(\mathfrak{e}_c)$$

$$- \frac{\Gamma(\gamma)}{(\mathscr{U}_{\varepsilon,\ \sigma}^{\rho}(\mathfrak{e}_d - m\mathfrak{e}_c))^{\gamma+1}} \mathrm{I}^{\gamma}_{(m\mathfrak{e}_c + \frac{1}{2}\Pi(\mathfrak{e}_d,\ \mathfrak{e}_c,\ m)^-} (\mathscr{H}\Phi)(m\mathfrak{e}_c). \tag{12}$$

From the second integral,

$$J_2 = \int_{\frac{1}{2}}^{1} \left(\int_1^{\mathfrak{c}} (1-u)^{\gamma-1} \Phi(m\mathfrak{e}_c + u\mathscr{U}_{\varepsilon,\ \sigma}^{\rho}(\mathfrak{e}_d - m\mathfrak{e}_c))du \right) \mathscr{H}'(m\mathfrak{e}_c + \mathfrak{c}\mathscr{U}_{\varepsilon,\ \sigma}^{\rho}(\mathfrak{e}_d - m\mathfrak{e}_c)) d\mathfrak{c}$$

$$= \frac{1}{\mathscr{U}_{\varepsilon,\ \sigma}^{\rho}(\mathfrak{e}_d - m\mathfrak{e}_c)} \left(\left(\int_1^{\mathfrak{c}} (1-u)^{\gamma-1} \Phi(m\mathfrak{e}_c + u\mathscr{U}_{\varepsilon,\ \sigma}^{\rho}(\mathfrak{e}_d - m\mathfrak{e}_c))du \right) \mathscr{H}(m\mathfrak{e}_c + \mathfrak{c}\mathscr{U}_{\varepsilon,\ \sigma}^{\rho}(\mathfrak{e}_d - m\mathfrak{e}_c)) \right) \Bigg|_{\frac{1}{2}}^{1}$$

$$-\frac{1}{\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\int_{\frac{1}{2}}^{1}(1-\mathfrak{c})^{\gamma-1}\mathscr{H}(m\mathfrak{e}_c+\mathfrak{c}\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))\Phi(m\mathfrak{e}_c+\mathfrak{c}\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))d\mathfrak{c}$$

$$=-\frac{\mathscr{H}\left(m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)\right)}{\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\int_{1}^{\frac{1}{2}}(1-\mathfrak{c})^{\gamma-1}\Phi(m\mathfrak{e}_c+\mathfrak{c}\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))d\mathfrak{c}$$

$$-\frac{1}{\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\int_{\frac{1}{2}}^{1}(1-\mathfrak{c})^{\gamma-1}\mathscr{H}(m\mathfrak{e}_c+\mathfrak{c}\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))\Phi(m\mathfrak{e}_c+\mathfrak{c}\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))d\mathfrak{c}. \tag{13}$$

Substituting $x=m\mathfrak{e}_c+\mathfrak{c}\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)$ in (13),

$$=-\frac{\mathscr{H}\left(m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)\right)}{(\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^{\gamma+1}}\int_{m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}^{m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)-x)^{\gamma-1}\Phi(x)dx$$

$$-\frac{1}{(\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^{\gamma+1}}\int_{m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}^{m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)-x)^{\gamma-1}\mathscr{H}(x)\Phi(x)dx$$

$$=\frac{\mathscr{H}\left(m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)\right)\Gamma(\gamma)}{(\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^{\gamma+1}}\mathtt{I}^{\gamma}_{(m\mathfrak{e}_c+\frac{1}{2}(\mathfrak{e}_d,\ \mathfrak{e}_c,\ m))^+}\Phi(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))$$

$$-\frac{\Gamma(\gamma)}{(\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^{2}}\mathtt{I}^{\gamma}_{(m\mathfrak{e}_c+\frac{1}{2}(\mathfrak{e}_d,\ \mathfrak{e}_c,\ m))^+}(\mathscr{H}\Phi)(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)). \tag{14}$$

Upon adding (12) and (14), we get the required result. □

Lemma 2 If $\Phi:[m\mathfrak{e}_c,\ m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)]\to\mathbf{R}$ is an integrable function which is also symmetric about $m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)$ with $m\mathfrak{e}_c<m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)$, then

$$\mathtt{I}^{\gamma}_{m\mathfrak{e}_c^+}\Phi(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))=\mathtt{I}^{\gamma}_{(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^-}\Phi(m\mathfrak{e}_c)$$

$$=\frac{1}{2}\left[\mathtt{I}^{\gamma}_{m\mathfrak{e}_c^+}\Phi(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))+\mathtt{I}^{\gamma}_{(m\mathfrak{e}_c+\Pi(\mathfrak{e}_d-m\mathfrak{e}_c))^-}\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(m\mathfrak{e}_c)\right], \tag{15}$$

where $\gamma>0$.

Proof. Since Φ is symmetric about $m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)$, we have $\Phi(2m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)-x)=\Phi(x)$, for all $x\in[m\mathfrak{e}_c,\ m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d,\ \mathfrak{e}_c,\ m)]$. Taking $2\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)-\mathfrak{c}=x$

$$\mathtt{I}^{\gamma}_{m\mathfrak{e}_c+}\Phi(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))=\frac{1}{\Gamma(\gamma)}\int_{m\mathfrak{e}_c}^{m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}[m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)-\mathfrak{c}]^{\gamma-1}\Phi(\mathfrak{c})d\mathfrak{c}$$

$$=\frac{1}{\Gamma(\gamma)}\int_{m\mathfrak{e}_c}^{m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}(x-m\mathfrak{e}_c)^{\gamma-1}\Phi(2m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)-x)dx$$

$$=\frac{1}{\Gamma(\gamma)}\int_{m\mathfrak{e}_c}^{m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}(x-m\mathfrak{e}_c)^{\gamma-1}\Phi(x)dx$$

$$=\mathtt{I}^{\gamma}_{(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^-}\Phi(\mathfrak{e}_c).\qquad\square$$

Lemma 3 Let A be an open m-convex set where $H\subset\mathbb{R}$ and $\Pi:A\times A\to\mathbb{R}$ is a mapping. Suppose there is a differentiable mapping $\mathscr{H}:A\to\mathbb{R}$ on H such that $\mathscr{H}'\in L[\mathfrak{e}_c,\ \mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)]$ and $\mathfrak{e}_c,\ \mathfrak{e}_d\in A$ with $m\mathfrak{e}_c<m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)$. If $\Phi:[m\mathfrak{e}_c,\ m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)]\to[0,\ \infty)$ is an integrable mapping, then $\forall\mathfrak{e}_c,\ \mathfrak{e}_d\in A$ with $\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)\neq 0$ the following equality holds:

$$\left[\frac{\mathscr{H}(m\mathfrak{e}_c)+\mathscr{H}(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))}{2}\right]\left[\mathtt{I}^{\gamma}_{(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^-}\Phi(m\mathfrak{e}_c)-\mathtt{I}^{\gamma}_{m\mathfrak{e}_c+}\Phi(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))\right]$$

$$-\left[\mathtt{I}^{\gamma}_{(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^-}(\mathscr{H}\Phi)(m\mathfrak{e}_c)-\mathtt{I}^{\gamma}_{m\mathfrak{e}_c+}(\mathscr{H}\Phi)(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))\right]$$

$$=\frac{(\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^{a+1}}{\Gamma(a)}\int_0^1 w(\mathfrak{c})\mathscr{H}'(m\mathfrak{e}_c+\mathfrak{c}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))d\mathfrak{c},\tag{16}$$

where

$$w(\mathfrak{c})=\int_1^{\mathfrak{c}}(1-u)^{a-1}\Phi(m\mathfrak{e}_c+u\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))du+\int_0^{\mathfrak{c}}(1-u)^{a-1}\Phi(m\mathfrak{e}_c+u\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))du,\qquad\mathfrak{c}\in[0,\ 1].$$

Proof. Let us consider

$$\int_0^1 w(\mathfrak{c})\mathscr{H}'(m\mathfrak{e}_c+\mathfrak{c}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))d\mathfrak{c}$$

$$=\int_0^1\left[\int_1^{\mathfrak{c}}(1-u)^{a-1}\Phi(m\mathfrak{e}_c+u\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))du\right.$$

$$\left.+\int_0^{\mathfrak{c}}(1-u)^{\gamma-1}\Phi(m\mathfrak{e}_c+u\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))du\right]\mathscr{H}'(m\mathfrak{e}_c+\mathfrak{c}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))d\mathfrak{c}$$

$$= \int_0^1 \left[\int_1^{\mathfrak{c}} (1-u)^{\gamma-1}\Phi(m\mathfrak{e}_c + u\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c))du\right]\mathscr{H}(m\mathfrak{e}_c + \mathfrak{c}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c))d\mathfrak{c}$$

$$+ \int_0^1 \left[\int_0^{\mathfrak{c}} (1-u)^{a-1}\Phi(m\mathfrak{e}_c + u\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c))du\right]\mathscr{H}'(m\mathfrak{e}_c + \mathfrak{c}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c))d\mathfrak{c}$$

$$= J_1 + J_2. \tag{17}$$

From the first integral,

$$J_1 = \int_0^1 \left(\int_1^{\mathfrak{c}} (1-u)^{\gamma-1}\Phi(m\mathfrak{e}_c + u\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c))\ du\right)\mathscr{H}'(m\mathfrak{e}_c + \mathfrak{c}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c))\ d\mathfrak{c}$$

$$= \frac{1}{\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c)}\left(\left(\int_1^{\mathfrak{c}} (1-u)^{\gamma-1}\Phi(m\mathfrak{e}_c + u\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c))\ du\right)\mathscr{H}(m\mathfrak{e}_c + \mathfrak{c}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c))\Big|_0^1\right.$$

$$\left. - \frac{1}{\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c)}\int_0^1 (1-\mathfrak{c})^{\gamma-1}\Phi(m\mathfrak{e}_c + \mathfrak{c}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c))\mathscr{H}(m\mathfrak{e}_c + \mathfrak{c}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c))\ d\mathfrak{c}\right)$$

$$= \frac{\mathscr{H}(m\mathfrak{e}_c)}{\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c)}\int_0^1 (1-\mathfrak{c})^{\gamma-1}\Phi(m\mathfrak{e}_c + \mathfrak{c}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c))\ d\mathfrak{c}$$

$$- \frac{1}{\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c)}\int_0^1 (1-\mathfrak{c})^{\gamma-1}\mathscr{H}(m\mathfrak{e}_c + \mathfrak{c}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c))\Phi(m\mathfrak{e}_c + \mathfrak{c}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c))\ d\mathfrak{c}. \tag{18}$$

Substituting $x = \mathfrak{e}_c + \mathfrak{c}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c)$ in (18),

$$J_1 = \frac{\mathscr{H}(m\mathfrak{e}_c)}{(\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c))^{\gamma+1}}\int_{m\mathfrak{e}_c}^{m\mathfrak{e}_c + \mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c)} ((m\mathfrak{e}_c + \mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c) - x)^{\gamma-1}\Phi(x)\ dx$$

$$- \frac{1}{(\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c))^{\gamma+1}}\int_{m\mathfrak{e}_c}^{m\mathfrak{e}_c + \mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c)} (m\mathfrak{e}_c + \mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c) - x)^{\gamma-1}\mathscr{H}(x)\Phi(x)\ dx$$

$$= \frac{\mathscr{H}(m\mathfrak{e}_c)\Gamma(\gamma)}{(\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c))^{\gamma+1}}\mathtt{I}^{\gamma}_{m\mathfrak{e}_c^+}\Phi(\mathfrak{e}_c + \mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c))$$

$$- \frac{\Gamma(\gamma)}{(\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c))^{\gamma+1}}\mathtt{I}^{\gamma}_{m\mathfrak{e}_c^+}(\Phi\mathscr{H})(m\mathfrak{e}_c + \mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c)). \tag{19}$$

Now for the second integral,

$$J_2 = \int_0^1 \left(\int_0^{\mathfrak{c}} (1-u)^{\gamma-1} \Phi(m\mathfrak{e}_c + u\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c))\ du \right) \mathscr{H}'(m\mathfrak{e}_c + \mathfrak{c}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c))\ d\mathfrak{c}$$

$$= \frac{1}{\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c)} \left(\left(\int_0^{t} (1-u)^{\gamma-1} \Phi(m\mathfrak{e}_c + u\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c))\ du \right) \mathscr{H}(m\mathfrak{e}_c + \mathfrak{c}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c)) \Big|_0^1 \right.$$

$$\left. - \int_0^1 (1-\mathfrak{c})^{\gamma-1} \mathscr{H}(m\mathfrak{e}_c + \mathfrak{c}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c)) \Phi(m\mathfrak{e}_c + \mathfrak{c}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c))\ d\mathfrak{c} \right)$$

$$= \frac{\mathscr{H}(m\mathfrak{e}_c + \mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c))}{\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c)} \int_0^1 (1-\mathfrak{c})^{\gamma-1} \Phi(m\mathfrak{e}_c + \mathfrak{c}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c))\ d\mathfrak{c}$$

$$- \frac{1}{\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c)} \int_0^1 (1-\mathfrak{c})^{\gamma-1} \mathscr{H}(m\mathfrak{e}_c + \mathfrak{c}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c)) \Phi(m\mathfrak{e}_c + \mathfrak{c}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c))\ d\mathfrak{c}. \tag{20}$$

Substituting $x = m\mathfrak{e}_c + \mathfrak{c}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c)$ in (20),

$$J_2 = \frac{\mathscr{H}(m\mathfrak{e}_c + \mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c))}{(\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c))^{\gamma+1}} \int_{m\mathfrak{e}_c}^{m\mathfrak{e}_c + \mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c)} ((m\mathfrak{e}_c + \mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c) - x)^{\gamma-1} \Phi(x)\ dx$$

$$- \frac{1}{(\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c))^{\gamma+1}} \int_{m\mathfrak{e}_c}^{m\mathfrak{e}_c + \mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c)} ((m\mathfrak{e}_c + \mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c) - x)^{\gamma-1} \mathscr{H}(x)\Phi(x)\ dx$$

$$= \frac{\mathscr{H}(m\mathfrak{e}_c + \mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c))\Gamma(\gamma)}{(\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c))^{\gamma+1}} \mathrm{I}^{\gamma}_{m\mathfrak{e}_c^+} \Phi(m\mathfrak{e}_c + \mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c))$$

$$- \frac{\Gamma(\gamma)}{(\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c))^{\gamma+1}} \mathrm{I}^{\gamma}_{m\mathfrak{e}_c^+} (\mathscr{H}\Phi)(m\mathfrak{e}_c + \mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c)). \tag{21}$$

By adding the results of (19) and (21) using (15), we get the required result. □

Theorem 1 Let A be an open m-invex set where $A \subseteq \mathbb{R}$ and Π is mapping such that $\Pi : A \times A \to \mathbb{R}$. Suppose there is a differentiable mapping $\mathscr{H} : A \to \mathbb{R}$ such that $\mathscr{H} \in L[m\mathfrak{e}_c,\ m\mathfrak{e}_c + \mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c)]$ and $\mathfrak{e}_c,\ \mathfrak{e}_d \in A$ with $m\mathfrak{e}_c < m\mathfrak{e}_c + \mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c)$. If $\Phi : [m\mathfrak{e}_c,\ m\mathfrak{e}_c + \mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c)] \to [0,\ \infty)$ is an integrable mapping that is also symmetric with respect to $m\mathfrak{e}_c + \frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c)$. If $[\mathscr{H}']$ is G_mCRM on A, then $\forall \mathfrak{e}_c,\ \mathfrak{e}_d \in A$ with $\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d - m\mathfrak{e}_c) \neq 0$ the following inequality holds:

$$\left|\Phi\left(m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)\right)\left[\mathtt{I}^{\gamma}_{(m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^-}\Phi(m\mathfrak{e}_c)+\mathtt{I}^{\gamma}_{(m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^+}\Phi(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))\right]\right.$$

$$\left.-\left[\mathtt{I}^{\gamma}_{(m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^-}(\mathscr{H}\Phi)(m\mathfrak{e}_c)+\mathtt{I}^{\gamma}_{(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^+}(\mathscr{H}\Phi)(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))\right]\right|$$

$$\leq\frac{1}{\Gamma(\gamma+2)}(m|\mathscr{H}'(m\mathfrak{e}_c)|+|\mathscr{H}'(\mathfrak{e}_d)|)||\Phi||_{\infty}\frac{(\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^{\gamma+1}}{2^{\gamma+1}}. \tag{22}$$

Proof. Applying modulus on both sides of (10),

$$\left|\frac{\mathscr{H}\left(m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)\right)\Gamma(\gamma)}{\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)^{\gamma+1}}\left[\mathtt{I}^{\gamma}_{(m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^-}\Phi(m\mathfrak{e}_c)\right.\right.$$

$$\left.+\mathtt{I}^{\gamma}_{(m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^+}\Phi(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))\right]$$

$$\left.-[\mathtt{I}^{\gamma}_{(m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^-}(\mathscr{H}\Phi)(m\mathfrak{e}_c)+\mathtt{I}^{\gamma}_{(m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^+}(\mathscr{H}\Phi)(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))]\right|$$

$$=\left|\int_0^{1/2}\left(\int_0^1 u^{\gamma-1}\Phi(m\mathfrak{e}_c+u\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))du\right)\mathscr{H}'(m\mathfrak{e}_c+\mathfrak{c}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))d\mathfrak{c}\right.$$

$$\left.+\int_{1/2}^1\left(-\int_{\mathfrak{c}}^1(1-u)^{\gamma-1}\Phi(m\mathfrak{e}_c+u\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))du\right)\mathscr{H}'(m\mathfrak{e}_c+\mathfrak{c}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))d\mathfrak{c}\right| \tag{23}$$

From G_mCRM of $|\mathscr{H}'|$ on A and Lemma 1, we have

$$\left|\frac{\mathscr{H}\left(m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)\right)\Gamma(\gamma)}{\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)^{\gamma+1}}\right.$$

$$\left[\mathtt{I}^{\gamma}_{(m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^-}\Phi(m\mathfrak{e}_c)+\mathtt{I}^{\gamma}_{(m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^+}\Phi(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))\right]$$

$$\left.-\left[\mathtt{I}^{\gamma}_{(m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^-}(\mathscr{H}\Phi)(m\mathfrak{e}_c)+\mathtt{I}^{\gamma}_{(m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^+}(\mathscr{H}\Phi)(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))\right]\right|$$

$$\leq \int_0^{1/2}\left(\int_0^{\mathfrak{c}} u^{\gamma-1}|\Phi(m\mathfrak{e}_c+u\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))|du\right)[m(1-\mathfrak{c})|\mathscr{H}'(m\mathfrak{e}_c)|+\mathfrak{c}|\mathscr{H}'(\mathfrak{e}_d)|]d\mathfrak{c}$$

$$+\int_{1/2}^1\left(\int_{\mathfrak{c}}^1 (1-u)^{\gamma-1}|\Phi(m\mathfrak{e}_c+u\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))|du\right)[m(1-\mathfrak{c})|\mathscr{H}'(m\mathfrak{e}_c)|+\mathfrak{c}|\mathscr{H}'(\mathfrak{e}_d)|]d\mathfrak{c}$$

$$=J_1+J_2. \tag{24}$$

By the change of the order of integration in first term of (24), we have

$$J_1=\int_0^{1/2}\left(\int_0^{\mathfrak{c}} u^{\gamma-1}|\Phi(m\mathfrak{e}_c+u\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))|du\right)[m(1-\mathfrak{c})|\mathscr{H}'(\mathfrak{e}_c)|+\mathfrak{c}|\mathscr{H}'(\mathfrak{e}_d)|]d\mathfrak{c}$$

$$=\int_0^{1/2} u^{\gamma-1}|\Phi(m\mathfrak{e}_c+u\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))|\int_u^{1/2}[m(1-\mathfrak{c})|\mathscr{H}'(\mathfrak{e}_c)|+\mathfrak{c}|\mathscr{H}'(\mathfrak{e}_d)|]d\mathfrak{c}du$$

$$=\int_0^{1/2} u^{\gamma-1}|\Phi(m\mathfrak{e}_c+u\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))|\left[m|\mathscr{H}'(\mathfrak{e}_c)|\left(\frac{(1-u)^2}{2}-\frac{1}{8}\right)+|\mathscr{H}'(\mathfrak{e}_d)|\left(\frac{1}{8}-\frac{u^2}{2}\right)\right]du.$$

Making the change of variable $x=m\mathfrak{e}_c+u\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)$ for $u\in[0,\ 1]$,

$$J_1=\frac{m|\mathscr{H}'(\mathfrak{e}_c)|}{\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\int_{m\mathfrak{e}_c}^{m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\left(\frac{1}{2}\left(1-\frac{x-m\mathfrak{e}_c}{\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\right)^2-\frac{1}{8}\right)\left(\frac{x-m\mathfrak{e}_c}{\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\right)^{\gamma-1}|\Phi(x)|dx$$

$$+\frac{|\mathscr{H}'(\mathfrak{e}_d)|}{\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\int_{m\mathfrak{e}_c}^{m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\left(\frac{1}{8}-\frac{1}{2}\left(\frac{x-m\mathfrak{e}_c}{\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\right)^2\right)\left(\frac{x-m\mathfrak{e}_c}{\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\right)^{\gamma-1}|\Phi(x)|dx.$$

Let $\|\Phi\|_\infty=\sup_{x\in[m\mathfrak{e}_c,\ \mathfrak{e}_d]}|\Phi(x)|$,

$$J_1\leq\frac{m|\mathscr{H}'(\mathfrak{e}_c)|}{\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\|\Phi\|_\infty\int_{m\mathfrak{e}_c}^{m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\left(\frac{1}{2}\left(1-\frac{x-m\mathfrak{e}_c}{\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\right)^2-\frac{1}{8}\right)\left(\frac{x-m\mathfrak{e}_c}{\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\right)^{\gamma-1}dx$$

$$+\frac{|\mathscr{H}'(\mathfrak{e}_d)|}{\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\|\Phi\|_\infty\int_{m\mathfrak{e}_c}^{m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\left(\frac{1}{8}-\frac{1}{2}\left(\frac{x-m\mathfrak{e}_c}{\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\right)^2\right)\left(\frac{x-m\mathfrak{e}_c}{\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\right)^{\gamma-1}dx. \tag{25}$$

Similarly, by changing the order of integration in the second term and using the fact that Φ is symmetric to $m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)$, we obtain

$$J_2 = \int_{1/2}^{1}\left(\int_{\mathfrak{c}}^{1}(1-u)^{\gamma-1}|\Phi(m\mathfrak{e}_c+(1-u)\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))|du\right)[m(1-\mathfrak{c})|\mathscr{H}'(\mathfrak{e}_c)|+\mathfrak{c}|\mathscr{H}'(\mathfrak{e}_d)|]d\mathfrak{c}$$

$$= \int_{1/2}^{1}(1-u)^{\gamma-1}|\Phi(m\mathfrak{e}_c+(1-u)\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))|\int_{1/2}^{u}[m(1-\mathfrak{c})|\mathscr{H}'(\mathfrak{e}_c)|+\mathfrak{c}|\mathscr{H}'(\mathfrak{e}_d)|]d\mathfrak{c}du$$

$$= \int_{1/2}^{1}(1-u)^{\gamma-1}|\Phi(m\mathfrak{e}_c+(1-u)\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))|\left[m|\mathscr{H}'(\mathfrak{e}_c)|\left(\frac{1}{8}-\frac{(1-u)^2}{2}\right)+|\mathscr{H}'(\mathfrak{e}_d)|\left(\frac{u^2}{2}-\frac{1}{8}\right)\right]du.$$

By the change of variable $x = m\mathfrak{e}_c+(1-u)\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)$,

$$J_2 = \frac{m|\mathscr{H}'(\mathfrak{e}_c)|}{\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\int_{m\mathfrak{e}_c}^{m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\left(\frac{1}{8}-\frac{1}{2}\left(\frac{x-m\mathfrak{e}_c}{\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\right)^2\right)\left(\frac{x-m\mathfrak{e}_c}{\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\right)^{\gamma-1}|\Phi(x)|dx$$

$$+\frac{|\mathscr{H}'(\mathfrak{e}_d)|}{\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\int_{m\mathfrak{e}_c}^{m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\left(\frac{1}{2}\left(1-\frac{x-m\mathfrak{e}_c}{\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\right)^2-\frac{1}{8}\right)\left(\frac{x-m\mathfrak{e}_c}{\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\right)^{\gamma-1}|\Phi(x)|dx.$$

Knowing that $\|\Phi\|_\infty = \sup_{x\in[m\mathfrak{e}_c,\ \mathfrak{e}_d]}|\Phi(x)|$,

$$J_2 \le \frac{m|\mathscr{H}'(\mathfrak{e}_c)|}{\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\|\Phi\|_\infty\int_{m\mathfrak{e}_c}^{m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\left(\frac{1}{8}-\frac{1}{2}\left(\frac{x-m\mathfrak{e}_c}{\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\right)^2\right)\left(\frac{x-m\mathfrak{e}_c}{\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\right)^{\gamma-1}dx$$

$$+\frac{|\mathscr{H}'(\mathfrak{e}_d)|}{\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}||\Phi|_\infty\int_{m\mathfrak{e}_c}^{m\mathfrak{e}_c+\frac{1}{2}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}$$

$$\left(\frac{1}{2}\left(1-\frac{x-m\mathfrak{e}_c}{\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\right)^2-\frac{1}{8}\right)\left(\frac{x-m\mathfrak{e}_c}{\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\right)^{\gamma-1}dx. \tag{26}$$

Adding Equations (25) to (26) based on (24), we get our required result. □

Remark 3 Employing Theorem 1, we investigate the following new generalized fractional approach of Fejér-type inequality involving the CMLF via RLFIO if we choose $\rho = (1,\ 1,\ \ldots)$ with $\varepsilon = \alpha$ and $\sigma = 1$:

$$\Bigg|\Phi(m\mathfrak{e}_c+\frac{1}{2}\mathscr{E}_\alpha(\mathfrak{e}_d-m\mathfrak{e}_c))\left[\mathtt{I}^{\gamma}_{(m\mathfrak{e}_c+\frac{1}{2}\mathscr{E}_\alpha(\mathfrak{e}_d-m\mathfrak{e}_c))^-}\Phi(m\mathfrak{e}_c)+\mathtt{I}^{\gamma}_{(m\mathfrak{e}_c+\frac{1}{2}\mathscr{E}_\alpha(\mathfrak{e}_d-m\mathfrak{e}_c))^+}\Phi(m\mathfrak{e}_c+\mathscr{E}_\alpha(\mathfrak{e}_d-m\mathfrak{e}_c))\right]$$

$$-\left[\mathtt{I}^{\gamma}_{(m\mathfrak{e}_c+\frac{1}{2}\mathscr{E}_\alpha(\mathfrak{e}_d-m\mathfrak{e}_c))^-}(\mathscr{H}\Phi)(m\mathfrak{e}_c)+\mathtt{I}^{\gamma}_{(m\mathfrak{e}_c+\mathscr{E}_\alpha(\mathfrak{e}_d-m\mathfrak{e}_c))^+}(\mathscr{H}\Phi)(m\mathfrak{e}_c+\mathscr{E}_\alpha(\mathfrak{e}_d-m\mathfrak{e}_c))\right]\Bigg|$$

$$\leq \frac{1}{\Gamma(\gamma+2)}(m|\mathscr{H}'(m\mathfrak{e}_c)|+|\mathscr{H}'(\mathfrak{e}_d)|)||\Phi||_\infty \frac{(\mathscr{E}_\alpha(\mathfrak{e}_d-m\mathfrak{e}_c))^{\gamma+1}}{2^{\gamma+1}}.$$

Theorem 2 Let A be the m-convex set where $A \subseteq \mathbb{R}$ and $\Pi : A\times A \to \mathbb{R}$ is a mapping. Suppose there is a differentiable mapping $\mathscr{H} : A \to \mathbb{R}$ on A such that $\mathscr{H}(m\mathfrak{e}_c)+\mathscr{H}(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))$ and $\mathfrak{e}_c,\ \mathfrak{e}_d \in A$ with $m\mathfrak{e}_c \leq m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)$. If $\Phi : [m\mathfrak{e}_c,\ m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)] \to [0,\infty)$ is an integrable mapping symmetric to $m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)$ and also $|\mathscr{H}|$ is a G_mCRM on A, then $\forall \mathfrak{e}_c,\ \mathfrak{e}_d \in A$ with $\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c) \neq 0$ the following inequality holds:

$$\Bigg|\frac{\mathscr{H}(m\mathfrak{e}_c)+\mathscr{H}(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))}{2}\left[\mathrm{I}^{\gamma}_{(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^-}\Phi(m\mathfrak{e}_c)+\mathrm{I}^{\gamma}_{m\mathfrak{e}_c^+}\Phi(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))\right]$$

$$-\left[\mathrm{I}^{\gamma}_{(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^-}(\mathscr{H}\Phi)(m\mathfrak{e}_c)+\mathrm{I}^{\gamma}_{m\mathfrak{e}_c^+}(\mathscr{H}\Phi)(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))\right]\Bigg|$$

$$\leq ||\Phi||_\infty \frac{(\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^{\gamma+1}}{\Gamma(\gamma+1)}\left(\frac{m|\mathscr{H}'(\mathfrak{e}_c)|+|\mathscr{H}'(\mathfrak{e}_d)|}{2}\right). \tag{27}$$

Proof. Applying modulus on both sides of (16),

$$\Bigg|\frac{\mathscr{H}(m\mathfrak{e}_c)+\mathscr{H}(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))}{2}\left[\mathrm{I}^{\gamma}_{(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^-}\Phi(m\mathfrak{e}_c)+\mathrm{I}^{\gamma}_{m\mathfrak{e}_c^+}\Phi(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))\right]$$

$$-\left[\mathrm{I}^{\gamma}_{(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^-}(\mathscr{H}\Phi)(m\mathfrak{e}_c)+\mathrm{I}^{\gamma}_{m\mathfrak{e}_c^+}(\mathscr{H}\Phi)(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))\right]\Bigg|$$

$$\leq \Bigg|\frac{(\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^{\gamma+1}}{\Gamma(\gamma)}\int_0^1\left(-\int_{\mathfrak{c}}^1(1-u)^{\gamma-1}\Phi(m\mathfrak{e}_c+u\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))du\right.$$

$$\left.+\int_0^{\mathfrak{c}}(1-u)^{\gamma-1}\Phi(m\mathfrak{e}_c+u\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))du\right)\mathscr{H}'(m\mathfrak{e}_c+\mathfrak{c}\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))d\mathfrak{c}\Bigg|.$$

From G_mCRM of $|\mathscr{H}'|$ on A, we have

$$\Bigg|\frac{\mathscr{H}(m\mathfrak{e}_c)+\mathscr{H}(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))}{2}\left[\mathrm{I}^{\gamma}_{(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^-}\Phi(m\mathfrak{e}_c)+\mathrm{I}^{\gamma}_{m\mathfrak{e}_c^+}\Phi(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))\right]$$

$$-\left[\mathrm{I}^{\gamma}_{(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^-}(\mathscr{H}\Phi)(m\mathfrak{e}_c)+\mathrm{I}^{\gamma}_{m\mathfrak{e}_c^+}(\mathscr{H}\Phi)(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))\right]\Bigg|$$

$$\leq \frac{(\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^{\gamma+1}}{\Gamma(\gamma)}\int_0^1\Bigg|-\int_{\mathfrak{c}}^1(1-u)^{\gamma-1}\Phi(m\mathfrak{e}_c+u\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))du$$

$$+\int_0^{\mathfrak{c}}(1-u)^{\gamma-1}\Phi(m\mathfrak{e}_c+u\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))du\Big|\left[m(1-\mathfrak{c})|\mathscr{H}'(m\mathfrak{e}_c)|+\mathfrak{c}|\mathscr{H}'(\mathfrak{e}_d)|\right]d\mathfrak{c}. \tag{28}$$

After simplification, (28) becomes

$$\Bigg|\frac{\mathscr{H}(m\mathfrak{e}_c)+\mathscr{H}(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))}{2}\left[\mathtt{I}^{\gamma}_{(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^-}\Phi(m\mathfrak{e}_c)+\mathtt{I}^{\gamma}_{m\mathfrak{e}_c^+}\Phi(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))\right]$$

$$-\left[\mathtt{I}^{\gamma}_{(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^-}(\mathscr{H}\Phi)(m\mathfrak{e}_c)+\mathtt{I}^{\gamma}_{m\mathfrak{e}_c^+}(\mathscr{H}\Phi)(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))\right]\Bigg|$$

$$\leq\frac{(\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^{\gamma+1}}{\Gamma(\gamma)}\int_0^1\left(\int_{\mathfrak{c}}^1(1-u)^{\gamma-1}|\Phi(m\mathfrak{e}_c+u\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))du\right)\left[m(1-\mathfrak{c})|\mathscr{H}'(\mathfrak{e}_c)|+\mathfrak{c}|\mathscr{H}'(\mathfrak{e}_d)|\right]d\mathfrak{c}$$

$$+\frac{(\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^{\gamma+1}}{\Gamma(\gamma)}\int_0^1\left(\int_0^{\mathfrak{c}}(1-u)^{\gamma-1}|\Phi(m\mathfrak{e}_c+u\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))du\right)\left[m(1-\mathfrak{c})|\mathscr{H}'(\mathfrak{e}_c)|+\mathfrak{c}|\mathscr{H}'(\mathfrak{e}_d)|\right]d\mathfrak{c}$$

By changing the order of integration, we have

$$\Bigg|\frac{\mathscr{H}(m\mathfrak{e}_c)+\mathscr{H}(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))}{2}\left[\mathtt{I}^{\gamma}_{(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^-}\Phi(m\mathfrak{e}_c)+\mathtt{I}^{\gamma}_{m\mathfrak{e}_c^+}\Phi(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))\right]$$

$$-\left[\mathtt{I}^{\gamma}_{(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^-}(\mathscr{H}\Phi)(m\mathfrak{e}_c)+\mathtt{I}^{\gamma}_{m\mathfrak{e}_c^+}(\mathscr{H}\Phi)(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))\right]\Bigg|$$

$$\leq\frac{(\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^{\gamma+1}}{\Gamma(a)}\int_0^1(1-u)^{\gamma-1}|\Phi(m\mathfrak{e}_c+u\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))|\int_0^u(m(1-\mathfrak{c})|\mathscr{H}'(\mathfrak{e}_c)|+\mathfrak{c}|\mathscr{H}'(\mathfrak{e}_d)|)d\mathfrak{c}du$$

$$+\frac{(\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^{\gamma+1}}{\Gamma(\gamma)}\int_0^1(1-u)^{\gamma-1}|\Phi(m\mathfrak{e}_c+u\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))|\int_0^1(m(1-\mathfrak{c})|\mathscr{H}'(\mathfrak{e}_c)|+\mathfrak{c}|\mathscr{H}'(\mathfrak{e}_d)|)d\mathfrak{c}du$$

$$=\frac{(\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^{\gamma+1}}{\Gamma(\gamma)}\int_0^1(1-u)^{\gamma-1}|\Phi(m\mathfrak{e}_c+u\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))|\int_0^1(m(1-\mathfrak{c})|\mathscr{H}'(\mathfrak{e}_c)|+\mathfrak{c}|\mathscr{H}'(\mathfrak{e}_d)|)d\mathfrak{c}du$$

$$=\frac{(\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^{\gamma+1}}{\Gamma(\gamma)}\left(\frac{m|\mathscr{H}'(\mathfrak{e}_c)|+|\mathscr{H}'(\mathfrak{e}_d)|}{2}\right)\int_0^1(1-u)^{\gamma-1}|\Phi(m\mathfrak{e}_c+u\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))|du.$$

By changing the variable $x=m\mathfrak{e}_c+u\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)$ and reminding that $\|\Phi\|_\infty=\sup_{x\in[\mathfrak{e}_c,\,\mathfrak{e}_d]}|\Phi(x)|$,

$$\Bigg|\frac{\mathscr{H}(m\mathfrak{e}_c)+\mathscr{H}(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))}{2}\left[\mathtt{I}^{\gamma}_{(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^-}\Phi(m\mathfrak{e}_c)+\mathtt{I}^{\gamma}_{m\mathfrak{e}_c^+}\Phi(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\,\sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))\right]$$

$$-\left[\mathrm{I}^{\gamma}_{(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^-}(\mathscr{H}\Phi)(m\mathfrak{e}_c)+\mathrm{I}^{\gamma}_{m\mathfrak{e}_c^+}(\mathscr{H}\Phi)(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))\right]\Bigg|$$

$$\leq \frac{(\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^{\gamma}}{\Gamma(\gamma)}\|\Phi\|_{\infty}\left(\frac{|\mathscr{H}'(m\mathfrak{e}_c)|+|\mathscr{H}'(\mathfrak{e}_d)|}{2}\right)\int_{m\mathfrak{e}_c}^{m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\left(\frac{(m\mathfrak{e}_c+\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)-x}{\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c)}\right)^{\gamma-1}dx$$

$$=\frac{(\mathscr{U}^{\rho}_{\varepsilon,\ \sigma}(\mathfrak{e}_d-m\mathfrak{e}_c))^{\gamma+1}}{\Gamma(\gamma+1)}\|\Phi\|_{\infty}\left(\frac{|\mathscr{H}'(m\mathfrak{e}_c)|+|\mathscr{H}'(\mathfrak{e}_d)|}{2}\right),$$

which is as required. □

Remark 4 Employing Theorem 2, we investigate the following new generalized fractional approach of Fejér-type inequality involving the CMLF via RLFIO if we choose $\rho=(1,\ 1,\ \ldots)$ with $\varepsilon=\alpha$ and $\sigma=1$:

$$\Bigg|\frac{\mathscr{H}(m\mathfrak{e}_c)+\mathscr{H}(m\mathfrak{e}_c+\mathscr{E}_{\alpha}(\mathfrak{e}_d-m\mathfrak{e}_c))}{2}\left[\mathrm{I}^{\gamma}_{(m\mathfrak{e}_c+\mathscr{E}_{\alpha}(\mathfrak{e}_d-m\mathfrak{e}_c))^-}\Phi(m\mathfrak{e}_c)+\mathrm{I}^{\gamma}_{m\mathfrak{e}_c^+}\Phi(m\mathfrak{e}_c+\mathscr{E}_{\alpha}(\mathfrak{e}_d-m\mathfrak{e}_c))\right]$$

$$-\left[\mathrm{I}^{\gamma}_{(m\mathfrak{e}_c+\mathscr{E}_{\alpha}(\mathfrak{e}_d-m\mathfrak{e}_c))^-}(\mathscr{H}\Phi)(m\mathfrak{e}_c)+\mathrm{I}^{\gamma}_{m\mathfrak{e}_c^+}(\mathscr{H}\Phi)(m\mathfrak{e}_c+\mathscr{E}_{\alpha}(\mathfrak{e}_d-m\mathfrak{e}_c))\right]\Bigg|$$

$$\leq ||\Phi||_{\infty}\frac{(\mathscr{E}_{\alpha}(\mathfrak{e}_d-m\mathfrak{e}_c))^{\gamma+1}}{\Gamma(\gamma+1)}\left(\frac{m|\mathscr{H}'(\mathfrak{e}_c)|+|\mathscr{H}'(\mathfrak{e}_d)|}{2}\right).$$

5. Conclusions

Scientists and researchers from many different fields have shown a great deal of interest in fractional calculus. On the other hand, convexity theory has become a potent instrument for creating novel numerical models that make it possible to solve challenging issues in the applied and pure sciences. Due to continuous advancements, extensions, and applications, convex analysis and the related inequalities are growing in popularity and gaining more attention from researchers.

In this work:

(1) First, we examined a new approach of Fejér inequality via G_mCRM over RLFIO.

(2) We constructed new equalities involving generalized preinvex mapping over RLFIO. Further, we explored new refinements of Fejér inequality with the support of G_mCRM over RLFIO.

6. Future directions

Quantum calculus and interval analysis can be used to study the inequalities under investigation. The study of integral inequality in particular is a rapidly expanding topic. Researchers will be fascinated by the integration of quantum calculus and interval-valued analysis in the study of integral inequalities since it offers intriguing avenues for future research. Examining possible links to optimization theory, where these inequalities can give objective functions tighter bounds under the premise of invexity. Applications in information geometry, machine learning, and entropy-based statistics are further investigated.

Conflict of interest

The authors declare no competing financial interest.

References

[1] Pelczyński J. Application of the theory of convex sets for engineering structures with uncertain parameters. *Applied Sciences*. 2020; 10(19): 6864. Available from: https://doi.org/10.3390/app10196864.

[2] Föllmer H, Schied A. Convex measures of risk and trading constraints. *Finance and Stochastics*. 2002; 6: 429-447. Available from: https://doi.org/10.1007/s007800200072.

[3] Chandrasekarana V, Jordan MI. Computational and statistical tradeoffs via convex relaxation. *Proceedings of the National Academy of Sciences*. 2013; 110(13): E1181-E1190. Available from: https://doi.org/10.1073/pnas.1302293110.

[4] Pennanen T. Convex duality in stochastic optimization and mathematical finance. *Mathematics of Operations Research*. 2011; 36(2): 340-362. Available from: https://doi.org/10.1287/moor.1110.0485.

[5] Tariq M, Ahmad H, Sahoo SK, Kashuri A, Nofal TA, Hsu CH. Inequalities of Simpson-Mercer-type including Atangana-Baleanu fractional operators and their applications. *AIMS Mathematics*. 2022; 7(8): 15159-15181. Available from: https://doi.org/10.3934/math.2022831.

[6] Sahoo SK, Mohammed PO, Kodamasingh B, Tariq M, Hamed YS. New fractional integral inequalities for convex functions pertaining to Caputo-Fabrizio operator. *Fractal and Fractional*. 2022; 6(3): 171. Available from: https://doi.org/10.3390/fractalfract6030171.

[7] Tariq M, Ahmad H, Shaikh AG, Sahoo SK, Khedher KM, Nguye TG. New fractional integral inequalities for preinvex functions involving Caputo-Fabrizio operator. *AIMS Mathematics*. 2021; 7(3): 3440-3455. Available from: https://doi.org/10.3934/math.2022191.

[8] Tariq M. Hermite-Hadamard type inequalities via p-harmonic exponential type convexity and applications. *Universal Journal of Mathematical Applications*. 2021; 4(2): 59-69.

[9] Axtell M, Bise ME. Fractional calculus application in control systems. In: *IEEE Conference on Aerospace and Electronics*. Dayton, OH, USA: IEEE; 1990. p.563-566.

[10] Butt SI, Umar M, Khan KA, Kashuri A, Emadifar H. Fractional Hermite-Jensen-Mercer integral inequalities with respect to another function and application. *Complexity*. 2021; 2021(1): 9260828. Available from: https://doi.org/10.1155/2021/9260828.

[11] Baleanu D, Güvenç ZB, Machado JT. *New Trends in Nanotechnology and Fractional Calculus Applications*. Dordrecht, Netherlands: Springer; 2010.

[12] El Shaed MA. Fractional calculus model of semilunar heart valve vibrations. In: *Proceedings of the International Mathematica Symposium*. London, UK: World Scientific; 2003. p.711-714.

[13] Moaaz O, Dassios I, Muhsin W, Muhib A. Oscillation theory for nonlinear neutral delay differential equations of third order. *Applied Sciences*. 2020; 10(14): 4855. Available from: https://doi.org/10.3390/app10144855.

[14] Moaaz O, Awrejcewicz J, Muhib A. Establishing new criteria for oscillation of odd-order nonlinear differential equations. *Mathematics*. 2020; 8(6): 937. Available from: https://doi.org/10.3390/math8060937.

[15] Moaaz O, Furuichi S, Muhib A. New comparison theorems for the nth-order neutral differential equations with delay inequalities. *Mathematics*. 2020; 8(3): 454. Available from: https://doi.org/10.3390/math8030454.

[16] Atangana A. Application of fractional calculus to epidemiology. In: *Fractional Dynamics*. Warsaw, Poland: De Gruyter Open; 2015. p.174-190. Available from: https://doi.org/10.1515/9783110472097-011.

[17] Baleanu D, Jajarmi A, Sajjadi SS, Mozyrska D. A new fractional model and optimal control of a tumor-immune surveillance with non-singular derivative operator. *Chaos*. 2019; 29(8): 083127. Available from: https://doi.org/10.1063/1.5096159.

[18] Butt SI, Umar M, Rashid S, Akdemir AO, Chu YM. New Hermite-Jensen-Mercer-type inequalities via k-fractional integrals. *Advances in Difference Equations*. 2020; 2020: 635. Available from: https://doi.org/10.1186/s13662-020-03093-y.

[19] Baleanu D, Jajarmi A, Defterli O, Wannan R, Sajjadi SS, Asad JH. Fractional investigation of time-dependent mass pendulum. *Journal of Low Frequency Noise, Vibration and Active Control*. 2024; 43(1): 196-207. Available from: https://doi.org/10.1177/14613484231187439.

[20] Niculescu CP, Persson LE. *Convex Functions and Their Applications*. New York: Springer; 2006. Available from: https://doi.org/10.1007/0-387-31077-0.

[21] Fejér L. Über die Fourierreihen, II. *Mathematische und Naturwissenschaftliche Berichte aus Ungarn*. 1906; 24: 369-390.

[22] Raina RK. On generalized Wright's hypergeometric functions and fractional calculus operators. *East Asian Mathematical Journal*. 2005; 21(2): 191-203.

[23] Cortez MJV, Liko R, Kashuri A, Hernández JEH. New quantum estimates of trapezium-type inequalities for generalized ϕ-convex functions. *Mathematics*. 2019; 7(11): 1047. Available from: https://doi.org/10.3390/math7111047.

[24] Cortez MJV, Kashuri A, Hernández JEH. Trapezium-type inequalities for Raina's fractional integrals operator using generalized convex functions. *Symmetry*. 2020; 12(6): 1034. Available from: https://doi.org/10.3390/sym12061034.

[25] Ahmad H, Tariq M, Sahoo SK, Baili J, Cesarano C. New estimations of Hermite-Hadamard type integral inequalities for special functions. *Fractal and Fractional*. 2021; 5(4): 144. Available from: https://doi.org/10.3390/fractalfract5040144.

[26] Kilbas AA, Srivastava HM, Trujillo JJ. *Theory and Applications of Fractional Differential Equations*. Amsterdam: Elsevier; 2006.

[27] Tariq M, Ntouyas SK, Shaikh AA. New variant of Hermite-Hadamard, Fejér and Pachpatte-type inequality and its refinements pertaining to fractional integral operator. *Fractal and Fractional*. 2023; 7(5): 405. Available from: https://doi.org/10.3390/fractalfract7050405.

[28] Park J. Inequalities of Hermite-Hadamard-Fejér type for convex functions via fractional integrals. *International Journal of Mathematical Analysis*. 2014; 8(59): 2927-2937.

[29] İşcan I, Kunt M. Hermite-Hadamard-Fejér type inequalities for harmonically s-convex functions via fractional integrals. *Australian Journal of Mathematical Analysis and Applications*. 2015; 12(1): 1-16.

[30] İşcan I, Turhan S. Generalized Hermite-Hadamard-Fejér type inequalities for GA-convex functions via fractional integral. *Moroccan Journal of Pure and Applied Analysis*. 2016; 2(1): 4. Available from: https://doi.org/10.7603/s40956-016-0004-2.

[31] Kunt M, İşcan I. Fractional Hermite-Hadamard-Fejér type inequalities for GA-convex functions. *Turkish Journal of Inequalities*. 2018; 2(1): 1-20.

[32] Kunt M, Işcan I. Hermite-Hadamard-Fejér type inequalities for p-convex functions via fractional integrals. *Iranian Journal of Science and Technology, Transactions A: Science*. 2018; 42(3): 2079-2089. Available from: https://doi.org/10.1007/s40995-017-0352-4.

[33] Baleanu D, Samraiz M, Perveen Z, Iqbal S, Nisar KS, Rahman G. Hermite-Hadamard-Fejér type inequalities via fractional integral of a function concerning another function. *AIMS Mathematics*. 2021; 6(5): 4280-4295. Available from: https://doi.org/10.3934/math.2021253.

[34] Hakiki MT, Wibowo A. Hermite-Hadamard-Fejér inequalities for s-convex functions in the second sense via Riemann-Liouville fractional integral. *Journal of Physics: Conference Series*. 2020; 1442: 012039. Available from: https://doi.org/10.1088/1742-6596/1442/1/012039.

[35] Jia W, Yussouf M, Farid G, Khan KA. Hadamard and Fejér-Hadamard inequalities for $(\alpha, h\text{-}m)$-p-convex functions via Riemann-Liouville fractional integrals. *Mathematical Problems in Engineering*. 2021; 2021(1): 9945114. Available from: https://doi.org/10.1155/2021/9945114.

[36] Chandola A, Agarwal R, Pandey RM. Some new Hermite-Hadamard, Hermite-Hadamard-Fejér weighted Hardy type inequalities involving $(k\text{-}p)$ R-L fractional integral operator. *Applied Mathematics & Information Sciences*. 2022; 16(2): 287-297. Available from: https://doi.org/10.18576/amis/160216.