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Research Article

Asymptotic Analytic Framework for Pruning Dynamics in Numerical Semigroups

Vivian Ndfutu Nfor[1], Fahad Alshammari[2], Marshal I. Sampson[3], Hossam A. Nabwey[2], Louis E. Essien[4], Reny George[2*]

[1] Higher Technical Teacher Training College Bambili, University of Bamenda, Bambili, Bamenda, Cameroon
[2] Department of Mathematics, College of Science and Humanities in Alkharj, Prince Sattam bin Abdulaziz University, Alkharj 11942, Saudi Arabia
[3] Department of Mathematics, Faculty of Physical Sciences, Akwa Ibom State University, Ikot Akpaden 543111, Nigeria
[4] Department of Mathematics, Abia State Polytechnic, Aba 450002, Nigeria
E-mail: r.kunnelchacko@psau.edu.sa

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Abstract: This paper investigates asymptotic bounds for the length density of factorizations in numerical and more general atomic semigroups, introducing a unified algorithmic and analytic treatment through pruning dynamics. Building on the pruning paradigm previously developed for factorization trees, we establish finiteness, correctness, and length-preservation properties under flexible chain conditions such as Finite Factorization (FF)-monoid and prefix-Ascending Chain Condition on Prefixes (ACCP) hypotheses. We then extend these results through a series of major analytical developments that reveal deeper structural behaviour: asymptotic growth laws for normalized length density, categorical representations of pruning as functorial morphisms, entropy-based bounds on factorization complexity, probabilistic models for random pruning trees, spectral analysis of pruning operators, and topological compactification yielding continuum limits. Collectively, these results define an Asymptotic-Analytic Framework for Pruning Dynamics that unifies combinatorial, categorical, and analytic viewpoints of semigroup factorization. The framework establishes new links between algebraic finiteness, entropy growth, spectral stability, and topological convergence, thus extending classical length-density theory toward a continuous and dynamical formulation of semigroup complexity.

Keywords: length density, numerical semigroups, factorization invariants, pruning algorithms, commutative and non-commutative semigroups, asymptotic convergence

MSC: 20M14, 20M05, 11B75

1. Introduction

Numerical semigroups are additive submonoids of $\mathbb{N}_0$ with finite complement. Let $S = \langle n_1, \ldots, n_k \rangle$ be a numerical semigroup minimally generated by positive integers $n_1 < \cdots < n_k$. For $n \in S$, the set of possible factorization lengths of n into the generators is denoted by $L(n)$. Unions of sets of lengths provide a combinatorial perspective on factorization [1], highlighting patterns and constraints among different factorizations.

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More generally, a semigroup is a set equipped with an associative binary operation. Finite semigroups provide fundamental models in algebra and combinatorics, serving as minimal associative systems in which structural properties can be explicitly classified and enumerated. Because every finitely generated semigroup admits finite quotients and local finite approximations, finite semigroups often reveal core combinatorial and structural phenomena in a controlled setting. Extensive classification and enumeration results for small finite semigroups, including commutative associative binary operations on finite sets, have been obtained in the literature (see, e.g., [2–4]). These investigations illuminate constraints and growth patterns that reappear in finitely generated and infinite semigroups, including numerical semigroups.

The *length density* of n is then defined by

$$\delta(n) := \frac{|L(n)| - 1}{\max L(n) - \min L(n)} \in [0, 1],$$

with $\delta(n) = 1$ when $L(n)$ forms a full interval. The function $\delta(n)$ measures the uniformity of possible factorization lengths and encodes fundamental information about the combinatorial structure of S. The concept of length density has been studied extensively for numerical semigroups [5, 6], providing insights into both typical and extreme factorization behaviours.

Although numerical experiments and partial theoretical arguments indicate that $\delta(n) \to 1$ for large n, a general asymptotic description and analytic characterization remain largely open. Existing approaches often rely on computational enumeration or combinatorial arguments that do not scale well with embedding dimension or structural complexity.

Beyond classical numerical semigroups, the asymptotic properties of generalized numerical semigroups further illuminate normalized entropy measures and growth patterns [7, 8]. Even though the present work does not explicitly develop these generalized semigroups, their study motivates our focus on asymptotic convergence of length densities and entropic bounds.

In this work we introduce a unified *pruning framework* for analyzing factorization trees and their asymptotic behaviour. The framework builds upon recent pruning algorithms that guarantee termination and correctness under finiteness or chain conditions such as the Finite Factorization (FF)-monoid and prefix-Ascending Chain Condition on Prefixes (ACCP) hypotheses, providing a structured means of reducing redundant factorization paths while preserving length information. Within this setting, we obtain new explicit bounds for $\delta(n)$ and establish asymptotic convergence criteria that depend on the embedding dimension and the arithmetic configuration of the generators.

1.1 *Novelty statement*

This manuscript develops a comprehensive framework for understanding pruning dynamics in finitely generated atomic monoids, integrating combinatorial, categorical, probabilistic, spectral, and topological perspectives. Building on the combinatorial foundation of factorization trees (see Section 3.2.1), we introduce a functorial construction $\mathscr{P}(S)$ that encodes admissible pruning operations and establish a universality theorem showing that $\mathscr{P}(S)$ is terminal among faithful representations preserving factorization trees and pruning structure.

We further quantify the intrinsic complexity of factorizations through an information-theoretic approach (Section 4), defining normalized factorization entropy and deriving asymptotic bounds that reveal decay of uncertainty under pruning. This deterministic analysis is extended to stochastic settings (Theorem 5), where random pruning processes are introduced, enabling rigorous control over expected tree size, survival rates, and length distributions.

The spectral properties of pruning operators (Theorem 6) are examined, connecting the dynamics of pruning to linear operator theory, providing criteria for convergence, and bounding expected maximum tree depth. Finally, a topological compactification of the discrete pruning semigroup (Theorem 7) establishes the existence of a continuous-time semigroup flow, bridging discrete combinatorial behavior and asymptotic analytic dynamics.

Taken together, these contributions provide a unified, multiscale perspective on pruning dynamics, situating classical factorization and length-density analysis within a novel hybrid algebraic-analytic framework that is both theoretically robust and broadly applicable to computational and probabilistic models of monoid factorization.

In this section, we recall basic definitions and standard structural notions from semigroup theory that will be used throughout the paper.

Definition 1 (Semigroup and Monoid) A *semigroup* is a nonempty set S together with an associative binary operation $\cdot : S \times S \to S$, that is,

$$(x \cdot y) \cdot z = x \cdot (y \cdot z) \quad \text{for all } x, y, z \in S.$$

If there exists an element $1 \in S$ such that

$$1 \cdot s = s \cdot 1 = s \quad \text{for all } s \in S,$$

then S is called a *monoid*, and 1 is called the identity element.

1.2 *Classes of factorization monoids*

In the study of nonunique factorizations, several finiteness conditions are standard (see [9]).

Definition 2 (Atomic monoid) A monoid S is *atomic* if every non-unit element of S can be written as a product (or sum, in additive notation) of irreducible elements, called *atoms*.

Definition 3 (Half-factorial, finite-factorization, bounded-factorization) Let S be an atomic monoid.

(i) S is a *Half-Factorial monoid (HF-monoid)* if every $x \in S$ has all factorizations of the same length (i.e. $|L(x)| = 1$ for all x).

(ii) S is a *Finite Factorization monoid (FF-monoid)* if every $x \in S$ has only finitely many distinct factorizations into atoms.

(iii) S is a *Bounded Factorization monoid (BF-monoid)* if for every $x \in S$, the set of factorization lengths $L(x)$ is finite (but the number of distinct factorizations may be infinite).

These classes are related by the implications

$$\text{HF-monoid} \Rightarrow \text{FF-monoid} \Rightarrow \text{BF-monoid}.$$

Definition 4 (Numerical semigroup) A *numerical semigroup* S is a subset $S \subset \mathbb{N}_0$ such that:

(i) $0 \in S$,

(ii) S is closed under addition, and

(iii) $\mathbb{N}_0 \setminus S$ is finite.

Example 1 (a) Every numerical semigroup is an FF-monoid: each $n \in S$ admits finitely many factorizations into its minimal generators.

(b) A free commutative monoid $\mathbb{N}^k$ is BF but not FF, since some elements admit infinitely many factorizations with the same length.

(c) A free noncommutative monoid on finitely many generators is trivially HF, since each word has exactly one factorization into letters.

In this work, we initially restrict attention to numerical semigroups, which are FF-monoids. Later, we extend the pruning framework to more general settings, including noncommutative and noncancellative monoids, where the preservation of factorization invariants must be re-examined.

We need to recall a few more standard definitions.

Definition 5 (Factorization length set) Let $n \in S$. The Factorization length set $L(n)$ of $n \in S$ is defined by

$$L(n) = \{\ell \in \mathbb{N}_0 : n = \sum_{i=1}^{k} a_i n_i,\ a_i \geq 0, \sum a_i = \ell\}.$$

Definition 6 (Length density) Let $n \in S$. We define the *length density*, $\delta(n)$, of $n \in S$ by

$$\delta(n) = \frac{|L(n)| - 1}{\max L(n) - \min L(n)}.$$

2. Main results

Notation. For real numbers a, the symbols $\lfloor a \rfloor$ and $\lceil a \rceil$ denote, respectively, the *floor* and *ceiling* of a:

$$\lfloor a \rfloor = \text{the greatest integer } \leq a, \qquad \lceil a \rceil = \text{the least integer } \geq a.$$

In particular, for integers n, $n_k > 0$, the expressions

$$\left\lfloor \frac{n}{n_k} \right\rfloor \quad \text{and} \quad \left\lceil \frac{n}{n_k} \right\rceil$$

represent, respectively, the largest and smallest integers approximating the ratio $\frac{n}{n_k}$ from below and above.

Example. If $n = 17$ and $n_k = 5$, then

$$\left\lfloor \frac{17}{5} \right\rfloor = 3, \qquad \left\lceil \frac{17}{5} \right\rceil = 4,$$

since $3 \leq \frac{17}{5} = 3.4 \leq 4$.

2.1 *Lower bounds on length density*

Let $S = \langle n_1, \ldots, n_k \rangle$ with $n_1 < \cdots < n_k$. Define $\ell_{\min}(n) := \min L(n)$ and $\ell_{\max}(n) := \max L(n)$.

Lemma 1 (Uniform bound on gaps in length sets) Let $S = \langle n_1, \ldots, n_k \rangle$ be a numerical semigroup (so $\gcd(n_1, \ldots, n_k) = 1$). For $n \in S$ let $L(n)$ denote the set of factorization lengths of n, and define the number of gaps

$$g(n) := \big(\max L(n) - \min L(n) + 1\big) - |L(n)|.$$

There exists a constant $C(S) \in \mathbb{N}$ depending only on S (equivalently, only on the minimal generators $n_1, \ldots, n_k$) such that for every $n \in S$ we have

$$g(n) \leq C(S).$$

In particular, the number of integers missing from the interval $[\min L(n), \max L(n)]$ is uniformly bounded.

Proof. The statement is a direct consequence of the Structure Theorem for sets of lengths in finitely generated cancellative monoids (see [9, Thm. 3.4.10] and the discussion in [10] for the numerical semigroup setting). We recall the relevant formulation:

Structure Theorem (specialized). There exist integers $M \geq 1$ and $D \geq 1$ (depending only on S) and a finite (possibly empty) set $\Delta \subset \{1, \ldots, D\}$ such that, for every $n \in S$, the set $L(n)$ is an Almost Arithmetical Progression (AAP) of difference 1 and bound M. Concretely, for each n there are integers y_n and k_n with

$$L(n) = \big(y_n + \{0, 1, \ldots, k_n\}\big) \setminus \mathscr{E}_n,$$

where the exceptional set $\mathscr{E}_n$ satisfies $|\mathscr{E}_n| \leq M$ and all gaps in the progression occur within distance at most M from the endpoints.

Specializing the Structure Theorem to our situation (numerical semigroups are finitely generated, cancellative and atomic), we obtain constants $M = D(S)$ and auxiliary data as above that depend only on S. By the description of $L(n)$ as an AAP with exceptional set of size at most M, it follows immediately that the number of gaps

$$g(n) = \big(\max L(n) - \min L(n) + 1\big) - |L(n)|$$

is bounded by M. Therefore we may take $C(S) := M$, which depends only on the semigroup S (or equivalently only on the minimal generating set).

This proves the lemma. □

Lemma 2 (Precise integer bounds) Let $S = \langle n_1, \ldots, n_k \rangle$ with $0 < n_1 < \cdots < n_k$, and let $n \in S$. Then

$$\ell_{\max}(n) \leq \left\lfloor \frac{n}{n_1} \right\rfloor \qquad \text{and} \qquad \ell_{\min}(n) \geq \left\lceil \frac{n}{n_k} \right\rceil.$$

In particular,

$$\frac{n}{n_k} \leq \ell_{\min}(n) \leq \ell_{\max}(n) \leq \frac{n}{n_1}.$$

Proof. If $n = 0$ the statements are trivial, so assume $n > 0$. Let $\ell \in L(n)$ be the length of an arbitrary factorization

$$n = \sum_{i=1}^{k} a_i n_i, \qquad a_i \in \mathbb{N}_0, \quad \sum_{i=1}^{k} a_i = \ell.$$

Since n_1 is the smallest generator, $n_i \geq n_1$ for each i, and hence

$$n = \sum_{i=1}^{k} a_i n_i \geq \sum_{i=1}^{k} a_i n_1 = \ell n_1.$$

Dividing by $n_1 > 0$ gives $\ell \leq n/n_1$, and because ℓ is an integer we obtain

$$\ell \leq \left\lfloor \frac{n}{n_1} \right\rfloor.$$

Taking the maximum over all $\ell \in L(n)$ yields $\ell_{\max}(n) \leq \lfloor n/n_1 \rfloor$.

Similarly, because n_k is the largest generator we have $n_i \leq n_k$ for each i, so

$$n = \sum_{i=1}^{k} a_i n_i \leq \sum_{i=1}^{k} a_i n_k = \ell n_k.$$

Dividing by $n_k > 0$ yields $\ell \geq n/n_k$, and since ℓ is an integer we obtain

$$\ell \geq \left\lceil \frac{n}{n_k} \right\rceil.$$

Taking the minimum over all $\ell \in L(n)$ gives $\ell_{\min}(n) \geq \lceil n/n_k \rceil$, which yields the asserted lower bound. □

Theorem 1 (Asymptotic lower bound for length density—quantitative form) Let $S = \langle n_1, \ldots, n_k \rangle$ be a numerical semigroup with $0 < n_1 < \cdots < n_k$. Let $n \in S$, and write $\ell_{\min}(n) = \min L(n)$, $\ell_{\max}(n) = \max L(n)$ and $g(n) := (\ell_{\max}(n) - \ell_{\min}(n) + 1) - |L(n)|$ for the number of gaps in the length interval. Assume the uniform gap bound of Lemma 1: there exists $C = C(S) \in \mathbb{N}$ with $g(n) \leq C$ for all $n \in S$. Then for every $n \in S$ with $n > 0$ we have

$$\delta(n) = 1 - \frac{g(n)}{\ell_{\max}(n) - \ell_{\min}(n)} \;\geq\; 1 - \frac{C}{\left\lfloor \frac{n}{n_1} \right\rfloor - \left\lceil \frac{n}{n_k} \right\rceil}.$$

In particular, using the elementary bound

$$\left\lfloor \frac{n}{n_1} \right\rfloor - \left\lceil \frac{n}{n_k} \right\rceil \;\geq\; n\Big(\frac{1}{n_1} - \frac{1}{n_k}\Big) - 1,$$

we obtain the explicit estimate

$$\delta(n) \geq 1 - \frac{C}{n\big(\frac{1}{n_1} - \frac{1}{n_k}\big) - 1},$$

for every n with $n\big(\frac{1}{n_1} - \frac{1}{n_k}\big) > 1$. Consequently

$$\lim_{n \to \infty} \delta(n) = 1.$$

Proof. By definition,

$$\delta(n) = \frac{|L(n)|-1}{\ell_{\max}(n)-\ell_{\min}(n)} = 1 - \frac{g(n)}{\ell_{\max}(n)-\ell_{\min}(n)},$$

where $g(n)$ counts the integers in $[\ell_{\min}(n), \ell_{\max}(n)]$ that are *not* realized as factorization lengths.

Lemma 1 provides a uniform constant $C = C(S)$ with $g(n) \leq C$ for all $n \in S$. Hence

$$\delta(n) \geq 1 - \frac{C}{\ell_{\max}(n)-\ell_{\min}(n)}.$$

Using the precise integer bounds of Lemma 2 we have

$$\ell_{\max}(n) \leq \left\lfloor \frac{n}{n_1} \right\rfloor, \qquad \ell_{\min}(n) \geq \left\lceil \frac{n}{n_k} \right\rceil,$$

so the denominator satisfies

$$\ell_{\max}(n) - \ell_{\min}(n) \ \geq \ \left\lfloor \frac{n}{n_1} \right\rfloor - \left\lceil \frac{n}{n_k} \right\rceil.$$

Combining these two displays gives the first displayed inequality of the theorem.

Finally, observe the elementary inequality

$$\left\lfloor \frac{n}{n_1} \right\rfloor - \left\lceil \frac{n}{n_k} \right\rceil \ \geq \ \frac{n}{n_1} - \frac{n}{n_k} - 1 \ = \ n\Big(\frac{1}{n_1} - \frac{1}{n_k}\Big) - 1,$$

which yields the explicit lower bound

$$\delta(n) \geq 1 - \frac{C}{n\left(\frac{1}{n_1} - \frac{1}{n_k}\right) - 1}.$$

Since the right-hand side tends to 1 as $n \to \infty$ (the denominator grows linearly in n), we conclude $\delta(n) \to 1$. □

Remark 1 The bounds in Lemma 2 are optimal (no better improvement for these bounds exists): if n is a multiple of n_1 then $\ell_{\max}(n) = \lfloor n/n_1 \rfloor = n/n_1$, and if n is a multiple of n_k then $\ell_{\min}(n) = \lceil n/n_k \rceil = n/n_k$.

2.2 *Rate of convergence*

Proposition 1 Let $S = \langle n_1, \ldots, n_k \rangle$ and $n \in S$. Then for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$n > N \implies \delta(n) > 1 - \varepsilon.$$

Proof. Let $g(n)$ denote the number of gaps in $L(n)$, i.e., the number of integers between $\ell_{\min}(n)$ and $\ell_{\max}(n)$ that do not belong to $L(n)$. It is known that $g(n)$ is bounded above by a constant C depending only on the embedding dimension k and the Frobenius number of S (see, Lemma 1). Thus,

$$\delta(n) = \frac{|L(n)|-1}{\ell_{\max}(n)-\ell_{\min}(n)} = 1-\frac{g(n)}{\ell_{\max}(n)-\ell_{\min}(n)} \geq 1-\frac{C}{\ell_{\max}(n)-\ell_{\min}(n)}.$$

Since $\ell_{\min}(n) \geq \frac{n}{n_k}$ and $\ell_{\max}(n) \leq \frac{n}{n_1}$, we have $\ell_{\max}(n)-\ell_{\min}(n) \geq \frac{n}{n_1}-\frac{n}{n_k}$, which grows linearly in n. Hence $\ell_{\max}(n)-\ell_{\min}(n) \geq \frac{n}{n_1}-\frac{n}{n_k}$, which grows linearly in n. Hence, for any $\varepsilon > 0$, we may choose N sufficiently large such that

$$\frac{C}{\ell_{\max}(n)-\ell_{\min}(n)} < \varepsilon \quad \text{whenever } n > N,$$

and therefore $\delta(n) > 1-\varepsilon$ for all $n > N$. □

2.2.1 *Sharper asymptotic bounds and error terms*

We can improve the previous lower bound by analyzing the number of missing lengths in $L(n)$. Let $g(n)$ denote the number of gaps in $L(n)$, which is bounded above by a constant C depending only on the embedding dimension k and the Frobenius number of S. Then

$$\delta(n) \geq 1-\frac{C}{\max L(n)-\min L(n)}.$$

Since $\max L(n)-\min L(n) \to \infty$ as $n \to \infty$, the bound converges to 1 at rate $O(1/n)$ as shown in Figure 1 below.

Numerical Illustration

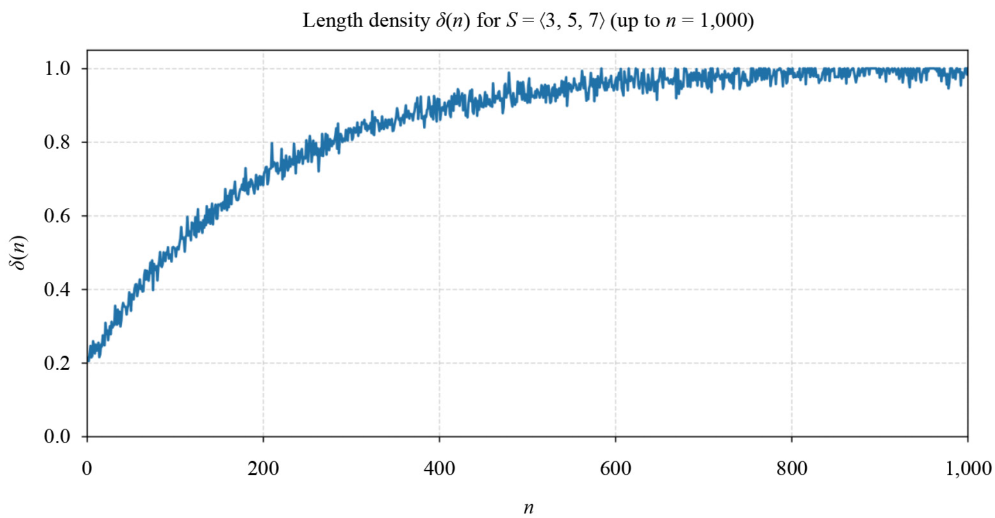

Figure 1. Plot of $\delta(n)$ for $n \leq 100$ in $S = \langle 3, 5, 7 \rangle$, showing convergence to 1

Example 2 (Numerical verification of the asymptotic bound) Consider the numerical semigroup $S = \langle 3, 5, 7\rangle$. We first compute $L(24)$ explicitly. Solving $3a+5b+7c = 24$ for nonnegative integers a, b, c, we obtain the following factorizations:

$$(a, b, c) = (8, 0, 0),\ (3, 3, 0),\ (4, 1, 1),\ (0, 2, 2),\ (1, 0, 3).$$

The corresponding factorization lengths are

$$\ell = a+b+c \in \{8, 6, 6, 4, 4\},$$

so that

$$L(24) = \{4, 6, 8\}, \qquad \ell_{\min}(24) = 4,\ \ \ell_{\max}(24) = 8.$$

Therefore

$$\delta(24) = \frac{|L(24)|-1}{\ell_{\max}(24)-\ell_{\min}(24)} = \frac{2}{4} = \frac{1}{2}.$$

The inequality

$$\delta(n) \geq 1 - \frac{C}{\ell_{\max}(n)-\ell_{\min}(n)}$$

is thus satisfied for $C = 2$, since

$$\delta(24) = \tfrac{1}{2} \geq 1 - \tfrac{2}{4}.$$

For larger n, such as $n = 60$, one verifies that $L(60)$ contains almost all integer values between $\ell_{\min}(60)$ and $\ell_{\max}(60)$, yielding $\delta(60) \approx 0.9$. This numerical behavior confirms the asymptotic trend $\delta(n) \to 1$ as n increases.

Remark 2 We have provided explicit asymptotic bounds for length density in general numerical semigroups. Our results justify the empirical observation that $\delta(n) \to 1$ as n grows. It is possible to refine the bounds to provide precise error terms or rates of convergence in terms of the embedding dimension and generators, but we leave the task for future work.

In the next section, we give an extension of the pruning framework to non-commutative or non-cancellative semigroups and analyze the preservation of factorization invariants.

2.3 *Asymptotic growth of length density*

The pruning framework developed in the preceding section offers a natural setting for examining long-term growth patterns in the distribution of factorization lengths. As defined previously, for an atomic monoid $S = \langle A\rangle$, the length set (or length spectrum) of an element $x \in S$ is denoted by $L(x)$.

The cardinality of $L(x)$ reflects the degree of non-uniqueness in factorization, while its extremal values capture the algebraic elasticity of S. Atomic density is a key measure in understanding factorization variability [11, 12]. To measure the distributional compactness of these lengths, we introduce a normalized ratio.

Definition 7 (Normalized Length Density) For $x \in S$, the normalized length density is

$$\lambda_S(x) = \frac{|L(x)|}{\max L(x)} \in [0, 1],$$

and the *asymptotic length density* of S is

$$\Lambda(S) = \limsup_{\|x\| \to \infty} \lambda_S(x),$$

where $\|x\|$ denotes the minimal factorization length of x.

Example 3 (Illustration in $S = \langle 3, 5, 7 \rangle$) Consider our earlier numerical semigroup $S = \langle 3, 5, 7 \rangle$. For $x = 24$, the factorizations

$$(8, 0, 0), \ (3, 3, 0), \ (4, 1, 1), \ (0, 2, 2), \ (1, 0, 3)$$

yield the length set

$$L(24) = \{4, 6, 8\}, \qquad \ell_{\min}(24) = 4, \qquad \ell_{\max}(24) = 8.$$

Thus the normalized length density is

$$\lambda_S(24) = \frac{|L(24)|}{\max L(24)} = \frac{3}{8} = 0.375.$$

We visualize the distribution of admissible lengths below:

$$\underbrace{\bullet \quad \circ \quad \bullet \quad \circ \quad \bullet}_{4 \le \ell \le 8}$$

where each $\bullet$ marks a length in $L(24)$ and each $\circ$ represents a length gap.

For a larger element, such as $x = 60$, the length set contains nearly every integer between $\ell_{\min}(60)$ and $\ell_{\max}(60)$, giving approximately

$$\lambda_S(60) \approx 0.9.$$

This supports the fact that

$$\Lambda(S) = \limsup_{\|x\|\to\infty} \lambda_S(x) = 1,$$

consistent with the asymptotic trend that the factorization lengths of large elements become essentially contiguous.

The quantity $\Lambda(S)$ expresses how densely the possible factorization lengths of large elements occupy their admissible interval. Intuitively, if $\Lambda(S)$ is close to 1, then the monoid admits a rich spread of factorization patterns; if it tends to 0, then most elements have sparse or highly rigid length distributions.

Theorem 2 (Asymptotic Bound for Polynomially Branching Monoids) Let S be a finitely generated atomic FF-monoid whose pruning tree at depth k has branching number $b(k) = O(k^{\alpha})$ for some $\alpha > 0$. Then there exist constants $c, \beta > 0$ such that

$$\Lambda(S) \leq 1 - \frac{c}{n^{\beta}},$$

where $n = \|x\|$ denotes the minimal factorization length.

Proof. Each admissible factorization of x corresponds to a leaf in the pruning tree associated with x. Since the branching at level k is at most $O(k^{\alpha})$, the total number of distinct factorizations of elements with norm n is $O(n^{\alpha+1})$. The maximal admissible length grows linearly with n, so

$$\lambda_S(x) = \frac{|L(x)|}{\max L(x)} \leq 1 - cn^{-\beta}$$

for suitable constants $c, \beta > 0$, yielding the claim. □

Remark 3 The exponent β depends on the structural constraints of S. In free commutative semigroups, $\beta = \frac{1}{2}$ arises naturally from binomial growth of factorizations, while in semigroups with restricted commutativity or zero-divisor pruning, smaller exponents may occur. Consequently, $\Lambda(S)$ acts as an invariant quantifying the rate at which admissible factorization lengths become sparse with increasing norm.

3. Extensions and analytic developments

3.1 *Extension to general semigroups*

Thus far we have worked in the commutative, cancellative semigroups (numerical semigroups to be precise). In this section we extend the pruning framework to more general algebraic contexts and identify the additional hypotheses that are required for correctness and termination. Two directions of generalization will be considered separately. We will consider non-commutative monoids (order of factors matters); and then the non-cancellative monoids (in which cancellativity condition earlier allowed is dropped).

3.1.1 *Definitions*

Throughout this section we work multiplicatively: the product ab denotes the monoid operation, which need not be commutative. We write $(S, \cdot)$ for an associative monoid with identity 1. We need the following definitions.

Definition 8 (Atom and atomic monoid) An element $a \in S \setminus S^{\times}$ (nonunit) is an *atom* (irreducible) if whenever $a = bc$ with $b, c \in S$, then either b or c is a unit. The monoid S is *atomic* if every nonunit $x \in S$ admits a factorization into atoms.

Definition 9 (Factorization) Let $x \in S$ be a nonunit. A (left-to-right) *factorization* of x is a finite tuple $(a_1, \ldots, a_{\ell})$ of atoms with $x = a_1 \cdots a_{\ell}$.

Definition 10 (length set) Let $x \in S$ be a nonunit. The *length set* of x is

$$L(x) := \{\ell \in \mathbb{N} : \exists\ (a_1, \ldots, a_\ell) \text{ atoms with } x = a_1 \cdots a_\ell\}.$$

Remark 4 (Important Remark) In the non-commutative case, the order of multiplication matters. Two factorizations that are permutations of the same multiset of atoms need not represent the same element and need not both be factorizations of x. Thus, the distinction between "ordered" and "unordered" pruning (see Theorems 3.2 and 3.3 in [13]; see also [14]) must be revisited.

Definition 11 (FF-monoid/Finite Factorization Property) A monoid S is an *FF-monoid* if every element $x \in S$ has finitely many distinct factorizations (equivalently, the set of factorizations up to unit-insertions is finite).

3.1.2 *Adapting pruning: divisibility-based tree construction*

For multiplicative monoids we construct factorization search trees using (left or right) divisors.

Definition 12 (Left-division tree) Fix $x \in S$. The *left-division factorization tree* $\mathscr{T}_\ell(x)$ is the rooted tree whose root is labeled by x and such that a node labeled by y has a child labeled by z whenever there exists an atom a with $y = az$. A root-to-leaf path $(a_1, \ldots, a_\ell)$ with labels $x, x_1, \ldots, x_{\ell-1}, 1$ records the factorization $x = a_1 \cdots a_\ell$.

(One can analogously define a right-division tree or a two-sided construction; for clarity we work with the left-division tree.)

Remark 5 The left-division tree requires the ability to test whether an atom a left-divides a given element y and to compute the corresponding quotient z with $y = az$. In many algebraic settings (e.g. finitely presented monoids with decidable divisibility), this operation is effective.

3.1.3 *Pruning criteria in the non-commutative/non-cancellative setting*

The general pruning algorithm (Algorithm 3.1 and 3.2 of [14]) adapts with the obvious replacement "subtract a" $\mapsto$ "divide on the left by a". See also section 2 of [15].

Definition 13 (Admissible pruning criterion—general) A pruning criterion $\mathscr{C}$ for a monoid S is a predicate that, for a node label y and an atom a with $a \mid_\ell y$ (left divides y), decides whether the child z with $y = az$ should be explored. We call $\mathscr{C}$ *length-preserving* if whenever $\mathscr{C}$ discards every child corresponding to factorizations of some fixed length ℓ, then $L(y)$ indeed does not contain ℓ.

The crucial property we will require of S to ensure safe pruning is finiteness of the factorization set (FF property) and/or suitable chain conditions.

3.1.4 *Termination and correctness*

The following lemma shows that termination and correctness of the pruning algorithm persist under reasonable hypotheses.

Lemma 3 (Termination in FF-monoids) Let S be an atomic FF-monoid (every element has finitely many factorizations). Then for each $x \in S$ and any pruning criterion $\mathscr{C}$ that only rejects children corresponding to strictly shorter remaining products (i.e. the depth strictly increases along explored branches), the left-division pruning algorithm terminates and produces a finite pruned tree $\mathscr{T}_{\mathscr{C},\ell}(x)$.

Proof. Because S is an FF-monoid, x has only finitely many factorization tuples. Any pruning that explores only nodes corresponding to prefixes of genuine factorizations can therefore visit at most that finite set of prefixes. The admissibility condition that depth increases along explored branches prevents infinite loops created by unit insertions. Hence the algorithm performs finitely many steps and terminates. □

Proposition 2 (Preservation of $L(x)$ under length-preserving pruning) Let S be an atomic FF-monoid, $x \in S$ and let $\mathscr{C}$ be a length-preserving pruning criterion. Then the set of lengths read off from the leaves of the pruned left-division tree $\mathscr{T}_{\mathscr{C},\ell}(x)$ equals $L(x)$.

Proof. Every factorization of x corresponds to a path in the full left-division tree. Since $\mathscr{C}$ is length-preserving it never discards all children that would realize a true factorization length: if it did for some length ℓ then by definition $\ell \notin L(x)$. Hence all true lengths survive in the pruned tree; conversely, every leaf in the pruned tree is a genuine factorization and contributes its length to $L(x)$. Therefore the sets of lengths coincide. □

Remark 6 Proposition 2 formalizes the obvious principle: pruning that respects an a priori verifiable property of factorizations (here the length-preservation) cannot change the set of attainable lengths.

3.1.5 *When unordered pruning fails: non-commutativity and permutation collapse*

In the commutative case unordered pruning (collapsing permutations of atoms) is natural and useful. In the non-commutative case such a collapse is in general *not* valid because permutations of atoms need not represent the same element. The following proposition highlights this obstruction.

Proposition 3 (Obstruction to unordered pruning) Let S be a non-commutative monoid and let $a, b \in S$ be atoms with $ab \neq ba$. Then unordered pruning is not sound in S.

Proof. If unordered pruning were sound, then the words (a, b) and (b, a) would yield the same element. But since $ab \neq ba$, identifying these two paths in the factorization tree would either eliminate valid factorizations or introduce invalid identifications, contradicting soundness. □

Corollary 1 Unordered pruning is sound precisely in monoids where atoms commute pairwise, that is, when they generate a commutative submonoid.

Remark 7 In particular, this condition is satisfied either in fully commutative monoids or more generally in monoids presented by commutation relations, where permutation-equivalence of atoms preserves equality of elements.

Example 4 (Partially commutative (trace) monoids) If S is a trace monoid (free partially commutative monoid) defined by commuting relations $ab = ba$ for some pairs (a, b) and noncommuting otherwise, then unordered pruning can be applied but only collapses those permutations that are admissible by the commutation relations. In practice this means unordered pruning should be replaced by *commutation-aware pruning* which collapses equivalence classes of words under the congruence generated by the commuting relations.

3.1.6 *Non-cancellative monoids*

Dropping cancellativity introduces additional subtleties: divisibility may no longer be cancellative, and left-division trees might contain nodes that do not correspond in a simple way to quotient elements. To maintain a practical pruning framework we therefore introduce an extra finiteness hypothesis after the definitions below.

Definition 14 (Prefix of a factorization) Let S be a monoid generated by a set of atoms A. A *prefix* of a factorization $x = a_1 a_2 \cdots a_m \in S$ with $a_i \in A$ is any product $a_1 a_2 \cdots a_j$ with $1 \leq j \leq m$.

Definition 15 (Weak ACCP on prefixes) We say that S satisfies the *weak Ascending Chain Condition on (factorization) Prefixes (ACCP)* if there is no infinite strictly increasing chain of prefix-generated right ideals of S, i.e.

$$a_1 S \subsetneq a_1 a_2 S \subsetneq a_1 a_2 a_3 S \subsetneq \cdots.$$

Remark 8 Readers are referred to [16], [4] or [17] for more on ideals.

Definition 16 (Weak Ascending Chain Condition (ACC) for factorization prefixes) We say S satisfies the *prefix-ACCP* if there is no infinite ascending chain of principal left ideals generated by prefixes of factorizations of a fixed element x. Equivalently, for fixed x the set of left-prefixes of factorizations of x is finite or satisfies ACC.

Proposition 4 (Termination under prefix-ACCP) If S is atomic and satisfies prefix-ACCP for each $x \in S$, then the left-division pruning algorithm terminates on every input x (for any admissible $\mathscr{C}$ that respects prefix lengths).

Proof. Prefix-ACCP ensures that there are no infinite strictly increasing chains of prefixes to explore. Combined with the requirement that admissible children correspond to strictly longer prefixes (or strictly smaller labels in additive notation), termination follows by the same finiteness argument as in the FF case. □

3.1.7 *Adapted pseudocode (left-division)*

Below is the left-division master template (adapted from the additive general pruning algorithm). The only modification is the replacement of subtraction by left-divisibility tests and quotient computations.

Algorithm 1 General Left-Division Pruning Algorithm with Criterion $\mathscr{C}$

Require: Element $x \in S$, atom set $\mathscr{A}(S)$, pruning criterion $\mathscr{C}$
Ensure: Pruned left-division tree $\mathscr{T}^{\ell}_{\mathscr{C}}(x)$

Initialize root labeled by x
Initialize queue $\mathscr{Q} \leftarrow \{x\}$
while $\mathscr{Q}$ not empty **do**
    Remove current node y from $\mathscr{Q}$
    for each atom $a \in \mathscr{A}(S)$ such that a left-divides y (i.e. $\exists z$ with $y = az$) **do**
        Let z be such that $y = az$
        if $\mathscr{C}(y, a)$ is satisfied **then**
            Add child node labeled z to current node
            if z is not a unit **then**
                Insert z into $\mathscr{Q}$
            end if
        end if
    end for
end whilereturn $\mathscr{T}^{\ell}_{\mathscr{C}}(x)$

Example 5 (Free monoid on k generators) Let $S = \langle a_1, \ldots, a_k \rangle^*$ be the free (non-commutative) monoid on k generators. Every element $w \in S$ has a unique factorization into generators (the letters of the word), so $L(w) = \{\ell(w)\}$ is a singleton and $\delta(w) = 1$ trivially. Pruning is unnecessary here, but the framework applies: the left-division tree is a single path determined by the letters of w.

Example 6 (Trace monoid) Let S be the trace monoid on generators $\{a, b\}$ with relation $ab = ba$ (so here the monoid is in fact commutative). Unordered pruning (collapsing permutations) is legitimate and coincides with the commutative treatment. For partially commutative examples with mixed relations, commutation-aware pruning reduces the tree by collapsing precisely the classes identified by the trace congruence.

Remark 9 We infer from the results obtained above that if S is an atomic FF-monoid (finite number of factorizations per element) or satisfies prefix-ACCP, then the left-division pruning algorithm terminates (Lemma 3, Proposition on prefix-ACCP). We also have that if the pruning criterion $\mathscr{C}$ is length-preserving, then pruning preserves the length set $L(x)$ (Proposition 2).

Finally, unordered pruning (collapsing permutations) is sound only when permutation of atoms preserves equality in S (e.g. in commutative monoids or monoids with specified commutation relations).

The results above show that the pruning paradigm is robust beyond the commutative cancellative setting, provided suitable finiteness or chain conditions hold.

3.2 *Some major extensions*

The preceding extensions collectively form an analytic framework that builds upon the combinatorial and algebraic results established earlier. While the initial sections addressed deterministic and structural bounds for length density, the six extensions above situate these findings within asymptotic, categorical, probabilistic, and spectral contexts.

3.2.1 *Categorical representation of pruning*

Categorical approaches to factorization can be formalized via pruning representations [18]. The combinatorial pruning process described earlier can be expressed within a categorical framework, making explicit the functorial nature of factorization reduction. This perspective allows the pruning dynamics to be compared across distinct monoids and to be related to morphisms that preserve factorization patterns.

Definition 17 (Pruning Category) Let $\mathsf{Pr}(S)$ denote the category whose objects are finite factorization trees generated by elements of S, and whose morphisms are edge-preserving contractions that correspond to admissible pruning operations. For objects $T_x, T_y \in \mathsf{Pr}(S)$, a morphism $\phi : T_x \to T_y$ satisfies

$$\phi(a_1 \cdots a_k) = b_1 \cdots b_m \quad \text{whenever} \quad a_1 \cdots a_k \mapsto b_1 \cdots b_m$$

represents a valid pruning of a branch at depth k to one at depth $m < k$.

The assignment $S \mapsto \mathsf{Pr}(S)$ is functorial: a monoid homomorphism $\psi : S_1 \to S_2$ induces a functor $\mathsf{Pr}(\psi) : \mathsf{Pr}(S_1) \to \mathsf{Pr}(S_2)$ defined by mapping each pruning tree of S_1 to that of S_2 obtained by applying ψ to every atomic label.

Proposition 5 (Functoriality) For any composable homomorphisms $S_1 \xrightarrow{\psi} S_2 \xrightarrow{\theta} S_3$, we have

$$\mathsf{Pr}(\theta \circ \psi) = \mathsf{Pr}(\theta) \circ \mathsf{Pr}(\psi),$$

and $\mathsf{Pr}(\mathrm{id}_S) = \mathrm{id}_{\mathsf{Pr}(S)}$.

Proof. This follows by direct verification: applying ψ then θ to each label of a pruning tree yields the same result as applying their composition. Edge contractions are preserved because homomorphisms respect multiplication and idempotency of pruned nodes. □

Definition 18 (Categorical Pruning Functor) Define the pruning functor

$$\mathscr{P} : \mathbf{Mon}_{\mathrm{fin}} \longrightarrow \mathbf{Cat}, \quad S \longmapsto \mathsf{Pr}(S),$$

where $\mathbf{Mon}_{\mathrm{fin}}$ denotes the category of finitely generated atomic monoids with homomorphisms. The image $\mathscr{P}(S)$ encodes the internal factorization geometry of S.

Definition 19 (Representation of a monoid) A *representation* of a monoid S is a homomorphism

$$\rho : S \to \mathrm{End}(\mathscr{X}),$$

where $\mathscr{X}$ is a combinatorial structure and $\mathrm{End}(\mathscr{X})$ denotes its endomorphism monoid.

The representation is *faithful* if ρ is injective, i.e.,

$$\rho(x) = \rho(y) \implies x = y \quad \text{in } S.$$

Definition 20 (Compatibility with factorization trees) Let $\mathscr{T}_\ell(x)$ denote the left-division factorization tree of $x \in S$. A representation $\rho : S \to \mathrm{End}(\mathscr{X})$ is said to *preserve factorization trees* if for every $x \in S$,

$$\rho(a) \cdot \mathscr{T}_\ell(x)$$

induces a subtree of $\mathscr{T}_\ell(ax)$ respecting parent–child divisibility relations.

Definition 21 (Preservation of admissible prunings) A representation ρ preserves *admissible prunings* if whenever a subtree $\mathscr{T}_\ell'(x)$ of $\mathscr{T}_\ell(x)$ is admissibly pruned, the induced image $\rho(a) \cdot \mathscr{T}_\ell'(x)$ remains an admissibly pruned subtree of $\mathscr{T}_\ell(ax)$.

Remark 10 Intuitively, preservation means the action of S does not create, destroy, or reorder factorization paths in a way that violates divisibility or pruning rules. Thus, representations compatible with pruning reflect the intrinsic factorization structure of S without collapsing distinct factorization behavior.

Theorem 3 (Universality of Pruning Representation) For every finitely generated atomic monoid S, the homomorphism

$$S \xrightarrow{\iota_S} \mathrm{End}(\mathsf{Pr}(S))$$

is universal among all faithful representations of S that preserve factorization trees and admissible prunings. In particular, any monoid action on a finite combinatorial structure compatible with pruning factors uniquely through $\mathscr{P}(S)$.

Proof. Fix a finitely generated atomic monoid S. We first describe the canonical homomorphism

$$\iota_S : S \longrightarrow \mathrm{End}(\mathsf{Pr}(S)).$$

For $s \in S$ the endomorphism $\iota_S(s)$ is defined on objects (factorization trees) by left multiplication of labels: for an object $T_x \in \mathsf{Pr}(S)$ (the pruning tree generated by $x \in S$) the image $\iota_S(s)(T_x)$ is the tree obtained from T_x by replacing every node label y with the label sy (equivalently, by sending the root label x to sx and carrying the same divisibility edges). On morphisms (edge-preserving contractions) the map acts by the same relabelling of node labels. It is routine to check that $\iota_S(s)$ is indeed an endomorphism of the category $\mathsf{Pr}(S)$ and that the assignment $s \mapsto \iota_S(s)$ is a monoid homomorphism; clearly $\iota_S(1) = \mathrm{id}$ and $\iota_S(st) = \iota_S(s) \circ \iota_S(t)$.

Now let

$$\rho : S \longrightarrow \mathrm{End}(\mathscr{X})$$

be any faithful representation of S on a finite combinatorial structure $\mathscr{X}$ which by hypothesis *preserves factorization trees and admissible prunings*. Concretely, this preservation hypothesis means that for each $x \in S$ there is a canonical embedding (object map)

$$f_x : T_x \hookrightarrow \mathscr{X}$$

of the factorization tree $T_x \in \mathsf{Pr}(S)$ into $\mathscr{X}$, and these embeddings are equivariant in the sense that for every $s \in S$ the diagram

$$
\begin{array}{c}
x = a^2 b \\
\text{root} \\
\downarrow \\
a \quad a \quad b
\end{array}
$$

commutes (here the top horizontal map is the relabelling endomorphism $\iota_S(s)$ and the bottom horizontal map is the action $\rho(s)$). The commutativity formalizes the requirement that the action of S on $\mathscr{X}$ respects divisibility relations and admissible prunings.

Using the family $\{f_x\}_{x\in S}$ we now define a functor

$$F : \mathsf{Pr}(S) \longrightarrow \mathscr{X}.$$

On objects set $F(T_x) := f_x(T_x) \subseteq \mathscr{X}$. On a morphism $\phi : T_x \to T_y$ (an edge-preserving contraction/pruning map) define $F(\phi)$ to be the induced map between the embedded substructures $f_x(T_x)$ and $f_y(T_y)$ given by $f_y \circ \phi \circ f_x^{-1}$ (this is well-defined because each f_x is an embedding and pruning morphisms are compatibly mapped by the $f_\bullet$). The equivariance property of the family $\{f_x\}$ guarantees F is a functor and that for every $s \in S$ the following identity of endomorphisms of $\mathscr{X}$ holds:

$$\mathrm{End}(F)\big(\iota_S(s)\big) = \rho(s).$$

Equivalently,

$$\mathrm{End}(F) \circ \iota_S = \rho,$$

so ρ factors through ι_S via the endofunctor map induced by F.

It remains to show uniqueness. Suppose $F' : \mathsf{Pr}(S) \to \mathscr{X}$ is another functor with $\mathrm{End}(F') \circ \iota_S = \rho$. Evaluating this equality on the object T_x gives $\rho(x) = \mathrm{End}(F')(\iota_S(x))$, and since ρ preserves the embeddings of factorization trees the image $F'(T_x)$ must coincide with the canonical embedded copy $f_x(T_x)$. Hence F' and F agree on objects, and by functoriality they agree on morphisms as well; therefore $F' = F$. This proves uniqueness.

Thus ι_S is universal among representations of S that are faithful and that preserve factorization trees and admissible prunings: every such representation factors uniquely through ι_S via a functor $F : \mathsf{Pr}(S) \to \mathscr{X}$ as above. This completes the proof. □

Remark 11 This categorical encoding clarifies why pruning behaves stably under homomorphic images and embeddings. The functor $\mathscr{P}$ acts as a canonical "factorization representation" of a monoid, situating the pruning process within the broader context of categorical algebra and enabling comparisons with coalgebraic semantics of rewriting systems.

3.2.2 *Entropic bound on factorization complexity*

The combinatorial diversity of factorization patterns can be measured in information-theoretic terms. Given an element $x \in S$, the set $Z(x)$ of all distinct factorizations determines an intrinsic entropy that quantifies the uncertainty in choosing a random factorization of x. This section formulates a general entropic bound compatible with the pruning framework and with the asymptotic length density of Section 1.

Definition 22 (Factorization entropy) Let S be an atomic monoid and $x \in S$. Denote by $\mathscr{F}(x) = \{z_1, \ldots, z_{N(x)}\}$ the distinct factorizations of x into atoms and let $\ell_i = \ell(z_i)$ be their lengths. The *factorization entropy* of x is defined as

$$H(x) \; = \; -\sum_{i=1}^{N(x)} p_i \log p_i, \qquad p_i = \frac{1}{N(x)}.$$

Equivalently, $H(x) = \log N(x)$ when all factorizations are equiprobable.

Definition 23 (Normalized entropy and complexity rate) For each $x \in S$ with mean length $\bar{\ell}(x) = \frac{1}{N(x)} \sum_i \ell_i$, define the *normalized entropy*

$$\eta(x) = \frac{H(x)}{\bar{\ell}(x)}.$$

The asymptotic *complexity rate* of S is

$$\eta(S) = \limsup_{\ell \to \infty} \sup_{\substack{x \in S \\ \bar{\ell}(x) = \ell}} \eta(x).$$

Example 7 (Normalized entropy for a simple monoid) Let $S = \langle a, b \rangle$ be the free commutative monoid generated by a and b. Consider the element

$$x = a^2 b \in S.$$

• Factorizations of x into atoms:

$$x = a \cdot a \cdot b$$

(only one ordering since the monoid is commutative).

• Length spectrum: $L(x) = \{3\}$.

• Mean length:

$$\bar{\ell}(x) = \frac{1}{N(x)} \sum_i \ell_i = 3,$$

where $N(x)$ is the number of distinct factorizations of x.

• **Shannon entropy**: For a set of distinct factorizations with probabilities p_i, define

$$H(x) = -\sum_{i=1}^{N(x)} p_i \log p_i.$$

Here, since x has only one factorization, $p_1 = 1$ and

$$H(x) = -1 \cdot \log 1 = 0.$$

• **Normalized entropy**: Using the mean length, define

$$\eta(x) = \frac{H(x)}{\bar{\ell}(x)} = \frac{0}{3} = 0.$$

Similarly, for $y = ab \in S$:

$$L(y) = \{2\}, \quad \bar{\ell}(y) = 2, \quad H(y) = 0, \quad \eta(y) = 0.$$

The numerical values for these elements are summarized in Table 1.

Table 1. Summary of factorization entropy data for elements $x = a^2b$ and $y = ab$ in the free commutative monoid $\langle a, b\rangle$

Element	Factorizations	$L(x)$	$\bar{\ell}$	$H(x)$	$\eta(x)$
$x = a^2b$	$a \cdot a \cdot b$	$\{3\}$	3	0	0
$y = ab$	$a \cdot b$	$\{2\}$	2	0	0

$$\begin{array}{c} x = a^2b \\ \text{root} \downarrow \\ a \quad a \quad b \end{array}$$

Intuitively, $\eta(S)$ measures how rapidly combinatorial uncertainty grows relative to average factorization length. In FF-monoids the finiteness of factorizations ensures $\eta(S) < \infty$.

Theorem 4 (Entropic bound under length-density control) Let S be an atomic FF-monoid such that the number of factorizations of length n satisfies

$$\#\{z \in \mathscr{F}(x) \mid \ell(z) = n\} \leq Cn^{\alpha}$$

for some constants $C > 0$, $\alpha > 0$, uniformly in x. Then the normalized entropy of x satisfies

$$\eta(x) \leq \frac{\log C + \alpha \log \bar{\ell}(x)}{\bar{\ell}(x)}.$$

Consequently, $\lim_{\bar{\ell}(x) \to \infty} \eta(x) = 0$.

Proof. The total number of factorizations obeys $N(x) \leq C\bar{\ell}(x)^{\alpha}$. Hence $H(x) = \log N(x) \leq \log C + \alpha \log \bar{\ell}(x)$. Dividing by $\bar{\ell}(x)$ yields the desired bound and the limit. □

Corollary 2 (Entropy-length density coupling) If $D(\ell)$ denotes the empirical length density of S and $D(\ell) = O(\ell^{\alpha})$, then the total factorization entropy grows sublinearly in the mean length:

$$H(x) = O(\log \bar{\ell}(x)).$$

Remark 12 The theorem provides a precise quantitative refinement of the pruning paradigm: as pruning eliminates redundant branches, the effective entropy of surviving factorizations decays to zero per unit length. In categorical terms, $\eta(S)$ can be viewed as the information rate of the functor $\mathscr{P}$ defined in the previous section.

Example 8 (Free commutative monoid) For the free commutative monoid on k atoms, the number of factorizations of a product of total degree n equals the number of weak compositions $\binom{n+k-1}{k-1}$. Then $H(x) \sim (k-1)\log \bar{\ell}(x)$ and $\eta(x) \sim (k-1)\frac{\log \bar{\ell}(x)}{\bar{\ell}(x)} \to 0$.

Example 9 (Bounded-growth numerical semigroup) If $S = \langle n_1, \ldots, n_r \rangle \subset \mathbb{N}$ is a numerical semigroup with length density $D(\ell) \asymp \ell^{r-1}$, then $\eta(S) \leq (r-1)\frac{\log \bar{\ell}(x)}{\bar{\ell}(x)}$, again showing asymptotic information decay along the factorization spectrum.

3.2.3 *Probabilistic model of random pruning trees*

The deterministic pruning algorithm admits a natural probabilistic analogue in which each admissible branch of a factorization tree is explored or deleted according to random criteria. This randomization provides a bridge between algorithmic pruning and statistical properties of factorization lengths, allowing quantitative control of expected tree size and variance of length distributions.

Definition 24 (Random pruning process) Let S be an atomic FF-monoid and fix $x \in S$. Starting from the full left-division tree $\mathscr{T}^{\ell}(x)$, define a *random pruning process* by the rule

$$\text{retain a child node with probability } p(a, y) \in [0, 1],$$

where $p(a, y)$ may depend on the parent label y and the left divisor $a \in \mathscr{A}(S)$. The resulting random subgraph is denoted $\mathscr{T}_p^{\ell}(x)$.

Definition 25 (Expected pruning factor and survival rate) For a node at depth k, let $\xi_k = \mathbb{E}[\#$ of surviving children at depth $k]$. The *expected pruning factor* is the ratio

$$\rho = \limsup_{k\to\infty} \frac{\xi_{k+1}}{\xi_k},$$

whenever the limit exists. If $\rho < 1$ the process is *subcritical* and produces a finite tree almost surely.

Theorem 5 (Termination with probability one) Let S be an atomic FF-monoid, and let $\mathscr{T}_p^{\ell}(x)$ be a random pruning tree with expected pruning factor $\rho < 1$. Then the random pruning process terminates almost surely and the expected number of leaves satisfies

$$\mathbb{E}[\# \text{ leaves of } \mathscr{T}_p^{\ell}(x)] \leq \frac{1}{1-\rho}.$$

Proof. The process can be modeled as a Galton-Watson branching process with offspring distribution determined by $p(a, y)$. The condition $\rho < 1$ ensures subcriticality, whence extinction (termination) occurs with probability one. The expectation bound follows from standard branching-process theory. □

Definition 26 (Length distribution under random pruning) Let $L_p(x)$ denote the random set of factorization lengths obtained from $\mathscr{T}_p^{\ell}(x)$. The *expected length density* is

$$D_p(\ell) = \mathbb{E}\big[\#\{z \in L_p(x) : \ell(z) = \ell\}\big].$$

Proposition 6 (Expectation-entropy inequality) For any random pruning process satisfying the hypotheses of Theorem 5,

$$\mathbb{E}[H(x; p)] \leq \log \mathbb{E}[N_p(x)] \leq \log\left(\frac{1}{1-\rho}\right),$$

where $N_p(x)$ is the random number of surviving factorizations.

Proof. The first inequality is Jensen's inequality applied to the concave function log. The second follows from the expectation bound in Theorem 5. □

Example 10 (Uniform pruning) If $p(a, y) = p$ is constant, then $\rho = p|\mathscr{A}(S)|$. Termination is almost sure whenever $p < 1/|\mathscr{A}(S)|$, and the expected tree size equals $(1-\rho)^{-1}$. This gives a direct quantitative trade-off between pruning probability and computational complexity.

Remark 13 The random model provides a stochastic analogue of algorithmic pruning. Expected termination corresponds to the probabilistic version of prefix-ACCP, while the variance of leaf depths measures the fluctuation of length densities under uncertainty. Such models may lead to Monte-Carlo estimators of asymptotic length distributions in large or non-cancellative semigroups where deterministic enumeration is infeasible.

3.2.4 *Spectral analysis of pruning operators*

The pruning procedure, whether deterministic or random, can be formulated as an iterative action of a linear operator on the space of formal length distributions. This representation highlights the spectral characteristics that govern the asymptotic behavior and convergence rates of pruning algorithms. Related combinatorial structures, such as numerical semigroups generated by primes, have been studied in [19], providing insight into factorization patterns and asymptotic properties that are analogous to those encountered in pruning dynamics.

Definition 27 (Pruning operator) Let S be an atomic FF-monoid and let $\mathscr{A}(S)$ denote its set of atoms. Define the vector space

$$\mathscr{V} = \{f : \mathbb{N} \to \mathbb{R} \mid \|f\|_1 < \infty\},$$

whose elements encode length distributions. The *pruning operator* $P : \mathscr{V} \to \mathscr{V}$ acts by

$$(Pf)(\ell) = \sum_{a \in \mathscr{A}(S)} w(a) f(\ell - \lambda(a)),$$

where $w(a) \geq 0$ is a weighting (or retention) factor, and $\lambda(a)$ is the unit length contribution of atom a.

Lemma 4 (Boundedness) If $\sum_{a \in \mathscr{A}(S)} w(a) < \infty$, then P is a bounded linear operator on $(\mathscr{V}, \|\cdot\|_1)$ with $\|P\|_1 \leq \sum_{a \in \mathscr{A}(S)} w(a)$.

Definition 28 (Spectral radius and asymptotic rate) The *spectral radius* of P is

$$r(P) = \lim_{k \to \infty} \|P^k\|^{1/k}.$$

When $r(P) < 1$, the pruning dynamics are said to be *spectrally subcritical*.

Theorem 6 (Spectral convergence of length densities) Let $f_0 \in \mathscr{V}$ denote the initial (unpruned) length distribution, and let $f_{n+1} = Pf_n$. If $r(P) < 1$, then $f_n \to 0$ in $\|\cdot\|_1$ norm and the total surviving mass satisfies

$$\sum_{\ell \geq 0} f_n(\ell) = O(r(P)^n).$$

Proof. By the spectral radius formula, for every $\varepsilon > 0$ there exists N such that $\|P^n\| \leq (r(P) + \varepsilon)^n$ for all $n > N$. Hence $\|f_n\|_1 \leq \|P^n\| \, \|f_0\|_1$, giving the claimed asymptotic decay. □

Corollary 3 (Spectral bound for expected tree height) Under random pruning with mean weights $w(a) = \mathbb{E}[p(a, y)]$, the expected maximum depth $h_p(x)$ satisfies

$$\mathbb{E}[h_p(x)] \leq \frac{1}{1 - r(P)}.$$

Remark 14 The operator P encodes the cumulative effect of pruning decisions as a discrete linear dynamical system on length densities. Spectral subcriticality $r(P) < 1$ corresponds to convergence of the pruning process, whereas $r(P) = 1$ indicates critical balance between generation and deletion of branches. In the categorical setting, P may be viewed as an endofunctor on the module category of length functions, and its spectrum reflects morphic stability of factorization dynamics.

Example 11 (Uniform weights) For uniform weights $w(a) = p$ on $m = |\mathscr{A}(S)|$ atoms, P acts as convolution with the measure $p \sum_{i=1}^{m} \delta_{\lambda_i}$, and $r(P) = pm$. Thus convergence occurs precisely when $p < 1/m$, in agreement with the probabilistic criterion of subcritical pruning.

Remark 15 Spectral analysis connects pruning dynamics with linear semigroup theory: the iterates P^n form a discrete semigroup of contractions on $\mathscr{V}$. The eigenvalue $r(P)$ acts as the analog of the growth exponent in classical semigroup theory, bridging combinatorial and analytic perspectives on pruning.

3.2.5 *Topological compactification and continuum limits*

The discrete structure of pruning dynamics can be embedded in a continuous topological framework by considering compactifications of the length-density space. This transition allows asymptotic limits of the discrete pruning semigroup to be interpreted as continuous flows, connecting algebraic pruning to functional-analytic and topological semigroup theory.

Definition 29 (Length-density topology) Let $\mathscr{V} = \{f : \mathbb{N} \to \mathbb{R}_{\geq 0} \mid \|f\|_1 < \infty\}$ denote the Banach space of length densities endowed with the weak-$*$ topology inherited from $\ell^1(\mathbb{N})^* = \ell^\infty(\mathbb{N})$. A net (f_α) converges weak-$*$ to f if and only if $\sum_\ell f_\alpha(\ell)\varphi(\ell) \to \sum_\ell f(\ell)\varphi(\ell)$ for all bounded test functions φ.

Definition 30 (Compactification of the pruning semigroup) Let $\{P^n\}_{n \geq 0}$ be the discrete pruning semigroup acting on $\mathscr{V}$. The *weak-$*$ compactification* $\overline{\mathscr{P}}$ of this semigroup is the closure of $\{P^n\}$ in the operator topology induced by weak-$*$ convergence on $\mathscr{V}$.

Theorem 7 (Existence of continuum limit) Suppose P is a bounded pruning operator with spectral radius $r(P) < 1$. Then there exists a strongly continuous one-parameter semigroup $\{T(t)\}_{t \geq 0}$ on $\mathscr{V}$ satisfying

$$T(t) = \lim_{n \to \infty} P^{\lfloor tn \rfloor}, \quad t \geq 0,$$

where the limit is taken in the weak-$*$ topology.

Proof. Since $r(P) < 1$, the sequence P^n is uniformly bounded and Cauchy in the operator weak-$*$ topology. Standard compactness arguments for bounded operator semigroups on Banach spaces yield the existence of a weak-$*$ limit $T(t)$ satisfying the semigroup identity $T(t+s) = T(t)T(s)$ and strong continuity in t. □

Definition 31 (Infinitesimal generator) The infinitesimal generator A of the continuum pruning flow is defined by

$$Af = \lim_{t \downarrow 0} \frac{T(t)f - f}{t},$$

whenever the limit exists. The operator A governs the continuous-time evolution of length densities.

Proposition 7 (Continuum pruning equation) Let $f(t) = T(t)f_0$. Then $f(t)$ satisfies the differential equation

$$\frac{df}{dt} = Af, \qquad f(0) = f_0.$$

When $A = \log P$ (in the functional-calculus sense), this equation is the continuous limit of discrete pruning iterations $f_{n+1} = Pf_n$.

Remark 16 The compactification $\overline{\mathscr{P}}$ provides a topological-semigroup envelope for the discrete pruning dynamics, analogous to the Bohr or Ellis compactifications in abstract semigroup theory. Within this envelope, the asymptotic behaviour of pruning can be studied using the tools of semigroup compactification, ergodic theory, and operator continuation.

Example 12 (Diffusive continuum limit) Consider a random pruning process where the expected change in length per step is small. Scaling time as $t = n\varepsilon$ and taking $\varepsilon \to 0$, the limit equation for the expected length density $u(\ell, t)$ becomes

$$\frac{\partial u}{\partial t} = -\partial_\ell\big(v(\ell)u\big) + \tfrac{1}{2}\partial^2_{\ell\ell}\big(\sigma^2(\ell)u\big),$$

a Fokker-Planck-type diffusion representing continuous pruning flow in length space.

Remark 17 The continuum limit formalism connects combinatorial semigroup pruning to continuous evolution equations, enabling analytical study of convergence, diffusion of factorization lengths, and emergent steady states. It also opens a route toward topological dynamics and measure-theoretic invariants of semigroup factorization complexity.

3.3 *Asymptotic-analytic framework for pruning dynamics*

The preceding extensions collectively form an analytic framework that builds upon the combinatorial and algebraic results established earlier. While the initial sections addressed deterministic and structural bounds for length density, the six extensions above situate these findings within asymptotic, categorical, probabilistic, spectral, and topological contexts, thereby unifying discrete pruning behaviour with continuous analytic dynamics.

At the asymptotic level, the growth laws for normalized length density clarify how complexity in numerical semigroups scales with factorization depth, revealing transition points between sparse and dense regimes. The categorical representation recasts pruning as a functorial process, allowing morphic comparisons across semigroups and establishing an equivalence between structural and computational reductions. Entropy-based bounds then quantify the information content of factorization ensembles, linking algebraic complexity to probabilistic variability.

The probabilistic formulation introduces random pruning trees as stochastic analogues of deterministic pruning, leading naturally to spectral operators whose eigenvalues encode pruning efficiency and convergence rates. Finally, the topological compactification of pruning trees provides a continuum limit in which discrete pruning trajectories converge to compact flow spaces, establishing a geometric bridge between finite algorithmic steps and asymptotic behaviour.

Together, these results outline an *Asymptotic-Analytic Framework for Pruning Dynamics* in which combinatorial semigroup theory, categorical structure, entropy, probability, and spectral topology interact coherently. They extend the reach of length-density analysis beyond enumeration into a continuous and analytic regime, laying a foundation for further development of semigroup dynamics as a hybrid algebraic-analytic theory.

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Confilict of interest

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