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Research Article

Applications of a Hurwitz-Lerch Zeta Linear Operator Involving Classes of Meromorphic Functions

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Abstract: The findings of this study are connected with geometric function theory and were acquired by using linear operator is applied to investigate specific classes of meromorphic functions defined using the subordination. The first class introduced and investigated here, $\Sigma_{\tau}^{\varepsilon+1}(\rho, \varkappa, \vartheta; \mu)$, is a generalized class of meromorphic functions. It is also shown that for particular choice of parameters for the new generalized class, the class of close-to-convex meromorphic functions emerges. Using the properties of the convolution and subordination, certain characterization properties of this class are proved involving combinations of the functions from the class. Further, three more classes, $\mathfrak{C}_{\tau}^{\varepsilon+1}(\rho, \varkappa, \vartheta; \mu)$, $\wp_{\tau}^{\varepsilon+1}(\rho, \varkappa, \vartheta; \delta; \mu)$, and $\mathfrak{R}_{\tau}^{\varepsilon+1}(\rho, \varkappa, \vartheta; \delta; \mu)$, are defined in connection to this first class, developing new applications of this operator. The connections between the classes are given in the definition or in associated remarks and characterization properties are also proved, including combinations of functions belonging to those classes and inclusion relations.

Keywords: analytic function, differential subordination, meromorphic function, Hurwitz-Lerch Zeta function

MSC: 30C80, 30C45

1. Introduction

Geometric Function Theory (GFT) is a significant branch of mathematics that integrates geometric ideas with tools from complex analysis. It primarily focuses on the study of analytic functions defined on the complex plane, particularly univalent and multivalent functions, which constitute two of the most fundamental and extensively investigated classes within GFT. Research in these areas represents one of the oldest and most active topics in complex analysis and has attracted considerable attention due to the remarkable geometric structures exhibited by these functions, as well as the wide range of theoretical and applied problems they generate.

Among the various areas of complex analysis involving one complex variables, the theory of univalent functions occupies a central position. These functions play a crucial role in understanding conformal mappings and geometric properties of analytic functions. Comprehensive treatments of univalent functions can be found in the classical monographs by Duren [1] and Goodman [2], which remain foundational references in the field.

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The concept of analytic function subordination, a powerful tool in GFT, was first introduced by Littlewood [3]. This notion was later formalized and extensively developed by Rogosinski [4], who established many of its fundamental results. In recent years, subordination theory has been effectively applied to study various subclasses of analytic functions. Notably, Srivastava and Owa [5] employed this concept to investigate several interesting properties of generalized hypergeometric functions.

A further extension of this idea led to the development of differential subordination theory, which generalizes classical differential inequalities. This theory was systematically studied by Miller and Mocanu [6], whose work laid the groundwork for numerous subsequent investigations. The theory is highly versatile, allowing researchers to define new, broader classes of meromorphic functions that often generalize well-known subclasses of starlike, convex, or close-to-convex functions. Over the past decades, several researchers have explored intriguing aspects of Geometric Function Theory, particularly those related to subordination, differential inequalities, and their applications, have been explored by several researchers (see, for example, [7–10]).

The investigation presented in this paper focuses on the class of meromorphic functions. The approach is motivated by the numerous recent studies concerned with applying operators in order to introduce and provide properties of new classes of meromorphic functions. It is widely known that, while the classical GFT relied on direct geometric arguments, the modern approach is profoundly shaped by the introduction of operator theory. With the introduction of operators in the studies, new classes of analytic and univalent functions have been easily defined, and the study of their specific properties, such as starlikeness, convexity, and close-to-convexity, which are otherwise difficult to characterize, has been facilitated. Recent studies have provided such applications. For example, in [11], a new subclass of meromorphic functions is introduced and investigated by applying a new convolution operator involving Hurwitz-Lerch Zeta type and Kummer functions. A linear complex operator and a special class of Hurwitz-Lerch-Zeta functions are applied for the study on meromorphic functions proposed in the work [12], while the study presented in [13] provides not only a new class of meromorphic functions defined by applying a new operator but also a potential application of the results in fluid mechanics. Hence, the approach involving operators proves to be beneficial in improving knowledge related to new subclasses of meromorphic functions and it will be taken within this study.

A prolific tool in generating new subclasses of meromorphic functions is the Hurwitz-Lerch Zeta function, as a mathematical function that generalizes some well-known functions in number theory and analysis. Named after mathematicians Adolf Hurwitz and Wilhelm Lerch, this function has important applications in various areas of mathematics, including number theory, complex analysis, and mathematical physics. Understanding this function helps us explore deeper concepts in mathematics and enables us to analyze problems that may not be apparent with more basic functions.

The Hurwitz-Lerch Zeta function is a fascinating mathematical concept that extends the idea of the classic Riemann Zeta function. This function is often denoted as $\phi(\xi, \varepsilon, \tau)$, where ξ is a complex variable, ε is a complex parameter that influences the beginning of the series, and τ is a positive real parameter related to the terms of the series. The function is defined for complex numbers and is formulated as an infinite series that converges under certain conditions. This makes it an important tool in complex analysis. What sets the Hurwitz-Lerch Zeta function apart from the Riemann Zeta function is its versatility and the additional dimensions it encompasses.

The applications of the Hurwitz-Lerch Zeta function in defining new meaningful operators and special classes of analytic functions are numerous in geometric function theory in recent investigations involving subordination techniques. For example, consideration is given to a novel convolution complex operator defined on meromorphic functions associated with the Kummer functions and Hurwitz-Lerch Zeta type functions in [14–16] and extensions to the well-known starlikeness and convexity properties are developed for this operator. A comprehensive study on a new class of meromorphic functions introduced using Hurwitz-Lerch Zeta function can be seen in [17]. In [18], investigations are conducted into the geometric properties of a subclass of meromorphic functions as they relate to a complex linear operator related to Hurwitz-Lerch Zeta and Kummer functions. New classes of analytic functions are defined in [19] by applying the Hurwitz-Lerch Zeta function, for which typical geometric properties are established.

Many fractional models, such as those in optimal control and chemical kinetics, have precise solutions that belong to meromorphic subclasses with particular polar characteristics. Additionally, it is possible to think of the operational matrices obtained for Bernstein and Chebyshev polynomials as finite-dimensional approximations of operators working on meromorphic function spaces (see [20, 21]). By defining and examining the starlikeness and convexity of meromorphic functions using fractional integral operators, recent developments in GFT have further closed this gap and offered a solid analytical basis for the numerical stability of the Galerkin and Finite Element frameworks (see [22, 23]).

The novelty of the present study lies in the definition and the investigation of the new subclasses that are presented in sections 3, 4, 5, and 6. Each section presents the connections of the new classes with previously investigated classes, proving that the classes considered here, generalize the known results, hence, adding knowledge to the topic. Also, the new results obtained in this paper are proved to be connected with known results by choosing particular values for the parameters involved or by choosing specific functions for the general results obtained here. The corollaries associated with the theorems containing the new results, clearly show that the new outcome presented here generalizes known results, hence contributing to the advancement of the studies on meromorphic functions associated with Hurwitz-Lerch Zeta function. The next section contains the preliminaries that set the context of the study by presenting the concepts utilized for the research and the methods used for obtaining the new results. The importance of the Hurwitz-Lerch Zeta function is clearly proved by citing a few of the numerous works that include it in the investigations. If for previous results listed above the Kummer hypergeometric function was applied, for the present approach Gaussian hypergeometric function is considered. Furthermore, the investigation described here presents new applications of a previously introduced operator, which is comprehensively presented in the next section. The current findings improve our understanding of the importance of this operator for GFT studies. As tools for the investigation, known lemmas related to differential subordination theory are listed. With new applications presented for those lemmas, the importance differential subordination theory is further highlighted. Furthermore, the known fact that operators facilitate the technique of differential subordination is reinforced.

2. Peliminaries

Let Φ be the class of analytic functions and univalent convex functions in $\mathfrak{D}$, with $\mu(0) = 1$ and $Re\mu(\xi) > 0$.

The class of meromorphic analytic functions indicated by Σ is given by

$$\mathfrak{f}(\xi) = \frac{1}{\xi} + \sum_{\kappa=0}^{\infty} a_\kappa \xi^\kappa \qquad (\xi \in \mathfrak{D}^* = \mathfrak{D} \setminus \{0\}), \tag{1}$$

where $\mathfrak{D}^*$ is the punctured unit disc defined by $\mathfrak{D}^* = \{\xi : \xi \in \mathbb{C} \text{ and } 0 < |\xi| < 1\}$.

We say that $\mathfrak{f}$ and $\mathfrak{l}$ are analytic in $\mathfrak{D}$, $\mathfrak{f}$ is *subordinate* to $\mathfrak{l}$, denoted $\mathfrak{f}(\xi) \prec \mathfrak{l}(\xi)$, if there exists an analytic function ϖ, with $\varpi(0) = 0$ and $|\varpi(\xi)| < 1$, for all $\xi \in \mathfrak{D}$, such that $\mathfrak{f}(\xi) = \mathfrak{l}(\varpi(\xi))$, $\xi \in \mathfrak{D}$. If the function $\mathfrak{l}$ is univalent in $\mathfrak{D}$, $\mathfrak{f}(\xi) \prec \mathfrak{l}(\xi)$ (cf., e.g. [24–26]) as:

$$\mathfrak{f}(0) = \mathfrak{l}(0) \text{ and } \mathfrak{f}(\mathfrak{D}) \subset \mathfrak{l}(\mathfrak{D}).$$

If $\mathfrak{f} \in \Sigma$ as in (1) and $\mathfrak{g}$ donated by

$$\mathfrak{g}(\xi) = \frac{1}{\xi} + \sum_{\kappa=0}^{\infty} b_\kappa \xi^\kappa.$$

The representation of the Hadamard product of $\mathfrak{f}$ and $\mathfrak{g}$ is (see [27])

$$(\mathfrak{f}*\mathfrak{g})(\xi)=\frac{1}{\xi}+\sum_{\kappa=0}^{\infty}a_\kappa b_\kappa\xi^\kappa.$$

The general Hurwitz-Lerch Zeta function $\phi(\xi,\,\varepsilon,\,\tau)$ is defined by (see [28])

$$\phi(\xi,\,\varepsilon,\,\tau)=\sum_{\kappa=0}^{\infty}\frac{\xi^\kappa}{(\kappa+\tau)^\varepsilon},$$

$$(\tau\in\mathbb{C}\backslash\mathbb{Z}_0^-=\{0,\,-1,\,-2,\,...\};\,\varepsilon\in\mathbb{C}\ \text{ where }\ |\xi|<1:Re\varepsilon>1\ \text{ when }\ |\xi|=1).$$

Several interesting features and characteristics of the Hurwitz-Lerch Zeta function $\phi(\xi,\,\varepsilon,\,\tau)$ have been discovered in early studies by several researchers who have sparked interest for further investigations on this prolific function (see [29–32]).

For the present investigation, the operator introduced in [33] is applied. The definition of the operator involving Hurwitz-Lerch Zeta function and Gaussian hypergeometric function requires a few steps and intermediate operators.

First, a function $\mathfrak{g}_\tau^\varepsilon(\xi)$ $(\tau\in\mathbb{C}\backslash\mathbb{Z}_0^-;\,\varepsilon\in\mathbb{C})$ is introduced as:

$$\mathfrak{g}_\tau^\varepsilon(\xi)=\frac{\tau^\varepsilon}{\xi}\phi(\xi,\,\varepsilon,\,\tau)\quad(\xi\in\mathfrak{D}^*).\tag{2}$$

Next, the linear operator $\mathscr{L}_\tau^\varepsilon\mathfrak{f}(\xi):\Sigma\to\Sigma$ is defined as:

$$\mathscr{L}_\tau^\varepsilon\mathfrak{f}(\xi)=\mathfrak{g}_\tau^\varepsilon(\xi)*\mathfrak{f}(\xi)\quad(\tau\in\mathbb{C}\backslash\mathbb{Z}_0^-;\,\varepsilon\in\mathbb{C};\,\xi\in\mathfrak{D}^*),$$

knowing that,

$$\mathscr{L}_\tau^\varepsilon\mathfrak{f}(\xi)=\frac{1}{\xi}+\sum_{\kappa=0}^{\infty}\left(\frac{\tau}{\kappa+\tau+1}\right)^\varepsilon a_\kappa\xi^\kappa.\tag{3}$$

It is shown in [33] that this operator is connected to previously investigated operators as:

(i) $\mathscr{L}_1^\varepsilon\mathfrak{f}(\xi)=P^\varepsilon\mathfrak{f}(\xi)$ $(\varepsilon>0)$ (see Aqlan et al. [34], with $p=1$);

(ii) $\mathscr{L}_\tau^\varepsilon\mathfrak{f}(\xi)=P_\tau^\varepsilon\mathfrak{f}(\xi)$ $(\varepsilon,\,\tau>0)$ (Lashin [35]);

(iii) $\mathscr{L}_\tau^1\mathfrak{f}(\xi)=\mathscr{F}_\tau\mathfrak{f}(\xi)$ $(\tau>0)$ (see [25], p.11 and 389).

For the next step, when $\mathfrak{f}(\xi)\in\Sigma$, ξ, $t_i\in\mathfrak{D}^*$ $(i=1,\,2,\,3,\,...,\,\kappa)$, $\kappa\in\mathbb{N}$ and $\tau\in\mathbb{C}\backslash\mathbb{Z}_0^-$, it is known that the following relations hold:

$$\mathscr{L}_1^o\mathfrak{f}(\xi)=\mathfrak{f}(\xi)\ \text{ and }\ \mathscr{L}_\tau^o\mathfrak{f}(\xi)=\mathfrak{f}(\xi);$$

$$\mathscr{L}_1^1\mathfrak{f}(\xi)=\frac{1}{\xi^2}\int_0^\xi t_1\mathfrak{f}(t_1)dt_1\quad(\mathfrak{f}\in\Sigma;\,\xi\in\mathfrak{D}^*);$$

$$\mathscr{L}_1^2\mathfrak{f}(\xi) = \frac{1}{\xi^2}\int_0^{\xi}\frac{1}{t_1}\int_0^{t_1} t_2\mathfrak{f}(t_2)dt_2dt_1 \quad (\mathfrak{f}\in\Sigma;\ \xi\in\mathfrak{D}^*);$$

$$\vdots$$

$$\mathscr{L}_1^{\kappa}\mathfrak{f}(\xi) = \frac{1}{\xi^2}\int_0^{\xi}\frac{1}{t_1}\int_0^{t_1}\frac{1}{t_2}\int_0^{t_2}\cdots\frac{1}{t_{\kappa-1}}\int_0^{t_{\kappa-1}} t_{\kappa}\mathfrak{f}(t_{\kappa})dt_{\kappa}dt_{\kappa-1}\ldots dt_2dt_1 \quad (\mathfrak{f}\in\Sigma;\ \xi\in\mathfrak{D}^*);$$

$$\mathscr{L}_{\tau}^1\mathfrak{f}(\xi) = \frac{\tau}{\xi^{\tau+1}}\int_0^{\xi} t^{\tau}\mathfrak{f}(t)dt \quad (\mathfrak{f}\in\Sigma;\ \xi\in\mathfrak{D}^*);$$

$$\mathscr{L}_{\tau}^2\mathfrak{f}(\xi) = \frac{\tau^2}{\xi^{\tau+1}}\int_0^{\xi}\frac{1}{t_1}\int_0^{t_1} t_2^{\tau}\mathfrak{f}(t_2)dt_2dt_1 \quad (\mathfrak{f}\in\Sigma;\ \xi\in\mathfrak{D}^*);$$

$$\vdots$$

$$\mathscr{L}_{\tau}^{\kappa}\mathfrak{f}(\xi) = \frac{\tau^{\kappa}}{\xi^{\tau+1}}\int_0^{\xi}\frac{1}{t_1}\int_0^{t_1}\frac{1}{t_2}\int_0^{t_2}\cdots\frac{1}{t_{\kappa-1}}\int_0^{t_{\kappa-1}} t_{\kappa}^{\tau}\mathfrak{f}(t_{\kappa})dt_{\kappa}dt_{\kappa-1}\ldots dt_2dt_1 \quad (\mathfrak{f}\in\Sigma;\ \xi\in\mathfrak{D}^*).$$

And the following relation is also known:

$$\mathscr{L}_{\tau}^{\varepsilon+1}\mathfrak{f}(\xi) = \frac{\tau}{\xi^{\tau+1}}\int_0^{\xi} t^{\tau}\mathscr{L}_{\tau}^{\varepsilon}\mathfrak{f}(t)dt \quad (\mathfrak{f}\in\Sigma;\ \xi\in\mathfrak{D}^*).$$

Develop the function:

$$\Psi(\rho,\varkappa;\xi) = \frac{1}{\xi} + \sum_{\kappa=0}^{\infty}\frac{(\rho)_{\kappa+1}}{(\varkappa)_{\kappa+1}}\xi^{\kappa} \quad (\rho\in\mathbb{C}^* = \mathbb{C}\backslash\{0\};\ \varkappa\in\mathbb{C}^* = \mathbb{C}\backslash\mathbb{Z}_0^-;\ \xi\in\mathfrak{D}^*), \tag{4}$$

where $(\zeta)_{\kappa}$ is given by

$$(\zeta)_{\kappa} = \frac{\Gamma(\zeta+\kappa)}{\Gamma(\zeta)} = \begin{cases} 1 & (\kappa = 0) \\ \zeta(\zeta+1)\ldots(\zeta+\kappa+1) & (\kappa\in\mathbb{N}). \end{cases}$$

It can be noted that

$$\Psi(\rho,\varkappa;\xi) = \frac{1}{\xi}\,{}_2\Gamma_1(\rho,1;\varkappa;\xi)\,,$$

where

$$
{}_2\Gamma_1(\rho, v; \varkappa; \xi) = \sum_{\kappa=0}^{\infty} \frac{(\rho)_\kappa (v)_\kappa}{(\varkappa)_\kappa (1)_\kappa} \xi^\kappa \quad (\rho, v, \varkappa \in \mathbb{C} \text{ and } \varkappa \notin \mathbb{Z}_0^-; \xi \in \mathfrak{D}),
$$

is the Gaussian hypergeometric function.

Using the definition for the convolution and the definition of the operator $\mathscr{L}_\tau^\varepsilon \mathfrak{f}(\xi)$ given by (3), by setting

$$
\mathscr{L}_\tau^\varepsilon * \mathfrak{k}_\tau^\varepsilon(\xi) = \frac{1}{\xi(1-\xi)},
$$

we have

$$
\mathfrak{k}_\tau^\varepsilon(\xi) = \frac{1}{\xi} + \sum_{\kappa=0}^{\infty} \left(\frac{\kappa+\tau+1}{\tau} \right)^\varepsilon \xi^\kappa.
$$

Finally, using the operator $\mathfrak{k}_\tau^\varepsilon(\xi)$, the operator $\mathfrak{k}_\tau^\varepsilon(\rho, \varkappa; \xi)$ is obtained as:

$$
\mathfrak{k}_\tau^\varepsilon(\xi) * \mathfrak{k}_\tau^\varepsilon(\rho, \varkappa; \xi) = \Psi(\rho, \varkappa; \xi) \quad (\xi \in \mathfrak{D}^*). \tag{5}
$$

The linear operator $\mathfrak{k}_\tau^\varepsilon(\rho, \varkappa; \xi) : \Sigma \to \Sigma$, introduced by El-Ashawh in [33] and applied to obtain the new results of the present study, is defined by:

$$
\mathfrak{k}_\tau^\varepsilon(\rho, \varkappa; \xi)\mathfrak{f}(\xi) = \mathfrak{k}_\tau^\varepsilon(\rho, \varkappa; \xi) * \mathfrak{f}(\xi) \quad (\tau, \varkappa \in \mathbb{C}\backslash\mathbb{Z}_0^-; \rho \in \mathbb{C}^*; \varepsilon \in \mathbb{C}, \xi \in \mathfrak{D}^*).
$$

Its series expansion for $\tau, \varkappa \in \mathbb{C}\backslash\mathbb{Z}_0^-; \rho \in \mathbb{C}^*; \varepsilon \in \mathbb{C}, \xi \in \mathfrak{D}^*$ and for $\mathfrak{f}$ as in (1) is given by:

$$
\mathfrak{k}_\tau^\varepsilon(\rho, \varkappa; \xi)\mathfrak{f}(\xi) = \frac{1}{\xi} + \sum_{\kappa=0}^{\infty} \left(\frac{\tau}{\kappa+\tau+1} \right)^\varepsilon \frac{(\rho)_{\kappa+1}}{(\varkappa)_{\kappa+1}} a_\kappa \xi^\kappa. \tag{6}
$$

It is easily verified from the definition of the operator $\mathfrak{k}_\tau^\varepsilon(\rho, \varkappa; \xi)$, that

$$
\xi \left(\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}(\xi) \right)' = \tau \mathfrak{k}_\tau^\varepsilon(\rho, \varkappa; \xi)\mathfrak{f}(\xi) - (\tau+1)\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}(\xi), \tag{7}
$$

and

$$
\xi \left(\mathfrak{k}_\tau^\varepsilon(\rho, \varkappa; \xi)\mathfrak{f}(\xi) \right)' = \rho \mathfrak{k}_\tau^\varepsilon(\rho+1, \varkappa; \xi)\mathfrak{f}(\xi) - (\rho+1)\mathfrak{k}_\tau^\varepsilon(\rho, \varkappa; \xi)\mathfrak{f}(\xi) \quad (\rho \in \mathbb{C}\backslash\{-1\}). \tag{8}
$$

We note that $\mathfrak{k}_\tau^\varepsilon(\mu, 1; \xi)\mathfrak{f}(\xi) = I_{\tau,\mu}^\varepsilon \mathfrak{f}(\xi)$ $(\tau, \mu \in \mathbb{R}^+, \varepsilon \in \mathbb{N}_0)$ (see Cho et al. [36]).

The next lemmas are used to demonstrate our results:

Lemma 1 [37] Let β, $\sigma \in \mathbb{C}$. Also let $\mu \in \Phi$ be convex univalent in $\mathfrak{D}$ with $Re[\beta\mu(\xi)+\sigma] > 0$ $(\xi \in \mathfrak{D})$, $\mu(0) = 1$ and $p(\xi) \in \Phi$ with $p(\xi) = 1 + p_1\xi + p_2\xi^2 + \ldots$ is analytic in $\mathfrak{D}$. If

$$p(\xi) + \frac{\xi p'(\xi)}{\beta p(\xi)+\sigma} \prec \mu(\xi) \quad (\xi \in \mathfrak{D}),$$

then $p(\xi) \prec \mu(\xi)$.

Lemma 2 [38] Let β, $\sigma \in \mathbb{C}$. Also suppose $\mu \in \Phi$ be convex univalent in $\mathfrak{D}$ with $\mu(0) = 1$ and $Re[\beta\mu(\xi)+\sigma] > 0$ $(\xi \in \mathfrak{D})$, and $q(\xi) \in \Phi$ with $q(0) = 1$ and $q(\xi) \prec \mu(\xi)$ $(\xi \in \mathfrak{D})$. If $p(\xi) = 1 + p_1\xi + p_2\xi^2 + \ldots$ is analytic in $\mathfrak{D}$,

$$p(\xi) + \frac{\xi p'(\xi)}{\beta q(\xi)+\sigma} \prec \mu(\xi) \quad (\xi \in \mathfrak{D}),$$

then

$$p(\xi) \prec \mu(\xi).$$

By using the operator $\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}(\xi)$, given by (6), in the following section, we investigate the class $\Sigma_{\tau}^{\varepsilon+1}(\rho, \varkappa, \vartheta; \mu)$, of analytic functions $\mathfrak{f} = \{\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{\vartheta}\}$ in the open unit disc $\mathfrak{D}$ satisfying

$$\frac{-\xi\left(\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}_i(\xi)\right)'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}_j(\xi)} \prec \mu(\xi) \quad (\mathfrak{f}_i \in \Sigma,\ i = 1, 2, \ldots, \vartheta,\ \xi \in \mathfrak{D}),$$

where $\xi\sum_{j=1}^{\vartheta}\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}_j(\xi) \neq 0$ and μ is a convex univalent function in $\mathfrak{D}$ with $\mu(0) = 1$.

Also, we define $\mathscr{F} = \{\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_{\vartheta}\}$ where $\mathscr{F}_i(\xi) = \frac{\varsigma+1}{\xi^{\varsigma+2}}\int_0^{\xi} t^{\varsigma+1}\mathfrak{f}_i(t)dt$, with $\varsigma > 0$ and $i = 1, 2, \ldots, \vartheta$ and we prove that $\mathscr{F} \in \Sigma_{\tau}^{\varepsilon+1}(\rho, \varkappa, \vartheta; \mu)$, whenever $\mathfrak{f} \in \Sigma_{\tau}^{\varepsilon+1}(\rho, \varkappa, \vartheta; \mu)$. It is understood with the notation $\mathfrak{f} = \{\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{\vartheta}\}$ that each one of these functions $\mathfrak{f}_i$ is connected with the arithmetic mean of all functions $\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{\vartheta}$ by division. The same remark holds true for the function $\mathscr{F} = \{\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_{\vartheta}\}$.

Additional classes of this type are indicated by $\mathfrak{C}_{\tau}^{\varepsilon+1}(\rho, \varkappa, \vartheta; \mu)$, $\wp_{\tau}^{\varepsilon+1}(\rho, \varkappa, \vartheta; \mu)$, and $\mathfrak{R}_{\tau}^{\varepsilon+1}(\rho, \varkappa, \vartheta; \delta, \mu)$. They are presented and examined in sections 4, 5, and 6 using convolutions and the subordination approach.

Throughout this paper, unless otherwise mentioned, we set τ, $\varkappa \in \mathbb{R}\backslash\mathbb{Z}_0^-$; $\rho \in \mathbb{R}^*$; $\varepsilon \in \mathbb{R}$, $\xi \in \mathfrak{D}^*$, and $\vartheta \in \mathbb{N}$.

3. The class $\Sigma_{\tau}^{\varepsilon+1}(\rho, \varkappa, \vartheta; \mu)$

Definition 1 Let $\mathfrak{f} = \{\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{\vartheta}\}$, $\mathfrak{f}_i \in \Sigma$, $1 \leq i \leq \vartheta$ be such that

$$\frac{-\xi\left(\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\,\varkappa;\,\xi)\mathfrak{f}_i(\xi)\right)'}{\dfrac{1}{\vartheta}\displaystyle\sum_{j=1}^{\vartheta}\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\,\varkappa;\,\xi)\mathfrak{f}_j(\xi)}\prec\mu(\xi)\quad(\xi\in\mathfrak{D};\,i=1,\,2,\,...,\,\vartheta),$$

where $\xi\sum_{j=1}^{\vartheta}\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\,\varkappa;\,\xi)\mathfrak{f}_j(\xi)\neq0$ in $\mathfrak{D}$, μ is convex univalent in $\mathfrak{D}$ with $\mu(0)=1$. Then we say that $\mathfrak{f}=\{\mathfrak{f}_1,\,\mathfrak{f}_2,\,...,\,\mathfrak{f}_\vartheta\}\in\Sigma_\tau^{\varepsilon+1}(\rho,\,\varkappa,\,\vartheta;\,\mu)$.

Remark 1 $\Sigma_\tau^0\left(1,\,1,\,1;\,\dfrac{1-\xi}{1+\xi}\right)=\Sigma^*$, the class of meromorphic starlike functions introduced by Clunie [39], Pommerenke [40] and others.

Theorem 1 Let $\mathfrak{f}=\{\mathfrak{f}_1,\,\mathfrak{f}_2,\,...,\,\mathfrak{f}_\vartheta\}\in\Sigma_\tau^{\varepsilon+1}(\rho,\,\varkappa,\,\vartheta;\,\mu)$ and $\mathscr{F}(\xi)=\dfrac{1}{\vartheta}\sum_{i=1}^{\vartheta}\mathfrak{f}_i(\xi)$. Then $\mathscr{F}(\xi)$ satisfies the condition

$$\frac{-\xi\left(\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\,\varkappa;\,\xi)\mathscr{F}(\xi)\right)'}{\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\,\varkappa;\,\xi)\mathscr{F}(\xi)}\prec\mu(\xi)\quad(\xi\in\mathfrak{D}).\tag{9}$$

Proof. Let $\mathfrak{f}=\{\mathfrak{f}_1,\,\mathfrak{f}_2,\,...,\,\mathfrak{f}_\vartheta\}\in\Sigma_\tau^{\varepsilon+1}(\rho,\,\varkappa,\,\vartheta;\,\mu)$. Then for any $\xi_0\in\mathfrak{D}$, we have

$$\frac{-\xi_0\left(\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\,\varkappa;\,\xi)\mathfrak{f}_i(\xi_0)\right)'}{\dfrac{1}{\vartheta}\displaystyle\sum_{j=1}^{\vartheta}\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\,\varkappa;\,\xi)\mathfrak{f}_j(\xi_0)}\prec\mu(\mathfrak{D}),$$

and hence equals to $\mu(w_i)$ (say) for some $w_i\in\mathfrak{D}$, $i=1,\,2,\,...,\,\vartheta$. So

$$-\frac{\displaystyle\sum_{i=1}^{\vartheta}\xi_0\left(\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\,\varkappa;\,\xi)\mathfrak{f}_i(\xi_0)\right)'}{\dfrac{1}{\vartheta}\displaystyle\sum_{j=1}^{\vartheta}\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\,\varkappa;\,\xi)\mathfrak{f}_j(\xi_0)}=\sum_{i=1}^{\vartheta}\mu(w_i).$$

Let $\mathfrak{f}(\xi)=\dfrac{1}{\xi}+\sum_{k=0}^{\infty}a_k\xi^k$. Then, from (6), we see that

$$\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\,\varkappa;\,\xi)\mathfrak{f}(\xi)=\mathfrak{f}(\xi)*\left\{\frac{1}{\xi}+\sum_{\kappa=0}^{\infty}\left(\frac{\tau}{\kappa+\tau+1}\right)^{\varepsilon+1}\frac{(\rho)_{\kappa+1}}{(\varkappa)_{\kappa+1}}\xi^\kappa\right\}$$

$$=(\mathfrak{f}*\mathfrak{K}_\tau^{\varepsilon+1}(\rho,\,\varkappa))(\xi),$$

where

$$\mathfrak{K}_{\tau}^{\varepsilon+1}(\rho,\, \varkappa;\, \xi) = \frac{1}{\xi} + \sum_{\kappa=0}^{\infty} \left(\frac{\tau}{\kappa+\tau+1}\right)^{\varepsilon+1} \frac{(\rho)_{\kappa+1}}{(\varkappa)_{\kappa+1}} \xi^{\kappa}. \tag{10}$$

Hence

$$\frac{-\xi_0 \left(\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho,\, \varkappa;\, \xi)\mathscr{F}(\xi_0)\right)'}{\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho,\, \varkappa;\, \xi)\mathscr{F}(\xi_0)} = \frac{-\xi_0 \left[\mathfrak{K}_{\tau}^{\varepsilon+1}(\rho,\, \varkappa;\, \xi_0) * \sum_{i=1}^{\vartheta} \mathfrak{f}_i(\xi_0)\right]'}{\mathfrak{K}_{\tau}^{\varepsilon+1}(\rho,\, \varkappa;\, \xi_0) * \sum_{j=1}^{\vartheta} \mathfrak{f}_j(\xi_0)}.$$

Since

$$\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho,\, \varkappa;\, \xi) \sum_{j=1}^{\vartheta} \mathfrak{f}_j(\xi) = \sum_{j=1}^{\vartheta} \mathfrak{k}_{\tau}^{\varepsilon+1}(\rho,\, \varkappa;\, \xi)\mathfrak{f}_j(\xi),$$

we have

$$\frac{-\xi_0 \left(\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho,\, \varkappa;\, \xi)\mathscr{F}(\xi_0)\right)'}{\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho,\, \varkappa;\, \xi)\mathscr{F}(\xi_0)} = \frac{1}{\vartheta} \left[\frac{-\xi_0 \sum_{i=1}^{\vartheta} \left(\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho,\, \varkappa;\, \xi)\mathfrak{f}_i(\xi_0)\right)'}{\frac{1}{\vartheta} \sum_{j=1}^{\vartheta} \mathfrak{k}_{\tau}^{\varepsilon+1}(\rho,\, \varkappa;\, \xi)\mathfrak{f}_j(\xi_0)}\right]$$

$$= \frac{1}{\vartheta} \sum_{i=1}^{\vartheta} \mu(w_i) = \mu(w_0),$$

for some $w_0 \in \mathfrak{D}$, since μ is convex in $\mathfrak{D}$. This completes the proof. □

Considering $\mu(\xi) = \frac{1-\xi}{1+\xi}$ in Theorem 1 we get:

Corollary 1 If $\mathfrak{f} = \{\mathfrak{f}_1,\, \mathfrak{f}_2,\, ...,\, \mathfrak{f}_\vartheta\} \in \Sigma_{\tau}^{\varepsilon+1}(\rho,\, \varkappa,\, \vartheta;\, \mu)$ and $\mu(\xi) = \frac{1-\xi}{1+\xi}$, then $\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho,\, \varkappa;\, \xi)\mathfrak{f}_i(\xi)$ are close-to-convex meromorphic functions (see [41, 42]).

Theorem 2 Let $\mathfrak{f} = \{\mathfrak{f}_1,\, \mathfrak{f}_2,\, ...,\, \mathfrak{f}_\vartheta\} \in \Sigma_{\tau}^{\varepsilon+1}(\rho,\, \varkappa,\, \vartheta;\, \mu)$. Define

$$\mathscr{F}_i(\xi) = \frac{\varsigma+1}{\xi^{\varsigma+2}} \int_0^{\xi} t^{\varsigma+1} \mathfrak{f}_i(t)dt \quad (\varsigma > 0;\, i = 1,\, 2,\, ...,\, \vartheta).$$

If μ is bounded in $\mathfrak{D}$ and $(\varsigma+2) > Re\mu(\xi)$, then $\mathscr{F} = \{\mathscr{F}_1,\, \mathscr{F}_2,\, ...,\, \mathscr{F}_\vartheta\} \in \Sigma_{\tau}^{\varepsilon+1}(\rho,\, \varkappa,\, \vartheta;\, \mu)$.

Proof. From the definition of $\mathscr{F}_i(\xi)$, it follows that

$$\xi \mathscr{F}_i'(\xi) + (\varsigma+2)\mathscr{F}_i(\xi) = (\varsigma+1)\mathfrak{f}_i(\xi),$$

and on taking convolution with $\mathfrak{K}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)$ given by (10), we get

$$\xi[\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathscr{F}_i(\xi)]' + (\varsigma+2)\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathscr{F}_i(\xi) = (\varsigma+1)\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}_i(\xi), \quad i = 1, 2, ..., \vartheta. \tag{11}$$

Let

$$p_i(\xi) = -\frac{\vartheta\xi[\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathscr{F}_i(\xi)]'}{\sum_{j=1}^{\vartheta}\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathscr{F}_j(\xi)}. \tag{12}$$

From (11), we have

$$-\frac{p_i(\xi)}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathscr{F}_j(\xi) + (\varsigma+2)\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathscr{F}_i(\xi) = (\varsigma+1)\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}_i(\xi). \tag{13}$$

Differentiating (13) with respect to ξ, we get

$$-\frac{p_i'(\xi)}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathscr{F}_j(\xi) - \frac{p_i(\xi)}{\vartheta}\sum_{j=1}^{\vartheta}[\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathscr{F}_j(\xi)]' + (\varsigma+2)[\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathscr{F}_i(\xi)]'$$

$$= (\varsigma+1)[\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}_i(\xi)]'.$$

From (12), we have

$$-p_i'(\xi)\frac{\sum_{j=1}^{\vartheta}\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathscr{F}_j(\xi)}{\vartheta} + \frac{p_i(\xi)}{\vartheta}\frac{\sum_{i=1}^{\vartheta}p_i(\xi)\sum_{j=1}^{\vartheta}\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathscr{F}_j(\xi)}{\vartheta\xi}$$

$$-(\varsigma+2)\frac{p_i(\xi)\sum_{j=1}^{\vartheta}\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathscr{F}_j(\xi)}{\vartheta\xi}$$

$$= (\varsigma+1)[\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}_i(\xi)]'.$$

Hence

$$-p_i'(\xi)+\frac{p_i(\xi)}{\vartheta\xi}\sum_{i=1}^{\vartheta}p_i(\xi)-(\varsigma+2)\frac{p_i(\xi)}{\xi}=\frac{(\varsigma+1)[\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\varkappa;\xi)\mathfrak{f}_i(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\varkappa;\xi)\mathscr{F}_j(\xi)}.$$

Then

$$\frac{-\xi p_i'(\xi)}{\frac{1}{\vartheta}\sum_{i=1}^{\vartheta}p_i(\xi)-(\varsigma+2)}+p_i(\xi)$$

$$=\frac{(\varsigma+1)\xi[\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\varkappa;\xi)\mathfrak{f}_i(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\varkappa;\xi)\mathscr{F}_j(\xi)}\cdot\frac{1}{\frac{1}{\vartheta}\sum_{i=1}^{\vartheta}p_i(\xi)-(\varsigma+2)}$$

$$=\frac{(\varsigma+1)\xi[\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\varkappa;\xi)\mathfrak{f}_i(\xi)]'}{\frac{-1}{\vartheta}\left\{\frac{-1}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\varkappa;\xi)\mathscr{F}_j(\xi)\cdot\sum_{i=1}^{\vartheta}p_i(\xi)+(\varsigma+2)\sum_{j=1}^{\vartheta}\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\varkappa;\xi)\mathscr{F}_j(\xi)\right\}}.$$

From (13), we have

$$\frac{-\xi p_i'(\xi)}{\frac{1}{\vartheta}\sum_{i=1}^{\vartheta}p_i(\xi)-(\varsigma+2)}+p_i(\xi)=\frac{(\varsigma+1)\xi[\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\varkappa;\xi)\mathfrak{f}_i(\xi)]'}{\frac{-1}{\vartheta}(\varsigma+1)\sum_{i=1}^{\vartheta}\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\varkappa;\xi)\mathfrak{f}_i(\xi)}\prec\mu(\xi), \tag{14}$$

since $\mathfrak{f}=\{\mathfrak{f}_1,\mathfrak{f}_2,....,\mathfrak{f}_\vartheta\}\in\Sigma_\tau^{\varepsilon+1}(\rho,\varkappa,\vartheta;\mu)$. Now we can write for any $\xi_0\in\mathfrak{D}$,

$$\frac{-\frac{1}{\vartheta}\xi_0 p_i'(\xi_0)}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}p_j(\xi_0)-(\varsigma+2)}+\frac{1}{\vartheta}p_i(\xi_0)=\frac{1}{\vartheta}\mu(w_i),$$

for some $w_i\in\mathfrak{D}$. This is true for $i=1,2,...,\vartheta$. Since μ is convex, there exists a $w_0\in\mathfrak{D}$ such that

$$\frac{\xi_0 Q'(\xi_0)}{-Q(\xi_0)+(\varsigma+2)}+Q(\xi_0)=\mu(w_0),$$

where $Q(\xi)=\frac{1}{\vartheta}\sum_{i=1}^{\vartheta}p_i(\xi)$. Hence

$$\frac{\xi Q'(\xi)}{-Q(\xi)+(\varsigma+2)}+Q(\xi)\prec\mu(\xi).$$

Since $Re\mu(\xi)$ is bounded and $(\varsigma+2)>Re\mu(\xi)$, from Lemma 1 we get $Q(\xi)\prec\mu(\xi)$ $(\xi\in\mathfrak{D})$. From (14), we have

$$\frac{\xi[p_i(\xi)]'}{-Q(\xi)+(\varsigma+2)}+p_i(\xi)\prec\mu(\xi),$$

where $Q(\xi)\prec\mu(\xi)$. The uses of Lemma 2 provides $p_i(\xi)\prec\mu(\xi)$ $(\xi\in\mathfrak{D})$, $i=1, 2, ..., \vartheta$, that is

$$\frac{-\xi[\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathscr{F}_i(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathscr{F}_j(\xi)}\prec\mu(\xi).$$

Now

$$\mathscr{F}_i(\xi)=\frac{\varsigma+1}{\xi^{\varsigma+2}}\int_0^\xi t^{\varsigma+1}\mathfrak{f}_i(t)dt, \quad Re(\varsigma)>0.$$

For every i, $1\leq i\leq\vartheta$,

$$\mathfrak{k}_\tau^{\varepsilon}(\rho, \varkappa; \xi)\mathscr{F}_i(\xi)=\frac{\varsigma+1}{\xi^{\varsigma+2}}\int_0^\xi t^{\varsigma+1}\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}_i(t)dt,$$

and hence

$$\sum_{i=1}^{\vartheta}\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathscr{F}_i(\xi)=\frac{\varsigma+1}{\xi^{\varsigma+2}}\int_0^\xi t^{\varsigma+1}\sum_{i=1}^{\vartheta}\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}_i(t)dt$$

$$=\frac{\varsigma+1}{\xi^{\varsigma+2}}\int_0^\xi t^{\varsigma}\mathfrak{g}(t)dt,$$

where $\mathfrak{g}(t)=t\sum_{i=1}^{\vartheta}\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}_i(t)\neq 0$, for $\xi\in\mathfrak{D}$.

Now, define

$$\Omega(\xi) = \sum_{\kappa=1}^{\infty} \frac{\varsigma+1}{\varsigma+\kappa} \xi^{k-1}, \quad Re(\varsigma) > 0.$$

Then an easy calculation shows that

$$\xi \sum_{i=1}^{\vartheta} \mathfrak{k}_{\tau}^{\varepsilon+1}(\rho, \varkappa; \xi) \mathscr{F}_i(\xi) = (\Omega * \mathfrak{g})(\xi) \neq 0.$$

Thus $\mathscr{F} = \{\mathscr{F}_1, \mathscr{F}_2, ..., \mathscr{F}_\vartheta\} \in \Sigma_{\tau}^{\varepsilon+1}(\rho, \varkappa, \vartheta; \mu)$. □

Theorem 3 If $\mathfrak{f} = \{\mathfrak{f}_1, \mathfrak{f}_2, ..., \mathfrak{f}_\vartheta\} \in \Sigma_{\tau}^{\varepsilon+1}(\rho, \varkappa, \vartheta; \mu)$, and $Re\mu(\xi)$ is bounded in $\mathfrak{D}$, then $\mathfrak{f} = \{\mathfrak{f}_1, \mathfrak{f}_2, ..., \mathfrak{f}_\vartheta\} \in \Sigma_{\tau}^{\varepsilon+1}$ $(\rho, \varkappa, \vartheta; \mu)$ holds for $(\tau+1) > Re\mu(\xi)$ in $\mathfrak{D}$.

Proof. Let

$$p_i(\xi) = -\frac{\vartheta \xi \left[\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho, \varkappa; \xi) \mathfrak{f}_i(\xi)\right]'}{\sum_{j=1}^{\vartheta} \mathfrak{k}_{\tau}^{\varepsilon+1}(\rho, \varkappa; \xi) \mathfrak{f}_j(\xi)} \quad (\xi \in \mathfrak{D}; i = 1, 2, ..., \vartheta). \tag{15}$$

From (8) and (15), we have

$$-\frac{1}{\vartheta} p_i(\xi) \sum_{j=1}^{\vartheta} \mathfrak{k}_{\tau}^{\varepsilon+1}(\rho, \varkappa; \xi) \mathfrak{f}_j(\xi) = \tau \mathfrak{k}_{\tau}^{\varepsilon}(\rho, \varkappa; \xi) \mathfrak{f}_i(\xi) - (\tau+1) \mathfrak{k}_{\tau}^{\varepsilon+1}(\rho, \varkappa; \xi) \mathfrak{f}_i(\xi). \tag{16}$$

Differentiating (16) with respect to ξ, we get

$$-\frac{\xi}{\vartheta} p_i'(\xi) \sum_{j=1}^{\vartheta} \mathfrak{k}_{\tau}^{\varepsilon+1}(\rho, \varkappa; \xi) \mathfrak{f}_j(\xi) - \frac{\xi}{\vartheta} p_i(\xi) \sum_{j=1}^{\varkappa} \left[\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho, \varkappa; \xi) \mathfrak{f}_j(\xi)\right]'$$

$$= \tau \xi \left[\mathfrak{k}_{\tau}^{\varepsilon}(\rho, \varkappa; \xi) \mathfrak{f}_i(\xi)\right]' - (\tau+1) \xi \left[\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho, \varkappa; \xi) \mathfrak{f}_i(\xi)\right]'.$$

Using (15), we obtain

$$-\frac{\xi}{\vartheta} p_i'(\xi) \sum_{j=1}^{\vartheta} \mathfrak{k}_{\tau}^{\varepsilon+1}(\rho, \varkappa; \xi) \mathfrak{f}_j(\xi) - p_i(\xi) \left[\frac{\xi}{\vartheta} \sum_{j=1}^{\vartheta} \left[\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho, \varkappa; \xi) \mathfrak{f}_j(\xi)\right]' \frac{(\tau+1)}{\vartheta} \sum_{j=1}^{\vartheta} \mathfrak{k}_{\tau}^{\varepsilon+1}(\rho, \varkappa; \xi) \mathfrak{f}_j(\xi)\right]$$

$$= \tau \xi \left[\mathfrak{k}_{\tau}^{\varepsilon}(\rho, \varkappa; \xi) \mathfrak{f}_i(\xi)\right]'.$$

Then

$$\frac{-\frac{\xi}{\vartheta} p_i'(\xi) \sum_{j=1}^{\vartheta} \mathfrak{k}_{\tau}^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}_j(\xi)}{\frac{\xi}{\vartheta} \sum_{j=1}^{\vartheta} [\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}_j(\xi)]' + \frac{(\tau+1)}{\vartheta} \sum_{j=1}^{\vartheta} \mathfrak{k}_{\tau}^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}_j(\xi)} - p_i(\xi)$$

$$= \frac{\tau\xi [\mathfrak{k}_{\tau}^{\varepsilon}(\rho, \varkappa; \xi)\mathfrak{f}_i(\xi)]'}{\frac{\xi}{\vartheta} \sum_{j=1}^{\vartheta} [\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}_j(\xi)]' + \frac{(\tau+1)}{\vartheta} \sum_{j=1}^{\vartheta} \mathfrak{k}_{\tau}^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}_j(\xi)}.$$

Using (8), we have

$$\frac{-\frac{\xi}{\vartheta} p_i'(\xi) \sum_{j=1}^{\vartheta} \mathfrak{k}_{\tau}^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}_j(\xi)}{\frac{\xi}{\vartheta} \sum_{j=1}^{\vartheta} [\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}_j(\xi)]' + \frac{(\tau+1)}{\vartheta} \sum_{j=1}^{\vartheta} \mathfrak{k}_{\tau}^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}_j(\xi)} - p_i(\xi) = \frac{\xi [\mathfrak{k}_{\tau}^{\varepsilon}(\rho, \varkappa; \xi)\mathfrak{f}_i(\xi)]'}{\frac{1}{\vartheta} \sum_{j=1}^{\vartheta} \mathfrak{k}_{\tau}^{\varepsilon}(\rho, \varkappa; \xi)\mathfrak{f}_j(\xi)}. \tag{17}$$

Using (15) in (17), we have

$$\frac{-\xi p_i'(\xi)}{\frac{-1}{\vartheta} \sum_{j=1}^{\vartheta} p_j(\xi) + (\tau+1)} - p_i(\xi) = \frac{\xi [\mathfrak{k}_{\tau}^{\varepsilon}(\rho, \varkappa; \xi)\mathfrak{f}_i(\xi)]'}{\frac{1}{\vartheta} \sum_{j=1}^{\vartheta} \mathfrak{k}_{\tau}^{\varepsilon}(\rho, \varkappa; \xi)\mathfrak{f}_j(\xi)}.$$

Since $\mathfrak{f} = \{\mathfrak{f}_1, \mathfrak{f}_2, ..., \mathfrak{f}_{\vartheta}\} \in \Sigma_{\tau}^{\varepsilon}(\rho, \varkappa, \vartheta; \mu)$, then we have

$$\frac{\xi p_i'(\xi)}{\frac{-1}{\vartheta} \sum_{j=1}^{\vartheta} p_j(\xi) + (\tau+1)} + p_i(\xi) = \frac{-\xi [\mathfrak{k}_{\tau}^{\varepsilon}(\rho, \varkappa; \xi)\mathfrak{f}_i(\xi)]'}{\frac{1}{\vartheta} \sum_{j=1}^{\vartheta} \mathfrak{k}_{\tau}^{\varepsilon}(\rho, \varkappa; \xi)\mathfrak{f}_j(\xi)} \prec \mu(\xi) \quad i = 1, 2, ..., \vartheta. \tag{18}$$

Therefore for any $\xi_0 \in \mathfrak{D}$, we have

$$\frac{\frac{1}{\vartheta}\xi_0 p_i'(\xi_0)}{\frac{-1}{\vartheta} \sum_{j=1}^{\vartheta} p_j(\xi_0) + (\tau+1)} + \frac{1}{\vartheta} p_i(\xi_0) = \frac{1}{\vartheta}\mu(w_i),$$

for some $w_0 \in \mathfrak{D}$. Since μ is convex, there exists a $w_i \in \mathfrak{D}$, such that

$$\frac{\frac{\xi_0}{\vartheta}\sum_{i=1}^{\rho} p_i'(\xi_0)}{\frac{-1}{\vartheta}\sum_{j=1}^{\vartheta} p_j(\xi_0)+(\tau+1)}+\frac{1}{\vartheta}\sum_{i=1}^{\vartheta} p_i(\xi_0)=\frac{1}{\vartheta}\sum_{i=1}^{\vartheta}\mu(w_i)=\mu(w_0).$$

Setting $Q(\xi)=\frac{1}{\vartheta}\sum_{i=1}^{\vartheta} p_i(\xi)$, we have

$$\frac{\xi Q'(\xi)}{-Q(\xi)+(\tau+1)}+Q(\xi)\prec\mu(\xi).$$

This shows that $Q(\xi)\prec\mu(\xi)$ by Lemma 1. Relation (18) gives us

$$\frac{\xi p_i'(\xi)}{-Q(\xi)+(\tau+1)}+p_i(\xi)\prec\mu(\xi),$$

where $Q(\xi)\prec\mu(\xi)$. By Lemma 2 we have that $p_i(\xi)\prec\mu(\xi)$, which further implies $\mathfrak{f}=\{\mathfrak{f}_1, \mathfrak{f}_2, ..., \mathfrak{f}_\vartheta\}\in\Sigma_\tau^{\varepsilon+1}(\rho, \varkappa, \vartheta; \mu)$. □

4. The class $\mathfrak{C}_\tau^{\varepsilon+1}(\rho, \varkappa, \vartheta; \mu)$

Definition 2 $\mathfrak{C}_\tau^{\varepsilon+1}(\rho, \varkappa, \vartheta; \mu)$ denote the class of functions $\mathfrak{f}\in\Sigma$ satisfying

$$\frac{-\xi[\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{g}_j(\xi)}\prec\mu(\xi)\quad(\xi\in\mathfrak{D}),$$

where $\mathfrak{g}=\{\mathfrak{g}_1, \mathfrak{g}_2, ..., \mathfrak{g}_\vartheta\}\in\Sigma_\tau^{\varepsilon+1}(\rho, \varkappa, \vartheta; \mu)$, and μ is a convex univalent function in $\mathfrak{D}$ with $\mu(0)=1$.

Remark 2 Let $\rho=\varkappa=1$, $\varepsilon=-1$ and $\mu(\xi)=\frac{1-\xi}{1+\xi}$ then $\mathfrak{C}_1^0\left(1, 1, 1; \frac{1-\xi}{1+\xi}\right)=\mathfrak{C}$, are close-to-convex meromorphic functions (see [41, 42]).

Theorem 4 Let $\mathfrak{f}\in\mathfrak{C}_\tau^{\varepsilon+1}(\rho, \varkappa, \vartheta; \mu)$. If $Re\mu(\xi)$ is bounded in $\mathfrak{D}$ and $(\upsilon+2)>Re\mu(\xi)$, then

$$\mathscr{F}(\xi)=\frac{\upsilon+1}{\xi^{\upsilon+2}}\int_0^{\xi} t^{\upsilon+1}\mathfrak{f}(t)dt\quad(\xi\in\mathfrak{D};\ \upsilon>0),$$

also belongs to $\mathfrak{C}_\tau^{\varepsilon+1}(\rho, \varkappa, \vartheta; \mu)$.

Proof. Since $\mathfrak{f}\in\mathfrak{C}_\tau^{\varepsilon+1}(\rho, \varkappa, \vartheta; \mu)$, then there exists $\mathfrak{g}=\{\mathfrak{g}_1, \mathfrak{g}_2, ..., \mathfrak{g}_\vartheta\}\in\Sigma_\tau^{\varepsilon+1}(\rho, \varkappa, \vartheta; \mu)$, such that

$$\frac{-\xi[\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\varkappa;\xi)\mathfrak{f}(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\varkappa;\xi)\mathfrak{g}_j(\xi)} \prec \mu(\xi) \quad (\xi\in\mathfrak{D}).$$

Let

$$G_i(\xi)=\frac{\upsilon+1}{\xi^{\upsilon+2}}\int_0^\xi t^{\upsilon+1}\mathfrak{g}_i(t)dt \quad (\upsilon>0).$$

Then by Theorem 2, we have $G=\{G_1, G_2, ..., G_\vartheta\}\in\Sigma_\tau^{\varepsilon+1}(\rho,\varkappa,\vartheta;\mu)$. Also let

$$p(\xi)=-\frac{\xi[\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\varkappa;\xi)\mathscr{F}(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\varkappa;\xi)G_j(\xi)} \quad (\xi\in\mathfrak{D}). \tag{19}$$

Now, from the definitions of G_i and $\mathscr{F}$, we have

$$\xi[\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\varkappa;\xi)G_i(\xi)]'+(\upsilon+2)\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\varkappa;\xi)G_i(\xi)=(\upsilon+1)\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\varkappa;\xi)\mathfrak{g}_i(\xi), \tag{20}$$

and

$$\xi[\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\varkappa;\xi)\mathscr{F}_i(\xi)]'+(\upsilon+2)\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\varkappa;\xi)\mathscr{F}(\xi)=(\upsilon+1)\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\varkappa;\xi)\mathfrak{f}(\xi). \tag{21}$$

From (19) to (21), we have

$$-\frac{1}{\vartheta}p(\xi)\sum_{j=1}^{\vartheta}\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\varkappa;\xi)G_j(\xi)+(\upsilon+2)\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\varkappa;\xi)\mathscr{F}(\xi)=(\upsilon+1)\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\varkappa;\xi)\mathfrak{f}(\xi). \tag{22}$$

Differentiating (22) with respect to ξ, and multiplying the resulting equation by ξ, we have

$$-\frac{\xi}{\vartheta}p'(\xi)\sum_{j=1}^{\vartheta}\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\varkappa;\xi)G_j(\xi)-\frac{\xi}{\vartheta}p(\xi)\sum_{j=1}^{\vartheta}[\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\varkappa;\xi)G_j(\xi)]'+(\upsilon+2)\xi[\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\varkappa;\xi)\mathscr{F}(\xi)]'$$

$$=(\upsilon+1)\xi[\mathfrak{k}_\tau^{\varepsilon+1}(\rho,\varkappa;\xi)\mathfrak{f}(\xi)]'. \tag{23}$$

From (19) into (23), we have

$$-\frac{\xi}{\vartheta}p'(\xi)\sum_{j=1}^{\vartheta}\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho,\varkappa;\xi)G_j(\xi)-\frac{\xi}{\vartheta}p(\xi)\sum_{j=1}^{\vartheta}[\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho,\varkappa;\xi)G_j(\xi)]'-(\upsilon+2)\frac{p(\xi)}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho,\varkappa;\xi)G_j(\xi)$$

$$=(\upsilon+1)\xi[\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho,\varkappa;\xi)\mathfrak{f}(\xi)]'.$$

Hence, we get

$$\frac{-\frac{\xi}{\vartheta}p'(\xi)\sum_{j=1}^{\vartheta}\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho,\varkappa;\xi)G_j(\xi)}{\frac{\xi}{\vartheta}\sum_{j=1}^{\vartheta}[\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho,\varkappa;\xi)G_j(\xi)]'+\frac{(\upsilon+2)}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho,\varkappa;\xi)G_j(\xi)}-p(\xi)$$

$$=\frac{(\upsilon+1)\xi[\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho,\varkappa;\xi)\mathfrak{f}(\xi)]'}{\frac{\xi}{\vartheta}\sum_{j=1}^{\vartheta}[\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho,\varkappa;\xi)G_j(\xi)]'+\frac{(\upsilon+2)}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho,\varkappa;\xi)G_j(\xi)}.$$

From the above, we have

$$\frac{\xi p'(\xi)}{\frac{-1}{\vartheta}\sum_{j=1}^{\vartheta}Q_j(\xi)+(\upsilon+2)}+p(\xi)=\frac{-\xi[\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho,\varkappa;\xi)\mathfrak{f}(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho,\varkappa;\xi)\mathfrak{g}_j(\xi)}\prec\mu(\xi),$$

where

$$Q_j(\xi)=\frac{-\xi[\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho,\varkappa;\xi)G_j(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho,\varkappa;\xi)G_j(\xi)}.$$

Now $(Q_j(\xi))\prec\mu(\xi)$, $j=1,2,\ldots,\vartheta$, since $G=\{G_1,G_2,\ldots,G_{\vartheta}\}\in\Sigma_{\tau}^{\varepsilon+1}(\rho,\varkappa,\vartheta;\mu)$ and μ is a convex univalent function. Since $(\upsilon+2)>Re\mu(\xi)$, Lemma 2 applied indicates that $p(\xi)\prec\mu(\xi)$, hence $\mathscr{F}\in\mathfrak{C}_{\tau}^{\varepsilon+1}(\rho,\varkappa,\vartheta;\mu)$. □

Theorem 5 If $\mathfrak{f}\in\mathfrak{C}_{\tau}^{\varepsilon}(\rho,\varkappa,\vartheta;\mu)$ and $Re\mu(\xi)$ is bounded in $\mathfrak{D}$, then $\mathfrak{f}\in\mathfrak{C}_{\tau}^{\varepsilon+1}(\rho,\varkappa,\vartheta;\mu)$ holds for $(\tau+1)>Re\mu(\xi)$ in $\mathfrak{D}$.

Proof. This theorem's proof is removed since it is comparable to that of Theorem 3. □

5. The class $\wp_{\tau}^{\varepsilon+1}(\rho,\varkappa,\vartheta;\delta;\mu)$

Definition 3 Let $\wp_{\tau}^{\varepsilon+1}(\rho,\varkappa,\vartheta;\delta;\mu)$, $\delta\geq 0$, denote the class of functions $\mathfrak{f}\in\Sigma$ satisfying:

$$\mathfrak{J}(\delta; \mathfrak{f}; \mathfrak{g}_1, \mathfrak{g}_2, ..., \mathfrak{g}_\vartheta)(\xi) = -\left\{ \delta \frac{\xi[\mathfrak{k}_\tau^\varepsilon(\rho, \varkappa; \xi)\mathfrak{f}(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{k}_\tau^\varepsilon(\rho, \varkappa; \xi)\mathfrak{g}_j(\xi)} + (1-\delta)\frac{\xi[\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{g}_j(\xi)} \right\},$$

$$\prec \mu(\xi) \quad (\xi \in \mathfrak{D}),$$

where $\mathfrak{g} = \{\mathfrak{g}_1, \mathfrak{g}_2, ..., \mathfrak{g}_\vartheta\} \in \Sigma_\tau^{\varepsilon+1}(\rho, \varkappa, \vartheta; \mu)$, $\xi \sum_{j=1}^{\vartheta} \mathfrak{k}_\tau^\varepsilon(\rho, \varkappa; \xi)\mathfrak{g}_j(\xi) \neq 0$ in $\mathfrak{D}$, μ is a convex univalent function in $\mathfrak{D}$ with $\mu(0) = 1$.

Remark 3 We note that $\wp_\tau^{\varepsilon+1}(\rho, \varkappa, \vartheta; 0; \mu) = \mathfrak{C}_\tau^{\varepsilon+1}(\rho, \varkappa, \vartheta; \mu)$.

Theorem 6 If $\mathfrak{f} \in \wp_\tau^{\varepsilon+1}(\rho, \varkappa, \vartheta; \delta; \mu)$ and $Re\{\mu\}$ is bounded in $\mathfrak{D}$, then $\mathfrak{f} \in \wp_\tau^{\varepsilon+1}(\rho, \varkappa, \vartheta; 0; \mu) = \mathfrak{C}_\tau^{\varepsilon+1}(\rho, \varkappa, \vartheta; \mu)$ hold for $(\tau+1) > Re\mu(\xi)$.

Proof. For $\delta = 0$, the theorem is trivial, we can presume that $\delta \neq 0$. Let

$$p(\xi) = \frac{-\xi[\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{g}_j(\xi)} \quad (\xi \in \mathfrak{D}).$$

Then an easy calculation shows that

$$\frac{\xi p'(\xi)}{\frac{-1}{\vartheta}\sum_{j=1}^{\vartheta} q_j(\xi) + (\tau+1)} + p(\xi) = \frac{-\xi[\mathfrak{k}_\tau^\varepsilon(\rho, \varkappa; \xi)\mathfrak{f}(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{k}_\tau^\varepsilon(\rho, \varkappa; \xi)\mathfrak{g}_j(\xi)},$$

where

$$q_j(\xi) = \frac{-\xi[\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{g}_i(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{g}_j(\xi)}.$$

Also $\frac{1}{\vartheta}\sum_{j=1}^{\vartheta} q_j(\xi) \prec \mu(\xi)$. Since $\mathfrak{f} \in \wp_\tau^{\varepsilon+1}(\rho, \varkappa, \vartheta; \delta; \mu)$, we have

$$\mathfrak{J}(\delta; \mathfrak{f}; \mathfrak{g}_1, \mathfrak{g}_2, ..., \mathfrak{g}_\vartheta)(\xi) = \frac{\delta \xi p'(\xi)}{-\frac{1}{\vartheta}\sum_{j=1}^{\vartheta} q_j(\xi) + (\tau+1)} + p(\xi) \prec \mu(\xi).$$

Now Lemma 2 gives $p(\xi) \prec \mu(\xi)$ which completes the proof. □

Theorem 7 If $\delta > \lambda \geq 0$ and $Re\mu(\xi)$ is bounded in $\mathfrak{D}$, then $\wp_\tau^{\varepsilon+1}(\rho, \varkappa, \vartheta; \delta; \mu) \subset \wp_\tau^{\varepsilon+1}(\rho, \varkappa, \vartheta; \lambda; \mu)$.

Proof. The case $\lambda = 0$ was treated in the previous theorem. Hence we assume that $\lambda \neq 0$. Suppose that $\mathfrak{f} \in \wp_{\tau}^{\varepsilon+1}(\rho, \varkappa, \vartheta; \delta; \mu)$. Then

$$\mathfrak{J}(\delta; \mathfrak{f}; \mathfrak{g}_1, \mathfrak{g}_2, ..., \mathfrak{g}_{\vartheta})(\xi) \prec \mu(\xi). \tag{24}$$

Let ξ_1 be any arbitrary point in $\mathfrak{D}$. Then

$$\mathfrak{J}(\delta; \mathfrak{f}; \mathfrak{g}_1, \mathfrak{g}_2, ..., \mathfrak{g}_{\vartheta})(\xi_1) \prec \mu(\mathfrak{D}).$$

From Theorem 6, we have

$$\frac{-\xi\left[\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}(\xi)\right]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{g}_j(\xi)} \prec \mu(\xi). \tag{25}$$

Now

$$\mathfrak{J}(\lambda; \mathfrak{f}; \mathfrak{g}_1, \mathfrak{g}_2, ..., \mathfrak{g}_{\vartheta})(\xi) = -\left(\left(1-\frac{\lambda}{\delta}\right)\frac{\xi\left[\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}(\xi)\right]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{g}_j(\xi)} + \frac{\lambda}{\delta}\mathfrak{J}(\delta; \mathfrak{f}; \mathfrak{g}_1, \mathfrak{g}_2, ..., \mathfrak{g}_{\vartheta})(\xi)\right).$$

From (24) and (25) it follows that

$$\frac{-\xi_1\left[\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}(\xi_1)\right]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{g}_j(\xi_1)} \prec \mu(\mathfrak{D}),$$

and

$$-\left(\delta\frac{\xi_1\left[\mathfrak{k}_{\tau}^{\varepsilon}(\rho, \varkappa; \xi)\mathfrak{f}(\xi_1)\right]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{k}_{\tau}^{\varepsilon}(\rho, \varkappa; \xi)\mathfrak{g}_j(\xi_1)} + (1-\delta)\frac{\xi_1\left[\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}(\xi_1)\right]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta}\mathfrak{k}_{\tau}^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{g}_j(\xi_1)}\right) \prec \mu(\mathfrak{D}).$$

Now $\mu(\mathfrak{D})$ is convex and $\frac{\lambda}{\delta} < 1$, hence we have $\mathfrak{J}(\lambda; \mathfrak{f}; \mathfrak{g}_1, \mathfrak{g}_2, ..., \mathfrak{g}_{\vartheta})(\xi_1) \prec \mu(\mathfrak{D})$, showing that $\mathfrak{f} \in \wp_{\tau}^{\varepsilon+1}(\rho, \varkappa, \vartheta; \lambda; \mu)$. □

6. The class $\mathfrak{R}_\tau^{\varepsilon+1}(\rho, \varkappa, \vartheta; \delta; \mu)$

Definition 4 Let $\mathfrak{R}_\tau^{\varepsilon+1}(\rho, \varkappa, \vartheta; \delta; \mu)$, $\delta \geq 0$, denote the class of function $\mathfrak{f} \in \Sigma$ satisfying

$$\mathfrak{J}(\delta; \mathfrak{f}; \mathfrak{f}_1, \mathfrak{f}_2, ..., \mathfrak{f}_\vartheta)(\xi) = -\left\{ \delta \frac{\xi [\mathfrak{k}_\tau^\varepsilon(\rho, \varkappa; \xi)\mathfrak{f}_i(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta} \mathfrak{k}_\tau^\varepsilon(\rho, \varkappa; \xi)\mathfrak{f}_j(\xi)} + (1-\delta) \frac{\xi [\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}_i(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta} \mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}_j(\xi)} \right\}$$

$$\prec \mu(\xi) \quad (\xi \in \mathfrak{D}),$$

where $\mathfrak{f} = \{\mathfrak{f}_1, \mathfrak{f}_{2,...,} \mathfrak{f}_\vartheta\} \in \mathfrak{R}_\tau^{\varepsilon+1}(\rho, \varkappa, \vartheta; \mu)$ and $\xi \sum_{j=1}^{\vartheta} \mathfrak{k}_\tau^\varepsilon(\rho, \varkappa; \xi)\mathfrak{f}_j(\xi) \neq 0$ in $\mathfrak{D}$, μ is a convex univalent function in $\mathfrak{D}$ with $\mu(0) = 1$.

Remark 4 We note that $\mathfrak{R}_\tau^{\varepsilon+1}(\rho, \varkappa, \vartheta; 0; \mu) = \mathfrak{R}_\tau^{\varepsilon+1}(\rho, \varkappa, \vartheta; \delta; \mu)$.

Theorem 8 If $\mathfrak{f} \in \mathfrak{R}_\tau^{\varepsilon+1}(\rho, \varkappa, \vartheta; \delta; \mu)$ and $Re\mu(\xi)$ is bounded in $\mathfrak{D}$, then $\mathfrak{f} \in \mathfrak{R}_\tau^{\varepsilon+1}(\rho, \varkappa, \vartheta; 0; \mu) = \mathfrak{R}_\tau^{\varepsilon+1}(\rho, \varkappa, \vartheta; \mu)$ hold for $(\tau+1) > Re\mu(\xi)$.

Proof. For $\delta = 0$, the theorem is unimportant. Hence, we can presume that $\delta \neq 0$. Let

$$p(\xi) = \frac{-\xi [\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}_i(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta} \mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}_j(\xi)} \quad (\xi \in \mathfrak{D}).$$

Then an easy calculation shows that

$$\frac{\xi p'(\xi)}{\frac{-1}{\vartheta}\sum_{j=1}^{\vartheta} q_j(\xi) + (\tau+1)} + p(\xi) = \frac{-\xi [\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}_i(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta} \mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}_j(\xi)},$$

where

$$q_j(\xi) = \frac{-\xi [\mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}_i(\xi)]'}{\frac{1}{\vartheta}\sum_{j=1}^{\vartheta} \mathfrak{k}_\tau^{\varepsilon+1}(\rho, \varkappa; \xi)\mathfrak{f}_j(\xi)}.$$

Also $\frac{1}{\vartheta}\sum_{j=1}^{\vartheta} q_j(\xi) \prec \mu(\xi)$.

Since $\mathfrak{f}(\xi) \in \mathfrak{R}_\tau^{\varepsilon+1}(\rho, \varkappa, \vartheta; \delta; \mu)$, we have

$$J(\delta; \mathfrak{f}; \mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_\vartheta)(\xi) = \frac{\delta \xi p'(\xi)}{\frac{-1}{\vartheta} \sum_{j=1}^{\vartheta} q_j(\xi) + (\tau+1)} + p(\xi) \prec \mu(\xi).$$

Now Lemma 2 gives $p(\xi) \prec \mu(\xi)$ which completes the proof. □

7. Conclusion

The Riemann zeta function and the Hurwitz zeta function are generalized by the Hurwitz-Lerch Zeta function, commonly referred to as Lerch's transcendence. Numerous branches of mathematics and science, such as number theory, quantum statistics, and geometric function theory, incorporate it. The Introduction presents the context of the research and highlights the motivation of the study and why the present tools of the investigation have been selected. The idea for the study, the convolution operator $\mathfrak{k}_\tau^\varepsilon(\rho, \varkappa; \xi)\mathfrak{f}(\xi)$, given in (6), and the reasoning behind the topic's investigation are all contained in Preliminaries part. In this investigation, the operator is applied to define and investigate certain specific classes of univalent functions using the theory of differential subordination. The lemmas used for proving the main outcome are also contained in Preliminaries. The main findings are found in sections 3, 4, 5, and 6, where classes $\Sigma_\tau^{\varepsilon+1}(\rho, \varkappa, \vartheta; \mu)$, given in Definition 1, $\mathfrak{C}_\tau^{\varepsilon+1}(\rho, \varkappa, \vartheta; \mu)$, described in Definition 2, $\wp_\tau^{\varepsilon+1}(\rho, \varkappa, \vartheta; \delta; \mu)$, seen in Definition 3 and $\mathfrak{R}_\tau^{\varepsilon+1}(\rho, \varkappa, \vartheta; \delta; \mu)$, presented in Definition 4 are investigated, respectively.

In future investigations, coefficient estimates could be developed for the classes introduced here as seen for the class investigated in [42]. The outcome of the present investigation could inspire the use of this operator for introducing other new classes of analytic functions as seen in [43]. Also, the dual theory of differential superordination could be applied to the operator in order to obtain sandwich type results as seen in the recent publication [44]. The particular forms of strong and fuzzy differential subordination could be applied to the operator $\mathfrak{k}_\tau^\varepsilon(\rho, \varkappa; \xi)\mathfrak{f}(\xi)$ considering the results established in [33, 45], respectively.

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Conflict of interest

The authors declare no competing financial interest.

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