Contemporary Mathematics
https://ojs.wiserpub.com/index.php/CM/

UNIVERSAL WISER PUBLISHER

Research Article

Cauchy Problem Involving Hybrid Proportional–Caputo Derivative in Banach Spaces

Abdelkader Moumen[1*], Hussien Albala[2], Oualid Zentar[3], Mohamed Ziane[3], Tayeb Mahrouz[3]

[1] Department of Mathematics, College of Science, University of Hail, Hail 55473, Saudi Arabia
[2] Department of Mathematics, Faculty of Sciences, King Khalid University, PO Box 9004 Abha, Saudi
[3] Department of Computer Science, Laboratory of Research in Artificial Intelligence and Systems (LRAIS), University of Ibn Khaldoun, Tiaret 14000, Algeria
E-mail: mo.abdelkader@uoh.edu.sa

Received: 17 November 2025; **Revised:** 28 January 2026; **Accepted:** 12 March 2026

Abstract: This paper studies a class of nonlinear Cauchy problems driven by hybrid proportional-Caputo fractional derivatives in Banach spaces. The nonlinear source term is assumed to satisfy combined Lipschitz and Carathéodory conditions. By employing Weissinger's fixed point theorem together with a fixed point principle for convex-power condensing operators, new quantitative results on existence and uniqueness are established. To illustrate the applicability of the theoretical findings, two examples involving infinite-dimensional fractional systems in spaces of tempered sequences are presented. Moreover, previously known results are recovered as special cases of the main theorems developed in this work.

***Keywords*:** hybrid proportional-Caputo derivatives, fixed point theorem, measure of noncompactness

MSC: 34A08, 26A33, 47H08

1. Introduction

The advent of Fractional Calculus (FC), first introduced more than three centuries ago, represents a key milestone in the transition from classical to modern analysis. Long regarded as a purely theoretical discipline, FC has since witnessed remarkable advances and now plays a significant role across a wide range of scientific and engineering fields [1, 2]. In recent years, research activity in FC has grown rapidly, leading to a substantial body of work addressing various classes of fractional differential equations and inclusions under diverse analytical frameworks and assumptions [3–7]. Related advances on existence, stability, and positive solutions for fractional models under nonlocal or integral boundary conditions can be found in [8–12].

The need to accurately model nonlocal phenomena has motivated the development of various fractional derivative operators, including the Riemann–Liouville, Caputo, and Hadamard definitions. A notable extension, known as the proportional derivative, was proposed in [13] as an alternative to these classical operators, offering enhanced modeling flexibility. This framework has demonstrated effectiveness in a wide range of applications, including building thermodynamics [14], magnetohydrodynamic flows [15], and nanofluid dynamics [16]. Building on these advances,

DOI: https://doi.org/10.37256/cm.7320269181

Abbas [17, 18] established existence, uniqueness, and Ulam–Hyers stability results for Cauchy problems involving hybrid proportional–Caputo derivatives via fixed point techniques. In the present work, we advance this direction by extending the analysis to forcing terms defined in general Banach spaces, thereby broadening the applicability of the existing theory.

On the other hand, measures of noncompactness serve as a fundamental tool in nonlinear functional analysis, with broad applications in the study of differential, integral, and functional integral equations, as well as in optimization theory. Within operator theory, they enable the identification of significant operator classes, such as those satisfying Darbo's condition or acting as condensing mappings, which generalize the notion of compact operators and play a crucial role in establishing existence results for equations in infinite-dimensional spaces. Therefore, numerous studies have been dedicated to various aspects of the nonlinear fractional Cauchy problem in Banach spaces, as evidenced by [3, 4, 19, 20].

Based on a thorough examination of the existing literature, a notable gap has been observed concerning the theoretical exploration of the hybrid proportional–Caputo fractional Cauchy problem. This recognition of an absence in current research highlights the need for additional scholarly investigation. In response to this need, our study employs the methodology of hybrid proportional fractional calculus to develop and prove new existence theorems, specifically for instances where the nonlinearity is defined by operators within infinite-dimensional Banach spaces.

Let $(\mathbb{T}, \|\cdot\|)$ be a real Banach space. In this paper, we consider the following problem:

$$\begin{cases} {}^{pc}\mathscr{D}_0^{\kappa}\mathfrak{r}(\vartheta) = \Omega(\vartheta, \mathfrak{r}(\vartheta)), \quad \vartheta \in \mathscr{L} := [0, b], \\ \mathfrak{r}(0) = \mathfrak{r}_0, \end{cases} \tag{1}$$

where ${}^{pc}\mathscr{D}_0^{\kappa}$ denotes the hybrid proportional-Caputo fractional derivative of order $0 < \kappa < 1$, $\Omega : \mathscr{L} \times \mathbb{T} \to \mathbb{T}$ is a function satisfying certain hypotheses to be specified later.

To the best of our knowledge, this work provides the first analysis of Cauchy problems driven by hybrid proportional–Caputo derivatives in infinite-dimensional Banach spaces. The main contributions of the paper are threefold.

(1) The unique solvability of the considered problem is established without requiring extra conditions.

(2) A new existence criterion is obtained under a Nagumo-type condition on the Carathéodory forcing term.

(3) An illustrative infinite system is analyzed within the framework of tempered sequence spaces to support the theoretical findings.

The paper is organized into five sections. Section 2 introduces the necessary background, notation, and preliminary results used throughout the analysis. In Section 3, we establish new existence results by applying Weissinger's fixed point theorem and a fixed point theorem for convex–power condensing operators. Two illustrative examples are then presented to highlight the applicability of the theoretical findings. The paper concludes with a discussion of possible directions for future research.

2. Preliminaries

Throughout this paper, we consider the space $C(\mathscr{L}, \mathbb{T})$ of continuous functionals $z : \mathscr{L} \to \mathbb{T}$ with the norm

$$\|z\| = \sup_{\vartheta \in \mathscr{L}} \|z(\vartheta)\|, \quad \forall z \in C(\mathscr{L}, \mathbb{T}).$$

Let $L^1(\mathscr{L}, \mathbb{T})$ denote the Banach space of Bochner integrable functions $z : \mathscr{L} \to \mathbb{T}$, equipped with the norm

$$\|z\|_{L^1(\mathscr{L}, \mathbb{T})} = \int_0^b \|z(\vartheta)\| \, d\vartheta, \quad \forall z \in L^1(\mathscr{L}, \mathbb{T}).$$

Furthermore, $L^{\infty}(\mathscr{L}, \mathbb{R}_{+})$ denotes the space of all essentially bounded functions, normed by

$$\|z\|_{L^{\infty}} = \operatorname*{ess\,sup}_{\vartheta\in\mathscr{L}} \|z(\vartheta)\| = \inf\{M > 0 : \|z(\vartheta)\| \le M \text{ for almost every } \vartheta \in \mathscr{L}\}.$$

Definition 1 [21] The Mittag-Leffler (ML) function $\mathbb{E}_{\kappa}(\cdot)$ is defined by

$$\mathbb{E}_{\kappa}(z) = \sum_{k=0}^{\infty} \frac{z^n}{\Gamma(k\kappa+1)}, \quad (\kappa > 0,\ z \in \mathbb{R}),$$

where $\Gamma(\cdot)$ is the gamma function .

Definition 2 [13] The Hybrid Proportional-Caputo Fractional Derivative (HPCFD) of order $\kappa \in (0,\ 1)$ of a differentiable function $f(\vartheta)$ can be defined in one of two possible ways. The following general way:

$$\begin{aligned} {}^{pc}\mathscr{D}_0^{\kappa} f(\vartheta) &= \frac{1}{\Gamma(1-\alpha)} \int_0^{\vartheta} \left(\eta_1(\kappa,\ \tau) f(\vartheta) + \eta_0(\eta,\ \tau) f'(\vartheta)\right)(\vartheta-\tau)^{-\kappa} d\tau \\ &= {}^{RL}\mathscr{I}_0^{1-\kappa}\left(\eta_1(\kappa,\ \tau) f(\vartheta) + \eta_0(\kappa,\ \tau) f'(\vartheta)\right), \end{aligned} \tag{2}$$

where ${}^{RL}\mathscr{I}_0^{1-\kappa}$ is the Riemann-Liouville fractional integral difined by for $1-\kappa > 0$,

$${}^{RL}\mathscr{I}_0^{1-\kappa} f(\vartheta) = \frac{1}{\Gamma(1-\kappa)} \int_0^{\vartheta} (\vartheta-\tau)^{-\kappa} f(\tau) d\tau,$$

and $\eta_0,\ \eta_1 : [0,\ 1] \times \mathbb{R} \to [0,\ \infty)$ are continuous with respect to the variable ϑ and the parameter $\kappa \in [0,\ 1]$, and satisfy the following conditions for all $\vartheta \in \mathbb{R}$.

$$\lim_{\kappa\to 0^+} \eta_0(\kappa,\ \vartheta) = 0, \quad \lim_{\kappa\to 1^-} \eta_0(\kappa,\ \vartheta) = 1, \quad \eta_0(\kappa,\ \vartheta) \neq 0, \quad \kappa \in (0,\ 1],$$

$$\lim_{\kappa\to 0^+} \eta_1(\kappa,\ \vartheta) = 1, \quad \lim_{\kappa\to 1^-} \eta_1(\kappa,\ \vartheta) = 0, \quad \eta_1(\kappa,\ \vartheta) \neq 0, \quad \kappa \in [0,\ 1).$$

Definition 3 [13] The inverse operator of the fractional proportional–Caputo derivative (2) is given by

$${}^{pc}\mathscr{I}_0^{\kappa} f(\vartheta) = \int_0^{\vartheta} \exp\left(-\int_y^{\vartheta} \frac{\eta_1(\kappa,\ s)}{\eta_0(\kappa,\ s)} ds\right) \frac{{}^{RL}\mathscr{D}_y^{1-\kappa} f(y)}{\eta_0(\kappa,\ y)} dy$$

where ${}^{RL}\mathscr{D}_y^{1-\kappa}$ stands for the Riemann–Liouville fractional derivative of order $1-\kappa$, which is defined as

$${}^{RL}\mathscr{D}_y^{1-\kappa} f(y) = \frac{1}{\Gamma(\kappa)} \frac{d}{dy} \int_0^{y} (y-\tau)^{\kappa-1} f(\tau) d\tau$$

Definition 4 [22] Let $\mathbb{F} \subset \mathbb{T}$ denote a bounded set. The Hausdorff measure of noncompactness of $\mathbb{F}$ is given by

$$\Upsilon(\mathbb{F}) = \inf\{\varepsilon > 0 : \mathbb{F} \text{ has a finite } \varepsilon - \text{net in } \mathbb{T}\}.$$

Lemma 1 [22] Let $\mathbb{F}$, $\mathbb{G} \subset \mathbb{T}$ be bounded. Then Υ satisfies
(1) $\Upsilon(\mathbb{F}) = 0 \Longleftrightarrow \mathbb{F}$ is relatively compact.
(2) $\mathbb{F} \subset \mathbb{G} \Longrightarrow \Upsilon(\mathbb{F}) \leq \Upsilon(\mathbb{G})$.
(3) $\Upsilon(\mathbb{F} \cup \mathbb{G}) = \max\{\Upsilon(\mathbb{F}), \Upsilon(\mathbb{G})\}$.
(4) $\Upsilon(\mathbb{F}) = \Upsilon(\overline{\mathbb{F}}) = \Upsilon(\text{conv}(\mathbb{F}))$. Here, conv $\mathbb{F}$ and $\overline{\mathbb{F}}$refer to the convex hull and the closure of $\mathbb{F}$, respectively.
(5) $\Upsilon(\mathbb{F} + \mathbb{G}) \leq \Upsilon(\mathbb{F}) + \Upsilon(\mathbb{G})$.
(6) $\Upsilon(\lambda\mathbb{F}) \leq |\lambda|\Upsilon(\mathbb{F})$, for any $\lambda \in \mathbb{R}$.
Lemma 2 [20] Assume that $\mathbb{F} \subset \mathbb{T}$ is bounded. Then, for each $\varepsilon > 0$, one can find a sequence $\{z_n\}_{n=1}^{\infty} \subset \mathbb{F}$, such that

$$\Upsilon(\mathbb{F}) \leq 2\Upsilon(\{z_n\}_{n=1}^{\infty}) + \varepsilon.$$

A set $\mathbb{F} \subset L^1(\mathscr{L}, \mathbb{T})$ is termed uniformly integrable if, for all $\rho \in \mathbb{F}$, the condition below is fulfilled:

$$\|\rho(\vartheta)\| \leq \varsigma(\vartheta), \quad \text{a.e. } \vartheta \in \mathscr{L},$$

with $\varsigma \in L^1(\mathscr{L}, \mathbb{R}^+)$.

Lemma 3 [23] Assume that $\{\rho_n\}_{n=1}^{\infty}$ involve $L^1(\mathscr{L}, \mathbb{T})$ is uniformly integrable, the map $\vartheta \longmapsto \Upsilon(\{\rho_n(\vartheta)\}_{n=1}^{\infty})$ is measurable, furthermore

$$\Upsilon\left(\left\{\int_a^{\vartheta} \rho_n(s)ds\right\}_{n=1}^{\infty}\right) \leq 2\int_a^{\vartheta} \Upsilon(\{\rho_n(s)\}_{n=1}^{\infty})\, ds.$$

We now recall, respectively, Weissinger's fixed point theorem and the theorem on Convex–Power Condensing (CPC) operators, which will be used in the sequel.

Theorem 1 ([24]) Let $(\mathbb{X}, \|\cdot\|)$ be a Banach space and $\Lambda_n \geq 0$ for every $n \in \mathbb{N}$ with $\sum_{n=0}^{\infty} \Lambda_n$ converges. If the operator $\mathscr{Z} : \mathbb{X} \longrightarrow \mathbb{X}$ satisfies

$$d(\mathscr{Z}^n u, \mathscr{Z}^n v) \leq \Lambda_n d(u, v), \quad u, v \in \mathbb{X},$$

for each $n \in \mathbb{N}$. It follows that $\mathscr{Z}$ possesses a unique fixed point u^*. Furthermore, for any $v_0 \in \mathbb{X}$, the sequence $\{\mathscr{Z}^n v_0\}_{n=1}^{\infty}$ converges to u^*.

Definition 5 Let $\mathbb{G}$ be a closed and convex set of a Banach space $\mathbb{T}$. An operator $\mathscr{Z} : \mathbb{G} \to \mathbb{G}$ said to be a Convex-Power Condensing (CPC) operator with respect to v_0 and n_0 if it is bounded and continuous, and if there exist $v_0 \in \mathbb{G}$ and $n_0 \in \mathbb{N}^*$ such that for every bounded, non-relatively compact subset $G \subset \mathbb{G}$, with

$$\Upsilon\left(\mathscr{Z}^{(n_0, v_0)}(G)\right) < \Upsilon(G),$$

here

$$\mathscr{Z}^{(1,\, v_0)}(G) \equiv \mathscr{Z}(G), \quad \mathscr{Z}^{(n,\, v_0)}(G) = \mathscr{Z}\left(\overline{co}\left\{\mathscr{Z}^{(n-1,\, v_0)}(G)\right\}\right), \quad n = 2,\, 3,\, \cdots$$

Theorem 2 [25] Consider a Banach space $\mathbb{T}$ and a bounded, closed, convex subset $G \subset \mathbb{T}$. If $\mathscr{Z} : G \to G$ is a Convex-Power Condensing (CPC) operator, then there exists at least one fixed point $\mathscr{Z}$ in G.

3. Main results

In this section, we investigate the existence of solutions for the problem (1).
Theorem 3 Let's impose the assumptions:
(H1) $\Omega : \mathscr{L} \times \mathbb{T} \to \mathbb{T}$ is a continuous function.
(H2) There exists a positive constant $\varpi > 0$ satisfying

$$\|\Omega(\vartheta,\, x_1) - \Omega(\vartheta,\, x_2)\| \leq \varpi \|x_1 - x_2\|, \quad \text{for any } x_1,\, x_2 \in \mathbb{T} \text{ and } \vartheta \in \mathscr{L}. \tag{3}$$

There exists a unique solution to problem (1) on $\mathscr{L}$.
Proof. According to [17, Lemma 4.1], let us introduce $\mathscr{Z} : C(\mathscr{L},\, \mathbb{T}) \to C(\mathscr{L},\, \mathbb{T})$ given by

$$\begin{aligned} \mathscr{Z}\mathfrak{r}(\vartheta) = {} & \exp\left(-\int_0^{\vartheta} \frac{\eta_1(\kappa,\, s)}{\eta_0(\kappa,\, s)} ds\right) \mathfrak{r}_0 + \frac{1}{\Gamma(\kappa - 1)} \int_0^{\vartheta} \int_0^{y} \exp\left(-\int_y^{\vartheta} \frac{\eta_1(\kappa,\, s)}{\eta_0(\kappa,\, s)} ds\right) \\ & \times \frac{(y-\tau)^{\kappa-2}}{\eta_0(\kappa,\, y)} \Omega(\tau,\, \mathfrak{r}(\tau)) d\tau dy, \quad \vartheta \in \mathscr{L}. \end{aligned} \tag{4}$$

It is clear that solutions of problem (1) correspond to fixed points of the operator $\mathscr{Z}$. By invoking Theorem 1 and introducing a suitably chosen equivalent norm, We show that $\mathscr{Z}$ possesses a unique fixed point.

Consider $\mathfrak{r},\, \mathfrak{s} \in C(\mathscr{L},\, \mathbb{T})$. Hence, for each $\vartheta \in \mathscr{L}$ for every $n \in \mathbb{N}$, given that

$$0 < \exp\left(-\int_0^{\vartheta} \frac{\eta_1(\kappa,\, s)}{\eta_0(\kappa,\, s)} ds\right) < 1,$$

we have

$$\begin{aligned} \|\mathscr{Z}\mathfrak{r}(\vartheta) - \mathscr{Z}\mathfrak{s}(\vartheta)\| & = \left\| \frac{1}{\Gamma(\kappa-1)} \int_0^{\vartheta} \int_0^{y} \exp\left(-\int_y^{\vartheta} \frac{\eta_1(\kappa,\, s)}{\eta_0(\kappa,\, s)} ds\right) \frac{(y-\tau)^{\kappa-2}}{\eta_0(\kappa,\, y)} (\Omega(\tau,\, \mathfrak{r}(\tau)) - \Omega(\tau,\, \mathfrak{s}(\tau))) d\tau dy \right\| \\ & \leq \frac{1}{\Gamma(\kappa-1)} \int_0^{\vartheta} \int_0^{y} \frac{(y-\tau)^{\kappa-2}}{|\eta_0(\kappa,\, y)|} \|\Omega(\tau,\, \mathfrak{r}(\tau)) - \Omega(\tau,\, \mathfrak{s}(\tau))\| d\tau dy. \end{aligned}$$

Using (H2), one gets

$$\|\mathscr{Z}\mathfrak{r}(\vartheta)-\mathscr{Z}\mathfrak{s}(\vartheta)\| \leq \frac{\varpi N_\eta}{\Gamma(\kappa-1)}\|\mathfrak{r}-\mathfrak{s}\|\int_0^\vartheta\int_0^y (y-\tau)^{\kappa-2}d\tau dy \leq \frac{\varpi N_\eta \vartheta^\kappa}{\Gamma(\kappa+1)}\|\mathfrak{r}-\mathfrak{s}\|, \tag{5}$$

where $N_\eta = \sup_{\vartheta\in\mathscr{L}} \frac{1}{\eta_0(\kappa,\,\vartheta)}$. Again, by (H2) and using (5), we obtain

$$\begin{aligned}
\|\mathscr{Z}^2\mathfrak{r}(\vartheta)-\mathscr{Z}^2\mathfrak{s}(\vartheta)\| &\leq \|\mathscr{Z}(\mathscr{Z}\mathfrak{r}(\vartheta))-\mathscr{Z}(\mathscr{Z}\mathfrak{s}(\vartheta))\| \\
&\leq \frac{1}{\Gamma(\kappa-1)}\int_0^\vartheta\int_0^y \frac{(y-\tau)^{\kappa-2}}{|\eta_0(\kappa,\,y)|}\|\Omega(\tau,\,\mathscr{Z}\mathfrak{r}(\tau))-\Omega(\tau,\,\mathscr{Z}\mathfrak{s}(\tau))\|d\tau dy \\
&\leq \frac{\varpi}{\Gamma(\kappa-1)}\int_0^\vartheta\int_0^y \frac{(y-\tau)^{\kappa-2}}{|\eta_0(\kappa,\,y)|}\|\mathscr{Z}\mathfrak{r}(\tau)-\mathscr{Z}\mathfrak{s}(\tau)\|d\tau dy \\
&\leq \frac{\varpi^2 N_\eta^2}{\Gamma(\kappa+1)\Gamma(\kappa-1)}\|\mathfrak{r}-\mathfrak{s}\|\int_0^\vartheta\int_0^y (y-\tau)^{\kappa-2}\tau^\kappa d\tau dy.
\end{aligned}$$

It is easy to see that

$$\int_0^y (y-\tau)^{\kappa-2}\tau^\kappa d\tau = \frac{\Gamma(\kappa-1)\Gamma(\kappa+1)}{\Gamma(2\kappa)}y^{2\kappa-1}. \tag{6}$$

Hence

$$\|\mathscr{Z}^2\mathfrak{r}(\vartheta)-\mathscr{Z}^2\mathfrak{s}(\vartheta)\| \leq \frac{\varpi^2 N_\eta^2}{\Gamma(2\kappa)}\|\mathfrak{r}-\mathfrak{s}\|\int_0^\vartheta y^{2\kappa-1}dy \leq \frac{\varpi^2 N_\eta^2 \vartheta^{2\kappa}}{\Gamma(2\kappa+1)}\|\mathfrak{r}-\mathfrak{s}\|.$$

Extending the process to $n = 3,\,4,\,\cdots$, for any $\vartheta\in\mathscr{L}$, it remains to establish that

$$\|\mathscr{Z}^n\mathfrak{r}(\vartheta)-\mathscr{Z}^n\mathfrak{s}(\vartheta)\| \leq \frac{(\varpi N_\eta \vartheta^\kappa)^n}{\Gamma(n\kappa+1)}\|\mathfrak{r}-\mathfrak{s}\|. \tag{7}$$

Assuming, by induction, that (7) holds for a particular n, we proceed to establish it for $n+1$.
We have

$$\begin{aligned}
\|\mathscr{Z}^{n+1}\mathfrak{r}(\vartheta)-\mathscr{Z}^{n+1}\mathfrak{s}(\vartheta)\| &\leq \|\mathscr{Z}(\mathscr{Z}^n\mathfrak{r}(\vartheta))-\mathscr{Z}(\mathscr{Z}^n\mathfrak{s}(\vartheta))\| \\
&\leq \frac{1}{\Gamma(\kappa-1)}\int_0^\vartheta\int_0^y \frac{(y-\tau)^{\kappa-2}}{|\eta_0(\kappa,\,y)|}\|\Omega(\tau,\,\mathscr{Z}\mathfrak{r}^n(\tau))-\Omega(\tau,\,\mathscr{Z}^n\mathfrak{s}(\tau))\|d\tau dy
\end{aligned}$$

$$\leq \frac{\varpi}{\Gamma(\kappa-1)} \int_0^{\vartheta} \int_0^{y} \frac{(y-\tau)^{\kappa-2}}{|\eta_0(\kappa, y)|} \|\mathscr{X}^n \mathfrak{r}(\tau) - \mathscr{X}^n \mathfrak{s}(\tau)\| \, d\tau dy$$

$$\leq \frac{(\varpi N_\eta \vartheta^\kappa)^{n+1}}{\Gamma((n+1)\kappa+1)} \|\mathfrak{r}-\mathfrak{s}\|$$

Hence the inequality (7) holds.

Consequently, for every $n \in \mathbb{N}$, it follows that

$$\|\mathscr{X}^n \mathfrak{r} - \mathscr{X}^n \mathfrak{s}\| \leq \frac{(\varpi N_\eta b^\kappa)^n}{\Gamma(n\kappa+1)} \|\mathfrak{r}-\mathfrak{s}\|, \quad \mathfrak{r}, \mathfrak{s} \in C(\mathscr{L}, \mathbb{T}).$$

By putting

$$\Lambda_n = \frac{(\varpi N_\eta b^\kappa)^n}{\Gamma(n\kappa+1)}, \tag{8}$$

we observe that

$$\sum_{n=0}^{\infty} \Lambda_n = \sum_{n=0}^{\infty} \frac{(\varpi N_\eta b^\kappa)^n}{\Gamma(n\kappa+1)} = \mathbb{E}_\kappa \left(\varpi N_\eta b^\kappa\right).$$

Finally, Theorem 1 guarantees that $\mathscr{X}$ has a unique fixed point, which is precisely the unique global solution of problem (1). □

We next prove another existence result, based on Theorem 2.

Theorem 4 We suppose:

(H3) $\Omega : \mathscr{L} \times \mathbb{T} \to \mathbb{T}$ is Carathéodory type function.

(H4) There exists a continuous nondecreasing function $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ and $\Psi \in L^\infty(\mathscr{L}, \mathbb{R}_+)$ such that

$$\|\Omega(\vartheta, u)\| \leq \Psi(\vartheta)\Phi(\|u\|), \quad u \in \mathbb{T}$$

for all $\vartheta \in \mathscr{L}$.

(H5) There exists constant $\mho > 0$ such that for each $\vartheta \in \mathscr{L}$, we have

$$\Upsilon(\Omega(\vartheta, \mathbb{A}(\vartheta))) \leq \mho\Upsilon(\mathbb{A}(\vartheta)),$$

where $\mathbb{A} \subset \mathbb{T}$ is a bounded and countable set.

Then the Equation (1) admits at least one solution.

Proof. Let us again consider the operator $\mathscr{X}$ represented by (4) and define the ball

$$\mathbb{B}_T = \{\mathfrak{r} \in C(\mathscr{L}, \mathbb{T}) : \|\mathfrak{r}\| \leq T\},$$

with

$$\|\mathfrak{r}_0\| + \frac{N_\eta \|\Psi\|_{L^\infty}\Phi(T)}{\Gamma(\kappa+1)} b^\kappa \leq T. \tag{9}$$

Step 1. We show that $\mathscr{Z}\mathbb{B}_T \subset \mathbb{B}_T$.
By (H4), for each $\mathfrak{r} \in \mathbb{B}_T$ and $\vartheta \in \mathscr{L}$, we have

$$\begin{aligned}
\|\mathscr{Z}\mathfrak{r}(\vartheta)\| \leq & \left\| \exp\left(-\int_0^\vartheta \frac{\eta_1(\kappa, s)}{\eta_0(\kappa, s)} ds\right) \mathfrak{r}_0 \right\| + \frac{1}{\Gamma(\kappa-1)} \int_0^\vartheta \int_0^y \exp\left(-\int_y^\vartheta \frac{\eta_1(\kappa, s)}{\eta_0(\kappa, s)} ds\right) \\
& \times \frac{(y-\tau)^{\kappa-2}}{|\eta_0(\kappa, y)|} \|\Omega(\tau, \mathfrak{r}(\tau))\| d\tau dy \\
\leq & \|\mathfrak{r}_0\| + \frac{1}{\Gamma(\kappa-1)} \int_0^\vartheta \int_0^y \frac{(y-\tau)^{\kappa-2}}{|\eta_0(\kappa, y)|} \|\Omega(\tau, \mathfrak{r}(\tau))\| d\tau dy \\
\leq & \|\mathfrak{r}_0\| + \frac{N_\eta \|\Psi\|_{L^\infty}\Phi(T)}{\Gamma(\kappa-1)} \int_0^\vartheta \int_0^y (y-\tau)^{\kappa-2} d\tau dy \\
\leq & \|\mathfrak{r}_0\| + \frac{N_\eta \|\Psi\|_{L^\infty}\Phi(T)}{\Gamma(\kappa+1)} \vartheta^\kappa.
\end{aligned}$$

Using (9), one gets

$$\|\mathscr{Z}\mathfrak{r}(\vartheta)\| \leq \|\mathfrak{r}_0\| + \frac{N_\eta \|\Psi\|_{L^\infty}\Phi(T)}{\Gamma(\kappa+1)} b^\kappa \leq T.$$

Thus

$$\|\mathscr{Z}\mathfrak{r}\| \leq T. \tag{10}$$

This shows that $\mathscr{Z}$ maps $\mathbb{B}_T$ into itself.
Step 2. We show the continuity of $\mathscr{Z}$.
For this, let $\{\mathfrak{r}_n\}$ be a sequence satisfying $\mathfrak{r}_n \to \mathfrak{r}$ in $\mathbb{B}_T$ as $n \to \infty$, since Ω is of Carathéodory, we have

$$\Omega(\cdot, \mathfrak{r}_n(\cdot)) \xrightarrow[n\to\infty]{} \Omega(\cdot, \mathfrak{r}(\cdot)).$$

For each $\vartheta \in \mathscr{L}$, in view of (H4), we get

$$\frac{(y-\tau)^{\kappa-2}}{\Gamma(\kappa-1)|\eta_0(\kappa,\, y)|}\|\Omega(\tau,\, \mathfrak{r}_n(\tau))-\Omega(\tau,\, \mathfrak{r}(\tau))\| \leq \frac{2\Phi(T)\Psi(\tau)}{\Gamma(\kappa-1)|\eta_0(\kappa,\, y)|}(y-\tau)^{\kappa-2}.$$

In conjunction with the Lebesgue dominated convergence theorem and the observation that the function

$$\tau \to \frac{2\Phi(T)\Psi(\tau)}{\Gamma(\kappa-1)|\eta_0(\kappa,\, y)|}(y-\tau)^{\kappa-2},$$

is Lebesgue integrable on $\mathscr{L}$

$$\|\mathscr{Z}(\mathfrak{r}_n(\vartheta))-\mathscr{Z}(\mathfrak{r}(\vartheta))\| \leq \frac{1}{\Gamma(\kappa-1)}\int_0^{\vartheta}\int_0^{y}\exp\left(-\int_y^{\vartheta}\frac{\eta_1(\kappa,\, s)}{\eta_0(\kappa,\, s)}ds\right)\times\frac{(y-\tau)^{\kappa-2}}{|\eta_0(\kappa,\, y)|}\|\Omega(\tau,\, \mathfrak{r}_n(\tau))-\Omega(\tau,\, \mathfrak{r}(\tau))\|\,d\tau dy$$

$$\leq \frac{1}{\Gamma(\kappa-1)}\int_0^{\vartheta}\int_0^{y}\frac{(y-\tau)^{\kappa-2}}{|\eta_0(\kappa,\, y)|}\|\Omega(\tau,\, \mathfrak{r}_n(\tau))-\Omega(\tau,\, \mathfrak{r}(\tau))\|\,d\tau dy.$$

Therefore

$$\|\mathscr{Z}(\mathfrak{r}_n)-\mathscr{Z}(\mathfrak{r})\| \to 0 \quad \text{as} \quad n\to\infty.$$

Step 3. We show that $\mathscr{Z}(\mathbb{B}_T)$ is equicontinuous.
On one hand, for all $\mathfrak{r}\in\mathbb{B}_T$, and $0<\vartheta_1<\vartheta_2<b$, we get

$$\|\mathscr{Z}(\mathfrak{r}(\vartheta_2))-\mathscr{Z}(\mathfrak{r}(\vartheta_1))\| \leq S_0+S_1+S_2, \tag{11}$$

where

$$S_0=\left\|\exp\left(-\int_0^{\vartheta_2}\frac{\eta_1(\kappa,\, s)}{\eta_0(\kappa,\, s)}ds\right)\mathfrak{r}_0-\exp\left(-\int_0^{\vartheta_1}\frac{\eta_1(\kappa,\, s)}{\eta_0(\kappa,\, s)}ds\right)\mathfrak{r}_0\right\|,$$

$$S_1=\frac{1}{\Gamma(\kappa-1)}\left\|\int_{\vartheta_1}^{\vartheta_2}\int_0^{y}\exp\left(-\int_y^{\vartheta_2}\frac{\eta_1(\kappa,\, s)}{\eta_0(\kappa,\, s)}ds\right)\frac{(y-\tau)^{\kappa-2}}{\eta_0(\kappa,\, y)}\Omega(\tau,\, \mathfrak{r}(\tau))d\tau dy\right\|,$$

and

$$S_2=\frac{1}{\Gamma(\kappa-1)}\left\|\int_0^{\vartheta_1}\int_0^{y}\left[\exp\left(-\int_y^{\vartheta_2}\frac{\eta_1(\kappa,\, s)}{\eta_0(\kappa,\, s)}ds\right)-\exp\left(-\int_y^{\vartheta_1}\frac{\eta_1(\kappa,\, s)}{\eta_0(\kappa,\, s)}ds\right)\right]\times\frac{(y-\tau)^{\kappa-2}}{\eta_0(\kappa,\, y)}\Omega(\tau,\, \mathfrak{r}(\tau))d\tau dy\right\|.$$

Now, let $f(\vartheta)=\exp\left(-\int_0^{\vartheta}\frac{\eta_1(\kappa,\, s)}{\eta_0(\kappa,\, s)}ds\right)$. By Lagrange Mean value theorem, there exist $\varsigma\in[\vartheta_1,\, \vartheta_2]\subset\mathscr{L}$, such that

$$\frac{f(\vartheta_2)-f(\vartheta_1)}{\vartheta_2-\vartheta_1} \leq f'(\varsigma).$$

Thus

$$|f(\vartheta_2)-f(\vartheta_1)| \leq \left| \frac{\eta_1(\kappa,\, \varsigma)}{\eta_0(\kappa,\, \varsigma)} \exp\left(-\int_0^{\varsigma} \frac{\eta_1(\kappa,\, s)}{\eta_0(\kappa,\, s)} ds \right) (\vartheta_2-\vartheta_1) \right|. \tag{12}$$

On the other hand, from (12) and (H4), one obtains

$$S_0 \leq \left\| \frac{\eta_1(\kappa,\, \varsigma)}{\eta_0(\kappa,\, \varsigma)} \mathfrak{r}_0 \right\| (\vartheta_2-\vartheta_1) \tag{13}$$

and

$$S_1 \leq \frac{N_\eta}{\Gamma(\kappa-1)} \int_{\vartheta_1}^{\vartheta_2} \int_0^y (y-\tau)^{\kappa-2} \left\| \Omega(\tau,\, \mathfrak{r}(\tau)) \right\| d\tau dy$$

$$\leq \frac{N_\eta \|\Psi\|_{L^\infty} \Phi(T)}{\Gamma(\kappa-1)} \int_{\vartheta_1}^{\vartheta_2} \int_0^y (y-\tau)^{\kappa-2} d\tau dy \tag{14}$$

$$\leq \frac{N_\eta \|\Psi\|_{L^\infty} \Phi(T)}{\Gamma(\kappa+1)} \left(\vartheta_2^{\kappa} - \vartheta_1^{\kappa} \right).$$

On the other side,

$$S_2 \leq \frac{1}{\Gamma(\kappa-1)} \int_0^{\vartheta_1} \int_0^y \left| \frac{\eta_1(\kappa,\, \varsigma)}{\eta_0(\kappa,\, \varsigma)} \exp\left(-\int_y^{\varsigma} \frac{\eta_1(\kappa,\, s)}{\eta_0(\kappa,\, s)} ds \right) (\vartheta_2-\vartheta_1) \right|$$

$$\times \frac{(y-\tau)^{\kappa-2}}{\eta_0(\kappa,\, y)} \|\Omega(\tau,\, \mathfrak{r}(\tau))\| d\tau dy \tag{15}$$

$$\leq \frac{N_\eta \|\Psi\|_{L^\infty} \Phi(T)}{\Gamma(\kappa+1)} \left| \frac{\eta_1(\kappa,\, \varsigma)}{\eta_0(\kappa,\, \varsigma)} \right| \vartheta_1^{\kappa} (\vartheta_2-\vartheta_1).$$

Equations (13)-(15) along with (11) imply

$$\|\mathscr{Z}\mathfrak{r}(\vartheta_2)-\mathscr{Z}\mathfrak{r}(\vartheta_1)\| \xrightarrow[\vartheta_2\to\vartheta_1]{} 0.$$

This proves that, $\mathscr{Z}(\mathbb{B}_T)$ is equicontinuous.

Step 4. We show that $\mathscr{Z}: G\to G$ is a CPC operator, where $G=\overline{\text{co}}\mathscr{Z}(\mathbb{B}_T)$.

Let $\mathfrak{r}_0\in G$. It remains to demonstrate that $\mathscr{Z}$ fulfills Definition 5. In order to do so, for each bounded subset $\mathbb{A}\subset C(\mathscr{L},\mathbb{T})$, we define the Measure of Noncompactness (MNC) by

$$\Upsilon_C\left(\mathscr{Z}^{(n,\,\mathfrak{r}_0)}(\mathbb{A})\right)=\sup_{\vartheta\in\mathscr{L}}\Upsilon\left(\mathscr{Z}^{(n,\,\mathfrak{r}_0)}(\mathbb{A})(\vartheta)\right),\quad n\in\mathbb{N}^*. \tag{16}$$

Next, fix $\varepsilon>0$. Lemma 2 yields the existence of $\{\mathfrak{r}_k\}_{k=1}^{\infty}\subset\mathbb{A}$ such that

$$\Upsilon\left(\mathscr{Z}^{(1,\,\mathfrak{r}_0)}(\mathbb{A})(\vartheta)\right)=\Upsilon\left(\mathscr{Z}(\mathbb{A})(\vartheta)\right)$$

$$\leq 2\Upsilon\left\{\int_0^{\vartheta}\int_0^{y}\frac{\exp\left(-\int_y^{\vartheta}\frac{\eta_1(\kappa,\,s)}{\eta_0(\kappa,\,s)}ds\right)}{\Gamma(\kappa-1)}\frac{(y-\tau)^{\kappa-2}}{\eta_0(\kappa,\,y)}\Omega(\tau,\,\{\mathfrak{r}_k(\tau)\}_{k=1}^{\infty})d\tau dy\right\}+\varepsilon.$$

Lemma 3 and the hypothesis (H5) imply that

$$\begin{aligned}\Upsilon\left(\mathscr{Z}^{(1,\,\mathfrak{r}_0)}(\mathbb{A})(\vartheta)\right)&\leq\frac{8\mho N_\eta}{\Gamma(\kappa-1)}\int_0^{\vartheta}\int_0^{y}(y-\tau)^{\kappa-2}\Upsilon\left(\{\mathfrak{r}_k(\tau)\}_{k=1}^{\infty}\right)ds+\varepsilon\\&\leq\frac{8\mho N_\eta\Upsilon(\mathbb{A})}{\Gamma(\kappa-1)}\int_0^{\vartheta}\int_0^{y}(y-\tau)^{\kappa-2}d\tau dy+\varepsilon\\&\leq\frac{8\mho N_\eta\Upsilon(\mathbb{A})}{\Gamma(\kappa+1)}\vartheta^{\kappa}+\varepsilon.\end{aligned}$$

Given that $\varepsilon>0$ was chosen arbitrarily, we conclude that

$$\Upsilon\left(\mathscr{Z}^{(1,\,\mathfrak{r}_0)}(\mathbb{A})(\vartheta)\right)\leq\frac{8\mho N_\eta\Upsilon(\mathbb{A})}{\Gamma(\kappa+1)}\vartheta^{\kappa}. \tag{17}$$

By applying Lemma 2 once more, let $\varepsilon>0$ be fixed. Then, there exists a sequence $\{\mathfrak{z}_k\}_{k=1}^{\infty}\subset\overline{\text{co}}\left\{\mathscr{Z}^{(1,\,\mathfrak{r}_0)}(\mathbb{A}),\,\mathfrak{r}_0\right\}$ such that

$$\Upsilon\left(\mathscr{Z}^{(2,\,\mathfrak{r}_0)}(\mathbb{A})(\vartheta)\right)=\Upsilon\left(\mathscr{Z}\left(\overline{\mathrm{co}}\left\{\mathscr{Z}^{(1,\,\mathfrak{r}_0)}(\mathbb{A}),\,\mathfrak{r}_0\right\}\right)(\vartheta)\right)$$

$$\leq 2\Upsilon\left\{\int_0^{\vartheta}\int_0^{y}\frac{\exp\left(-\int_y^{\vartheta}\frac{\eta_1(\kappa,\,s)}{\eta_0(\kappa,\,s)}ds\right)}{\Gamma(\kappa-1)}\frac{(y-\tau)^{\kappa-2}}{\eta_0(\kappa,\,y)}\Omega(\tau,\,\{\mathfrak{z}_k(\tau)\}_{k=1}^{\infty})d\tau dy\right\}+\varepsilon.$$

Another recalling of Lemma 3 and (H5), it yields

$$\Upsilon\left(\mathscr{Z}^{(2,\,\mathfrak{r}_0)}(\mathbb{A})(\vartheta)\right)\leq\frac{8N_\eta\mho}{\Gamma(\kappa-1)}\int_0^{\vartheta}\int_0^{y}(y-\tau)^{\kappa-2}\Upsilon(\{\mathfrak{z}_k(\tau)\}_{k=1}^{\infty})\,d\tau dy+\varepsilon$$

$$\leq\frac{8N_\eta\mho}{\Gamma(\kappa-1)}\int_0^{\vartheta}\int_0^{y}(y-\tau)^{\kappa-2}\Upsilon\left\{\overline{\mathrm{co}}\left\{\mathscr{Z}^{(1,\,\mathfrak{r}_0)}(\mathbb{A})(\tau),\,\mathfrak{r}_0\right\}\right\}d\tau dy+\varepsilon$$

$$\leq\frac{8N_\eta\mho}{\Gamma(\kappa-1)}\int_0^{\vartheta}\int_0^{y}(y-\tau)^{\kappa-2}\Upsilon\left(\mathscr{Z}^{(1,\,\mathfrak{r}_0)}(\mathbb{A})(\tau)\right)d\tau dy+\varepsilon.$$

By (17), (6) and since $\varepsilon>0$ is arbitrary, it follows that

$$\Upsilon\left(\mathscr{Z}^{(2,\,\mathfrak{r}_0)}(\mathbb{A})(\vartheta)\right)\leq\frac{8N_\eta\mho}{\Gamma(\kappa-1)}\int_0^{\vartheta}\int_0^{y}(y-\tau)^{\kappa-2}\Upsilon\left(\mathscr{Z}^{(1,\,\mathfrak{r}_0)}(\mathbb{A})(\tau)\right)d\tau dy$$

$$\leq\frac{(8\mho N_\eta)^2\,\Upsilon(\mathbb{A})}{\Gamma(\kappa-1)\Gamma(\kappa+1)}\int_0^{\vartheta}\int_0^{y}(y-\tau)^{\kappa-2}\tau^{\kappa}d\tau dy$$

$$\leq\frac{(8\mho N_\eta)^2\,\Upsilon(\mathbb{A})}{\Gamma(2\kappa+1)}\vartheta^{2\kappa}.$$

Proceeding by induction, we obtain

$$\Upsilon\left(\mathscr{Z}^{(n,\,\mathfrak{r}_0)}(\mathbb{A})(\vartheta)\right)\leq\frac{(8\mho N_\eta)^n\,\vartheta^{n\kappa}}{\Gamma(n\kappa+1)}\Upsilon(\mathbb{A}),\quad n\in\mathbb{N}^*.$$

Now, we prove that the series $\sum_{n=0}^{\infty}\frac{(8\mho N_\eta)^n\,\vartheta^{n\kappa}}{\Gamma(n\kappa+1)}$ converges, and the ratio test yields

$$\lim_{n\to\infty}8\mho N_\eta\,\vartheta^{\kappa}\frac{\Gamma(n\kappa+1)}{\Gamma(n\kappa+1+\kappa)}=0.$$

(We note that (see Eq. (1) in [26])

$$\frac{\Gamma(n\kappa+1)}{\Gamma(n\kappa+1+\kappa)}=\frac{1}{((n+1)\kappa+1)^{\kappa}}\left(1-\frac{\kappa(\kappa-1)}{2((n+1)\kappa+1)}+O(((n+1)\kappa+1)^{-2})\right),$$

here, O denotes the Landau symbol.

It follows that there exists a positive integer n_0 such that

$$\frac{(8\mho N_\eta)^{n_0}\,\vartheta^{n_0\kappa}}{\Gamma(n_0\kappa+1)}<1. \tag{18}$$

Therefore, Definition 5 is satisfied, verified, we conclude that $\mathscr{X}:G\to G$ is a CPC operator.

Then, Theorem 2 entails that $\mathscr{X}$ admits a fixed point $\mathfrak{r}\in G$ and it is the solution of (1). □

4. Examples

In this section we introduce some examples that illustrate our theoretical results. Let c_0 be Banach space with sup norms,

$$c_0:=\left\{\mathfrak{r}=(\mathfrak{r}_1,\mathfrak{r}_2,\cdots,\mathfrak{r}_n,\cdots):\lim_{n\to\infty}\mathfrak{r}_n=0,\quad\|\mathfrak{r}\|_{c_0}=\sup_{n\geq1}|\mathfrak{r}_n|\right\}$$

Definition 6 [27] Let $\xi=(\xi_n)$ where each ξ_n is positive and nonincreasing sequence. Accordingly, ξ is termed a tempering sequence. Consider the set $\mathbb{K}$ consisting of all real sequences $\mathfrak{r}=(\mathfrak{r}_n)_{n=1}^{\infty}$ for which $\lim_{n\to\infty}\xi_n\mathfrak{r}_n=0$. Clearly, $\mathbb{K}$ a vector space over the real numbers. We denote the space by c_0^{ξ}. Moreover, it can be seen that c_0^{ξ} is a Banach space with the norm

$$\|\mathfrak{r}\|_{c_0^{\xi}}=\sup_{n\geq1}\{\xi_n|\mathfrak{r}_n|\}.$$

Next, consider the function spaces $C\left(\mathscr{L},c_0^{\xi}\right)$, where, $\mathscr{L}=[0,b]$ for $b>0$ the spaces of all continuous functions on $\mathscr{L}$ with values in c_0^{ξ}. It follows that $C\left(\mathscr{L},c_0^{\xi}\right)$ forms a Banach space under the norms defined in [6],

$$\|\mathfrak{r}\|_{C\left(\mathscr{L},c_0^{\xi}\right)}=\max\left\{\|\mathfrak{r}(\vartheta)\|_{c_0^{\xi}}:\vartheta\in\mathscr{L}\right\},\quad\mathfrak{r}\in C\left(\mathscr{L},c_0^{\xi}\right).$$

Example 1 Consider the following problem:

$$\begin{cases}{}^{pc}\mathscr{D}_0^{\kappa}\mathfrak{r}(\vartheta)=\left\{\frac{e^{-n\vartheta}-e^{-3n\vartheta}}{(1+e^{-2n\vartheta})(n+1)^4}+\sum_{r=n}^{\infty}\frac{\mathfrak{r}_n(\vartheta)}{(r+1)^2}\right\}_{n\geq1},\ \vartheta\in\mathscr{L}:=[0,b],\\ \mathfrak{r}(0)=(0,0,\cdots,0,\cdots).\end{cases} \tag{19}$$

Note that, the problem (19) is a particular case of (1), where: $\Omega : [0, b] \times c_0^{\xi} \to c_0^{\xi}$ given by,

$$\Omega(\vartheta, \mathfrak{r}) = \{\Omega_n(\vartheta, \mathfrak{r})\}_{n\geq 1} = \left\{ \frac{e^{-n\vartheta} - e^{-3n\vartheta}}{(1+e^{-2n\vartheta})(n+1)^4} + \sum_{r=n}^{\infty} \frac{\mathfrak{r}_n(\vartheta)}{(r+1)^2} \right\}_{n\geq 1}, \tag{20}$$

for $\vartheta \in [0, b]$, $\mathfrak{r} = \{\mathfrak{r}_n\}_{n\geq 1} \in c_0^{\xi}$.

It is clear (H1) is satisfied. Moreover, let $\xi_n = 1/n^2$ for all $n \in \mathbb{N}$, for any $\mathfrak{r}, \mathfrak{s} \in c_0^{\xi}$ and $\vartheta \in [0, b]$, we have

$$\begin{aligned} \|\Omega(\vartheta, \mathfrak{r}(\vartheta)) - \Omega(\vartheta, \mathfrak{s}(\vartheta))\|_{c_0^{\xi}} &= \sup_{n\in\mathbb{N}} \left\{ \xi_n |\Omega_n(\vartheta, \mathfrak{r}(\vartheta)) - \Omega_n(\vartheta, \mathfrak{s}(\vartheta))| \right\} \\ &= \sup_{n\in\mathbb{N}} \left\{ \frac{1}{n^2} \sum_{r=n}^{\infty} \frac{1}{(r+1)^2} |\mathfrak{r}_n(\vartheta) - \mathfrak{s}_n(\vartheta)| \right\} \\ &\leq \frac{\pi^2}{6} \|\mathfrak{r}_n(\vartheta) - \mathfrak{s}_n(\vartheta)\|_{c_0^{\xi}}. \end{aligned}$$

So condition (H2) is satisfied with:

$$\varpi = \frac{\pi^2}{6}.$$

Thus, by Theorem 3, problem (19) admits a unique solution $\mathfrak{r} \in C([0, b], c_0^{\xi})$.

Example 2 Consider the following problem posed in c_0^{ξ}:

$$\begin{cases} {}^{pc}\mathscr{D}_0^{\kappa}\mathfrak{r}(\vartheta) = \left\{ \frac{e^{-n\vartheta}\sin(n\vartheta)}{\theta} \left(\frac{\pi^2 n^2}{24} + \sum_{r=n}^{\infty} \frac{\mathfrak{r}_n(\vartheta)}{4r^2} \right) \right\}_{n\geq 1}, \ \vartheta \in \mathscr{L} := [0, b], \\ \mathfrak{r}(0) = (0, 0, \cdots, 0, \cdots), \end{cases} \tag{21}$$

where $\theta = \frac{(N_\eta+13)b^{\kappa}}{\Gamma(\kappa+1)}$ and $\Omega : [0, b] \times c_0^{\xi} \to c_0^{\xi}$ given by,

$$\Omega(\vartheta, \mathfrak{r}) = \{\Omega_n(\vartheta, \mathfrak{r})\}_{n\geq 1} = \left\{ \frac{e^{-n\vartheta}\sin(n\vartheta)}{\theta} \left(\frac{\pi^2 n^2}{24} + \sum_{r=n}^{\infty} \frac{\mathfrak{r}_n(\vartheta)}{4r^2} \right) \right\}_{n\geq 1}, \tag{22}$$

for $\vartheta \in [0, b]$, $\mathfrak{r} = \{\mathfrak{r}_n\}_{n\geq 1} \in c_0^{\xi}$.

Evidently (H3) holds and for $\xi_n = 1/n^2$ for all $n \in \mathbb{N}$, and $\mathfrak{r} = \{\mathfrak{r}_n\}_{n\geq 1} \in c_0^{\xi}$,

$$\|\Omega(\vartheta, \mathfrak{r})\|_{c_0^{\xi}} = \sup_{n\in\mathbb{N}} \left\{ \xi_n |\Omega_n(\vartheta, \mathfrak{r})| \right\}$$

$$\leq \frac{e^{-n\vartheta}|\sin(n\vartheta)|}{\theta}\left(\frac{\pi^2}{24}+\frac{1}{n^2}\sum_{r=n}^{\infty}\frac{1}{4r^2}|\mathfrak{r}_n(\vartheta)|\right)$$

$$\leq \frac{\pi^2 e^{-n\vartheta}}{24\theta}(\|\mathfrak{r}_n\|_{c_0^\xi}+1).$$

Hence, condition (H4) is satisfied with

$$\Psi(\vartheta)=\frac{\pi^2 e^{-n\vartheta}}{24\theta}, \quad \vartheta\in[0,\, b] \quad \text{and} \quad \Phi(u)=u+1, \quad u\in[0,\, \infty).$$

On the other side, the Hausdorff measure of noncompactness in $(c_0,\, \|\cdot\|_{c_0})$ is defined as follows (see [28])

$$\Upsilon_{c_0}(\mathbb{T})=\lim_{n\to\infty}\sup_{\mathfrak{r}\in\mathbb{T}}\|(I-P_n)\mathfrak{r}\|_\infty.$$

Here, P_n stands for the projection operator onto the linear span of the first n vectors forming the standard basis. Since c_0 and c_0^ξ are isometric to each other [27], the Hausdorff MNC $\Upsilon_{c_0^\xi}$ is defined by the formula

$$\Upsilon_{c_0^\xi}(\mathbb{T})=\lim_{n\to\infty}\sup_{\mathfrak{r}\in\mathbb{T}}\|(I-P_n)\xi\mathfrak{r}\|_\infty \tag{23}$$

For any bounded set $\mathbb{A}\subset c_0^\xi$, (see Example in [3, 4]) we obtain

$$\Upsilon_{c_0^\xi}(\Omega(\vartheta,\, \mathbb{A}))\leq\frac{\pi^2}{24\theta}\Upsilon_{c_0^\xi}(\mathbb{A}), \quad \text{for any } \vartheta\in[0,\, b].$$

Hence, (H4) is verified. Now, we can choose T such that

$$\frac{N_\eta\pi^2 T+N_\eta\pi^2}{24(N_\eta+13)}\leq T.$$

We then choose $T>0$ such that

$$T\geq\frac{N_\eta\pi^2}{24(N_\eta+13)-N_\eta\pi^2}.$$

Finally, since all assumptions of Theorem 4 are satisfied, problem (21) admits at least one solution $\mathfrak{r}\in C([0,\, b],\, c_0^\xi)$.

5. Conclusion

This paper has investigated a class of nonlinear Cauchy problems involving hybrid proportional–Caputo fractional derivatives in Banach spaces. By employing fixed point techniques and the theory of measures of noncompactness, we established new existence and uniqueness results under mild assumptions. Our work extends and refines several earlier contributions in the literature, notably those in [17, 18]. The applicability of the theoretical framework was illustrated through a concrete infinite-dimensional system in the space of tempered sequences.

In addition to the analytical results presented, our study opens several avenues for further inquiry. First, the framework developed here can be directly applied to more concrete models involving Partial Differential Equation (PDE) systems with memory, demographic models with delays, or economic systems exhibiting hysteresis effects. Second, natural extensions of this work include the study of associated control problems, the investigation of stability in the sense of Lyapunov, and the analysis of related inverse problems. Finally, the development of suitable numerical discretisations for problems of this type remains an important and challenging direction for future research.

Beyond these immediate prospects, qualitative properties of solutions, such as periodicity, attractivity, and other forms of asymptotic behaviour, offer rich and promising areas for further investigation.

Author contributions

All authors contributed to the content and writing of the main manuscript. All authors reviewed the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this research.

Funding

This research was funded by King Khalid University through large group research project under grant number RGP2/458/46.

Conflict of interest

The authors have no conflicts of interest to declare.

References

[1] Kilbas AA, Srivastava HM, Trujillo JJ. *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, Vol. 204. Elsevier; 2006.

[2] Samko SG, Kilbas AA, Marichev OI. *Fractional Integrals and Derivatives: Theory and Applications*. Gordon and Breach; 1993.

[3] Zentar O, Ziane M, Al Horani M. Theoretical study of a ϕ-Hilfer fractional differential system in Banach spaces. *Canadian Mathematical Bulletin*. 2024; 67(3): 742–759. Available from: https://doi.org/10.4153/S0008439524000134.

[4] Zentar O, Al Horani M, Ziane M. A study of a coupled system involving tempered Caputo derivatives with respect to functions. *Rendiconti del Circolo Matematico di Palermo Series 2*. 2025; 74(1): 17. Available from: https://doi.org/10.1007/s12215-024-01167-4.

[5] Norouzi F, N'Guérékata GM. A study of ψ-Hilfer fractional differential system with application in financial crisis. *Chaos Solitons and Fractals: X*. 2021; 6: 100056. Available from: https://doi.org/10.1016/j.csfx.2021.100056.

[6] Das A, Hazarika B, Agarwal RP, Nashine HK. Solvability of infinite systems of fractional differential equations in the spaces of tempered sequences. *Filomat*. 2019; 33(17): 5519–5530. Available from: https://doi.org/10.2298/FIL1917519D.

[7] Abbas S, Benchohra M, N'Guérékata GM. *Advanced Fractional Differential and Integral Equations*. Nova Science Publishers; 2015.

[8] Abdellouahab N, Tellab B, Zennir K. Existence and stability results of a nonlinear fractional integro-differential equation with integral boundary conditions. *Kragujevac Journal of Mathematics*. 2022; 46(5): 685–699. Available from: https://doi.org/10.46793/KgJMat2205.685A.

[9] Azzaoui B, Tellab B, Zennir K. Positive solutions for integral nonlinear boundary value problem in fractional Sobolev spaces. *Mathematical Methods in the Applied Sciences*. 2023; 46(3): 3115–3131. Available from: https://doi.org/10.1002/mma.7623.

[10] Naimi A, Brahim T, Zennir K. Existence and stability results for the solution of neutral fractional integro-differential equation with nonlocal conditions. *Tamkang Journal of Mathematics*. 2022; 53(3): 239–257. Available from: https://doi.org/10.5556/j.tkjm.53.2022.3550.

[11] Azzaoui B, Tellab B, Zennir K. Positive solutions for a fractional configuration of the Riemann-Liouville semilinear differential equation. *Mathematical Methods in the Applied Sciences*. 2022; 1–12. Available from: https://doi.org/10.1002/mma.8110.

[12] Bidi Y, Beniani A, Zennir K, Himadan A. Global existence and dynamic structure of solutions for damped wave equation involving the fractional Laplacian. *Demonstratio Mathematica*. 2021; 54(1): 245–258. Available from: https://doi.org/10.1515/dema-2021-0022.

[13] Baleanu D, Fernandez A, Akgül A. On a fractional operator combining proportional and classical differintegrals. *Mathematics*. 2020; 8: 360. Available from: https://doi.org/10.3390/math8030360.

[14] Acay B, Inc M. Fractional modeling of temperature dynamics of a building with singular kernels. *Chaos Solitons and Fractals*. 2021; 142: 110482. Available from: https://doi.org/10.1016/j.chaos.2020.110482.

[15] Ahmad M, Imran MA, Baleanu D, Alshomrani AS. Thermal analysis of magnetohydrodynamic viscous fluid with innovative fractional derivative. *Thermal Science*. 2020; 24(1): 351–359. Available from: https://doi.org/10.2298/TSCI20S1351A.

[16] Ikram MD, Asjad MI, Akgül I, Baleanu D. Effects of hybrid nanofluid on novel fractional model of heat transfer flow between two parallel plates. *Alexandria Engineering Journal*. 2021; 60(4): 3593–3604. Available from: https://doi.org/10.1016/j.aej.2021.01.054.

[17] Abbas MI. Existence results and the Ulam stability for fractional differential equations with hybrid proportional-Caputo derivatives. *Journal of Nonlinear Functional Analysis*. 2020; 2020: 1–14. Available from: https://doi.org/10.23952/jnfa.2020.48.

[18] Abbas MI. Ulam stability and existence results for fractional differential equations with hybrid proportional-Caputo derivatives. *Journal of Interdisciplinary Mathematics*. 2022; 25(2): 213–231. Available from: https://doi.org/10.1080/09720502.2021.1889156.

[19] Banaś J, Jleli M, Mursaleen M, Samet B, Vetro C. *Advances in Nonlinear Analysis via the Concept of Measure of Noncompactness*. Springer; 2017. Available from: https://doi.org/10.1007/978-981-10-3722-1.

[20] Aghajani A, Pourhadi E, Trujillo JJ. Application of measure of noncompactness to a Cauchy problem for fractional differential equations in Banach spaces. *Fractional Calculus and Applied Analysis*. 2013; 16: 962–977. Available from: https://doi.org/10.2478/s13540-013-0059-y.

[21] Gorenflo R, Kilbas AA, Mainardi F, Rogosin S. *Mittag-Leffler Functions, Related Topics and Applications*. Springer Berlin, Heidelberg; 2020. Available from: https://doi.org/10.1007/978-3-662-61550-8.

[22] Akhmerov RR, Kamenskii MI, Potapov AS, Rodkina AE, Sadovskii BN. *Measures of Noncompactness and Condensing Operators*. Birkhäuser Basel; 1992. Available from: https://doi.org/10.1007/978-3-0348-5727-7.

[23] Kamenskii M, Obukhovskii V, Zecca P. *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*. De Gruyter; 2001. Available from: https://doi.org/10.1515/9783110889888.

[24] Diethelm K, Ford N. Analysis of fractional differential equations. *Journal of Mathematical Analysis and Applications*. 2002; 265: 229–248. Available from: https://doi.org/10.1006/jmaa.2000.7194.

[25] Sun J, Zhang X. The fixed point theorem of convex-power condensing operator and applications. *Acta Mathematica Sinica*. 2005; 48: 439–446.

[26] Tricomi F, Erdélyi A. The asymptotic expansion of a ratio of gamma functions. *Pacific Journal of Mathematics*. 1951; 1: 133–142. Available from: https://doi.org/10.2140/pjm.1951.1.133.

[27] Banaś J, Krajewska M. Existence of solutions for infinite systems of differential equations in spaces of tempered sequences. *Electronic Journal of Differential Equations*. 2017; 2017(60): 1–28

[28] Banaś J, Goebel K. *Measure of Noncompactness in Banach Spaces*. Marcel Dekker; 1980.