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Research Article

w_b-Distance and Fixed Point Results on Metric Type Spaces

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Abstract: In this paper, we introduce some applications of w_b-distance concerning the existence of fixed point for single valued mappings on metric type spaces under some suitable conditions. To support our main findings, we include several illustrative examples. In conclusion, our results either extend or enhance notable fixed point theorems of metric fixed point theory.

***Keywords*:** metric type space, fixed point, w_b-distance, singlevalued mapping

MSC: 47H09, 54H25

1. Introduction

Metric fixed point theory is one of the most important and applicable research areas of nonlinear analysis. The most celebrated and pioneer metric fixed point (FP) result for singlevalued mappings involving classical metric, known as the Banach contraction principle (BCP). This classical result appeared as a powerful tool for solving several scientific problems related to a number of scientific areas. Due to its importance, this result has been extended and generalized in many directions, see [1–29] and references therein. The Caristi FP result is known as one of the wounderful and useful generalizations of the BCP for self-mappings on a complete metric space (MS). Inspired by the Caristi's FP result some interesting metric FP results for singlevalued have been appeared in the literature for example see [2, 10, 18] and others.

A classical concept of metric has been extended either by reducing or modifying the metric axioms. One of its generalization, known as a b-metric (here we call it metric type) introduced and studied in [4, 7, 8]. Compared to the metric, the b-metric is not necessarily continuous. Since the class of b-metric spaces (bMS's) is significantly bigger than the class of MS's, a number of FP results of MS's have been studied and extended to bMS's for singlevalued and multivalued mappings, see [1, 9, 10, 17, 26, 28] and references therein. It is worth to mention that not every FP theorem of MS's can be extended fully to bMS's, for example, see [9, Examples 20 and 21] and also see [10].

Kada et al. [14] proposed the notion of w-distance on MS's and ameliorate several results by applying the w-distance. Applying w-distance, Suzuki and Takahashi [29] extended some known classical FP findings. In this direction, much work has appeared, for example, see [21]. In [13], the wt-distance was defined by Husain et al. on metric type spaces (MTS's), and exhibited some results regarding FP for singlevalued mappings involving wt-distance. For further findings, see [11, 12, 15, 19, 20] and others.

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Despite the considerable number of noteworthy and well-established fixed point results reported in the literature (see [21]. In [13], the wt-distance was defined by Husain et al. on MTS's, and exhibited some results regarding FP for singlevalued mappings involving wt-distance. For further findings, see [11, 12, 15, 19, 20]), many fundamental questions in this area remain open. In particular, several unresolved issues persist concerning the existence of fixed points for generalized nonlinear mappings within the framework of generalized distance spaces.

Specifically, two main questions naturally arise. First, can one derive general fixed point results for singlevalued mappings under weaker contractive-type conditions in generalized metric spaces?

Second, what kinds of sufficient conditions, imposed either on the mappings themselves or on the underlying spaces, ensure the existence of fixed points for singlevalued mappings governed by general inequality conditions in metric fixed point theory?

In this article, we introduce several FP findings for singlevalued mappings involving w_b-distance on MTS's. Some supporting examples are also presented. As a conclusion, our results either improve or generalize a number of known FP results.

The results obtained in this work aim to contribute to the ongoing development of metric fixed point theory in nonlinear functional analysis. Moreover, the positive outcomes presented here are expected to be beneficial to researchers actively working in this field.

2. Preliminaries

Here, we recall some useful facts.

Let (V, Δ) be a classical MS. An element $\vartheta \in V$ is called a FP of a singlevalued mapping $f : V \to V$ if $\vartheta = f(\vartheta)$. We denote $\mathscr{F}(f) = \{\vartheta \in V : \vartheta = f(\vartheta)\}$. A point $\vartheta \in V$ is termed a coincidence point of f and g if $f(\vartheta) = g(\vartheta)$. A sequence $\{\vartheta_n\}$ in V is called an orbit of f at $\vartheta_0 \in V$ if $\vartheta_n = f(\vartheta_{n-1}) = f^n(\vartheta_0)$ for all $n \geq 1$. The set of orbit of f at ϑ_0 is signified by $O_{\vartheta_0}(f) = \{f^n(\vartheta_0) : n = 0, 1, 2, \ldots\}$. A map f is termed orbitally continuous at $\vartheta \in V$ if $f(s) = \lim_{n\to\infty} f(\rho_n)$ for any subsequence $\rho_n \in O_\vartheta(f)$ with $\lim_{n\to\infty} \rho_n = s \in V$. A map f is called orbitally continuous if it is orbitally continuous at all $\vartheta \in V$. A mapping f is called Weakly Orbitally Continuous (WOC) if the set $\{\rho \in V : f \text{ is orbitally continuous at } \rho\}$ is nonempty whenever the set $\left\{\vartheta \in V : \lim_{n\to\infty} \rho_n = s\right\}$ is nonempty. A mapping f is called k-continuous, $k = 1, 2, 3, \ldots$ if $\lim_{n\to\infty} f^k(\vartheta) = f(s)$ whenever $\{\vartheta_n\}$ is a sequence in V such that $\lim_{n\to+\infty} f^{k-1}(\vartheta_n) = s$. A map $G : V \to \mathbb{R}$ is depicted as Lower Semi-Continuous (LSC) if for any sequence $\{\vartheta_n\} \subset V$ with $\vartheta_n \to \vartheta \in V$, we have $G(\vartheta) \leq \liminf_{n\to\infty} G(\vartheta_n)$.

In Caristi [6] generalized the BCP as follows.

Theorem 1 [6] For a singlevalued mapping $f : V \to V$ along with a LSC function $\varphi : V \longrightarrow [0, +\infty)$, $\mathscr{F}(f) \neq \emptyset$ provided V is complete space and for any $\vartheta \in V$ the following hold:

$$\Delta(\vartheta, f(\vartheta)) \leq \varphi(\vartheta) - \varphi(f(\vartheta)). \tag{1}$$

Later, Khojasteh et al. [18] obtained the following result which is a generalization of Theorem 1.

Theorem 2 [18] For a singlevalued mapping $f : V \to V$ along with a LSC function with respect to first variable $\psi : V \times V \longrightarrow [0, +\infty)$, f has a unique FP provided V is complete space and for any $\vartheta, \rho \in V$ we have

$$\Delta(\vartheta, \rho) \leq \psi(\vartheta, \rho) - \psi(f(\vartheta), f(\rho)). \tag{2}$$

Applying the concept of weak orbital continuity, Pant et al. [24] generalized Theorem 1 as follows:

Theorem 3 [24] Let (V, Δ) be a complete MS and let $f : V \to V$, $\varphi : V \longrightarrow [0, +\infty)$ be mappings such that

$$\Delta(f(\vartheta),\ f(\rho)) \leq \varphi(\vartheta) - \varphi(f(\vartheta)) + \varphi(\rho) - \varphi(f(\rho)), \tag{3}$$

for all $\vartheta,\ \rho \in V$. Then f has a unique FP provided one of the following conditions, hold:

(a) f is WOC;

(b) f is orbitally continuous;

(c) f is k-continuous.

In Khojasteh et al. [18] introduced the partial answers to Reich, Mizoguchi and Takahashi's and Amini-Harandi's conjectures by using a light version of Caristi's FP result. In addition, they have shown that some known FP theorems can be obtained from Theorem 2. In Pant et al. [24] have shown that Theorem 3 generalizes FP results of Banach, Kannan, Chatterjea, Ćirić and Suzuki. In addition, Theorem 3 is independent of the result of Caristi.

The concept of w-distance on MS's was clarified by Kada et al. [14] as follows The function $\Theta : V \times V \to [0,\ +\infty)$ is termed a w-distance if the following conditions are fulfilled for all $\vartheta,\ \rho,\ z \in V$:

(i) $\Theta(\vartheta,\ z) \leq \Theta(\vartheta,\ \rho) + \Theta(\rho,\ z)$;

(ii) The map $\Theta(\vartheta,\ \cdot) : V \to [0,\ +\infty)$ is LSC;

(iii) For any $\iota > 0$, there is $\sigma > 0$ with $\Theta(z,\ \vartheta) \leq \sigma$ and $\Theta(z,\ \rho) \leq \sigma$ we have $\Delta(\vartheta,\ \rho) \leq \iota$.

Clearly, any metric Δ is a w-distance on V. The w-distance Θ on V is termed a w^0-distance if $\Theta(\vartheta,\ \vartheta) = 0$ for all $\vartheta \in V$. For more details on the w-distance, see [14].

In Czerwik [7] established the subsequent concept of MTS or bMS.

Let $V \neq \phi$ with $b \geq 1$ and $\Theta : V \times V \to [0,\ +\infty)$ be a map that meets the following criteria for all $\vartheta,\ \rho,\ z \in V$:

(i) $\Theta(\vartheta,\ \rho) = 0 \Leftrightarrow \vartheta = \rho$;

(ii) $\Theta(\vartheta,\ \rho) = \Theta(\rho,\ \vartheta)$;

(iii) $\Theta(\vartheta,\ \rho) \leq b[\Theta(\vartheta,\ z) + \Theta(z,\ \rho)]$.

Thus, Θ is termed a b-metric on V, and the pair $(V,\ \Theta)$ is referred to as a bMS. Hereafter, we will also refer it as a MTS.

It has been observed that, any MS becomes a MTS with $b = 1$. However, the converse is not true in general, see [7].

Example 1 [5, 9] Let $0 < q \leq 1$. Then,

$$\ell_q = \left\{ \{\vartheta_n\} : \vartheta_n \in \mathbb{R};\ n \in \mathbb{N},\ \sum_{n=1}^{\infty} \mid \vartheta_n \mid^q < \infty \right\},$$

with $\Theta(\vartheta,\ \rho) = \left(\sum_{n=1}^{\infty} \mid \vartheta_n - \rho_n \mid^q \right)^{\frac{1}{q}}$ for all $\vartheta = \{\vartheta_n\},\ \rho = \{\rho_n\} \in \ell_q$, is a MTS with $b = 2^{\frac{1}{q}-1} \geq 1$. Also the space

$$_q = \{g : [0,\ 1] \to \mathbb{R} : \int_0^1 |g(t)|^q\ dt < \infty\},$$

with $\Theta(g,\ h) = \left(\int_0^1 |g(t) - h(t)|^q\ dt \right)^{\frac{1}{q}}$ is a MTS with $b = 2^{\frac{1}{q}-1} \geq 1$.

For further examples and properties of MTS's, see [7–9, 13].

In fact, for MTS's the notions of convergent sequence, Cauchy sequence and completeness can be defined similarly as in MS's, see [1, 7, 17]. Further, a MTS can be endowed with a topology induced by its convergence and almost all the concepts and results which are valid for MS's can be extended to the framework of MTS's, (see [13, 17]).

Commonly, Θ may fail to be continuous in each variable, for example, see [3]. However, it has been noted that a topology can be established based on convergence in these types of spaces [3].

A set M in $(V,\ \Theta)$ is termed open if for all $\vartheta\in M$, there is $\sigma>0$ such that the open ball $B_o(\vartheta,\ \sigma)\subset M$. We describe τ as the family of all open subsets of V, which forms a topology on $(V,\ \Theta)$. Moreover, a subset M of V is viewed as closed if, for any sequence $\{\rho_n\}$ in M such that $\{\rho_n\}$ converges to ρ, it follows that $\rho\in M$, as noted in [17]. In addition, a real-valued map f on V is called b-LSC if, for every sequence $\{\vartheta_n\}$ in V with $\vartheta_n\to\vartheta\in V$, we have $f(\vartheta)\le\liminf\limits_{n\to\infty}(bf(\vartheta_n))$.

In Hussain et al. [13] and Kada et al. [14] presented a w-distance in MTS's, as wt-distance (we will call it w_b-distance in the following section).

Let $(V,\ \Theta)$ be a MTS. A function $\Theta_b : V\times V\to[0,\ +\infty)$ is called w_b-distance on V if it meets the following constraints for all $\vartheta,\ \rho,\ z\in V$:

(i) $\Theta_b(\vartheta,\ z)\le b[\Theta_b(\vartheta,\ \rho)+\Theta_b(\rho,\ z)]$;

(ii) The map $\Theta_b(\vartheta,\ \cdot): V\to[0,\ +\infty)$ is b-LSC;

(iii) For any $\iota>0$, there is $\sigma>0$ with $\Theta_b(z,\ \vartheta)\le\sigma$ and $\Theta_b(z,\ \rho)\le\sigma$ implies $\Theta(\vartheta,\ \rho)\le\iota$.

Noted that each w_b-distance becomes w-distance when $b=1$. The w_b-distance Θ_b on V is called a w_b^0-distance if $\Theta_b(\vartheta,\ \vartheta)=0$ for any $\vartheta\in V$ and we denote w_b^0-distance as Θ_b^0.

Example 2 [13] Let $V=\mathbb{R}$ and $\Theta_b(\vartheta,\ \rho)=(\vartheta-\rho)^2,\ \vartheta,\ \rho\in V$. Then, the maps $\Theta_{b_1},\ \Theta_{b_2} : V\times V\to\mathbb{R}^+$ defined by $\Theta_{b_1}(\vartheta,\ \rho)=|\vartheta|^2+|\rho|^2$ and $\Theta_{b_2}(\vartheta,\ \rho)=|\rho|^2$ for every $\vartheta,\ \rho\in V$ are w_b-distances on V.

Note that each metric type Θ is a w_b-distance on V. However, in general the opposite may not hold [12].

The following results concerning w_b-distance are essential for deduce our main results.

Lemma 1 [13] Consider $(V,\ \Theta)$ is a MTS, with a w_b-distance Θ_b on V. Let $\{\vartheta_n\}$ and $\{\rho_n\}$ be sequences within V. Let $\{\sigma_n\}$ and $\{\beta_n\}$ be sequences in $\mathbb{R}^+$ such that σ_n and β_n converge to 0. Then the following Asseverations are met for all $\vartheta,\ \rho,\ z\in V$.

(a) If $\Theta_b(\vartheta_n,\ \rho)\le\sigma_n$ and $\Theta_b(\vartheta_n,\ y)\le\beta_n$ for any $n\in\mathbb{N}$, then $\rho=y$. Particularly, if $\Theta_b(\vartheta,\ \rho)=0$ and $\Theta_b(\vartheta,\ y)=0$, then $\rho=y$;

(b) If $\Theta_b(\vartheta_n,\ \rho_n)\le\sigma_n$ and $\Theta_b(\vartheta_n,\ y)\le\beta_n$ for any $n\in\mathbb{N}$, then $\Theta(\rho_n,\ y)\to 0$;

(c) If $\Theta_b(\vartheta_n,\ \vartheta_m)\le\sigma_n$ for any $n,\ m\in\mathbb{N}$ with $m>n$, then $\{\vartheta_n\}$ is a Cauchy sequence;

(d) If $w_b(\rho,\ \vartheta_n)\le\sigma_n$ for any $n\in\mathbb{N}$, then $\{\vartheta_n\}$ is a Cauchy sequence.

Lemma 2 [27] Consider $(V,\ \Theta)$ is a MTS with a w_b-distance Θ_b. Let $\{\vartheta_n\}$ be a sequence in V. If there exists $r\in[0,\ 1)$ such that for $n\ge 1$,

$$\Theta_b(\vartheta_n,\ \vartheta_{n+1})\le r\Theta_b(\vartheta_{n-1},\ \vartheta_n),$$

then $\{\vartheta_n\}$ is a Cauchy sequence.

3. Main results

Throughout this section, $(V,\ \Theta)$ is a complete MTS. First, we prove key lemma in the setting of MTS's.

Lemma 3 Let $f : V\longrightarrow V$ be WOC mapping. Assume that there exist $s\in V$ and $\vartheta_0\in V$ such that $s=\lim\limits_{n\to+\infty}f^n(\vartheta_0)$, then $\mathscr{F}(f)\neq\emptyset$.

Proof. Let $s\in V$ such that $s=\lim\limits_{n\to+\infty}f^n(\vartheta_0)$. Then, $s=\lim\limits_{n\to+\infty}f^{n+1}(\vartheta_0)=\lim\limits_{n\to+\infty}f(f^n(\vartheta_0))$. Since f is a WOC, we obtain

$$\lim_{n\to+\infty}f^n(\vartheta_0)=s=\lim_{n\to+\infty}f(f^n(\vartheta_0))=f(s).$$

Hence, $s=f(s)$. □

Now, we present some interesting applications of w_b-distance for the existence of fixed points in metric type spaces.

Theorem 4 Let $f : V \to V$ be WOC mapping such that

$$\Theta_b(\vartheta,\ f(\vartheta)) \leq \varphi(\vartheta) - h^{b-1}\varphi(f^r\vartheta), \tag{4}$$

for all $\vartheta \in V$, where $r \in \mathbb{N}$ and $h > 1$ and $\varphi : V \to [0,\ +\infty)$. Then, $\mathscr{F}(f) \neq \emptyset$.

Proof. Let $\vartheta_0 \in V$ and $\vartheta_n = f^n(\vartheta_0),\ n \in \mathbb{N}$. Put $\lambda = h^{\frac{b-1}{r}}$. Note that $\lambda \geq 1$. From (4), we have

$$\Theta_b(\vartheta_0,\ \vartheta_1) \leq \varphi(\vartheta_0) - \lambda^r\varphi(\vartheta_r)$$

$$\lambda\Theta_b(\vartheta_1,\ \vartheta_2) \leq \lambda\varphi(\vartheta_1) - \lambda^{r+1}\varphi(\vartheta_{r+1})$$

$$\vdots$$

$$\lambda^r\Theta_b(\vartheta_r,\ \vartheta_{r+1}) \leq \lambda^r\varphi(\vartheta_r) - \lambda^{2r}\varphi(\vartheta_{2r})$$

$$\vdots$$

$$\lambda^n\Theta_b(\vartheta_n,\ \vartheta_{n+1}) \leq \lambda^n\varphi(\vartheta_n) - \lambda^{n+r}\varphi(\vartheta_{n+r}).$$

Thus, we have

$$\sum_{k=0}^{n}\lambda^k\Theta_b(\vartheta_k,\ \vartheta_{k+1}) \leq \sum_{k=0}^{n}\lambda^k\varphi(\vartheta_k) - \sum_{k=0}^{n}\lambda^{k+r}\varphi(\vartheta_{k+r})$$

$$\leq \sum_{k=0}^{r-1}\lambda^k\varphi(\vartheta_k)$$

Next, for all $n \in \mathbb{N}$ we have

$$\Theta_b(\vartheta_n,\ \vartheta_{n+1}) \leq \sum_{k=0}^{n}\lambda^k\Theta_b(\vartheta_k,\ \vartheta_{k+1}) \leq \varphi(\vartheta_0) + \lambda\varphi(\vartheta_1) + \cdots + \lambda^{r-1}\varphi(\vartheta_{r-1}). \tag{5}$$

Thus, for any $n,\ m \in \mathbb{N}$ with $m > n$, we get

$$\begin{aligned}\Theta_b(\vartheta_n,\ \vartheta_m) &\le b\left[\Theta_b(\vartheta_n,\ \vartheta_{n+1})+\Theta_b(\vartheta_{n+1},\ \vartheta_m)\right]\\ &\le b\left[\varphi(\vartheta_0)+\lambda\varphi(\vartheta_1)+\cdots+\lambda^{r-1}\varphi(\vartheta_{r-1})+\varphi(\vartheta_0)+\lambda\varphi(\vartheta_1)+\cdots+\lambda^{r-1}\varphi(\vartheta_{r-1})\right]\\ &=2b\sum_{i=0}^{r-1}\lambda^i\varphi(\vartheta_i).\end{aligned}\tag{6}$$

From Lemma 1c, we have that $\{\vartheta_n\}=\{f^n(\vartheta_0)\}$ is a Cauchy sequence in V. The completeness of V guarantee the existence of $s_0\in V$ such that $\lim_{n\to\infty}f^n(\vartheta_0)=s_0$. Therefore, $\mathscr{F}(f)\neq\emptyset$, by Lemma 3. □

Putting $r=1$ in Theorem 4, we obtain the following generalization of [22, Theorem 3.1].

Corollary 1 Suppose that all the hypotheses of Theorem 4 hold except inequality (4). Assume that for any $\vartheta\in V$ satisfy the following

$$\Theta_b(\vartheta,\ f(\vartheta))\le\varphi(\vartheta)-h^{b-1}\varphi(f(\vartheta)).\tag{7}$$

Then, $\mathscr{F}(f)\neq\emptyset$.

Remark 1 If we consider $r=1$ and $\Theta_b=\Theta$ in Theorem 4, then it reduces to [22, Theorem 3.1]. Moreover, by setting $r=1,\ b=1$ and $\Theta_b=\Delta$ we obtain [25, Theorem 2.10]. Further, Theorem 4 contains [23, Theorem 4] as a special case, when $\Theta_b=\Theta$.

Replacing the weakly orbitally continuity of the mapping f of Theorem 4 with another condition, we get the subsequent FP result.

Theorem 5 Suppose that all the hypotheses of Theorem 4 hold except the weakly orbitally continuity of f. Assume that for any $s\in V$, $\inf\{\Theta_b(\vartheta,\ s)+\Theta_b(\vartheta,\ f(\vartheta)):\vartheta\in V\}>0$ with $s\neq f(s)$. Then $\mathscr{F}(f)\neq\emptyset$.

Proof. Following the same manner in the proof of Theorem 4, we get an orbit $\{\vartheta_n\}$ of f, which is a Cauchy sequence in V. The completeness of V guarantee the existence of $s_0\in V$ such that $\lim_{n\to\infty}\vartheta_n=s_0$. Since $\Theta_b(\vartheta_n,\ \cdot)$ is b-LSC and $\lim_{m\to\infty}\vartheta_m=s_0\in V$, then we get

$$\Theta_b(\vartheta_n,\ s_0)\le\liminf_{m\to\infty}b\Theta_b(\vartheta_n,\ \vartheta_m)\le 2b\sum_{i=0}^{r-1}\lambda^i\varphi(\vartheta_i),$$

and

$$\Theta_b(\vartheta_n,\ f(\vartheta_n))\le\Theta_b(\vartheta_n,\ \vartheta_{n+1})\le\sum_{i=0}^{r-1}\lambda^i\varphi(\vartheta_i).$$

Letting $s_0\neq f(s_0)$, we have

$$0 < \inf\{\Theta_b(\vartheta,\ s_0) + \Theta_b(\vartheta,\ f(\vartheta)) : \vartheta \in V\}$$

$$\leq \inf\{\Theta_b(\vartheta_n,\ s_0) + \Theta_b(\vartheta_n,\ f(\vartheta_n)) : n \geq n_0\}$$

$$\leq \inf\left\{2b\sum_{i=0}^{r-1}\lambda^i\varphi(\vartheta_i) + \sum_{i=0}^{r-1}\lambda^i\varphi(\vartheta_i)\right\} = 0,$$

which is impossible, and hence $s_0 \in \mathscr{F}(f)$. □

Theorem 6 Suppose that all the hypotheses of Theorem 4 hold except inequality (4). Assume that Θ_b is a w_b^0-distance on V satisfying the following

$$\Theta_b^0(\vartheta,\ \rho) \leq \psi(\vartheta,\ \rho) - h^{b-1}\psi(f^r(\vartheta),\ f^r(\rho)), \tag{8}$$

for every $\vartheta,\ \rho \in V$, where ψ is the mapping from $V \times V$ to $[0,\ +\infty)$. Then, f has a unique FP.

Proof. Suppose that $\rho = f(\vartheta)$ and $\varphi(\vartheta) = \psi(\vartheta,\ f(\vartheta))$, for all $\vartheta \in V$. Then from Theorem 4 there exists $s \in V$ such that $s \in \mathscr{F}(f)$. If f has another FP $v \in V$, then from (8) we obtain that

$$\Theta_b^0(s,\ v) \leq \psi(s,\ v) - h^{b-1}\psi(s,\ v) \leq 0,$$

then, $\Theta_b^0(s,\ v) = 0$. Since Θ_b is a w_b^0-distance on V, we get that $\Theta_b^0(s,\ s) = 0$. Therefore $s = v$ by Lemma 1a. □

Remark 2 Indeed, if $r = 1$, $b = 1$ and $\Theta_b = \Delta$ in Theorem 6 then we obtain the results of [18, Theorem 5]. Further Theorem 6 includes the fixed point result [23, Theorem 5] as a special case when $\Theta_b = \Theta$.

Following the technique of the proof of Theorem 4, we have the following generalization of [24, Theorem 3].

Theorem 7 Let $f : V \longrightarrow V$ and $\psi : V \times V \longrightarrow [0,\ +\infty)$ be mappings satisfies that

$$\Theta_b(f(\vartheta),\ f(\rho)) \leq \psi(\vartheta,\ \rho) - h^{b-1}\psi(f(\vartheta),\ f(\rho)), \tag{9}$$

for all $\vartheta,\ \rho \in V$, where $h > 1$. Then f has a unique FP, under any one of the following conditions:

(i) f is WOC;

(ii) f is orbitally continuous;

(iii) f is k-continuous.

Following the proof of Theorem 7, we get the following extension of [24, Theorem 3].

Corollary 2 Let $f : V \longrightarrow V$ and $\varphi_i : V \to [0,\ +\infty)$, $i = 1,\ 2$ be mappings such that f is WOC. Assume that

$$\Theta_b(f(\vartheta),\ f(\rho)) \leq \varphi_1(\vartheta) - h^{b-1}\varphi_1(f(\vartheta)) + \varphi_2(\rho) - h^{b-1}\varphi_2(f(\rho)), \tag{10}$$

for each $\vartheta,\ \rho \in V$, such that $h > 1$. Then f has a unique FP.

Proof. Putting $\psi(\vartheta,\ \rho) = \varphi_1(\vartheta) + \varphi_2(\rho)$, $\vartheta,\ \rho \in V$ in Theorem 7, we obtain the proof. □

Remark 3 (1) If $\varphi_1 = \varphi_2$, $b = 1$ and $\Theta_b = \Delta$ from Corollary 2, we obtain [24, Theorem 3].

(2) Theorem 7 generalizes the result of [23, Theorem 6]. Further, Corollary 2 contains [23, Corollary 1] as a special case when $\Theta_b = \Theta$.

Now, we prove a coincidence point result for the singlevalued mappings, which extends the associated result in [16, Theorem 1].

Theorem 8 Let $f,\ g : V \longrightarrow V$ and $\psi : V \times V \longrightarrow \mathbb{R}$ be mappings. Assume that Θ_b is a w_b^0-distance on V and $\Theta_b^0(\vartheta,\ \rho) = \Theta_b^0(\rho,\ \vartheta)$ for any $\vartheta,\ \rho \in V$ such that

(a) $\inf\limits_{\vartheta,\ \rho \in V} \psi(\vartheta,\ \rho) > -\infty$;

(b) f and g are commuting mappings;

(c) The range of g contains the range of f;

(d) g is continuous;

(e) If $\Theta_b^0(g(\vartheta),\ f(\vartheta)) > 0$ then for any $\vartheta,\ \rho \in V$

$$\Theta_b^0(f(\vartheta),\ f(\rho)) \leq t\ \Theta_b^0(g(\vartheta),\ g(\rho)), \tag{11}$$

where $t = [\psi(g(\vartheta),\ g(\rho)) - \psi(f(\vartheta),\ f(\rho)] > 0$. Then f and g have a coincidence point.

Proof. Let $\vartheta_0 \in V$. Since $f(\vartheta_0) \in g(V)$, there is an $\vartheta_1 \in V$ such that $g(\vartheta_1) = f(\vartheta_0)$. Similarly, for any given $\vartheta_n \in V$, there is $\vartheta_{n+1} \in V$ such that $g(\vartheta_{n+1}) = f(\vartheta_n)$. If $\Theta_b^0(g(\vartheta_n),\ f(\vartheta_n)) = 0$ for some $n \in \mathbb{N}$, then ϑ_n is a coincidence point. Suppose that

$$\Theta_b^0(g(\vartheta_n),\ f(\vartheta_n)) > 0, \tag{12}$$

for each $n \in \mathbb{N}$. From (11), we obtain

$$\Theta_b^0(f(\vartheta_{n+1}),\ f(\vartheta_n)) \leq (\psi(g(\vartheta_{n+1}),\ g(\vartheta_n)) - \psi(f(\vartheta_{n+1}),\ f(\vartheta_n)))\,\Theta_b^0(g(\vartheta_{n+1}),\ g(\vartheta_n)).$$

Using definition of the iterative sequence $g(\vartheta_{n+2}) = f(\vartheta_{n+1})$, we get

$$\Theta_b^0(g(\vartheta_{n+2}),\ g(\vartheta_{n+1})) \leq (\psi(g(\vartheta_{n+1}),\ g(\vartheta_n)) - \psi(g(\vartheta_{n+2}),\ g(\vartheta_{n+1})))\,\Theta_b^0(g(\vartheta_{n+1}),\ g(\vartheta_n)).$$

Since $\Theta_b^0(g(\vartheta_{n+1}),\ g(\vartheta_n)) = \Theta_b^0(g(\vartheta_n),\ g(\vartheta_{n+1})) = \Theta_b^0(g(\vartheta_n),\ f(\vartheta_n)) > 0$, then for each $n \in \mathbb{N}$ we get

$$\frac{\Theta_b^0(g(\vartheta_{n+2}),\ g(\vartheta_{n+1}))}{\Theta_b^0(g(\vartheta_{n+1}),\ g(\vartheta_n))} \leq \psi(g(\vartheta_{n+1}),\ g(\vartheta_n)) - \psi(g(\vartheta_{n+2}),\ g(\vartheta_{n+1})).$$

Applying condition (a), we obtain

$$\sum_{j=1}^{n} \frac{\Theta_b^0\left(g(\vartheta_{j+2}),\ g(\vartheta_{j+1})\right)}{\Theta_b^0\left(g(\vartheta_{j+1}),\ g(\vartheta_j)\right)} < +\infty. \tag{13}$$

Therefore, the series $\sum_{j=1}^{+\infty} \frac{\Theta_b^0\left(g(\vartheta_{j+2}),\ g(\vartheta_{j+1})\right)}{\Theta_b^0\left(g(\vartheta_{j+1}),\ g(\vartheta_j)\right)}$ converges and

$$\lim_{j\to+\infty} \frac{\Theta_b^0\left(g(\vartheta_{j+2}),\ g(\vartheta_{j+1})\right)}{\Theta_b^0\left(g(\vartheta_{j+1}),\ g(\vartheta_j)\right)} = 0. \tag{14}$$

Then, we conclude that for $k \in (0,\ 1)$, there exists $n_0 \in \mathbb{N}$ where

$$\Theta_b^0\left(g(\vartheta_{j+2}),\ g(\vartheta_{j+1})\right) \leq k\Theta_b^0\left(g(\vartheta_{j+1}),\ g(\vartheta_j)\right), \tag{15}$$

for all $j \geq n_0$. Now, by applying Lemma 2 and using the symmetric of Θ_b^0, we conclude that $g(\vartheta_n)$ is Cauchy sequence. The completeness of V guarantee the existence of $s \in V$ such that $\lim_{n\to\infty} g(\vartheta_n) = s$. that is

$$s = \lim_{n\to+\infty} g(\vartheta_n) = \lim_{n\to+\infty} f(\vartheta_{n-1}). \tag{16}$$

Since g is continuous, and from (11) we get that f is continuous. On the other hand, f and g commute and thus

$$g(s) = g\left(\lim_{n\to+\infty} f(\vartheta_n)\right) = \lim_{n\to+\infty} g(f(\vartheta_n)) = \lim_{n\to+\infty} f(g(\vartheta_n)) = f\left(\lim_{n\to+\infty} g(\vartheta_n)\right) = f(s). \tag{17}$$

Therefore s is a coincidence point for f and g. □

From Theorem 8, we realize the next corollary.

Corollary 3 Let $f,\ g : V \longrightarrow V$ and $\varphi : V \times V \longrightarrow \mathbb{R}$ be mappings. Assume that Θ_b is a w_b^0-distance on V with $\Theta_b^0(\vartheta,\ \rho) = \Theta_b^0(\rho,\ \vartheta)$ and for any $\vartheta,\ \rho \in V$ we have

(a) $\inf_{\vartheta\in V} \varphi(\vartheta) > -\infty$;

(b) If $\Theta_b^0(\vartheta,\ f(\vartheta)) > 0$ then for any $\vartheta,\ \rho \in V$

$$\Theta_b^0(f(\vartheta),\ f(\rho)) \leq [\varphi(\vartheta) - \varphi(f(\vartheta))]\, \Theta_b^0(\vartheta,\ \rho).$$

Then $\mathscr{F}(f) \neq \emptyset$.

Proof. Put $g(\vartheta) = \vartheta$ and $\psi(\vartheta,\ \rho) = \varphi(\vartheta)$ in Theorem 8, then the proof is completed. □

Remark 4 If we consider $\Theta_b = \Theta$ in Theorem 8 then it reduces to [23, Theorem 7]. Further, Corollary 3 improves [16, Theorem 1] and [23, Corollary 2].

Now we present some examples in support of our main findings.

Example 3 Let $V = \left[0,\ \frac{1}{2}\right]$. For any $\vartheta,\ \rho$ of V, define $\Theta(\vartheta,\ \rho) = (\vartheta - \rho)^2$ and $\Theta_b(\vartheta,\ \rho) = \rho^2$. Note that $(V,\ \Theta)$ becomes a MTS with $b = 2$ and Θ_b is a w_b-distance on V. Define the functions $f : V \longrightarrow V,\ \varphi : V \longrightarrow [0,\ +\infty)$ by $f(\vartheta) = \vartheta^2,\ \varphi(\vartheta) = \sqrt{\vartheta}$. Now putting $r = 2$ in Theorem 4, we arrive at

$$\varphi(\vartheta) - \varphi\left(f^2(\vartheta)\right) = \sqrt{\vartheta} - \sqrt{\vartheta^4} = \sqrt{\vartheta} - \vartheta^2 \geq \vartheta^4 = \Theta_b(\vartheta,\ \vartheta^2) = \Theta_b(\vartheta,\ f(\vartheta)).$$

Example 4 Let $V = \mathbb{R}$. For each $\vartheta,\ \rho$ of V, we define $\Theta(\vartheta,\ \rho) = (\vartheta - \rho)^2$ and

$$\Theta_b(\vartheta,\ \rho) = \begin{cases} \rho^2, & \text{if } \vartheta \neq \rho \\ \\ 0, & \text{if } \vartheta = \rho. \end{cases}$$

Note that $(V,\ \Theta)$ becomes a complete MTS with $b = 2$ and Θ_b is a w_b-distance on V. $f : V \longrightarrow V$ is a WOC defined by $f(\vartheta) = \dfrac{\vartheta}{2}$ and $\psi : V \times V \longrightarrow [0,\ +\infty)$ defined by $\psi(\vartheta,\ \rho) = 2\rho^2$. Next, let us consider $h = 2$ and $r \in \mathbb{N}$. For $\vartheta,\ \rho \in \mathbb{R}$ and $\vartheta \neq \rho$, we have

$$\psi(\vartheta,\ \rho) - h^{b-1}\psi(f^r(\vartheta),\ f^r(\rho)) = \rho^2\left(2 - \frac{1}{2^{2(r-1)}}\right) \geq \rho^2 = \Theta_b(\vartheta,\ \rho).$$

Now if $\vartheta = \rho$, then it is easy to see that $\psi(\vartheta,\ \rho) - h^{b-1}\psi(f^r(\vartheta),\ f^r(\rho)) = \Theta_b(\vartheta,\ \rho)$. Therefore, the conditions of Theorem 7 are fulfilled.

Example 5 Let $V = [0,\ 1]$ with Θ and Θ_b as in Example 3. Define $f : V \longrightarrow V$ by $f(\vartheta) = \dfrac{\vartheta}{3}$ and $\psi : V \times V \longrightarrow [0,\ +\infty)$ by $\psi(\vartheta,\ \rho) = 2\rho^2$. Obviously, $(V,\ \Theta)$ is a complete MTS. Let $h = 2$. We obtain

$$\Theta_b(f(\vartheta),\ f(\rho)) = (f(\rho))^2 = \frac{\rho^2}{9}.$$

On the other side,

$$\psi(\vartheta,\ \rho) - h^{b-1}\psi(f(\vartheta),\ f(\rho)) = \frac{14\rho^2}{9}.$$

When

$$\frac{\rho^2}{9} \leq \frac{14\rho^2}{9},$$

for all $\vartheta,\ \rho \in [0,\ 1]$, we deduce that the conditions for Theorem 8 are met.

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Conflict of interest

The authors declare no competing financial interest.

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