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Research Article

Some Application of Fuzzy Third-Order Subordination and Superordination Associated with Zeta-Riemann Fractional Differential Operator

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Abstract: This work lays under the fuzzy third-order differential subordination. The development of the key ideas required to produce fresh findings on this subject marked the beginning of third-order fuzzy differential subordination research. The current work presents the twin idea of third-order fuzzy differential subordination. Investigating the best dominant for such fuzzy differential subordination when it is possible, and identifying best subordinant of a third-order fuzzy differential superordination are both made possible by the study's results.

***Keywords*:** fuzzy set, third-order fuzzy differential subordination and superordination, fuzzy subordinant, best fuzzy subordinant, admissible functions, Zeta-Riemann fractional differential operator

MSC: 30C45, 30C80

1. Introduction

The present work examines the dual concept of fuzzy differential superordination, viewed as a natural extension and counterpart of fuzzy differential subordination within the broader framework of fuzzy geometric function theory. The theory of fuzzy sets, introduced by Zadeh in 1965 [1], established a mathematical foundation for modeling graded, rather than binary, membership. This framework became instrumental in contexts where uncertainty, vagueness, or partial inclusion properties must be formalized with analytical precision. In [2, 3], the fuzzy set paradigm was incorporated into the theory of differential subordination, enabling the classical notion of analytic function domination to be expressed in terms of membership degrees rather than strict functional inclusion.

Classical differential subordination [4] serves as the conceptual precursor to its fuzzy analog. Whereas classical subordination demands a precise geometric containment between analytic functions, fuzzy differential subordination introduces a softer, more flexible structure wherein containment is achieved up to a graded level of compatibility. This distinction allows the theory to capture a richer set of analytic behaviors, particularly in situations where the boundaries of functional inclusion are inherently uncertain or cannot be sharply delineated. Many studies have examined the the methods

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of fuzzy differential subordination and superordination continue to yield significant and stimulating results as in [5–11] and also several new spaces as the generalizations of fuzzy triple controlled metric space as in [12–14].

Earlier works (e.g., Antonino & Miller [15]) have demonstrated te differential methods produce structurally rich and often nontrivial conclusions that mirror, extend, and refine classical geometric function theory. The third-order formulations introduced in this literature established the essential definitions, admissibility criteria, and analytical tools necessary to investigate dual problems such as third-order differential superordination.

The present study builds directly on these contributions, advancing the dual theory and broadening the understanding of analytic inequalities under higher-order differential operators.

The notion of third-order fuzzy differential superordination was examined in depth by Oros et al. [16, 17], who established the essential theoretical framework, including dual admissibility conditions, fundamental lemmas, and superordination theorems that parallel and complement existing results on third-order fuzzy subordination and also [18, 19].

Building on previous advances, this paper addresses third-order fuzzy differential superordination, thereby completing the dual framework corresponding to third-order fuzzy differential subordination. This extension yields new structural results relevant to the study of analytic functions under fuzzy constraints.

$\mathscr{H}(\mathbb{D})$ be the class of analytic functions in $\mathbb{D},$ where $\mathbb{D}=\{\xi\colon \xi\in\mathbb{C} \text{ and } |\xi|<1\}$ with $\overline{\mathbb{D}}=\{\xi\in\mathbb{C}\colon\ |\xi|\leq 1\}$ and $\partial\mathbb{D}=\{\xi\in\mathbb{C}\colon |\xi|=1\}$. The following subclasses of $\mathscr{H}(\mathbb{D})$ are:

$$\mathscr{A}_{\kappa}=\left\{\mathfrak{f}\in\mathscr{H}(\mathbb{D})\colon \mathfrak{f}(\xi)=\xi+a_{\kappa+1}\xi^{\kappa+1}+\cdots \qquad \xi\in\mathbb{D}\right\},$$

where $\mathscr{A}_1=\mathscr{A}$,

$$\mathscr{H}[a,\,\kappa]=\left\{\mathfrak{f}\in\mathscr{H}(\mathbb{D})\colon \mathfrak{f}(\xi)=a+a_{\kappa}\xi^{\kappa}+a_{\kappa+1}\xi^{\kappa+1}+\cdots \qquad \xi\in\mathbb{D}\right\},$$

where $\mathscr{H}_0=\mathscr{H}[0,\,1],$ with $a\in\mathbb{C}$ and $\kappa\in\mathbb{N}=\{1,\,2,\,3,\,\cdots\}$. Let

$$\mathfrak{K}=\left\{\mathfrak{f}\in\mathscr{H}(\mathbb{D})\,:\,Re\left(\frac{\xi\mathfrak{f}^{''}(\xi)}{\mathfrak{f}^{'}(\xi)}+1\right)>0,\ \mathfrak{f}^{'}(0)\neq 0,\ \xi\in\mathbb{D}\right\}.$$

Furthermore,

$$\mathscr{S}=\left\{\mathfrak{f}\in\mathscr{A}\colon \mathfrak{f}(\xi)=\xi+a_2\xi^2+\cdots,\ \mathfrak{f}(0)=0,\mathfrak{f}^{'}(0)=1,\ \ \mathfrak{f}\text{ is univalent for }\ \xi\in\mathbb{D}\right\}.$$

We say $\mathfrak{f}$ and $\mathfrak{g}$ are analytic in $\mathbb{D}$, $\mathfrak{f}$ is *subordinate* to $\mathfrak{g}$, denoted $\mathfrak{f}(\xi)\prec\mathfrak{g}(\xi)$, if there exists an analytic function w, with $w(0)=0$ and $|w(\xi)|<1$ for all $\xi\in\mathbb{D}$, such that $f(\xi)=\mathfrak{g}(w(\xi))$, $\xi\in\mathbb{D}$. If $\mathfrak{g}$ is univalent in $\mathbb{D}$, $\mathfrak{f}(\xi)\prec\mathfrak{g}(\xi)$ is define as (see [4, 20, 21]):

$$\mathfrak{f}(0)=\mathfrak{g}(0) \text{ and } \mathfrak{f}(\mathbb{D})\subset\mathfrak{g}(\mathbb{D}).$$

For two functions $\mathfrak{f}_{\iota}(\xi)\in\mathscr{A}(\iota=1,\,2)$ are denoted by

$$\mathfrak{f}_\iota(\xi) = \xi + \sum_{\kappa=2}^{\infty} a_{\kappa,\,\iota}\,\xi^{\kappa}, \tag{1}$$

we define the convolution of $\mathfrak{f}_1(\xi)$ and $\mathfrak{f}_2(\xi)$ as (see, for example [22], p.246)

$$(\mathfrak{f}_1 * \mathfrak{f}_2)(\xi) = \xi + \sum_{\kappa=2}^{\infty} a_{\kappa,\,1} a_{\kappa,\,2}\,\xi^{\kappa} = (\mathfrak{f}_2 * \mathfrak{f}_1)(\xi).$$

The Zeta-Riemann fractional differential operator is an interesting generalization of classical differential operators. It is made up of ideas from the Riemann zeta function and the field of fractional calculus. Its development is motivated by the need to model and analyze non-local phenomena that arise in mathematical physics, engineering, and applied mathematics contexts in which traditional integer-order derivatives often fall short. Fractional calculus, which permits derivatives and integrals of arbitrary real or complex order, provides enhanced flexibility for describing systems with memory, hereditary properties, or non-integer dynamical behavior. By integrating the analytic depth of the Riemann zeta function, well known for its profound connections to number theory and complex analysis, the Zeta-Riemann fractional differential operator inherits additional structural richness. This fusion yields a versatile and powerful tool capable of capturing subtler behaviors than those addressed by classical fractional operators alone.

This hybrid operator is especially relevant to fuzzy third-order subordination and superordination because such problems frequently involve function classes whose behavior depends on multi-scale or nonlocal geometric constraints. The non-integer order nature of the operator provides a flexible mechanism for controlling growth, distortion, and convexity properties of analytic functions within fuzzy environments. Moreover, the presence of the zeta-based kernel introduces a tunable structure that can amplify or dampen geometric features in a way that classical or simpler fractional operators cannot. Consequently, the Zeta–Riemann fractional operator not only broadens the analytic scope of fuzzy differential inequalities but also offers deeper control over the hierarchical relationships between subordinate and superordinate functions, making it an advantageous tool for the study of fuzzy third-order subordination/superordination problems.

Tayyah and Atshan [23] investigate the following fractional differential operator

$$\mathfrak{D}^{\nu}_{(\sigma,\,k,\,w)}\mathfrak{f}(\xi) = \sum_{\kappa=1}^{\infty} \frac{(w+1)^{\frac{\sigma(\nu-1)+1}{k}}\,\Gamma\left(\frac{\kappa}{w+1}+1\right)\Gamma\left(\frac{\delta+\eta+1}{2}\right)}{k^{\frac{\sigma(\nu-1)+1}{k}}\,\Gamma\left(\frac{\kappa}{w+1}-\frac{\sigma(\nu-1)+1}{k}+1\right)} a_{\kappa}\,\xi^{(w+1)(1-\frac{\sigma(\nu-1)+1}{k})+\kappa-1},$$

$$\sigma \in \mathbb{N},\ k>0,\, w\geq 0,\ 0 \leq \frac{\sigma(\nu-1)+1}{k} < 1. \tag{2}$$

In [24] the Hurwitz-Lerch Zeta function is defined as following

$$\Phi(\xi,\,\sigma,\,\nu) = \sum_{\kappa=0}^{\infty} \frac{\xi^{\kappa}}{(\kappa+\nu)^{\sigma}}, \tag{3}$$

$\sigma \in \mathbb{C},\ \nu \in \mathbb{C}\backslash\mathbb{Z}^{-},\ Re(\sigma) > 1,$ when $|\xi| = 1$.

Numerous authors have examined and explored the aforementioned Hurwitz-Lerech Zeta function's intriguing qualities and peculiarities [25–28].

We define the new Hadamard product fractional differential operator.

$$\mathfrak{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi) = \left[D^{\nu}_{(\sigma,1,0)}\mathfrak{f}(\xi)\right] * \left[\xi^{-(\sigma(\nu-1)+1)}(\Phi(\xi,\sigma-1,\nu+\varepsilon)-\nu^{-\sigma}\right]$$

$$= \sum_{\kappa=1}^{\infty} \frac{\kappa!}{\Gamma(\kappa-\sigma(\nu-1))(\kappa+\nu+\varepsilon)^{\sigma-1}} a_{\kappa}\xi^{\kappa-[\sigma(\nu-1)+1]}. \tag{4}$$

We note that if $\sigma = 1,$ then we have Srivastava fractional differential operator in [29] as:

$$\mathfrak{D}_{1}^{(\nu,\varepsilon)}\mathfrak{f}(\xi) = \sum_{\kappa=1}^{\infty} \frac{\kappa!}{\Gamma(\kappa-\nu+1)} a_{\kappa}\xi^{\kappa-\nu} \tag{5}$$

Shexo et al. [30] introduced a new operator (Zeta-Riemann Fractional differential operator) $\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}$: $\mathscr{A} \to \mathscr{A}$ as follows:

$$\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi) = (1+\nu+\varepsilon)^{\sigma-1}\Gamma(1-\sigma(\nu-1))\,\xi^{[\sigma(\nu-1)+1]}\mathfrak{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi)$$

$$= \xi + \sum_{\kappa=2}^{\infty} \frac{\Gamma(1-\sigma(\nu-1))(1+\nu+\varepsilon)^{\sigma-1}\kappa!}{\Gamma(\kappa-\sigma(\nu-1))(\kappa+\nu+\varepsilon)^{\sigma-1}} a_{\kappa}\xi^{\kappa}. \tag{6}$$

The operator $\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi),$ satisfying

$$\xi\left(\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi)\right)' = \sigma(1-\nu)\mathscr{D}_{\sigma}^{(\nu+\frac{1}{\sigma},\varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi) - [\sigma(1-\nu)-1]\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi) \tag{7}$$

The Zeta–Riemann fractional operator is not a single universally standardized object, but the term is commonly used to describe fractional operators inspired by the Riemann zeta function and Riemann–Liouville–type fractional calculus. Its properties and limitations are best understood by examining how zeta-function structures interact with fractional differentiation and integration.

In most contexts, a Zeta–Riemann fractional operator involves:

1. Fractional-order differentiation or integration (in the Riemann–Liouville or Caputo sense),
2. With kernels, spectra, or weights related to the Riemann zeta function or Dirichlet series.

Such operators often appear in spectral theory, analytic number theory and fractional differential equations with nonlocal or scale-invariant behavior.

Like all fractional operators the operator depends on the entire history or domain of the function local behavior cannot be isolated from global structure. This is often enhanced by zeta-related kernels, which encode long-range correlations.

Analytic Continuation: The zeta function requires analytic continuation the operator may only be well-defined in certain regions of the complex plane, Poles (e.g., at $\sigma = 1$) can introduce singular behavior.

• Behavioral Characteristics

Global sensitivity: Local perturbations can influence the solution globally.

Anomalous dynamics: Often associated with anomalous diffusion or non-Gaussian processes.

Arithmetic sensitivity: Behavior can change dramatically depending on how zeta-function zeros or poles enter the formulation.

• Limitations

1. Non-uniqueness

There is no universally accepted definition, leading to multiple formulations depending on context.

2. Domain Restrictions

Requires functions with strong smoothness or decay conditions.

Not all classical function spaces are suitable.

3. Computational Difficulty

Involves infinite series, integrals, or spectral decompositions.

Numerical implementation is expensive and sensitive to truncation.

• Scope of Applicability

The Zeta–Riemann fractional operator is best suited for:

1. Theoretical studies in analytic number theory,
2. Fractional PDEs with scale-invariant or fractal features,
3. Mathematical physics (quantum systems, statistical mechanics),
4. Spectral and operator-theoretic investigations.

It is not well-suited for:

1. Standard engineering models requiring simple local operators,
2. Real-time numerical simulations without heavy approximation.

Example 1 (i) $\mathfrak{f}(\xi) = \xi + \xi^2$

$$\mathscr{D}_1^{(\frac{1}{2},0)}\mathfrak{f}(\xi) = \xi + \sum_{\kappa=2}^{\infty} \frac{\Gamma(2)\,(\frac{3}{2})\kappa!}{\Gamma(\kappa+1))\,(\kappa+\frac{1}{2})}\xi^{\kappa}$$

$$= \xi + \frac{4}{3}\xi^2$$

$$\mathscr{D}_2^{(\frac{1}{2},0)}\mathfrak{f}(\xi) = \xi + \frac{3}{5}\xi^2$$

(ii) $\mathfrak{f}(\xi) = \xi + \xi^4$

$$\mathscr{D}_1^{(\frac{1}{2},0)}\mathfrak{f}(\xi) = \xi + \frac{8.4!}{115}\xi^4$$

$$\mathscr{D}_2^{(\frac{1}{2},0)}\mathfrak{f}(\xi) = \xi + \frac{2}{9}\xi^4$$

In the subsequent sections, building on the third-order differential subordination framework of Antonino and Miller [15] in the unit disk $\mathbb{D}$ and the third-order fuzzy differential subordination and superordination theories developed by Oros et al. [16, 17], we introduce appropriate classes of admissible functions and investigate third-order fuzzy differential subordination and superordination properties of univalent functions associated with the Zeta–Riemann fractional differential operator $\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi)$ defined by (6). The results derived from these two complementary approaches are unified through the establishment of corresponding sandwich-type conclusions at the end of the study

2. Preliminaries

The ensuing investigations will corroborate the new discoveries in the next section.

Definition 1 ([15], p.441, Definition 2) Consider $\mathfrak{Q}$ the set of all functions $\mathfrak{q}$ that are regular and injective on $\overline{\mathbb{D}}\backslash E(\mathfrak{q})$ where

$$E(\mathfrak{q}) = \left\{\xi \in \partial\mathbb{D}: \lim_{\xi\to\zeta} \mathfrak{q}(\xi) = \infty\right\},$$

and are such that $\min|\mathfrak{q}'(\xi)| = \rho > 0$ for $\zeta \in \partial\mathbb{D}\backslash E(\mathfrak{q})$. $E(\mathfrak{q})$ is called the exception set, and the subclass of $\mathfrak{Q}$ when $\mathfrak{q}(0) = a$ be indicated by $\mathfrak{Q}(a)$, $\mathfrak{Q}(0) = \mathfrak{Q}_0$ and $\mathfrak{Q}(1) = \mathfrak{Q}_1$.

Using the following as a generalization of fuzzy subordination:

Definition 2 [1] A fuzzy set is pair $(\varsigma, \mathscr{F})$, where ς is a set , $\varsigma \neq \phi$ and $\mathscr{F}$: $\varsigma \to [0, 1]$ a membership function.

Definition 3 [2] A Fuzzy subset of ς is a pair $((\upsilon, \mathscr{F}_\upsilon)$, where $\mathscr{F}_\upsilon$: $\varsigma \to [0, 1]$ is known as the membership function of the fuzzy set $(\upsilon, \mathscr{F}_\upsilon)$ and $\upsilon = \{Y \in \varsigma: 0 < \mathscr{F}_\upsilon(Y) \leq 1\} = \sup(\upsilon, \mathscr{F}_\upsilon)$ is called fuzzy subset.

The investigation suggested here makes use of the notations given below. Let $\mathscr{F}$: $\mathbb{C} \to [0, 1]$ indicate by

(i) $(\upsilon, \mathscr{F}_\upsilon) = (\mathbb{D}, \mathscr{F}_\mathbb{D})$ then $\upsilon = \mathbb{D}$ where $\upsilon = \sup(\upsilon, \mathscr{F}_\upsilon)$ and $\mathbb{D} = \sup(\mathbb{D}, \mathscr{F}_\mathbb{D})$.

(ii) If $(\upsilon, \mathscr{F}_\upsilon) \subseteq (\mathbb{D}, \mathscr{F}_\mathbb{D})$, then $I \subseteq \mathbb{D}$ where $I = \sup(I, \mathscr{F}_\upsilon)$ and $\mathbb{D} = \sup(\mathbb{D}, \mathscr{F}_\mathbb{D})$.

Let $\mathfrak{f}, \mathfrak{g} \in \mathscr{H}(\mathbb{D})$. We say

$$\mathfrak{f}(\mathbb{D}) = \sup(\mathfrak{f}(\mathbb{D}), \mathscr{F}_{\mathfrak{f}(\mathbb{D})}) = \{\mathfrak{f}(\xi): 0 < \left|\mathscr{F}_{\mathfrak{f}(\mathbb{D})}(\mathfrak{f}(\xi))\right| \leq 1,\ \xi \in \mathbb{D} \tag{8}$$

and

$$\mathfrak{g}(\mathbb{D}) = \sup(\mathfrak{g}(\mathbb{D}), \mathscr{F}_{\mathfrak{g}(\mathbb{D})}) = \{\mathfrak{g}(\xi): 0 < \left|\mathscr{F}_{\mathfrak{g}(\mathbb{D})}(\mathfrak{g}(\xi))\right| \leq 1,\ \xi \in \mathbb{D}. \tag{9}$$

Definition 4 [2] Assume that two fuzzy subsets of η, $(\upsilon_1, \mathscr{F}_{\upsilon_1})$ and $(\upsilon_2, \mathscr{F}_{\upsilon_2})$. We declare that the fuzzy subsets υ_1 and υ_2 are equal iff $\mathscr{F}_{\upsilon_1}(\eta) = \mathscr{F}_{\upsilon_2}(\eta)$, $\eta \in \varsigma$ and denote by $(\upsilon_1, \mathscr{F}_{\upsilon_1}) = (\upsilon_2, \mathscr{F}_{\upsilon_2})$. The fuzzy subset $(\upsilon_1, \mathscr{F}_{\upsilon_1})$ is contained in the fuzzy subset $(\upsilon_2, \mathscr{F}_{\upsilon_2})$ iff $\mathscr{F}_{\upsilon_1}(\eta) \leq \mathscr{F}_{\upsilon_2}(\eta)$, $\eta \in S$ and denote by $(\upsilon_1, \mathscr{F}_{\upsilon_1}) \subseteq (\upsilon_2, \mathscr{F}_{\upsilon_2})$.

Definition 5 [2] Let $\xi_0 \in \mathbb{D}$ and $\mathfrak{f}$, $\mathfrak{g} \in \mathscr{H}(\mathbb{D})$. $\mathfrak{f}$ be fuzzy subordinate to $\mathfrak{g}$ and writen as $\mathfrak{f} \prec_\mathscr{F} \mathfrak{g}$ or $\mathfrak{f}(\boldsymbol{\xi}) \prec_\mathscr{F} \mathfrak{g}(\xi)$ if each of the subsequent needed is satisfied

$$\mathfrak{f}(\xi_0) = \mathfrak{g}(\xi_0) \text{ and } \left|\mathscr{F}_{\mathfrak{f}(\mathbb{D})}(\mathfrak{f}(\xi))\right| \leq \left|\mathscr{F}_{\mathfrak{g}(\mathbb{D})}(\mathfrak{g}(\xi))\right|,\quad \xi \in \mathbb{D},$$

Proposition 1 [2] Let $\xi_0 \in \mathbb{D}$ and f , $\mathfrak{g} \in \mathscr{H}(\mathbb{D})$. if $f(\xi) \prec_\mathscr{F} g(\xi)$, $\xi \in \mathbb{D}$, then

(i) $\mathfrak{f}(\xi_0) = \mathfrak{g}(\xi_0)$,

(ii) $\mathfrak{f}(\mathbb{D}) \subseteq \mathfrak{g}(\mathbb{D})$ and $\left|\mathscr{F}_{\mathfrak{f}(\mathbb{D})}(\mathfrak{f}(\xi))\right| \leq \left|\mathscr{F}_{\mathfrak{g}(\mathbb{D})}(\mathfrak{g}(\xi))\right|,\quad \xi \in \mathbb{D},$

where $\mathfrak{f}(\mathbb{D})$ and $\mathfrak{g}(\mathbb{D})$ are given by (8) and (9), respectively.

Definition 6 [11] Consider $\Omega \subset \mathbb{C}$, where, $\mathfrak{q} \in \mathfrak{Q}$ and $\ell \geq 2$. Functions ψ: $\mathbb{C}^4 \times \overline{\mathbb{D}} \to \mathbb{C}$, are called admissible and are members of the known as class of admissible functions denoted by $\Psi_{\kappa}[\Omega, \mathfrak{q}]$ if

$$\psi(r, s, t, u; \xi) \notin \Omega,$$

is satisfied, with

$$r = \mathfrak{q}(\xi), \ s = \frac{\xi \mathfrak{q}'(\xi)}{\ell},$$

$$Re\left\{\frac{t}{s} + 1\right\} \geq \ell Re\left\{1 + \frac{\xi \mathfrak{q}''(\xi)}{\mathfrak{q}'(\xi)}\right\},$$

$$Re\left\{\frac{u}{s}\right\} \geq \ell^2 Re\left\{\frac{\xi^2 \mathfrak{q}'''(\xi)}{\mathfrak{q}'(\xi)}\right\},$$

where $\xi \in \mathbb{D}$, $\xi \in \partial\mathbb{D} \backslash E(\mathfrak{q})$ and $\ell \in \mathbb{N} \backslash \{1\}$.

Definition 7 ([16], Definition 5) Assume that Ω be a set in $\mathbb{C}$, $\mathfrak{q} \in \mathfrak{Q}$, $\kappa \geq 2$. Let $\Psi_{\kappa}[\Omega, \mathfrak{q}]$ the set of functions ψ: $\mathbb{C}^4 \times \overline{\mathbb{D}} \to \mathbb{C}$, satisfying:

$$\mathscr{F}_{\psi(\mathbb{C}^4 \times \overline{\mathbb{D}})} \psi(r, s, t, u; \xi) < \mathscr{F}_{\Omega}(\xi), \quad 0 < \mathscr{F}_{\Omega}(\xi) \leq 1, \tag{10}$$

where

$$r = \mathfrak{q}(\xi), \ s = \frac{\xi \mathfrak{q}'(\xi)}{\ell},$$

$$Re\left\{\frac{t}{s} + 1\right\} \leq \frac{1}{\ell} Re\left\{1 + \frac{\xi \mathfrak{q}''(\xi)}{\mathfrak{q}'(\xi)}\right\},$$

$$Re\left\{\frac{u}{s}\right\} \leq \frac{1}{\ell^2} Re\left\{\frac{\xi^2 \mathfrak{q}'''(\xi)}{\mathfrak{q}'(\xi)}\right\},$$

where $\xi \in \mathbb{D}$, $\xi \in \partial\mathbb{D}$ and $\ell \geq \kappa \geq 2$. Condition (10) is called the admissibility condition.

For the solutions of a third-order fuzzy differential subordination, [16] defines fuzzy dominance in terms of a function $\mathfrak{q} \in \mathscr{H}(\mathbb{D})$ that satisfies $\omega \prec_{\mathscr{F}} \mathfrak{q}$ whenever ω satisfies

$$\mathscr{F}_{\psi(\mathbb{C}^4 \times \mathbb{D})}(\psi(\omega(\xi), \xi\omega'(\xi), \xi^2\omega''(\xi), \xi^3\omega'''(\xi); \xi)) \leq \mathscr{F}_{\lambda(\mathbb{D})}\lambda(\xi)$$

i.e

$$\psi(\omega(\xi), \xi\omega'(\xi), \xi^2\omega''(\xi), \xi^3\omega'''(\xi); \xi) \prec_{\mathscr{F}} \lambda(\xi), \tag{11}$$

with ψ: $\mathbb{C}^4 \times \mathbb{D} \longrightarrow \mathbb{C}$, $\psi \in \Psi_{\kappa}[\Omega, \mathfrak{q}]$ and let $\lambda \in \mathscr{H}(\mathbb{D})$, ω is referred to as a fuzzy differential subordination solution (11). A fuzzy dominant $\mathfrak{q}$ that satisfies the fuzzy subordination $\widetilde{\mathfrak{q}} \prec_{\mathscr{F}} \mathfrak{q}$ is the fuzzy best dominant for all dominants $\mathfrak{q}$ of (11). Up to a rotation of $\mathbb{D}$, the fuzzy best dominant is known to be unique.

In [17], the dual notion of a fuzzy subordinant of a third-order fuzzy differential superordination is presented. as a function $\mathfrak{q} \in \mathscr{S}$ satisfying $\mathfrak{q}(\xi) \prec_{\mathscr{F}} \omega(\xi)$ or, equivalently written, as $\mathscr{F}_{\mathfrak{q}(\mathbb{D})}\mathfrak{q}(\xi) \leq \mathscr{F}_{\omega(\mathbb{D})}(\omega(\xi))$, $\xi \in \mathbb{D}$, whenever ω satisfies

$$\mathscr{F}_{\lambda(\mathbb{D})}\lambda(\xi) \leq \mathscr{F}_{\psi(\mathbb{C}^4\times\mathbb{D})}(\psi(\omega(\xi), \xi\omega'(\xi), \xi^2\omega''(\xi), \xi^3\omega'''(\xi); \xi)), \ \xi \in \mathbb{D}$$

i.e

$$\lambda(\xi) \prec_{\mathscr{F}} \psi(\omega(\xi), \xi\omega'(\xi), \xi^2\omega''(\xi), \xi^3\omega'''(\xi); \xi), \tag{12}$$

with ψ: $\mathbb{C}^4 \times \mathbb{D} \longrightarrow \mathbb{C}$, $\psi \in \Psi_{\kappa}[\Omega, \mathfrak{q}]$ and let $\lambda \in \mathscr{H}(\mathbb{D})$, ω is referred to as the fuzzy differential superordination solution (12). The best fuzzy subordinant for all fuzzy subordinants $\mathfrak{q}$ of (12) is a fuzzy subordinant $\widetilde{\mathfrak{q}} \in \mathscr{S}$ satisfying $\mathfrak{q}(\xi) \prec_{\mathscr{F}} \widetilde{\mathfrak{q}}(\xi)$ or, equivalently written, as $\mathscr{F}_{\mathfrak{q}(\mathbb{D})}\mathfrak{q}(\xi) \leq \mathscr{F}_{\omega(\mathbb{D})}(\omega(\xi))$, $\xi \in \mathbb{D}$.

The new result of the next sections is obtained by using the next two results mentioned as lemmas, which are proved in [16] and [17], respectively.

Lemma 1 ([16], Theorem 3.4) Let $p \in \mathscr{H}[a, \kappa]$ with $\kappa \geq 2$, suppose that $\mathscr{F}$: $\mathbb{C} \to [0, 1]$ and a function $\mathfrak{q} \in \mathfrak{Q}(a)$ for which:

$$Re\left\{\frac{\xi\mathfrak{q}''(\xi)}{\mathfrak{q}'(\xi)}\right\} \geq 0, \quad \left|\frac{\xi p'(\xi)}{\mathfrak{q}'(\xi)}\right| \leq \ell,$$

where $\xi \in \mathbb{D}$, $\xi \in \partial\mathbb{D}\backslash E(\mathfrak{q})$ and $\ell \geq \kappa$. If Ω is a set in $\mathbb{C}$, $\psi \in \Psi_{\kappa}[\Omega, \mathfrak{q}]$ satisfying

$$\{\psi(p(\xi), \xi p'(\xi), \xi^2 p''(\xi), \xi^3 p'''(\xi); \xi)\} \subset \Omega,$$

then

$$p(\xi) \prec_{\mathscr{F}} \mathfrak{q}(\xi) \text{ or } \mathscr{F}_{p(\mathbb{D})}p(\xi) \leq \mathscr{F}_{\mathfrak{q}(\mathbb{D})}\mathfrak{q}(\xi).$$

Lemma 2 ([17], Theorem 1) Consider $\Omega \in \mathbb{C}$, $\mathfrak{q} \in \mathscr{H}[a, \kappa]$, $\kappa \geq 2$, function $\psi \in \Psi_{\kappa}[\Omega, \mathfrak{q}]$, $\mathscr{F}$: $\mathbb{C} \to [0, 1]$ given by $F(\xi) = \frac{1}{1+|\xi|}$ and $p \in \mathfrak{Q}(a)$, $p(0) = \mathfrak{q}(0) = a$, satisfies:

$$Re\left\{\frac{\xi\mathfrak{q}''(\xi)}{\mathfrak{q}'(\xi)}\right\} \geq 0, \quad \left|\frac{\xi p'(\xi)}{\mathfrak{q}'(\xi)}\right| \leq \ell,$$

where $\xi \in \mathbb{D}$, $\xi \in \partial\mathbb{D}\backslash E(\mathfrak{q})$, $\ell \geq \kappa \geq 2$, if $p(\xi)$ and $\psi(p(\xi), \xi p'(\xi), \xi^2 p''(\xi), \xi^3 p'''(\xi); \xi)$ are univalent in $\mathbb{D}$ then

$$\Omega \subset \left\{\psi(p(\xi), \xi p'(\xi), \xi^2 p''(\xi), \xi^3 p'''(\xi); \xi): \xi \in \mathbb{D}\right\},$$

or equivalently,

$$\frac{1}{1+|\xi|} \leq \frac{1}{1+|\psi(p(\xi), \xi p'(\xi), \xi^2 p''(\xi), \xi^3 p'''(\xi); \xi)|}$$

implies that

$$\mathfrak{q}(\xi) \prec_{\mathscr{F}} p(\xi) \text{ or } \frac{1}{1+|\mathfrak{q}(\xi)|} \leq \frac{1}{1+|p(\xi)|} \quad (\xi \in \mathbb{D}).$$

3. Results on fuzzy third-order differential subordination

Unless otherwise noted, we will assume that $\sigma, \varepsilon, \nu \in \mathbb{R}$ throughout the investigation.

Some third-order fuzzy differential subordination results are obtained in this section. The class of admissible functions is defined as:

Definition 8 Let Ω be a set in $\mathbb{C}$ and $\mathfrak{q} \in \mathfrak{Q}_0 \cap \mathscr{H}_0$. The class of admissible functions $\Psi_\kappa[\Omega, \mathfrak{q}]$ consists of $\psi: \mathbb{C}^4 \times \mathbb{D} \to \mathbb{C}$ that satisfy:

$$\psi(\alpha, \beta, \gamma, \delta; \xi) \notin \Omega \tag{13}$$

whenever

$$\alpha = \mathfrak{q}(\xi), \quad \beta = \frac{\ell\xi\mathfrak{q}'(\xi) + (\sigma(1-\nu)-1)\mathfrak{q}(\xi)}{\sigma(1-\nu)}, \tag{14}$$

$$Re\left\{\frac{\sigma^2(1-\nu)^2[\gamma+\alpha-2\beta]-\alpha}{\sigma(1-\nu)\beta-[\sigma(1-\nu)-1]\alpha}+2\right\} \geq \ell Re\left\{\frac{\xi\mathfrak{q}''(\xi)}{\mathfrak{q}'(\xi)}+1\right\}, \tag{15}$$

and

$$Re\left\{\frac{(\sigma(1-\nu))^3[\delta-3(\gamma-\beta)-\alpha]+\alpha}{\sigma(1-\nu)\beta-[\sigma(1-\nu)-1]\alpha}-1\right\}$$

$$\geq \ell^2 Re\left\{\frac{\xi^2\mathfrak{q}'''(\xi)}{\mathfrak{q}'(\xi)}\right\}, \tag{16}$$

where $\ell \in \mathbb{N}\backslash\{1\}$, $\xi \in \partial\mathbb{D}\backslash E(\mathfrak{q})$ and $\xi \in \mathbb{D}$.

We now present and verify our first result as Theorem 1 below.

Theorem 1 Let $\psi \in \Psi_{\kappa}[\Omega, \mathfrak{q}]$. If $\mathfrak{f} \in \mathscr{A}$ and $\mathfrak{q} \in \mathfrak{Q}_0 \cap \mathscr{H}_0$ satisfies:

$$Re\left\{\frac{\xi \mathfrak{q}''(\xi)}{\mathfrak{q}'(\xi)}\right\} \geq 0, \quad \left|\sigma(1-\nu)\mathscr{D}_{\sigma}^{(\nu+\frac{1}{\sigma}, \varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi) - [\sigma(1-\nu)-1]\mathscr{D}_{\sigma}^{(\nu, \varepsilon)}\mathfrak{f}(\xi)\right| \leq \ell\left|\mathfrak{q}'(\xi)\right|, \tag{17}$$

and

$$\mathscr{F}_{\psi(\mathbb{C}^4\times\mathbb{D})}\psi(\mathscr{D}_{\sigma}^{(\nu, \varepsilon)}\mathfrak{f}(\xi), \mathscr{D}_{\sigma}^{(\nu+\frac{1}{\sigma}, \varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi), \mathscr{D}_{\sigma}^{(\nu+\frac{2}{\sigma}, \varepsilon-\frac{2}{\sigma})}\mathfrak{f}(\xi), \mathscr{D}_{\sigma}^{(\nu+\frac{3}{\sigma}, \varepsilon-\frac{3}{\sigma})}\mathfrak{f}(\xi); \xi) \leq \mathscr{F}_{\Omega}(\xi)$$

$$\ell \in \mathbb{N}\setminus\{1\}, \ \xi \in \partial\mathbb{D}\setminus E(\mathfrak{q}) \text{ and } \xi \in \mathbb{D}$$

then

$$\mathscr{D}_{\sigma}^{(\nu, \varepsilon)}\mathfrak{f}(\xi) \prec_{\mathscr{F}} \mathfrak{q}(\xi) \ (\xi \in \mathbb{D}). \tag{18}$$

or

$$\mathscr{F}_{\mathscr{D}_{\sigma}^{(\nu, \varepsilon)}\mathfrak{f}(\mathbb{D})}(\mathscr{D}_{\sigma}^{(\nu, \varepsilon)}\mathfrak{f}(\xi)) \leq \mathscr{F}_{\mathfrak{q}(\mathbb{D})}\mathfrak{q}(\xi). \tag{19}$$

Proof. Define

$$\mathfrak{g}(\xi) = \mathscr{D}_{\sigma}^{(\nu, \varepsilon)}\mathfrak{f}(\xi). \tag{20}$$

Utilizing of (7) and (20), we've

$$\mathscr{D}_{\sigma}^{(\nu+\frac{1}{\sigma}, \varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi) = \frac{\xi\mathfrak{g}'(\xi) + (\sigma(1-\nu)-1)\mathfrak{g}(\xi)}{\sigma(1-\nu)}. \tag{21}$$

By differentiation (21) and multiply by ξ then

$$\xi\left(\mathscr{D}_{\sigma}^{(\nu+\frac{1}{\sigma}, \varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi)\right)' = \frac{\xi^2\mathfrak{g}''(\xi) + (\sigma(1-\nu)-1)\xi\mathfrak{g}'(\xi)}{\sigma(1-\nu)},$$

using (7) again we obtain

$$\mathscr{D}_{\sigma}^{(\nu+\frac{2}{\sigma}, \varepsilon-\frac{2}{\sigma})}\mathfrak{f}(\xi) = \frac{\xi^2\mathfrak{g}''(\xi) + [2\sigma(1-\nu)-1]\xi\mathfrak{g}'(\xi) + [\sigma(1-\nu)-1]^2\mathfrak{g}(\xi)}{[\sigma(1-\nu)]^2}, \tag{22}$$

repeating the same calculation for (22) we obtain

$$\mathscr{D}_{\sigma}^{(\nu+\frac{3}{\sigma},\varepsilon-\frac{3}{\sigma})}\mathfrak{f}(\xi)=\frac{\xi^3\mathfrak{g}'''(\xi)+3\sigma(1-\nu)\xi^2\mathfrak{g}''(\xi)+[3\sigma(1-\nu)[\sigma(1-\nu)-1]+1]\xi\mathfrak{g}'(\xi)+[\sigma(1-\nu)-1]^3\mathfrak{g}(\xi)}{[\sigma(1-\nu)]^3} \tag{23}$$

We now define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$\alpha(t,u,\upsilon,w)=t,\quad \beta(t,u,\upsilon,w)=\frac{u+[\sigma(1-\nu)-1]t}{\sigma(1-\nu)}, \tag{24}$$

$$\gamma(t,u,\upsilon,w)=\frac{\upsilon+[2\sigma(1-\nu)-1]u+[\sigma(1-\nu)-1]^2t}{[\sigma(1-\nu)]^2}, \tag{25}$$

and

$$\delta(t,u,\upsilon,w;\xi)=\frac{w+3\sigma(1-\nu)\upsilon+[3\sigma(1-\nu)[\sigma(1-\nu)-1]+1]u+[\sigma(1-\nu)-1]^3t}{[\sigma(1-\nu)]^3}. \tag{26}$$

Let

$$\begin{aligned}\psi(t,u,\upsilon,w;\xi)&=\psi(\alpha,\beta,\gamma,\delta;\xi)\\&=\psi\left(t,\frac{u+[\sigma(1-\nu)-1]t}{\sigma(1-\nu)},\frac{\upsilon+[2\sigma(1-\nu)-1]u+[\sigma(1-\nu)-1]^2t}{[\sigma(1-\nu)]^2},\right.\\&\qquad\left.\frac{w+3\sigma(1-\nu)\upsilon+[3\sigma(1-\nu)[\sigma(1-\nu)-1]+1]u+[\sigma(1-\nu)-1]^3t}{[\sigma(1-\nu)]^3};\xi\right)\end{aligned} \tag{27}$$

Using Lemma 1, (20)–(23) and (24)–(27), we have

$$\begin{aligned}&\psi\left(\mathfrak{g}(\xi),\xi\mathfrak{g}'(\xi),\xi^2\mathfrak{g}''(\xi),\xi^3\mathfrak{g}'''(\xi);\xi\right)\\&=\psi\left(\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi),\mathscr{D}_{\sigma}^{(\nu+\frac{1}{\sigma},\varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi),\mathscr{D}_{\sigma}^{(\nu+\frac{2}{\sigma},\varepsilon-\frac{2}{\sigma})}\mathfrak{f}(\xi),\mathscr{D}_{\sigma}^{(\nu+\frac{3}{\sigma},\varepsilon-\frac{3}{\sigma})}\mathfrak{f}(\xi);\xi\right).\end{aligned} \tag{28}$$

Hence, (18) leads to

$$\mathscr{F}_{\psi(\mathbb{C}^4\times\mathbb{D})}\left(\psi\left(\mathfrak{g}(\xi),\xi\mathfrak{g}'(\xi),\xi^2\mathfrak{g}''(\xi),\xi^3\mathfrak{g}'''(\xi);\xi\right)\right)\leq\mathscr{F}_{\Omega}(\xi). \tag{29}$$

Additionally, applying (24)–(26), it is simple to obtain

$$\frac{\upsilon}{u}+1=\frac{\sigma^2(1-\nu)^2[\gamma+\alpha-2\beta]-\alpha}{\sigma(1-\nu)\beta-[\sigma(1-\nu)-1]\alpha}+2, \tag{30}$$

and

$$\frac{w}{u}=\frac{(\sigma(1-\nu))^3[\delta-3(\gamma-\beta)-\alpha]+\alpha}{\sigma(1-\nu)\beta-[\sigma(1-\nu)-1]\alpha}-1.$$

Thus, the admissibility condition of $\psi\in\Psi_\kappa[\Omega,\mathfrak{q}]$ in Definition 8 is equivalent to the admissibility condition of $\psi\in\Psi_2[\Omega,\mathfrak{q}]$ as given in Definition 6. Consequently, by using (17) and Lemma 1, we've $\mathscr{F}_{\mathfrak{g}(\mathbb{D})}\mathfrak{g}(\xi)\leq\mathscr{F}_{\mathfrak{q}(\mathbb{D})}\mathfrak{q}(\xi)$ or equivalently $\mathscr{F}_{\mathscr{D}_\sigma^{(\nu,\varepsilon)}\mathfrak{f}(\mathbb{D})}(\mathscr{D}_\sigma^{(\nu,\varepsilon)}\mathfrak{f}(\xi))\leq\mathscr{F}_{\mathfrak{q}(\mathbb{D})}\mathfrak{q}(\xi)$, i.e.,

$$\mathscr{D}_\sigma^{(\nu,\varepsilon)}\mathfrak{f}(\xi)\prec_{\mathscr{F}}\mathfrak{q}(\xi),$$

The proof is completed. □

Theorem 1 instantly yields the following result when the behavior of $\mathfrak{q}$ on $\partial\mathbb{D}$ is unknown.

Corollary 1 Let $\Omega\subset\mathbb{C}$ and let $\mathfrak{q}$ be univalent function in $\mathbb{D}$ with $\mathfrak{q}(0)=0$. Let $\psi\in\Psi_\kappa[\Omega,\mathfrak{q}_\rho]$ for some $\rho\in(0,1)$ where $\mathfrak{q}_\rho(\xi)=\mathfrak{q}(\rho\xi)$. If $\mathfrak{f}\in\mathscr{A}$ and $\mathfrak{q}_\rho$ satisfies

$$Re\left\{\frac{\xi\mathfrak{q}_\rho''(\xi)}{\mathfrak{q}_\rho'(\xi)}\right\}\geq 0,$$

$$\left|\sigma(1-\nu)\mathscr{D}_\sigma^{(\nu+\frac{1}{\sigma},\varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi)-[\sigma(1-\nu)-1]\mathscr{D}_\sigma^{(\nu,\varepsilon)}\mathfrak{f}(\xi)\right|\leq\ell\left|\mathfrak{q}_\rho'(\xi)\right| \tag{31}$$

$$(\xi\in\partial\mathbb{D}\backslash E(\mathfrak{q}_\rho);\ \xi\in\mathbb{D}),$$

and

$$\mathscr{F}_{\psi(\mathbb{C}^4\times\mathbb{D})}\psi(\mathscr{D}_\sigma^{(\nu,\varepsilon)}\mathfrak{f}(\xi),\mathscr{D}_\sigma^{(\nu+\frac{1}{\sigma},\varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi),\mathscr{D}_\sigma^{(\nu+\frac{2}{\sigma},\varepsilon-\frac{2}{\sigma})}\mathfrak{f}(\xi),\mathscr{D}_\sigma^{(\nu+\frac{3}{\sigma},\varepsilon-\frac{3}{\sigma})}\mathfrak{f}(\xi);\xi)\leq\mathscr{F}_\Omega(\xi),$$

then

$$\mathscr{D}_\sigma^{(\nu,\varepsilon)}\mathfrak{f}(\xi)\prec_{\mathscr{F}}\mathfrak{q}(\xi).$$

Proof. By applying Theorem 1, we get

$$\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi) \prec_{\mathscr{F}} \mathfrak{q}_{\rho}(\xi).$$

The Corollary 1 result is comes from

$$\mathfrak{q}_{\rho}(\xi) \prec_{\mathscr{F}} \mathfrak{q}(\xi) \quad (\xi \in \mathbb{D}).$$

□

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = \lambda(\mathbb{D})$ for some conformal mapping $\lambda(\xi)$ of $\mathbb{D}$ onto Ω. In this case the class $\Psi_{\kappa}[\mathscr{H}(\mathbb{D}), \mathfrak{q}]$ is denoted by $\Psi_{\kappa}[\lambda, \mathfrak{q}]$. The following two results follow directly from of Theorem 1 and Corollary 1.

Example 2 (i) Putting $\psi = \frac{\beta}{\alpha} - 1$, in Theorem 1 we get

$$\psi(\alpha, \beta, \gamma, \delta; \xi) = \frac{u + (\sigma(1-\nu)-1)t}{\sigma(1-\nu)t} - 1$$

$$= \frac{\sigma(1-\nu)\left[\mathscr{D}_{\sigma}^{(\nu+\frac{1}{\sigma}, \varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi) - t\right]}{\sigma(1-\nu)t}$$

$$= \frac{\mathscr{D}_{\sigma}^{(\nu+\frac{1}{\sigma}, \varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi)}{\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi)} - 1$$

which is analytic in U and

$$\mathscr{F}_{\psi(\mathbb{C}^4\times\mathbb{D})}\psi\left(\frac{\mathscr{D}_{\sigma}^{(\nu+\frac{1}{\sigma}, \varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi)}{\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi)} - 1\right) \leq \mathscr{F}_{\Omega}(\xi), \tag{32}$$

then

$$\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi) \prec_{\mathscr{F}} \mathfrak{q}(\xi) \ (\xi \in \mathbb{D}),$$

or

$$\mathscr{F}_{\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\mathbb{D})}(\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi)) \leq \mathscr{F}_{\mathfrak{q}(\mathbb{D})}\mathfrak{q}(\xi).$$

(ii) Taking $\sigma = 1$ and $\nu = \varepsilon = 0$, in equation (32), we obtain

$$\mathscr{F}_{\psi(\mathbb{C}^4\times\mathbb{D})}\psi\left(\frac{\xi \mathfrak{f}'(\xi)}{\mathfrak{f}(\xi)}-1\right)\leq \mathscr{F}_{\Omega}(\xi),$$

then

$$\mathfrak{f}(\xi)\prec_{\mathscr{F}}\mathfrak{q}(\xi)\ (\xi\in\mathbb{D}),$$

or

$$\mathscr{F}_{\mathfrak{f}(\mathbb{D})}(\mathfrak{f}(\xi))\leq \mathscr{F}_{\mathfrak{q}(\mathbb{D})}\mathfrak{q}(\xi).$$

Theorem 2 Suppose that $\psi\in\Psi_{\kappa}[\lambda,\mathfrak{q}]$. If $\mathfrak{f}\in\mathscr{A}$ and $\mathfrak{q}\in\mathfrak{Q}_0$ satisfy the condition (17), then

$$\mathscr{F}_{\psi(\mathbb{C}^4\times\mathbb{D})}\psi(\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi),\mathscr{D}_{\sigma}^{(\nu+\frac{1}{\sigma},\varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi),\mathscr{D}_{\sigma}^{(\nu+\frac{2}{\sigma},\varepsilon-\frac{2}{\sigma})}\mathfrak{f}(\xi),\mathscr{D}_{\sigma}^{(\nu+\frac{3}{\sigma},\varepsilon-\frac{3}{\sigma})}\mathfrak{f}(\xi);\xi)\leq \mathscr{F}_{\lambda(\Omega)}\lambda(\xi), \tag{33}$$

implies that

$$\mathscr{F}_{\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\mathbb{D})}(\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi))\leq \mathscr{F}_{\lambda(\Omega)}\lambda(\xi),$$

i.e.,

$$\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi)\prec_{\mathscr{F}}\lambda(\xi)\quad(\xi\in\mathbb{D}).$$

Corollary 2 Let $\Omega\subset\mathbb{C}$ and let $\mathfrak{q}$ be univalent function in $\mathbb{D}$ with $\mathfrak{q}(0)=0$. Let $\psi\in\Psi_{\kappa}[\lambda,\mathfrak{q}_{\rho}]$ for some $\rho\in(0,1)$ where $\mathfrak{q}_{\rho}(\xi)=\mathfrak{q}(\rho\xi)$. If $\mathfrak{f}\in\mathscr{A}$ and $\mathfrak{q}_{\rho}$ satisfies:

$$Re\left\{\frac{\xi \mathfrak{q}_{\rho}''(\xi)}{\mathfrak{q}_{\rho}'(\xi)}\right\}\geq 0,$$

$$\left|\sigma(1-\nu)\mathscr{D}_{\sigma}^{(\nu+\frac{1}{\sigma},\varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi)-[\sigma(1-\nu)-1]\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi)\right|\leq \ell\left|\mathfrak{q}_{\rho}'(\xi)\right| \tag{34}$$

$$(\xi\in\partial\mathbb{D}\backslash E(\mathfrak{q}_{\rho});\ \xi\in\mathbb{D}),$$

and

$$\mathscr{F}_{\psi(\mathbb{C}^4\times\mathbb{D})}\psi(\mathscr{D}_\sigma^{(\nu,\varepsilon)}\mathfrak{f}(\xi),\mathscr{D}_\sigma^{(\nu+\frac{1}{\sigma},\varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi),\mathscr{D}_\sigma^{(\nu+\frac{2}{\sigma},\varepsilon-\frac{2}{\sigma})}\mathfrak{f}(\xi),\mathscr{D}_\sigma^{(\nu+\frac{3}{\sigma},\varepsilon-\frac{3}{\sigma})}\mathfrak{f}(\xi);\xi)\leq\mathscr{F}_{\lambda(\Omega)}\lambda(\xi)$$

then

$$\mathscr{D}_\sigma^{(\nu,\varepsilon)}\mathfrak{f}(\xi)\prec_{\mathscr{F}}\mathfrak{q}(\xi)$$

Our next theorem provides the fuzzy best dominant of the fuzzy differential subordination (18) or (33).

Theorem 3 Let $\lambda(\xi)$ be univalent in $\mathbb{D}$. Also, let ψ: $\mathbb{C}^4\times\mathbb{D}\longrightarrow\mathbb{C}$ and ψ be given by (27). Let

$$\psi\left(\mathfrak{q}(\xi),\xi\mathfrak{q}'(\xi),\xi^2\mathfrak{q}''(\xi),\xi^3\mathfrak{q}'''(\xi);\xi\right)=\lambda(\xi)\tag{35}$$

has a solution $\mathfrak{q}(\xi)\in\mathfrak{Q}_0\cap\mathscr{H}_0$, which satisfies the conditions in (17). If $f\in\mathscr{A}$ satisfies condition (31) and

$$\mathscr{F}_{\psi(\mathbb{C}^4\times\mathbb{D})}\psi\{\mathscr{D}_\sigma^{(\nu,\varepsilon)}\mathfrak{f}(\xi),\mathscr{D}_\sigma^{(\nu+\frac{1}{\sigma},\varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi),\mathscr{D}_\sigma^{(\nu+\frac{2}{\sigma},\varepsilon-\frac{2}{\sigma})}\mathfrak{f}(\xi),\mathscr{D}_\sigma^{(\nu+\frac{3}{\sigma},\varepsilon-\frac{3}{\sigma})}\mathfrak{f}(\xi);\xi\}\leq\mathscr{F}_\Omega(\xi)$$

is analytic in $\mathbb{D}$, then

$$\mathscr{D}_\sigma^{(\nu,\varepsilon)}\mathfrak{f}(\xi)\prec_{\mathscr{F}}\mathfrak{q}(\xi),$$

and $\mathfrak{q}(\xi)$ is the fuzzy best dominant.

Proof. Using Theorem 1, we conclude that $\mathfrak{q}$ is a dominant of (33). Since $\mathfrak{q}$ satisfies (35), it is also a solution of (33). Therefore, $\mathfrak{q}$ will be dominated by all dominants. Consequently, $\mathfrak{q}$ is the fuzzy best dominating. □

Next, we defined a new admissible class $\widetilde{\Psi}_\kappa[\Omega,\mathfrak{q}]$ as follows:

Definition 9 Suppose that Ω be a set in $\mathbb{C}$ and $\mathfrak{q}\in\mathfrak{Q}_1\cap\mathscr{H}_1$. The class of admissible functions $\widetilde{\Psi}_\kappa[\Omega,\mathfrak{q}]$ consists of those functions ψ: $\mathbb{C}^4\times\mathbb{D}\to\mathbb{C}$ that satisfies:

$$\psi(\alpha,\beta,\gamma,\delta;\xi)\notin\Omega$$

whenever

$$\alpha=\mathfrak{q}(\xi),\quad\beta=\frac{\ell\xi\mathfrak{q}'(\xi)+\sigma(1-\nu)\mathfrak{q}(\xi)}{\sigma(1-\nu)},$$

$$Re\left\{\frac{\sigma(1-\nu)(\alpha+\gamma-2\beta)}{\beta-\alpha}\right\}\geq\ell Re\left\{\frac{\xi\mathfrak{q}''(\xi)}{\mathfrak{q}'(\xi)}+1\right\},$$

$$Re\left\{\frac{(\sigma(1-\nu))^2(\delta-\alpha-3\gamma+3\beta)-3\sigma(1-\nu)(\gamma+\alpha-2\beta)}{\beta-\alpha}+2\right\}\geq \ell^2 Re\left\{\frac{\xi^2\mathfrak{q}'''(\xi)}{\mathfrak{q}'(\xi)}\right\},$$

where $\ell\in\mathbb{N}\setminus\{1\}$, $\xi\in\partial\mathbb{D}\backslash E(\mathfrak{q})$ and $\xi\in\mathbb{D}$.

Theorem 4 Let $\psi\in\widetilde{\Psi}_{\kappa}[\Omega,\mathfrak{q}]$. If $\mathfrak{f}\in\mathscr{A}$ and $\mathfrak{q}\in\mathfrak{Q}_1$ satisfies:

$$Re\left\{\frac{\xi\mathfrak{q}''(\xi)}{\mathfrak{q}'(\xi)}\right\}\geq 0,\quad \left|\sigma(1-\nu)\mathscr{D}_{\sigma}^{(\nu+\frac{1}{\sigma},\varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi)-[\sigma(1-\nu)-1]\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi)\right|\leq \ell\left|\xi\mathfrak{q}'(\xi)\right|, \tag{36}$$

and

$$\mathscr{F}_{\psi(\mathbb{C}^4\times\mathbb{D})}\left(\psi\left(\frac{\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi)}{\xi},\frac{\mathscr{D}_{\sigma}^{(\nu+\frac{1}{\sigma},\varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi)}{\xi},\frac{\mathscr{D}_{\sigma}^{(\nu+\frac{2}{\sigma},\varepsilon-\frac{2}{\sigma})}\mathfrak{f}(\xi)}{\xi},\frac{\mathscr{D}_{\sigma}^{(\nu+\frac{3}{\sigma},\varepsilon-\frac{3}{\sigma})}\mathfrak{f}(\xi)}{\xi};\xi\right):\xi\in\mathbb{D}\right)\leq\mathscr{F}_{\Omega}(\xi), \tag{37}$$

then

$$\frac{\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi)}{\xi}\prec_{\mathscr{F}}\mathfrak{q}(\xi)\quad(\xi\in\mathbb{D}).$$

Proof. Let

$$\mathfrak{g}(\xi)=\frac{\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi)}{\xi}. \tag{38}$$

Utilizing of (7) and (38), we've

$$\frac{\mathscr{D}_{\sigma}^{(\nu+\frac{1}{\sigma},\varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi)}{\xi}=\frac{\xi\mathfrak{g}'(\xi)+\sigma(1-\nu)\mathfrak{g}(\xi)}{\sigma(1-\nu)}. \tag{39}$$

By differentiation (39) and multiply by ξ then

$$\xi\left(\frac{\mathscr{D}_{\sigma}^{(\nu+\frac{1}{\sigma},\varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi)}{\xi}\right)'=\frac{\xi^2\mathfrak{g}''(\xi)+\sigma(1-\nu)\xi\mathfrak{g}'(\xi)}{\sigma(1-\nu)}$$

using (7) again we obtain

$$\frac{\mathscr{D}_{\sigma}^{(\nu+\frac{2}{\sigma},\varepsilon-\frac{2}{\sigma})}\mathfrak{f}(\xi)}{\xi}=\frac{\xi^2\mathfrak{g}''(\xi)+[2\sigma(1-\nu)+1]\xi\mathfrak{g}'(\xi)+(\sigma(1-\nu))^2\mathfrak{g}(\xi)}{(\sigma(1-\nu))^2}, \tag{40}$$

repeating the same calculation for (40) we obtain

$$\frac{\mathscr{D}_{\sigma}^{(\nu+\frac{3}{\sigma},\,\varepsilon-\frac{3}{\sigma})}\mathfrak{f}(\xi)}{\xi}=\frac{\xi^{3}\mathfrak{g}'''(\xi)+3\left(\sigma(1-\nu)+1\right)\xi^{2}\mathfrak{g}''(\xi)}{\left(\sigma(1-\nu)\right)^{3}}+$$

$$\frac{\left[3\sigma(1-\nu)\left(\sigma(1-\nu)+1\right)+1\right]\xi\mathfrak{g}'(\xi)+\left(\sigma(1-\nu)\right)^{3}\mathfrak{g}(\xi)}{\left(\sigma(1-\nu)\right)^{3}}. \tag{41}$$

We now define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$\alpha(t,\,u,\,v,\,w)=t,\qquad \beta(t,\,u,\,v,\,w)=\frac{u+\sigma(1-\nu)t}{\sigma(1-\nu)}, \tag{42}$$

$$\gamma(t,\,u,\,\upsilon,\,w)=\frac{\upsilon+\left(2\sigma(1-\nu)+1\right)u+\left(\sigma(1-\nu)\right)^{2}t}{\left(\sigma(1-\nu)\right)^{2}},$$

and

$$\delta(t,\,u,\,\upsilon,\,w)=\frac{w+3\left(\sigma(1-\nu)+1\right)\upsilon+\left[3\sigma(1-\nu)\left(\sigma(1-\nu)+1\right)+1\right]u+\left(\sigma(1-\nu)\right)^{3}t}{\left(\sigma(1-\nu)\right)^{3}}.$$

Let

$$\psi(t,\,u,\,v,\,w;\,\xi)=\psi(\alpha,\,\beta,\,\gamma,\,\delta;\,\xi)$$

$$=\psi\left(t,\,\frac{u+\sigma(1-\nu)t}{\sigma(1-\nu)},\,\frac{\upsilon+\left(2\sigma(1-\nu)+1\right)u+\left(\sigma(1-\nu)\right)^{2}t}{\left(\sigma(1-\nu)\right)^{2}},\right.$$

$$\left.\frac{w+3\left(\sigma(1-\nu)+1\right)\upsilon+\left[3\sigma(1-\nu)\left(\sigma(1-\nu)+1\right)+1\right]u+\left(\sigma(1-\nu)\right)^{3}t}{\left(\sigma(1-\nu)\right)^{3}};\,\xi\right). \tag{43}$$

Using Lemma 1, (39)–(41) and (42)–(43), we have

$$\psi\left(\mathfrak{g}(\xi),\,\xi\mathfrak{g}'(\xi),\,\xi^{2}\mathfrak{g}''(\xi),\,\xi^{3}\mathfrak{g}'''(\xi);\,\xi\right)$$

$$=\left(\frac{\mathscr{D}_{\sigma}^{(\nu,\,\varepsilon)}\mathfrak{f}(\xi)}{\xi},\,\frac{\mathscr{D}_{\sigma}^{(\nu+\frac{1}{\sigma},\,\varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi)}{\xi},\,\frac{\mathscr{D}_{\sigma}^{(\nu+\frac{2}{\sigma},\,\varepsilon-\frac{2}{\sigma})}\mathfrak{f}(\xi)}{\xi},\,\frac{\mathscr{D}_{\sigma}^{(\nu+\frac{3}{\sigma},\,\varepsilon-\frac{3}{\sigma})}\mathfrak{f}(\xi)}{\xi};\,\xi\right). \tag{44}$$

Hence, (44) leads to

$$\mathscr{F}_{\psi(\mathbb{C}^4\times\mathbb{D})}\left(\psi\mathfrak{g}(\xi),\, \xi\mathfrak{g}'(\xi),\, \xi^2\mathfrak{g}''(\xi),\, \xi^3\mathfrak{g}'''(\xi);\, \xi\right) \leq \mathscr{F}_{\Omega}(\xi).$$

Using (42)–(43), then we have

$$\frac{v}{u}+1 = \frac{\sigma(1-\nu)(\alpha+\gamma-2\beta)}{\beta-\alpha},$$

and

$$\frac{w}{u} = \frac{(\sigma(1-\nu))^2(\delta-\alpha-3\gamma+3\beta)-3\sigma(1-\nu)(\gamma+\alpha-2\beta)}{\beta-\alpha}+2.$$

Thus, the admissibility condition for $\psi \in \widetilde{\Psi}_{\kappa}[\Omega,\, \mathfrak{q}]$ in Definition 9 is equivalent to the admissibility condition for $\psi \in \Psi_2[\Omega,\, \mathfrak{q}]$ as given in Definition 6. Consequently, by using (36) and Lemma 1, we have $\mathscr{F}_{\mathfrak{g}(\mathbb{D})}\mathfrak{g}(\xi) \leq \mathscr{F}_{\mathfrak{q}(\mathbb{D})}\mathfrak{q}(\xi)$ or equivalently $\mathscr{F}_{\frac{1}{\xi}\mathscr{D}_{\sigma}^{(\nu,\,\varepsilon)}\mathfrak{f}(\mathbb{D})}(\frac{\mathscr{D}_{\sigma}^{(\nu,\,\varepsilon)}\mathfrak{f}(\xi)}{\xi}) \leq \mathscr{F}_{\mathfrak{q}(\mathbb{D})}\mathfrak{q}(\xi)$, i.e.,

$$\frac{\mathscr{D}_{\sigma}^{(\nu,\,\varepsilon)}\mathfrak{f}(\xi)}{\xi} \prec_{\mathscr{F}} \mathfrak{q}(\xi).$$

The proof is completed. □

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = \lambda(\mathbb{D})$ for some conformal mapping $\lambda(\xi)$ of $\mathbb{D}$ onto Ω. In this case the class $\widetilde{\Psi}_{\kappa}[\lambda(\mathbb{D}),\, \mathfrak{q}]$ is written as $\widetilde{\Psi}_{\kappa}[\lambda,\, \mathfrak{q}]$. The next two results are immediate consequences of Theorem 4

Theorem 5 Assume that $\psi \in \widetilde{\Psi}_{\kappa}[\lambda,\, \mathfrak{q}]$. If $f \in \mathscr{A}$ and $\mathfrak{q} \in \mathfrak{Q}_1$ satisfies the condition (36), then

$$\mathscr{F}_{\psi(\mathbb{C}^4\times\mathbb{D})}\left(\psi\left(\frac{\mathscr{D}_{\sigma}^{(\nu,\,\varepsilon)}\mathfrak{f}(\xi)}{\xi}, \frac{\mathscr{D}_{\sigma}^{(\nu+\frac{1}{\sigma},\,\varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi)}{\xi}, \frac{\mathscr{D}_{\sigma}^{(\nu+\frac{2}{\sigma},\,\varepsilon-\frac{2}{\sigma})}\mathfrak{f}(\xi)}{\xi}, \frac{\mathscr{D}_{\sigma}^{(\nu+\frac{3}{\sigma},\,\varepsilon-\frac{3}{\sigma})}\mathfrak{f}(\xi)}{\xi};\, \xi\right) : \xi \in \mathbb{D}\right) \leq \mathscr{F}_{\lambda(\Omega)}\lambda(\xi),$$

implies that

$$\frac{\mathscr{D}_{\sigma}^{(\nu,\,\varepsilon)}\mathfrak{f}(\xi)}{\xi} \prec_{\mathscr{F}} \mathfrak{q}(\xi) \quad (\xi \in \mathbb{D}).$$

Example 3 Consider $p(\xi) = 1+\xi^2,$ hence $p \in \mathscr{H}[1,\, 2]$. Compute $p'(\xi) = 2\xi,$ and futher obtain $p''(\xi) = 2$ and $p'''(\xi) = 0$. Let

$$\psi\colon \mathbb{C}^4\times\mathbb{D} \to \mathbb{C}, \quad \psi(t,\, u,\, v,\, w;\, \xi) = t+\frac{1}{2}u+\frac{1}{2}v+\frac{1}{5}w.$$

Taking $t = p(\xi) = 1+\xi^2,\ u = \xi p'(\xi) = 2\xi^2,\ v = \xi^2 p''(\xi) = 2\xi^2,$ and $w = \xi^3 p'''(\xi) = 0,$ we get

$$\psi\left(1+\xi^2, 2\xi^2, 2\xi^2, 0; \xi\right) = 1+\xi^2+\frac{1}{2}2\xi^2+\frac{1}{2}2\xi^2+\frac{1}{5}0 = 1+3\xi^2.$$

Using Lemma 1 we have the following. Let $p(\xi) = 1+\xi^2,\ p \in \mathscr{H}[1, 2]$ and $\mathfrak{q} \in \mathfrak{Q}(1),\ \mathfrak{q}(\xi) = 1+\xi,$ an univalent function in $\mathbb{D}$ which satisfies

$$Re\left\{\frac{\xi \mathfrak{q}''(\xi)}{\mathfrak{q}'(\xi)}\right\} \geq 0, \quad \left|\frac{\xi p'(\xi)}{\mathfrak{q}'(\xi)}\right| \leq 2.$$

Function $\mathfrak{h}(\xi) = \frac{1-\xi}{1+\xi}$ maps conformally $\mathbb{D}$ into the half plane $\{\xi \in \mathbb{D}:\ Re\xi > 0\}$. Let $\mathscr{F}: \varsigma \to [0, 1]$. If $\psi \in \Psi_\kappa[\Omega, \mathfrak{q}],$

$$\psi\left(1+\xi^2, 2\xi^2, 2\xi^2, 0; \xi\right) = 1+3\xi^2$$

and the following third-order fuzzy differential subordination holds

$$1+3\xi^2 \prec_{\mathscr{F}} \frac{1-\xi}{1+\xi},\ \xi \in \mathbb{D},$$

or

$$\mathscr{F}_{\psi(\mathbb{C}^4\times\mathbb{D})}\psi\left(1+\xi^2, 2\xi^2, 2\xi^2, 0; \xi\right)) \leq \mathscr{F}_{\mathfrak{h}(\mathbb{D})}\mathfrak{h}(\xi),$$

then

$$p(\xi) = 1+\xi^2 \prec_{\mathscr{F}} \mathfrak{q}(\xi) = 1+\xi,\quad \xi \in \mathbb{D},$$

or

$$\mathscr{F}_{p(\mathbb{D})}p(\xi) \leq \mathscr{F}_{\mathfrak{q}(\mathbb{D})}\mathfrak{q}(\xi),\quad \xi \in \mathbb{D}.$$

4. Results on fuzzy third-order differential superordination

In this part, we obtain several results for third-order fuzzy differential superordination. For this purpose, the class of admissible functions is specified as follows:

Definition 10 Consider Ω be a set in $\mathbb{C}$, $\mathfrak{q} \in \mathscr{H}_0$ with $\mathfrak{q}'(\xi) \neq 0$ and $\vartheta \in \mathbb{N}\backslash\{1\}$. The class of admissible functions $\Psi'_\kappa[\Omega, \mathfrak{q}]$ consists of those functions $\psi: \mathbb{C}^4 \times \overline{\mathbb{D}} \to \mathbb{C}$ that satisfies:

$$\psi(\alpha, \beta, \gamma, \delta; \xi) \in \Omega \tag{45}$$

whenever

$$\alpha = \mathfrak{q}(\xi), \quad \beta = \frac{\xi \mathfrak{q}'(\xi) + (\sigma(1-\nu)-1)\mathfrak{q}(\xi)}{\sigma(1-\nu)}, \tag{46}$$

$$Re\left\{ \frac{\sigma^2(1-\nu)^2[\gamma+\alpha-2\beta]-\alpha}{\sigma(1-\nu)\beta-[\sigma(1-\nu)-1]\alpha}+2 \right\} \leq \frac{1}{\vartheta} Re\left\{ 1+\frac{\xi \mathfrak{q}''(\xi)}{\mathfrak{q}'(\xi)} \right\}, \tag{47}$$

$$Re\left\{ \frac{(\sigma(1-\nu))^3[\delta-3(\gamma-\beta)-\alpha]+\alpha}{\sigma(1-\nu)\beta-[\sigma(1-\nu)-1]\alpha}-1 \right\} \leq \frac{1}{\vartheta^2} Re\left\{ \frac{\xi^2 \mathfrak{q}'''(\xi)}{\mathfrak{q}'(\xi)} \right\} \tag{48}$$

where $\vartheta \geq 2$, $\xi \in \partial\mathbb{D}$ and $\xi \in \mathbb{D}$.

Theorem 6 Let $\psi \in \Psi_{\kappa}'[\Omega, \mathfrak{q}]$. If $\mathfrak{f} \in \mathscr{A}$ and $\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi) \in \mathfrak{Q}_0$ satisfies:

$$Re\left\{ \frac{\xi \mathfrak{q}''(\xi)}{\mathfrak{q}'(\xi)} \right\} \geq 0, \quad \left| \sigma \mathscr{D}_{\sigma}^{(\nu+\frac{1}{\sigma}, \varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi) - (\sigma-1)\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi) \right| \leq \vartheta \left| \mathfrak{q}'(\xi) \right|, \tag{49}$$

and

$$\psi\left(\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi), \mathscr{D}_{\sigma}^{(\nu+\frac{1}{\sigma}, \varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi), \mathscr{D}_{\sigma}^{(\nu+\frac{2}{\sigma}, \varepsilon-\frac{2}{\sigma})}\mathfrak{f}(\xi), \mathscr{D}_{\sigma}^{(\nu+\frac{3}{\sigma}, \varepsilon-\frac{3}{\sigma})}\mathfrak{f}(\xi); \xi \right)$$

is univalent in $\mathbb{D}$. Then

$$\mathscr{F}_{\Omega}(\xi) \leq \mathscr{F}_{\psi(\mathbb{C}^4\times\mathbb{D})}\left(\psi\left(\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi), \mathscr{D}_{\sigma}^{(\nu+\frac{1}{\sigma}, \varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi), \mathscr{D}_{\sigma}^{(\nu+\frac{2}{\sigma}, \varepsilon-\frac{2}{\sigma})}\mathfrak{f}(\xi), \mathscr{D}_{\sigma}^{(\nu+\frac{3}{\sigma}, \varepsilon-\frac{3}{\sigma})}\mathfrak{f}(\xi); \xi \right) : \xi \in \mathbb{D} \right),$$

implies

$$\mathfrak{q}(\xi) \prec_{\mathscr{F}} \mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi) \ (\xi \in \mathbb{D}). \tag{50}$$

Proof. Let $\mathfrak{g}(\xi)$ be given by (20) and ψ be given by (27). Since $\psi \in \Psi_{\kappa}'[\Omega, \mathfrak{q}]$, (28) and (50) yield

$$\mathscr{F}_{\Omega}(\xi) \leq \mathscr{F}_{\psi(\mathbb{C}^4\times\mathbb{D})}\left(\psi\left(\mathfrak{g}(\xi), \xi\mathfrak{g}'(\xi), \xi^2\mathfrak{g}''(\xi), \xi^3\mathfrak{g}'''(\xi); \xi \right) : \xi \in \mathbb{D} \right). \tag{51}$$

From (27), we deduce that the admissible condition for $\psi \in \Psi_{\kappa}'[\Omega, \mathfrak{q}]$ in Definition 10 is equivalent to the admissible condition for ψ as given in Definition 8. Consequently, using Lemma 2 and the words in (49), we have

$$\mathfrak{q}(\xi) \prec_{\mathscr{F}} \mathfrak{g}(\xi), \tag{52}$$

or, equivalently,

$$\mathfrak{q}(\xi) \prec_{\mathscr{F}} \mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi)\ (\xi \in \mathbb{D}), \tag{53}$$

this completes the proof. □

Example 4 $\psi = 2\alpha - \beta,$ in Theorem 6 we obtain

$$\psi(\alpha, \beta, \gamma, \delta; \tau) = 2t - \frac{u + (\sigma(1-\nu) - 1)t}{\sigma(1-\nu)}$$

$$= \frac{\sigma(1-\nu)t - \sigma(1-\nu)\left[\mathscr{D}_{\sigma}^{(\nu+\frac{1}{\sigma}, \varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi) - t)\right]}{\sigma(1-\nu)}$$

$$= 2t - \mathscr{D}_{\sigma}^{(\nu+\frac{1}{\sigma}, \varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi)$$

$$= 2\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi) - \mathscr{D}_{\sigma}^{(\nu+\frac{1}{\sigma}, \varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi)$$

is analytic in $\mathfrak{U}$ and

$$\mathscr{F}_{\Omega}(\xi) \leq \mathscr{F}_{\psi(\mathbb{C}^4 \times \mathfrak{U})}\psi\left(2\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi) - \mathscr{D}_{\sigma}^{(\nu+\frac{1}{\sigma}, \varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi)\right), \tag{54}$$

then

$$\mathfrak{q}(\xi) \prec_{\mathscr{F}} \mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi)\ (\xi \in \mathfrak{U}),$$

or

$$\mathscr{F}_{\mathfrak{q}(\mathfrak{D})}\mathfrak{q}(\xi) \leq \mathscr{F}_{\mathfrak{f}(\mathfrak{D})}(\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi)).$$

(ii) Taking $\sigma = 1$ and $\nu = \varepsilon = 0$, in equation (54), we obtain

$$\mathscr{F}_{\Omega}(\zeta) \leq \mathscr{F}_{\psi(\mathbb{C}^4 \times \mathfrak{U})}\psi\left(2\mathfrak{f}(\xi) - \xi\mathfrak{f}'(\xi)\right),$$

then

$$\mathfrak{q}(\zeta) \prec_{\mathscr{F}} \mathfrak{f}(\xi)\ (\zeta \in \mathfrak{U}),$$

or

$$\mathscr{F}_{\mathfrak{q}(\mathfrak{D})}\mathfrak{q}(\zeta) \leq \mathscr{F}_{\zeta\mathfrak{f}(\mathfrak{D})}\mathfrak{f}(\xi).$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain and $\Omega = \lambda(\mathbb{D})$ for some conformal mapping $\lambda(\xi)$ of $\mathbb{D}$ onto Ω, then the class $\Psi_{\kappa}^{'}[\lambda(\mathbb{D}), \mathfrak{q}]$ is written simply as $\Psi_{\kappa}^{'}[\lambda, \mathfrak{q}]$. With proceedings similar as in the preceding section, the coming Theorem is followes from Theorem 6.

Theorem 7 Consider $\psi \in \Psi_{\kappa}^{'}[\lambda, \mathfrak{q}]$, λ be analytic in $\mathbb{D}$. If $\mathfrak{f} \in \mathscr{A}$ and $\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi) \in \mathfrak{Q}_0$ satisfy the conditions in (49) and

$$\psi\left(\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi), \mathscr{D}_{\sigma}^{(\nu+\frac{1}{\sigma},\varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi), \mathscr{D}_{\sigma}^{(\nu+\frac{2}{\sigma},\varepsilon-\frac{2}{\sigma})}\mathfrak{f}(\xi), \mathscr{D}_{\sigma}^{(\nu+\frac{3}{\sigma},\varepsilon-\frac{3}{\sigma})}\mathfrak{f}(\xi); \xi\right) \tag{55}$$

is univalent in $\mathbb{D}$. Then

$$\mathscr{F}_{\lambda(\Omega)}\lambda(\xi) \leq \mathscr{F}_{\psi(\mathbb{C}^4\times\mathbb{D})}\psi\left(\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi), \mathscr{D}_{\sigma}^{(\nu+\frac{1}{\sigma},\varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi), \mathscr{D}_{\sigma}^{(\nu+\frac{2}{\sigma},\varepsilon-\frac{2}{\sigma})}\mathfrak{f}(\xi), \mathscr{D}_{\sigma}^{(\nu+\frac{3}{\sigma},\varepsilon-\frac{3}{\sigma})}\mathfrak{f}(\xi); \xi\right),$$

implies

$$\mathfrak{q}(\xi) \prec_{\mathscr{F}} \mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi) \ (\xi \in \mathbb{D}). \tag{56}$$

The following theorem proves that the best subordinant of (55) exists for a sufficiently selected ψ.

Theorem 8 Assume that λ be analytic in $\mathbb{D}$, ψ: $\mathbb{C}^4 \times \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ and ψ be given by (27). Let the differential equation

$$\psi\left(\mathfrak{q}(\xi), \xi\mathfrak{q}'(\xi), \xi^2\mathfrak{q}''(\xi), \xi^3\mathfrak{q}'''(\xi); \xi\right) = \lambda(\xi)$$

has a solution $\mathfrak{q}(\xi) \in \mathfrak{Q}_0$. If the $\mathfrak{f} \in \mathscr{A}$ and $\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi) \in \mathfrak{Q}_0$ satisfy the condition (49) and

$$\psi\left(\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi), \mathscr{D}_{\sigma}^{(\nu+\frac{1}{\sigma},\varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi), \mathscr{D}_{\sigma}^{(\nu+\frac{2}{\sigma},\varepsilon-\frac{2}{\sigma})}\mathfrak{f}(\xi), \mathscr{D}_{\sigma}^{(\nu+\frac{3}{\sigma},\varepsilon-\frac{3}{\sigma})}\mathfrak{f}(\xi); \xi\right)$$

is univalent in $\mathbb{D}$, then

$$\mathscr{F}_{\lambda(\Omega)}\lambda(\xi) \leq \mathscr{F}_{\psi(\mathbb{C}^4\times\mathbb{D})}\psi\left(\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi), \mathscr{D}_{\sigma}^{(\nu+\frac{1}{\sigma},\varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi), \mathscr{D}_{\sigma}^{(\nu+\frac{2}{\sigma},\varepsilon-\frac{2}{\sigma})}\mathfrak{f}(\xi), \mathscr{D}_{\sigma}^{(\nu+\frac{3}{\sigma},\varepsilon-\frac{3}{\sigma})}\mathfrak{f}(\xi); \xi\right),$$

implies

$$\mathfrak{q}(\xi) \prec_{\mathscr{F}} \mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi) \ (\xi \in \mathbb{D}),$$

and $\mathfrak{q}$ is the fuzzy best subordinant.

Proof. The proof of the theorem is similar to the proof of Theorem 3. □

Next, we introduce a new admissible class, $\widetilde{\Psi}'_{\kappa}[\Omega, \mathfrak{q}]$, as follows:

Definition 11 Let Ω be a set in $\mathbb{C}$, $\mathfrak{q} \in \mathscr{H}_1$ with $\mathfrak{q}'(\xi) \neq 0$ and $\vartheta \in \mathbb{N}\backslash\{1\}$. The class of admissible functions $\widetilde{\Psi}'_{\kappa}[\Omega, \mathfrak{q}]$ consists of ψ: $\mathbb{C}^4 \times \overline{\mathbb{D}} \to \mathbb{C}$ that satisfies:

$$\psi(\alpha, \beta, \gamma, \delta; \xi) \in \Omega$$

whenever

$$\alpha = \mathfrak{q}(\xi), \quad \beta = \frac{\xi\mathfrak{q}'(\xi) + \vartheta\sigma(1-\nu)\mathfrak{q}(\xi)}{\vartheta\sigma(1-\nu)},$$

$$Re\left\{\frac{\sigma(1-\nu)(\alpha+\gamma-2\beta)}{\beta-\alpha}\right\} \leq \frac{1}{\vartheta} Re\left\{1 + \frac{\xi\mathfrak{q}''(\xi)}{\mathfrak{q}'(\xi)}\right\},$$

$$Re\left\{\frac{(\sigma(1-\nu))^2(\delta-\alpha-3\gamma+3\beta) - 3\sigma(1-\nu)(\gamma+\alpha-2\beta)}{\beta-\alpha} + 2\right\} \leq \frac{1}{\vartheta^2} Re\left\{\frac{\xi^2\mathfrak{q}'''(\xi)}{\mathfrak{q}'(\xi)}\right\},$$

where $\vartheta \geq 2$, $\xi \in \partial\mathbb{D}$ and $\xi \in \mathbb{D}$.

If $\Omega \neq \mathbb{C}$ is a simply connected domain and $\Omega = \lambda(\mathbb{D})$ for some conformal mapping $\lambda(\xi)$ of $\mathbb{D}$ onto Ω, then the class $\widetilde{\Psi}'_{\kappa}[\lambda(\mathbb{D}), \mathfrak{q}]$ is denoted by $\widetilde{\Psi}'_{\kappa}[\lambda, \mathfrak{q}]$. With proceedings similar as in the preceding section, the coming Theorem is followes from Theorem 8.

Theorem 9 Consider λ be analytic in $\mathbb{D}$, ψ: $\mathbb{C}^4 \times \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ and ψ be given by (27). Let the differential equation

$$\psi\left(\mathfrak{q}(\xi), \xi\mathfrak{q}'(\xi), \xi^2\mathfrak{q}''(\xi), \xi^3\mathfrak{q}'''(\xi); \xi\right) = \lambda(\xi)$$

has a solution $\mathfrak{q}(\xi) \in \mathfrak{Q}_1$. If $f \in A$ and $\frac{\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi)}{\xi} \in \mathfrak{Q}_1$ satisfy the conditions in (36) and

$$\psi\left(\frac{\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi)}{\xi}, \frac{\mathscr{D}_{\sigma}^{(\nu+\frac{1}{\sigma},\varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi)}{\xi}, \frac{\mathscr{D}_{\sigma}^{(\nu+\frac{2}{\sigma},\varepsilon-\frac{2}{\sigma})}\mathfrak{f}(\xi)}{\xi}, \frac{\mathscr{D}_{\sigma}^{(\nu+\frac{3}{\sigma},\varepsilon-\frac{3}{\sigma})}\mathfrak{f}(\xi)}{\xi}; \xi\right)$$

is univalent in $\mathbb{D}$, then

$$\mathscr{F}_{\lambda(\Omega)}\lambda(\xi) \leq \mathscr{F}_{\psi(\mathbb{C}^4\times\mathbb{D})}\psi\left(\frac{\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi)}{\xi}, \frac{\mathscr{D}_{\sigma}^{(\nu+\frac{1}{\sigma},\varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi)}{\xi}, \frac{\mathscr{D}_{\sigma}^{(\nu+\frac{2}{\sigma},\varepsilon-\frac{2}{\sigma})}\mathfrak{f}(\xi)}{\xi}, \frac{\mathscr{D}_{\sigma}^{(\nu+\frac{3}{\sigma},\varepsilon-\frac{3}{\sigma})}\mathfrak{f}(\xi)}{\xi}; \xi\right),$$

implies

$$\mathfrak{q}(\xi) \prec_{\mathscr{F}} \frac{\mathscr{D}_{\sigma}^{(\nu,\varepsilon)}\mathfrak{f}(\xi)}{\xi} \ (\xi \in \mathbb{D}).$$

and $\mathfrak{q}$ is the fuzzy best subordinant.

Proof. The proof of Theorem is similar to that of Theorem 4. □

Example 5 Let $\psi(t, u, v, w; \xi): \mathbb{C}^4 \times \mathbb{D} \to \mathbb{C}$ given by $\psi(t, u, v, w; \xi) = t + \frac{1}{2}u + \frac{1}{4}v + \frac{1}{3}w$ and let $\mathfrak{q}(\xi) = 1 + \frac{2}{3}\xi^3$, a solution of the differential equation

$$\mathfrak{h}(\xi) = \psi\left(\mathfrak{q}(\xi), \xi\mathfrak{q}'(\xi), \xi^2\mathfrak{q}''(\xi), \xi^3\mathfrak{q}'''(\xi); \xi\right) = \psi\left(1 + \frac{2}{3}\xi^3, \frac{1}{2}2\xi^3, \frac{1}{4}4\xi^3, \frac{1}{3}4\xi^3\right) = 1 + 4\xi^3$$

$p \in \mathscr{H}[1, 2]$. Compute $p'(\xi) = 2\xi,$ and futher obtain $p''(\xi) = 2$ and $p'''(\xi) = 0$. Let

$$\psi\colon \mathbb{C}^4 \times \mathbb{D} \to \mathbb{C}, \quad \psi(t, u, v, w; \xi) = t + \frac{1}{2}u + \frac{1}{2}v + \frac{1}{3}w.$$

Taking $t = p(\xi) = 1 + \xi^2,\ u = \xi p'(\xi) = 2\xi^2,\ v = \xi^2 p''(\xi) = 2\xi^2,$ and $w = \xi^3 p'''(\xi) = 0,$ we get

$$\psi\left(1 + \xi^2, 2\xi^2, 2\xi^2, 0; \xi\right) = 1 + \xi^2 + \frac{1}{2}2\xi^2 + \frac{1}{4}2\xi^2 + \frac{1}{3}0 = 1 + \frac{5}{2}\xi^2.$$

Consider the functions $p(\xi) = 1 + 2\xi + \xi^2,$ and

$$\psi(p(\xi), \xi p'(\xi), \xi^2 p''(\xi), \xi^3 p'''(\xi); \xi \in \mathbb{D}) = 1 + 2\xi + \xi^2 + \xi(2 + 2\xi) + 2\xi^2 = 1 + 4\xi + 5\xi^2$$

univalent function in $\mathbb{D}$. Furthermore, take $\mathscr{F}\colon \varsigma \to [0, 1]$ F given by $F(\xi) = \frac{1}{1+|\xi|}$. Using the findings given by Lemma 2, the following numerical illustration is obtained. Let $\mathfrak{q}(\xi) = 1 + \frac{2}{3}\xi^3,$ a univalent solution of the differential equation

$$\mathfrak{h}(\xi) = (\psi\left(\mathfrak{q}(\xi), \xi\mathfrak{q}'(\xi), \xi^2\mathfrak{q}''(\xi), \xi^3\mathfrak{q}'''(\xi); \xi\right) = \psi\left(1 + \frac{2}{3}\xi^3, \frac{1}{2}2\xi^3, \frac{1}{4}4\xi^3, \frac{1}{3}4\xi^3\right) = 1 + 4\xi^3$$

Consider the functions $p(\xi) = 1 + 2\xi + \xi^2,$ and

$$\psi(p(\xi), \xi p'(\xi), \xi^2 p''(\xi), \xi^3 p'''(\xi); \xi \in \mathbb{D}) = 1 + 2\xi + \xi^2 + \xi(2 + 2\xi) + 2\xi^2 = 1 + 4\xi + 5\xi^2$$

univalent function in $\mathbb{D}$. If the following conditions are satisfied:

$$Re\left\{\frac{\xi\mathfrak{q}''(\xi)}{\mathfrak{q}'(\xi)}\right\} \geq 0, \quad \left|\frac{\xi(2 + 2\xi)}{\xi^3}\right| \leq \ell,$$

then

$$1+4\xi^3 \prec_{\mathscr{F}} 1+4\xi+5\xi^2,\ \ \xi\in\mathbb{D},$$

or

$$\frac{1}{1+|1+4\xi^3|}\leq\frac{1}{1+|1+4\xi+5\xi^2|}$$

whivh implies

$$1+\frac{2}{3}\xi^3 \prec 1+2\xi+\xi^2,\ \ \xi\in\mathbb{D},$$

or

$$\frac{1}{1+\left|1+\frac{2}{3}\xi^3\right|}\leq\frac{1}{1+|1+2\xi+\xi^2|}$$

and $\mathfrak{q}(\xi)=1+\frac{2}{3}\xi^3$ is the best fuzzy subordinant.

5. Sandwich-type results

This section displays two sandwich-style results. When Theorems 1 and 6 are combined, the following sandwich-type result is obtained:

Theorem 10 Let the functions λ_1 and $\mathfrak{q}_1$ be analytic functions in $\mathbb{D}$. Also let λ_2 be univalent in $\mathbb{D}$, $\mathfrak{q}_2\in\mathfrak{Q}_0$ with $\mathfrak{q}_1(0)=\mathfrak{q}_2(0)=0$ and $\psi\in\Psi_\kappa[\lambda_2,\mathfrak{q}_2]\cap\Psi'_\kappa[\lambda_1,\mathfrak{q}_1]$. If $\mathfrak{f}\in\mathscr{A}$ and $\mathscr{D}_\sigma^{(\nu,\varepsilon)}\mathfrak{f}(\xi)\in\mathfrak{Q}_0\cap\mathscr{H}_0$ and

$$\psi\left(\mathscr{D}_\sigma^{(\nu,\varepsilon)}\mathfrak{f}(\xi),\mathscr{D}_\sigma^{(\nu+\frac{1}{\sigma},\varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi),\mathscr{D}_\sigma^{(\nu+\frac{2}{\sigma},\varepsilon-\frac{2}{\sigma})}\mathfrak{f}(\xi),\mathscr{D}_\sigma^{(\nu+\frac{3}{\sigma},\varepsilon-\frac{3}{\sigma})}\mathfrak{f}(\xi);\xi\right)$$

is univalent in $\mathbb{D}$, and the conditions in (17) and (49) are satisfied, then

$$\mathscr{F}_{\lambda_1(\Omega)}\lambda_1(\xi)\leq\mathscr{F}_{\psi(\mathbb{C}^4\times\mathbb{D})}\psi\left(\mathscr{D}_\sigma^{(\nu,\varepsilon)}\mathfrak{f}(\xi),\mathscr{D}_\sigma^{(\nu+\frac{1}{\sigma},\varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi),\mathscr{D}_\sigma^{(\nu+\frac{2}{\sigma},\varepsilon-\frac{2}{\sigma})}\mathfrak{f}(\xi),\mathscr{D}_\sigma^{(\nu+\frac{3}{\sigma},\varepsilon-\frac{3}{\sigma})}\mathfrak{f}(\xi);\xi\right)$$

$$\leq\mathscr{F}_{\lambda_2(\Omega)}\lambda_2(\xi),$$

implies that

$$\lambda_1(\xi)\prec_{\mathscr{F}}\mathscr{D}_\sigma^{(\nu,\varepsilon)}\mathfrak{f}(\xi)\prec_{\mathscr{F}}\lambda_2(\xi).$$

Likewise, by combining Theorems 4 and 9, we arrive at:

Theorem 11 Let the functions $\widetilde{\lambda}_1$ and $\widetilde{\mathfrak{q}}_1$ be analytic functions in $\mathbb{D}$. Also let $\widetilde{\lambda}_2$ be univalent in $\mathbb{D}$, $\widetilde{\mathfrak{q}}_2 \in \mathfrak{Q}_1$ with $\widetilde{\mathfrak{q}}_1(0) = \widetilde{\mathfrak{q}}_2(0) = 1$ and $\psi \in \widetilde{\Psi}_\kappa[\widetilde{\lambda}_2, \widetilde{\mathfrak{q}}_2] \cap \widetilde{\Psi}'_\kappa[\widetilde{\lambda}_1, \widetilde{\mathfrak{q}}_1]$. If $\mathfrak{f} \in \mathscr{A}$ and $\mathscr{D}_\sigma^{(\nu+\frac{1}{\sigma}, \varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi) \in \mathfrak{Q}_1 \cap \mathscr{H}_1$ and

$$\psi\left(\frac{\mathscr{D}_\sigma^{(\nu,\varepsilon)}\mathfrak{f}(\xi)}{\xi}, \frac{\mathscr{D}_\sigma^{(\nu+\frac{1}{\sigma},\varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi)}{\xi}, \frac{\mathscr{D}_\sigma^{(\nu+\frac{2}{\sigma},\varepsilon-\frac{2}{\sigma})}\mathfrak{f}(\xi)}{\xi}, \frac{\mathscr{D}_\sigma^{(\nu+\frac{3}{\sigma},\varepsilon-\frac{3}{\sigma})}\mathfrak{f}(\xi)}{\xi}; \xi\right)$$

is univalent in $\mathbb{D}$, and the conditions (44) and (36) are satisfied, then

$$\mathscr{F}_{\widetilde{\lambda}_1(\Omega)}\widetilde{\lambda}_1(\xi) \leq \mathscr{F}_{\psi(\mathbb{C}^4\times\mathbb{D})}\psi\left(\frac{\mathscr{D}_\sigma^{(\nu,\varepsilon)}\mathfrak{f}(\xi)}{\xi}, \frac{\mathscr{D}_\sigma^{(\nu+\frac{1}{\sigma},\varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi)}{\xi}, \frac{\mathscr{D}_\sigma^{(\nu+\frac{2}{\sigma},\varepsilon-\frac{2}{\sigma})}\mathfrak{f}(\xi)}{\xi}, \frac{\mathscr{D}_\sigma^{(\nu+\frac{3}{\sigma},\varepsilon-\frac{3}{\sigma})}\mathfrak{f}(\xi)}{\xi}; \xi\right)$$

$$\leq \mathscr{F}_{\widetilde{\lambda}_2(\Omega)}\widetilde{\lambda}_2(\xi),$$

implies that

$$\widetilde{\lambda}_1(\xi) \prec_{\mathscr{F}} \mathscr{D}_\sigma^{(\nu+\frac{1}{\sigma},\varepsilon-\frac{1}{\sigma})}\mathfrak{f}(\xi) \prec_{\mathscr{F}} \widetilde{\lambda}_2(\xi).$$

6. Conclusion

The findings of this paper offer substantive advancements to the theories of differential subordination and superordination. Section 1 introduces the underlying framework of the study, including the Zeta–Riemann fractional differential operator $\mathscr{D}_\sigma^{(\nu,\varepsilon)}\mathfrak{f}(\xi)$, and the motivation for exploring its analytical implications. Section 2 establishes the preliminary concepts required for the subsequent development. Section 3 presents the principal results on third-order fuzzy differential subordination, with rigorous proofs of the main theorems. In parallel, Section 4 expands the theory by formulating and analyzing corresponding third-order fuzzy differential superordination results. Building upon these developments, Section 5 synthesizes the outcomes of the preceding sections to derive comprehensive sandwich-type conclusions.

To the best of our knowledge, the dual frameworks introduced here have not previously been utilized to obtain third-order fuzzy sandwich-type results. Accordingly, this work deepens and broadens an emerging line of investigation in the field. Furthermore, the methodology adopted suggests that the use of alternative fractional operators may yield additional meaningful contributions, thereby further advancing the study of third-order fuzzy differential subordination and superordination.

Since fuzzy differential subordination remains a relatively young and evolving theory, its potential applications across other scientific disciplines are not yet fully understood. The present work aims to encourage further research in this direction, particularly through the development of new operators within the classes of univalent and multivalent functions. Moreover, the interaction between fractional calculus and the analytical characteristics of the Riemann zeta function opens promising avenues for future exploration. These include potential connections with chaos theory, fractal analysis, complex networks, and other emerging areas of mathematical research.

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Data availability statement

The data used to support the findings of this study are included within the article.

Conflict of interest

The authors declare no competing financial interest.

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