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Research Article

Some Novel Integral Inequalities via the Generalized (k, Ψ)-Conformable Fractional Integrals

Warda[1], Gauhar Rahman[1*], Muhammad Samraiz[2], Salma Haque[3], Nabil Mlaiki[3]

[1] Department of Mathematics & Statistics, Hazara University, Mansehra 21300, Pakistan
[2] Department of Mathematics, University of Sargodha, Sargodha, Pakistan
[3] Department of Mathematics and Sciences, Prince Sultan University, Riyadh 11586, Saudi Arabia
E-mail: drgauhar.rahman@hu.edu.pk

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Abstract: The stability and uniqueness of the solutions depend on the inequalities. Bounding, simulating non-local behaviors, and resolving systems that integer-order algorithms are unable to handle are among their uses. In this paper, we examine the generalized (k, Ψ)-Conformable Fractional Integrals (CFIs), which are the k-analogues of the recently published generalized conformable fractional integrals. These integrals are reducible to other fractional integrals under specific parameter values. Finally, we present several integral inequalities pertaining to the generalized (k, Ψ)- Conformable Fractional Integrals. The classical integral inequalities can easily be restored by applying certain conditions on parameters k and Ψ.

Keywords: (k, Ψ)-Conformable Fractional Integral, Conformable Fractional Integral (CFI), inequalities

MSC: 26A33, 26D10, 35A23, 47A63

1. Introduction

Non-local fractional derivatives are known as conformable derivatives. Since we can derive them up to any arbitrary order, they can be referred to as fractional. The ability to use conformable derivatives, as well as other forms of local fractional derivatives or modified conformable derivatives in [1], to produce more generalized non-local fractional derivatives with singular kernels [2–4] can make them more significant. Fractional calculus is a generalized variant of classical derivatives and integrals that studies derivatives and integrals of non-integer order. Due to its applications in a variety of domains, including physics, biology, fluid dynamics, control theory, image processing, signal processing, and computer networking, it is as antique as classical calculus but has gained increased prominence in the last 20 years, refer to [5–10]. Through creative concepts and methods of fractional calculus, research has advanced in recent years to generalize existing inequalities. Using fractional integral operators is one of the most widely used strategies among scholars. The integral inequalities involving fractional integrals are significant because of their potential to be used to determine the presence of nontrivial and positive solutions of several classes of fractional differential equations.

The applications and generalizations of various inequalities using fractional integral operators are presented by [11–15]. The better form of generalized Grüss-type integral inequalities for k-Riemann-Liouville fractional integrals

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was presented by Mubeen and Iqbal [16], who contributed to the current study. Hermite-Hadamard type inequalities for generalized k-fractional integrals were obtained by Agarwal et al. [17]. An integral identity and generalized Hermite-Hadamard type inequalities for the Riemann-Liouville fractional integral were introduced by Set et al. [18]. Integral inequalities of Ostrowski type for k-fractional Riemann-Liouville integrals were defined by Mubeen et al. [19]. To obtain a generalized inequality, Sarikaya and Budak [20] used local fractional integrals. For fractional conformable integrals, Khan et al. [21] generated some significant generalized inequalities for a finite class of positive decreasing functions. A Hartman-Winter type inequality involving a fractional derivative associated with another function was identified by Jleli et al. [22]. Additional details on this subject can be found in the works [23–27] and their closely connected references.

Qi et al. [28] presented generalized k-fractional conformable integral and Sadek et al. [29] defined the ϑ-analogue of conformable fractional differential and integral operators.

This study aims to extend some integral inequalities for a finite class of positive and decreasing functions described in [30] to ones utilizing k-fractional conformable integrals. We suggest readers refer to [31–36] for specifics on those inequalities, their uses, and their stability.

2. Notations

Abdeljawad [37] developed the idea of left and right fractional conformable derivatives for a differentiable function f, which may be written as

$$I_{\mathbb{a}}^{\lambda} f(\theta) = (\theta - \mathbb{a})^{1-\lambda} f'(\theta),$$

and

$$I_{\mathbb{b}}^{\lambda} f(\theta) = (\mathbb{b} - \theta)^{1-\lambda} f'(\theta).$$

Accordingly, fractional conformable integrals on the left and right for $0 < \lambda < 1$ can be expressed as

$$\mathcal{M}_{\mathbb{a}}^{\lambda} f(\theta) = \int_{\mathbb{a}}^{\theta} \frac{f(\mathsf{x})}{(\mathsf{x} - \mathbb{a})^{1-\lambda}} d\mathsf{x},$$

and

$$\mathcal{M}_{\mathbb{b}}^{\lambda} f(\theta) = \int_{\mathbb{a}}^{\theta} \frac{f(\mathsf{x})}{(\mathbb{b} - \mathsf{x})^{1-\lambda}} d\mathsf{x}.$$

Definition 2.1 [4] The left and right Conformable Fractional Integral (CFI) operators of order $\varpi \in C$ for $\Re(\varpi) > 0$ are respectively given by

$$^{\varpi}_{\mathbb{a}}\mathcal{M}^{\lambda} f(\mathsf{x}) = \frac{1}{\Gamma(\varpi)} \int_{\mathbb{a}}^{\mathsf{x}} \left(\frac{(\mathsf{x} - \mathbb{a})^{\lambda} - (\theta - \mathbb{a})^{\lambda}}{\lambda} \right)^{\varpi - 1} \frac{f(\theta) d\theta}{(\theta - \mathbb{a})^{1-\lambda}},$$

and

$$
{}^{\varpi}_{\mathbb{b}}\mathcal{M}^{\lambda} f(\mathsf{x}) = \frac{1}{\Gamma(\varpi)} \int_{\mathsf{x}}^{\mathbb{b}} \left(\frac{(\mathbb{b}-\mathsf{x})^{\lambda} - (\mathbb{b}-\theta)^{\lambda}}{\lambda} \right)^{\varpi-1} \frac{f(\theta)d\theta}{(\mathbb{b}-\theta)^{1-\lambda}},
$$

where $\Gamma(\varpi)$ for $\Re(\varpi) > 0$ represents the classical gamma function [37, 38].

Definition 2.2 [28] Assume that f is a continuous function on the finite real interval $[\mathbb{a}, \mathbb{b}]$. Next, for $\Re(\varpi) > 0$, the generalized left and right k-conformable fractional integrals of order $\varpi \in C$ are defined as

$$
{}^{\varpi}_{\mathbb{a}}\mathcal{M}^{\lambda}_{\mathtt{k}} f(\mathsf{x}) = \frac{1}{\mathtt{k}\Gamma_{\mathtt{k}}(\varpi)} \int_{\mathbb{a}}^{\mathsf{x}} \left(\frac{(\mathsf{x}-\mathbb{a})^{\lambda} - (\theta-\mathbb{a})^{\lambda}}{\lambda} \right)^{\frac{\varpi}{\mathtt{k}}-1} \frac{f(\theta)d\theta}{(\theta-\mathbb{a})^{1-\lambda}},
$$

and

$$
{}^{\varpi}_{\mathbb{b}}\mathcal{M}^{\lambda}_{\mathtt{k}} f(\mathsf{x}) = \frac{1}{\mathtt{k}\Gamma_{\mathtt{k}}(\varpi)} \int_{\mathsf{x}}^{\mathbb{b}} \left(\frac{(\mathbb{b}-\mathsf{x})^{\lambda} - (\mathbb{b}-\theta)^{\lambda}}{\lambda} \right)^{\frac{\varpi}{\mathtt{k}}-1} \frac{f(\theta)d\theta}{(\mathbb{b}-\theta)^{1-\lambda}},
$$

where $\lambda \in \dfrac{\mathbb{R}}{\{0\}}$, $\mathtt{k} > 0$.

The semi-group property, the derivative of functions, the Laplace transforms of functions utilizing this derivative, and the solution of Initial Value Problems (IVP) are some characteristics of those concepts that are described in Definition 2.1 and can be seen in [37].

Definition 2.3 [29] Let $\varpi \in \mathbb{C}$, $\Re(\varpi) > 0$. The left and right Ψ-CFI are respectively given by

$$
{}^{\varpi}_{\mathbb{a}}\mathcal{M}^{\Psi}_{\lambda} f(\mathsf{x}) = \frac{1}{\Gamma(\varpi)} \int_{\mathbb{a}}^{\mathsf{x}} \left(\frac{(\Psi(\mathsf{x})-\Psi(\mathbb{a}))^{\lambda} - (\Psi(\theta)-\Psi(\mathbb{a}))^{\lambda}}{\lambda} \right)^{\varpi-1} \frac{f(\theta)\Psi'(\theta)d\theta}{(\Psi(\theta)-\Psi(\mathbb{a}))^{1-\lambda}},
$$

and

$$
{}^{\varpi}_{\mathbb{b}}\mathcal{M}^{\Psi}_{\lambda} f(\mathsf{x}) = \frac{1}{\Gamma(\varpi)} \int_{\mathsf{x}}^{\mathbb{b}} \left(\frac{(\Psi(\mathbb{b})-\Psi(\mathsf{x}))^{\lambda} - (\Psi(\mathbb{b})-\Psi(\theta))^{\lambda}}{\lambda} \right)^{\varpi-1} \frac{f(\theta)\Psi'(\theta)d\theta}{(\Psi(\mathbb{b})-\Psi(\theta))^{1-\lambda}}.
$$

Next, we consider the generalized (k, Ψ)-CFI recently defined by Warda et al. [39].

Definition 2.4 Let $\varpi \in \mathbb{C}$, $\Re(\varpi) > 0$. Then we define the (k, Ψ)-CFI is defined as

$$
{}^{\varpi}_{\mathbb{a}}\mathcal{M}^{\lambda,\Psi}_{\mathtt{k}} f(\mathsf{x}) = \frac{(\lambda)^{1-\frac{\varpi}{\mathtt{k}}}}{k\Gamma_{\mathtt{k}}(\varpi)} \int_{\mathbb{a}}^{\mathsf{x}} \left((\Psi(\mathsf{x})-\Psi(\mathbb{a}))^{\lambda} - (\Psi(\theta)-\Psi(\mathbb{a}))^{\lambda} \right)^{\frac{\varpi}{\mathtt{k}}-1} (\Psi(\theta)-\Psi(\mathbb{a}))^{\varrho-1} f(\theta)\Psi'(\theta)d\theta,
$$

and

$$ {}^{\varpi}_{\mathbb{b}}\mathcal{M}_{\mathrm{k}}^{\lambda,\Psi} f(\mathrm{x}) = \frac{(\lambda)^{1-\frac{\varpi}{\mathrm{k}}}}{\mathrm{k}\Gamma_{\mathrm{k}}(\varpi)} \int_{\mathrm{x}}^{\mathbb{b}} \left((\Psi(\mathbb{b})-\Psi(\mathrm{x}))^{\lambda} - (\Psi(\mathbb{b})-\Psi(\theta))^{\lambda}\right)^{\frac{\varpi}{\mathrm{k}}-1} (\Psi(\mathbb{b})-\Psi(\theta))^{\lambda-1} f(\theta)\Psi'(\theta)d\theta. $$

Remark 1

1. Let us choose $k=1$ in Definition 2.4, we get Definition 2.3 recently defined by [29].
2. Let us choose $\Psi(\mathrm{x})=\mathrm{x}$ in Definition 2.4, we get Definition 2.2.
3. Let us choose $k=1$ and $\Psi(\mathrm{x})=\mathrm{x}$ in Definition 2.4, we get Definition 2.1.
4. Let us choose $\Psi(\mathrm{x})=\mathrm{x}, \lambda=1$ in Definition 2.4, we get the k-Riemann-Liouville (RL) fractional operators.
5. Let us choose $\Psi(\mathrm{x})=\mathrm{x}, \lambda=1$, and $k=1$ in Definition 2.4, we get the RL fractional operators.
6. If we take $a=0$, $\eta\to 0$, $\Psi(\mathrm{x})=\mathrm{x}$, $k=1$ in Definition 2.4, then we get the Hadamard fractional operators.
7. If we take $a=0$, $\Psi(\mathrm{x})=\mathrm{x}$, and $k=1$ in Definition 2.4, then we get the Katugampola fractional operators [40].

3. Some inequalities related to generalized (k, Ψ)-CFI's

There are numerous practical applications for the analysis of fractional integral inequalities. The occurrence of nontrivial solutions to fractional differential equations is among the most practical uses of such inequalities. Numerous applications for the presence of eigenvalue challenges with non-trivial solutions by inequalities can be found in the literature; see [13, 15]. Applying fractional integral operators to generalize pre-existing inequalities is getting more common in the field of research and view, for instance [38, 41–44].

In this section, we provide some refinements of integral inequalities for generalized (k, Ψ)-CFI.

Theorem 3.1 Suppose that the function $\hbar(\mathrm{x})$ be a continuous and increasing and let the sequence of continuous and positive decreasing functions be $\{\mathrm{g}_j,\ 1\le j\le m\}$ on the interval $[\mathbb{a}, \mathbb{b}]$. Assume that $\mathbb{a}<\mathrm{x}\le\mathbb{b}$, $\varrho>0$, $\zeta\ge\delta_\rho>0$ for $1\le\rho\le m$, then for the left (k, Ψ)-CFI operator ${}^{\varpi}_{\mathbb{a}}\mathcal{M}_{\mathrm{k}}^{\sigma,\Psi}$, the following inequality holds:

$$ \frac{{}^{\varpi}_{\mathbb{a}}\mathcal{M}_{\mathrm{k}}^{\lambda,\Psi}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_\rho^{\zeta}(\mathrm{x})\right)}{{}^{\varpi}_{\mathbb{a}}\mathcal{M}_{\mathrm{k}}^{\lambda,\Psi}\left(\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)} \ge \frac{{}^{\varpi}_{\mathbb{a}}\mathcal{M}_{\mathrm{k}}^{\lambda,\Psi}\left(\hbar^{\varrho}(\mathrm{x})\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_\rho^{\zeta}(\mathrm{x})\right)}{{}^{\varpi}_{\mathbb{a}}\mathcal{M}_{\mathrm{k}}^{\lambda,\Psi}\left(\hbar^{\varrho}(\mathrm{x})\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)}. \tag{1} $$

Proof. Under the specified conditions, we have

$$ [\hbar^{\varrho}(\vartheta)-\hbar^{\varrho}(\theta)]\left[\mathrm{g}_\rho^{\zeta-\delta_\rho}(\theta)-\mathrm{g}_\rho^{\zeta-\delta_\rho}(\vartheta)\right]\ge 0. $$

Define a function

$$ {}^{\varpi}_{\mathbb{a}}\mathbb{N}_{\mathrm{k}}^{\lambda,\Psi}(\Psi(\mathrm{x}), \Psi(\wp), \Psi(\theta)) = \frac{1}{\mathrm{k}\Gamma_{\mathrm{k}}(\varpi)}\left(\frac{(\Psi(\mathrm{x})-\Psi(\mathbb{a}))^{\lambda}-(\Psi(\theta)-\Psi(\mathbb{a}))^{\lambda}}{\sigma}\right)^{\frac{\varpi}{\mathrm{k}}-1} $$

$$ \frac{\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})}{(\Psi(\theta)-\Psi(\mathbb{a}))^{1-\sigma}}[\hbar^{\varrho}(\vartheta)-\hbar^{\varrho}(\theta)]\left[\mathrm{g}_\rho^{\zeta-\delta_\rho}(\theta)-\mathrm{g}_\rho^{\zeta-\delta_\rho}(\vartheta)\right]\Psi'(\theta). \tag{2} $$

Therefore, the function ${}^{\varpi}_{\mathbb{a}}\mathbb{N}^{\lambda,\Psi}_{\mathrm{k}}(\Psi(\mathrm{x}),\Psi(\vartheta),\Psi(\theta))$ is positive $\forall\theta\in(\mathbb{a},\mathbb{b}]$. Integrating from $\mathbb{a}$ to x on both sides of Equation (2) w. r. t. θ reveals

$$0\leq\int_{\mathbb{a}}^{\mathrm{x}}{}^{\varpi}_{\mathbb{a}}\mathbb{N}^{\lambda,\Psi}_{\mathrm{k}}(\Psi(\mathrm{x}),\Psi(\vartheta),\Psi(\theta))$$

$$=\frac{1}{\mathrm{k}\Gamma_{\mathrm{k}}(\varpi)}\int_{\mathbb{a}}^{\mathrm{x}}\left(\frac{(\Psi(\mathrm{x})-\Psi(\mathbb{a}))^{\lambda}-(\Psi(\theta)-\Psi(\mathbb{a}))^{\lambda}}{\lambda}\right)^{\frac{\varpi}{\mathrm{k}}-1}$$

$$\times\prod_{j=1}^{m}\mathrm{g}_{j}^{\delta_j}(\mathrm{x})\left[\hbar^{\varrho}(\vartheta)-\hbar^{\varrho}(\theta)\right]\left[\mathrm{g}_{\rho}^{\zeta-\delta_\rho}(\theta)-\mathrm{g}_{\rho}^{\zeta-\delta_\rho}(\wp)\right]\frac{\Psi'(\theta)d\theta}{(\Psi(\theta)-\Psi(\mathbb{a}))^{1-\lambda}}.$$

In view of Definition 2.4, it follows that

$$=\hbar^{\varrho}(\vartheta)\left[{}^{\varpi}_{\mathbb{a}}\mathcal{M}^{\lambda,\Psi}_{\mathrm{k}}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_{j}^{\delta_j}\mathrm{g}_{\rho}^{\zeta}(\mathrm{x})\right)\right]+\mathrm{g}_{\rho}^{\zeta-\delta_\rho}(\vartheta)\left[{}^{\varpi}_{\mathbb{a}}\mathcal{M}^{\lambda,\Psi}_{\mathrm{k}}\left(\hbar^{\varrho}(\mathrm{x})\prod_{j=1}^{m}\mathrm{g}_{j}^{\delta_j}(\mathrm{x})\right)\right]$$

$$-\hbar^{\varrho}(\vartheta)\mathrm{g}_{\rho}^{\zeta-\delta_\rho}(\vartheta)\left[{}^{\varpi}_{\mathbb{a}}\mathcal{M}^{\lambda,\Psi}_{\mathrm{k}}\left(\prod_{j=1}^{m}\mathrm{g}_{j}^{\delta_j}(\mathrm{x})\right)\right]-\left[{}^{\varpi}_{\mathbb{a}}\mathcal{M}^{\lambda,\Psi}_{\mathrm{k}}\left(\hbar^{\varrho}(\mathrm{x})\prod_{j\neq\rho}^{m}\mathrm{g}_{j}^{\delta_j}(\mathrm{x})\right)\right].\tag{3}$$

By conducting the product of (3) with the following relation

$$\frac{1}{\mathrm{k}\Gamma_{\mathrm{k}}(\varpi)}\left(\frac{(\Psi(\mathrm{x})-\Psi(\mathbb{a}))^{\sigma}-(\Psi(\vartheta)-\Psi(\mathbb{a}))^{\sigma}}{\sigma}\right)^{\frac{\varpi}{\mathrm{k}}-1}\frac{\prod_{j=1}^{m}\mathrm{g}_{j}^{\delta_j}(\vartheta)\Psi'(\vartheta)}{(\Psi(\vartheta)-\Psi(\mathbb{a}))^{1-\sigma}}.\tag{4}$$

Additionally, integrating both sides with regard to ϑ from $\mathbb{a}$ to x yields

$$0\leq\left[{}^{\varpi}_{\mathbb{a}}\mathcal{M}^{\lambda,\Psi}_{\mathrm{k}}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_{j}^{\delta_j}\mathrm{g}_{\rho}^{\zeta}(\mathrm{x})\right)\right]\left[{}^{\varpi}_{\mathbb{a}}\mathcal{M}^{\lambda,\Psi}_{\mathrm{k}}\left(\hbar^{\varrho}(\mathrm{x})\prod_{j=1}^{m}\mathrm{g}_{j}^{\delta_j}(\mathrm{x})\right)\right]$$

$$-\left[{}^{\varpi}_{\mathbb{a}}\mathcal{M}^{\lambda,\Psi}_{\mathrm{k}}\left(\hbar^{\varrho}(\mathrm{x})\prod_{j\neq\rho}^{m}\mathrm{g}_{j}^{\delta_j}\mathrm{g}_{\rho}^{\zeta}(\mathrm{x})\right)\right]\left[{}^{\varpi}_{\mathbb{a}}\mathcal{M}^{\lambda,\Psi}_{\mathrm{k}}\left(\prod_{j=1}^{m}\mathrm{g}_{j}^{\delta_j}(\mathrm{x})\right)\right].\tag{5}$$

Now, dividing (5) by

$${}^{\varpi}_{\mathbb{a}}\mathcal{M}^{\lambda,\Psi}_{\mathrm{k}}\left(\hbar^{\varrho}(\mathrm{x})\prod_{j=1}^{m}\mathrm{g}_{j}^{\delta_j}(\mathrm{x})\right)\left[{}^{\varpi}_{\mathbb{a}}\mathcal{M}^{\lambda,\Psi}_{\mathrm{k}}\left(\prod_{j=1}^{m}\mathrm{g}_{j}^{\delta_j}(\mathrm{x})\right)\right],$$

completes the proof of Theorem 3.1.

Corollary 3.1 Let the continuous and positive decreasing functions be $\{g_j, 1 \le j \le m\}$ on the interval $[a, b]$. If $a < x \le b$, $\varrho > 0$, $\zeta \ge \delta_\rho > 0$ for $1 \le \rho \le m$. Then for the left (k, Ψ)-CFI operator ${}^{\varpi}_{a}\mathcal{M}^{\lambda,\Psi}_{k}$, the following inequality holds:

$$\frac{{}^{\varpi}_{a}\mathcal{M}^{\lambda,\Psi}_{k}\left(\prod_{j\neq\rho}^{m} g_j^{\delta_j} g_\rho^{\zeta}(x)\right)}{{}^{\varpi}_{a}\mathcal{M}^{\lambda,\Psi}_{k}\left(\prod_{j=1}^{m} g_j^{\delta_j}(x)\right)} \ge \frac{{}^{\varpi}_{a}\mathcal{M}^{\lambda,\Psi}_{k}\left((\Psi(x)-\Psi(a))^{\varrho}\prod_{j\neq\rho}^{m} g_j^{\delta_j} g_\rho^{\zeta}(x)\right)}{{}^{\varpi}_{a}\mathcal{M}^{\lambda,\Psi}_{k}\left((\Psi(x)-\Psi(a))^{\varrho}\prod_{j=1}^{m} g_j^{\delta_j}(x)\right)}. \tag{6}$$

Proof. By taking $\hbar^{\varrho}(x) = \Psi(x) - \Psi(a)$ in Theorem 3.1, we get Corollary 3.1.

Corollary 3.2 Suppose that the sequence of continuous and positive decreasing functions be $\{g_j, 1 \le j \le m\}$ on the interval $[a, b]$. If $a < x \le b$, $\varrho > 0$, $\zeta \ge \delta_\rho > 0$ for $1 \le \rho \le m$. Then for the left (k, Ψ)-CFI operator ${}^{\varpi}_{a}\mathcal{M}^{\lambda,\Psi}_{k}$, the following inequality holds:

$$\left[{}^{\omega}_{a}\mathcal{M}^{\lambda,\Psi}_{k}\left((\Psi(x)-\Psi(a))^{\varrho}\prod_{j=1}^{m} g_j^{\delta_j}(x)\right)\right]\left[{}^{\varpi}_{a}\mathcal{M}^{\lambda,\Psi}_{k}\left(\prod_{j\neq\rho}^{m} g_j^{\delta_j} g_\rho^{\zeta}(x)\right)\right]$$

$$+\left[{}^{\omega}_{a}\mathcal{M}^{\lambda,\Psi}_{k}\left(\prod_{j\neq\rho}^{m} g_j^{\delta_j} g_\rho^{\zeta}(x)\right)\right]\left[{}^{\varpi}_{a}\mathcal{M}^{\lambda,\Psi}_{k}\left((\Psi(x)-\Psi(a))^{\varrho}\prod_{j=1}^{m} g_j^{\delta_j}(x)\right)\right]$$

$$\ge\left[{}^{\omega}_{a}\mathcal{M}^{\lambda,\Psi}_{k}\left((\Psi(x)-\Psi(a))^{\varrho}\prod_{j\neq\rho}^{m} g_j^{\delta_j} g_\rho^{\zeta}(x)\right)\right]\left[{}^{\varpi}_{a}\mathcal{M}^{\lambda,\Psi}_{k}\left(\prod_{j\neq\rho}^{m} g_j^{\delta_j} g_\rho^{\zeta}(x)\right)\right]$$

$$+\left[{}^{\omega}_{a}\mathcal{M}^{\lambda,\Psi}_{k}\left(\prod_{j=1}^{m} g_j^{\delta_j}(x)\right)\right]\left[{}^{\varpi}_{a}\mathcal{M}^{\lambda,\Psi}_{k}\left((\Psi(x)-\Psi(a))^{\varrho}\prod_{j=1}^{m} g_j^{\delta_j}(x)\right)\right]. \tag{7}$$

Proof. Consider

$$0 \le \int_a^x {}^{\varpi}_{a}\mathbb{N}^{\lambda,\Psi}_{k}(\Psi(x), \Psi(\vartheta), \Psi(\theta))d\theta$$

$$= \frac{1}{k\Gamma_k(\varpi)}\int_a^x \left(\frac{(\Psi(x)-\Psi(a))^{\lambda}-(\Psi(\theta)-\Psi(a))^{\lambda}}{\lambda}\right)^{\frac{\varpi}{k}-1}$$

$$\times \prod_{j=1}^{m} g_j^{\delta_j}(\theta)\left[(\Psi(\vartheta)-\Psi(a))^{\varrho}-(\Psi(\theta)-\Psi(a))^{\varrho}\right]$$

$$\left[g_\rho^{\zeta-\delta_\rho}(\theta)-g_\rho^{\zeta-\delta_\rho}(\vartheta)\right]\frac{\Psi'(\theta)d\theta}{(\Psi(\theta)-\Psi(a))^{1-\lambda}}$$

$$= (\Psi(\wp) - \Psi(\mathrm{a}))^{\varrho} \left[{}_{\mathrm{a}}^{\varpi}\mathcal{M}_{\mathrm{k}}^{\lambda,\Psi} \left(\prod_{j\neq\rho}^{m} \mathrm{g}_j^{\delta_j} \mathrm{g}_\rho^{\zeta} (\mathrm{x}) \right) \right]$$

$$+ \mathrm{g}_\rho^{\zeta-\delta_\rho}(\vartheta) \left[{}_{\mathrm{a}}^{\varpi}\mathcal{M}_{\mathrm{k}}^{\lambda,\Psi} \left((\Psi(\mathrm{x}) - \Psi(\mathrm{a}))^{\varrho} \prod_{j=1}^{m} \mathrm{g}_j^{\delta_j} (\mathrm{x}) \right) \right]$$

$$- (\Psi(\vartheta) - \Psi(\mathrm{a}))^{\varrho} \mathrm{g}_\rho^{\zeta-\delta_\rho}(\vartheta) \left[{}_{\mathrm{a}}^{\varpi}\mathcal{M}_{\mathrm{k}}^{\lambda,\Psi} \left(\prod_{j=1}^{m} \mathrm{g}_j^{\delta_j} (\mathrm{x}) \right) \right]$$

$$- \left[{}_{\mathrm{a}}^{\varpi}\mathcal{M}_{\mathrm{k}}^{\lambda,\Psi} \left((\Psi(\mathrm{x}) - \Psi(\mathrm{a}))^{\varrho} \prod_{j\neq\rho}^{m} \mathrm{g}_j^{\delta_j} (\mathrm{x}) \right) \right]. \tag{8}$$

Multiplying (8) by

$$\frac{1}{\mathrm{k}\Gamma_{\mathrm{k}}(\omega)} \left(\frac{(\Psi(\mathrm{x}) - \Psi(\mathrm{a}))^{\lambda} - (\Psi(\vartheta) - \Psi(\mathrm{a}))^{\lambda}}{\lambda} \right)^{\frac{\omega}{\mathrm{k}}-1} \frac{\prod_{j=1}^{m} \mathrm{g}_j^{\delta_j}(\vartheta)\Psi'(\vartheta)}{(\Psi(\vartheta) - \Psi(\mathrm{a}))^{1-\lambda}}, \tag{9}$$

and integrating both sides from a to x with respect to ϑ, we get in view of Definition 2.4

$$0 \leq \left[{}_{\mathrm{a}}^{\omega}\mathcal{M}_{\mathrm{k}}^{\lambda,\Psi} \left((\Psi(\mathrm{x}) - \Psi(\mathrm{a}))^{\varrho} \prod_{j=1}^{m} \mathrm{g}_j^{\delta_j} (\mathrm{x}) \right) \right] \left[{}_{\mathrm{a}}^{\varpi}\mathcal{M}_{\mathrm{k}}^{\lambda,\Psi} \left(\prod_{j\neq\rho}^{m} \mathrm{g}_j^{\delta_j} \mathrm{g}_\rho^{\zeta} (\mathrm{x}) \right) \right]$$

$$+ \left[{}_{\mathrm{a}}^{\omega}\mathcal{M}_{\mathrm{k}}^{\lambda,\Psi} \left(\prod_{j\neq\rho}^{m} \mathrm{g}_j^{\delta_j} \mathrm{g}_\rho^{\zeta} (\mathrm{x}) \right) \right] \left[{}_{\mathrm{a}}^{\varpi}\mathcal{M}_{\mathrm{k}}^{\lambda,\Psi} \left((\Psi(\mathrm{x}) - \Psi(\mathrm{a}))^{\varrho} \prod_{j=1}^{m} \mathrm{g}_j^{\delta_j} (\mathrm{x}) \right) \right]$$

$$- \left[{}_{\mathrm{a}}^{\omega}\mathcal{M}_{\mathrm{k}}^{\lambda,\Psi} \left((\Psi(\mathrm{x}) - \Psi(\mathrm{a}))^{\varrho} \prod_{j\neq\rho}^{m} \mathrm{g}_j^{\delta_j} \mathrm{g}_\rho^{\zeta} (\mathrm{x}) \right) \right] \left[{}_{\mathrm{a}}^{\varpi}\mathcal{M}_{\mathrm{k}}^{\lambda,\Psi} \left(\prod_{j\neq\rho}^{m} \mathrm{g}_j^{\delta_j} \mathrm{g}_\rho^{\zeta} (\mathrm{x}) \right) \right]$$

$$- \left[{}_{\mathrm{a}}^{\omega}\mathcal{M}_{\mathrm{k}}^{\lambda,\Psi} \left(\prod_{j\neq\rho}^{m} \mathrm{g}_j^{\delta_j} \mathrm{g}_\rho^{\zeta} (\mathrm{x}) \right) \right] \left[{}_{\mathrm{a}}^{\varpi}\mathcal{M}_{\mathrm{k}}^{\lambda,\Psi} \left((\Psi(\mathrm{x}) - \Psi(\mathrm{a}))^{\varrho} \prod_{j=1}^{m} \mathrm{g}_j^{\delta_j} (\mathrm{x}) \right) \right]. \tag{10}$$

Now, dividing (10) by

$$\left[{}_{\mathrm{a}}^{\omega}\mathcal{M}_{\mathrm{k}}^{\lambda,\Psi} \left((\Psi(\mathrm{x}) - \Psi(\mathrm{a}))^{\varrho} \prod_{j\neq\rho}^{m} \mathrm{g}_j^{\delta_j} \mathrm{g}_\rho^{\zeta} (\mathrm{x}) \right) \right] \left[{}_{\mathrm{a}}^{\varpi}\mathcal{M}_{\mathrm{k}}^{\lambda,\Psi} \left(\prod_{j\neq\rho}^{m} \mathrm{g}_j^{\delta_j} \mathrm{g}_\rho^{\zeta} (\mathrm{x}) \right) \right]$$

$$+\left[{}_{\mathrm{a}}^{\omega}\mathcal{M}_{\mathrm{k}}^{\lambda,\,\Psi}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_\rho^{\zeta}(\mathrm{x})\right)\right]\left[{}_{\mathrm{a}}^{\varpi}\mathcal{M}_{\mathrm{k}}^{\lambda,\,\Psi}\left((\Psi(\mathrm{x})-\Psi(\mathrm{a}))^{\varrho}\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right].$$

Gives the proof of Corollary 3.2.

Corollary 3.3 Let continuous increasing function be $\hbar(\mathrm{x})$ and let the sequence of continuous and positive decreasing functions be $\left\{\mathrm{g}_j,\, 1\leq j\leq m\right\}$ on the interval $[\mathrm{a},\,\mathrm{b}]$. If $\mathrm{a}<\mathrm{x}\leq\mathrm{b}$, $\varrho>0$, $\zeta\geq\delta_\rho>0$ for $1\leq\rho\leq m$. Then for the left $(k,\,\Psi)$-CFI operator ${}_{\mathrm{a}}^{\varpi}\mathcal{M}_{\mathrm{k}}^{\lambda,\,\Psi}$, the following inequality holds:

$$\left[{}_{\mathrm{a}}^{\omega}\mathcal{M}_{\mathrm{k}}^{\lambda,\,\Psi}\left(\hbar^{\varrho}(\mathrm{x})\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right]\left[{}_{\mathrm{a}}^{\varpi}\mathcal{M}_{\mathrm{k}}^{\lambda,\,\Psi}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_\rho^{\zeta}(\mathrm{x})\right)\right]$$

$$+\left[{}_{\mathrm{a}}^{\omega}\mathcal{M}_{\mathrm{k}}^{\lambda,\,\Psi}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_\rho^{\zeta}(\mathrm{x})\right)\right]\left[{}_{\mathrm{a}}^{\varpi}\mathcal{M}_{\mathrm{k}}^{\lambda,\,\Psi}\left(\hbar^{\varrho}(\mathrm{x})\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right]$$

$$\geq\left[{}_{\mathrm{a}}^{\omega}\mathcal{M}_{\mathrm{k}}^{\lambda,\,\Psi}\left(\hbar^{\varrho}(\mathrm{x})\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_\rho^{\zeta}(\mathrm{x})\right)\right]\left[{}_{\mathrm{a}}^{\varpi}\mathcal{M}_{\mathrm{k}}^{\lambda,\,\Psi}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_\rho^{\zeta}(\mathrm{x})\right)\right]$$

$$+\left[{}_{\mathrm{a}}^{\omega}\mathcal{M}_{\mathrm{k}}^{\lambda,\,\Psi}\left(\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right]\left[{}_{\mathrm{a}}^{\varpi}\mathcal{M}_{\mathrm{k}}^{\lambda,\,\Psi}\left(\hbar^{\varrho}(\mathrm{x})\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right].$$

Proof. By conducting the product of (3) and (9) and then integrating the resultant inequality from a to x with respect to ϑ yields,

$$0\leq\left[{}_{\mathrm{a}}^{\omega}\mathcal{M}_{\mathrm{k}}^{\lambda,\,\Psi}\left(\hbar^{\varrho}(\mathrm{x})\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right]\left[{}_{\mathrm{a}}^{\varpi}\mathcal{M}_{\mathrm{k}}^{\lambda,\,\Psi}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_\rho^{\zeta}(\mathrm{x})\right)\right]$$

$$+\left[{}_{\mathrm{a}}^{\omega}\mathcal{M}_{\mathrm{k}}^{\lambda,\,\Psi}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_\rho^{\zeta}(\mathrm{x})\right)\right]\left[{}_{\mathrm{a}}^{\varpi}\mathcal{M}_{\mathrm{k}}^{\lambda,\,\Psi}\left(\hbar^{\varrho}(\mathrm{x})\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right]$$

$$-\left[{}_{\mathrm{a}}^{\omega}\mathcal{M}_{\mathrm{k}}^{\lambda,\,\Psi}\left(\hbar^{\varrho}(\mathrm{x})\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_\rho^{\zeta}(\mathrm{x})\right)\right]\left[{}_{\mathrm{a}}^{\varpi}\mathcal{M}_{\mathrm{k}}^{\lambda,\,\Psi}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_\rho^{\zeta}(\mathrm{x})\right)\right]$$

$$-\left[{}_{\mathrm{a}}^{\omega}\mathcal{M}_{\mathrm{k}}^{\lambda,\,\Psi}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_\rho^{\zeta}(\mathrm{x})\right)\right]\left[{}_{\mathrm{a}}^{\varpi}\mathcal{M}_{\mathrm{k}}^{\lambda,\,\Psi}\left(\hbar^{\varrho}(\mathrm{x})\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right]. \tag{11}$$

Dividing (11) by

$$\left[{}_{\mathrm{a}}^{\omega}\mathcal{M}_{\mathrm{k}}^{\lambda,\Psi}\left(\hbar^{\varrho}(\mathrm{x})\prod_{j\neq\rho}^{m}\mathrm{g}_{j}^{\delta_{j}}\mathrm{g}_{\rho}^{\zeta}(\mathrm{x})\right)\right]\left[{}_{\mathrm{a}}^{\varpi}\mathcal{M}_{\mathrm{k}}^{\lambda,\Psi}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_{j}^{\delta_{j}}\mathrm{g}_{\rho}^{\zeta}(\mathrm{x})\right)\right]$$

$$+\left[{}_{\mathrm{a}}^{\omega}\mathcal{M}_{\mathrm{k}}^{\lambda,\Psi}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_{j}^{\delta_{j}}\mathrm{g}_{\rho}^{\zeta}(\mathrm{x})\right)\right]\left[{}_{\mathrm{a}}^{\varpi}\mathcal{M}_{\mathrm{k}}^{\lambda,\Psi}\left(\hbar^{\varrho}(\mathrm{x})\prod_{j=1}^{m}\mathrm{g}_{j}^{\delta_{j}}(\mathrm{x})\right)\right],$$

yields the proof of Corollary 3.3.

Theorem 3.2 Let the sequence of continuous and positive decreasing functions be $\{\mathrm{g}_j,\ 1\leq j\leq m\}$ on the interval $[\mathrm{a},\mathrm{b}]$. If $\mathrm{a}<\mathrm{x}\leq\mathrm{b}$, $\varrho>0$, $\zeta\geq\delta_\rho>0$ for $1\leq\rho\leq m$. Then for the right (k,Ψ)-CFI operator ${}_{\mathrm{k}}^{\varpi}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}$, the following inequality holds:

$$\frac{{}_{\mathrm{k}}^{\varpi}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_{j}^{\delta_{j}}\mathrm{g}_{\rho}^{\zeta}(\mathrm{x})\right)}{{}_{\mathrm{k}}^{\varpi}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left(\prod_{j=1}^{m}\mathrm{g}_{j}^{\delta_{j}}(\mathrm{x})\right)}\geq\frac{{}_{\mathrm{k}}^{\varpi}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left((\Psi(\mathbb{b})-\Psi(\mathrm{x}))^{\varrho}\prod_{j\neq\rho}^{m}\mathrm{g}_{j}^{\delta_{j}}\mathrm{g}_{\rho}^{\zeta}(\mathrm{x})\right)}{{}_{\mathrm{k}}^{\varpi}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left((\Psi(\mathbb{b})-\Psi(\mathrm{x}))^{\varrho}\prod_{j=1}^{m}\mathrm{g}_{j}^{\delta_{j}}(\mathrm{x})\right)}.$$

Proof. Under the specified condition, there is

$$[(\Psi(\mathbb{b})-\Psi(\vartheta))^{\varrho}-(\Psi(\mathbb{b})-\Psi(\theta))^{\varrho}]\left[\mathrm{g}_{\rho}^{\zeta-\delta_{\rho}}(\theta)-\mathrm{g}_{\rho}^{\zeta-\delta_{\rho}}(\vartheta)\right]\geq 0.$$

Let us consider

$$\begin{aligned}&{}_{\mathrm{k}}^{\varpi}\mathbb{N}_{\mathbb{b}}^{\lambda,\Psi}(\Psi(\mathrm{x}),\Psi(\vartheta),\Psi(\theta))\\&=\frac{1}{\mathrm{k}\Gamma_{\mathrm{k}}(\varpi)}\left(\frac{(\Psi(\mathbb{b})-\Psi(\mathrm{x}))^{\lambda}-(\Psi(\mathbb{b})-\Psi(\theta))^{\lambda}}{\lambda}\right)^{\frac{\varpi}{\mathrm{k}}-1}\frac{\prod_{j=1}^{m}\mathrm{g}_{j}^{\delta_{j}}(\theta)}{(\Psi(\mathbb{b})-\Psi(\theta))^{1-\lambda}}\\&[(\Psi(\mathbb{b})-\Psi(\vartheta))^{\varrho}-(\Psi(\mathbb{b})-\Psi(\theta))^{\varrho}]\left[\mathrm{g}_{\rho}^{\zeta-\delta_{\rho}}(\theta)-\mathrm{g}_{\rho}^{\zeta-\delta_{\rho}}(\vartheta)\right]\Psi'(\theta).\end{aligned}\tag{12}$$

Obviously, the function defined in (12) is positive $\forall\theta\in(\mathrm{a},\mathbb{b}]$. Integrating both sides of (12) with regard to θ from x to $\mathbb{b}$ yields

$$\begin{aligned}0&\leq\int_{\mathrm{x}}^{\mathbb{b}}{}_{\mathrm{k}}^{\varpi}\mathbb{N}_{\mathbb{b}}^{\lambda,\Psi}(\Psi(\mathrm{x}),\Psi(\vartheta),\Psi(\theta))\\&=\frac{1}{\mathrm{k}\Gamma_{\mathrm{k}}(\varpi)}\int_{\mathrm{x}}^{\mathbb{b}}\left(\frac{(\Psi(\mathbb{b})-\Psi(\mathrm{x}))^{\lambda}-(\Psi(\mathbb{b})-\Psi(\theta))^{\lambda}}{\lambda}\right)^{\frac{\varpi}{\mathrm{k}}-1}\end{aligned}$$

$$\times \prod_{j=1}^{m} \mathrm{g}_j^{\delta_j}(\theta)\left[(\Psi(\mathbb{b})-\Psi(\vartheta))^{\varrho}-(\Psi(\mathbb{b})-\Psi(\theta))^{\varrho}\right]$$

$$\left[\mathrm{g}_\rho^{\zeta-\delta_\rho}(\theta)-\mathrm{g}_\rho^{\zeta-\delta_\rho}(\wp)\right]\frac{\Psi'(\theta)d\theta}{(\Psi(\mathbb{b})-\Psi(\theta))^{1-\lambda}}$$

$$=(\Psi(\mathbb{b})-\Psi(\vartheta))^{\varrho}\left[{}^{\varpi}_{\mathrm{k}}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_\rho^{\zeta}(\mathrm{x})\right)\right]$$

$$+\mathrm{g}_\rho^{\zeta-\delta_\rho}(\vartheta)\left[{}^{\varpi}_{\mathrm{k}}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left((\Psi(\mathbb{b})-\Psi(\mathrm{x}))^{\varrho}\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right]$$

$$-(\Psi(\mathbb{b})-\Psi(\vartheta))^{\varrho}\mathrm{g}_\rho^{\zeta-\delta_\rho}(\wp)\left[{}^{\varpi}_{\mathrm{k}}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left(\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right]$$

$$-\left[{}^{\varpi}_{\mathrm{k}}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left((\Psi(\mathbb{b})-\Psi(\mathrm{x}))^{\varrho}\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right]. \tag{13}$$

By conducting the product of (13) with

$$\frac{1}{\mathrm{k}\Gamma_{\mathrm{k}}(\varpi)}\left(\frac{(\Psi(\mathbb{b})-\Psi(\mathrm{x}))^{\lambda}-(\Psi(\mathbb{b})-\Psi(\vartheta))^{\lambda}}{\lambda}\right)^{\frac{\varpi}{\mathrm{k}}-1}\frac{\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\vartheta)\Psi'(\vartheta)}{(\Psi(\mathbb{b})-\Psi(\vartheta))^{1-\lambda}}, \tag{14}$$

and integrating the resultant relation from x to $\mathbb{b}$ with respect to $\wp$ yields

$$0\leq\left[{}^{\varpi}_{\mathrm{k}}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_\rho^{\zeta}(\mathrm{x})\right)\right]\left[{}^{\varpi}_{\mathrm{k}}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left((\Psi(\mathbb{b})-\Psi(\mathrm{x}))^{\varrho}\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right]$$

$$-\left[{}^{\varpi}_{\mathrm{k}}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left((\Psi(\mathbb{b})-\Psi(\mathrm{x}))^{\varrho}\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_\rho^{\zeta}(\mathrm{x})\right)\right]\left[{}^{\varpi}_{\mathrm{k}}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left(\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right]. \tag{15}$$

Dividing both sides of (15) by

$${}^{\varpi}_{\mathrm{k}}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left((\Psi(\mathbb{b})-\Psi(\mathrm{x}))^{\varrho}\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\left[{}^{\varpi}_{\mathrm{k}}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left(\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right],$$

gives the proof of Theorem 3.2.

Corollary 3.4 Let sequence of continuous positive decreasing functions be $\{\mathrm{g}_j,\, 1 \le j \le m\}$ on the interval $[\mathbb{a}, \mathbb{b}]$. If $\mathbb{a} < \mathrm{x} \le \mathbb{b}$, $\varrho > 0$, $\zeta \ge \delta_\rho > 0$ for $1 \le \rho \le m$. Then for the right (k, Ψ)-CFI operator ${}^{\varpi}_{\mathrm{k}}\mathcal{M}^{\lambda,\Psi}_{\mathbb{b}}$, the following inequality holds:

$$\left[{}^{\omega}_{\mathrm{k}}\mathcal{M}^{\lambda,\Psi}_{\mathbb{b}}\left((\Psi(\mathbb{b})-\Psi(\mathrm{x}))^{\varrho}\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right]\left[{}^{\varpi}_{\mathrm{k}}\mathcal{M}^{\lambda,\Psi}_{\mathbb{b}}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_\rho^{\zeta}(\mathrm{x})\right)\right]$$

$$+\left[{}^{\omega}_{\mathrm{k}}\mathcal{M}^{\lambda,\Psi}_{\mathbb{b}}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_\rho^{\zeta}(\mathrm{x})\right)\right]\left[{}^{\varpi}_{\mathrm{k}}\mathcal{M}^{\lambda,\Psi}_{\mathbb{b}}\left((\Psi(\mathbb{b})-\Psi(\mathrm{x}))^{\varrho}\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right]$$

$$\ge\left[{}^{\omega}_{\mathrm{k}}\mathcal{M}^{\lambda,\Psi}_{\mathbb{b}}\left((\Psi(\mathbb{b})-\Psi(\mathrm{x}))^{\varrho}\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_\rho^{\zeta}(\mathrm{x})\right)\right]\left[{}^{\varpi}_{\mathrm{k}}\mathcal{M}^{\lambda,\Psi}_{\mathbb{b}}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_\rho^{\zeta}(\mathrm{x})\right)\right]$$

$$+\left[{}^{\omega}_{\mathrm{k}}\mathcal{M}^{\lambda,\Psi}_{\mathbb{b}}\left(\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right]\left[{}^{\varpi}_{\mathrm{k}}\mathcal{M}^{\lambda,\Psi}_{\mathbb{b}}\left((\Psi(\mathbb{b})-\Psi(\mathrm{x}))^{\varrho}\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right].$$

Proof. By conducting the multiplication of (14) with

$$\frac{1}{\mathrm{k}\Gamma_{\mathrm{k}}(\varpi)}\left(\frac{(\Psi(\mathbb{b})-\Psi(\mathrm{x}))^{\lambda}-(\Psi(\mathbb{b})-\Psi(\vartheta))^{\lambda}}{\lambda}\right)^{\frac{\varpi}{\mathrm{k}}-1}\frac{\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\vartheta)\Psi'(\vartheta)}{(\Psi(\mathbb{b})-\Psi(\vartheta))^{1-\lambda}}, \tag{16}$$

and then integrating about ϑ from x to $\mathbb{b}$ yields

$$0\le\left[{}^{\omega}_{\mathrm{k}}\mathcal{M}^{\lambda,\Psi}_{\mathbb{b}}\left((\Psi(\mathbb{b})-\Psi(\mathrm{x}))^{\varrho}\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right]\left[{}^{\varpi}_{\mathrm{k}}\mathcal{M}^{\lambda,\Psi}_{\mathbb{b}}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_\rho^{\zeta}(\mathrm{x})\right)\right]$$

$$+\left[{}^{\omega}_{\mathrm{k}}\mathcal{M}^{\lambda,\Psi}_{\mathbb{b}}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_\rho^{\zeta}(\mathrm{x})\right)\right]\left[{}^{\varpi}_{\mathrm{k}}\mathcal{M}^{\lambda,\Psi}_{\mathbb{b}}\left((\Psi(\mathbb{b})-\Psi(\mathrm{x}))^{\varrho}\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right]$$

$$-\left[{}^{\omega}_{\mathrm{k}}\mathcal{M}^{\lambda,\Psi}_{\mathbb{b}}\left((\Psi(\mathbb{b})-\Psi(\mathrm{x}))^{\varrho}\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_\rho^{\zeta}(\mathrm{x})\right)\right]\left[{}^{\varpi}_{\mathrm{k}}\mathcal{M}^{\lambda,\Psi}_{\mathbb{b}}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_\rho^{\zeta}(\mathrm{x})\right)\right]$$

$$-\left[{}^{\omega}_{\mathrm{k}}\mathcal{M}^{\lambda,\Psi}_{\mathbb{b}}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_\rho^{\zeta}(\mathrm{x})\right)\right]\left[{}^{\varpi}_{\mathrm{k}}\mathcal{M}^{\lambda,\Psi}_{\mathbb{b}}\left((\Psi(\mathbb{b})-\Psi(\mathrm{x}))^{\varrho}\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right]. \tag{17}$$

Dividing both sides of (17) by the following

$$\left[{}_{\mathrm{k}}^{\omega}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left((\Psi(\mathbb{b})-\Psi(\mathrm{x}))^{\varrho}\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_\rho^{\zeta}(\mathrm{x})\right)\right]\left[{}_{\mathrm{k}}^{\varpi}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_\rho^{\zeta}(\mathrm{x})\right)\right]$$

$$+\left[{}_{\mathrm{k}}^{\omega}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_\rho^{\zeta}(\mathrm{x})\right)\right]\left[{}_{\mathrm{k}}^{\varpi}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left((\Psi(\mathbb{b})-\Psi(\mathrm{x}))^{\varrho}\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right],$$

yields the proof of Corollary 3.4.

Theorem 3.3 Let the continuous and increasing function be $\hbar(\mathrm{x})$ and let the sequence of continuous and positive decreasing functions be $\{\mathrm{g}_j,\ 1\le j\le m\}$ on the intervals $[\mathbb{a},\mathbb{b}]$. If $\mathbb{a}<\mathrm{x}\le\mathbb{b}$, $\varrho>0$, $\zeta\ge\delta_\rho>0$ for $1\le\rho\le m$. Then for the right (k,Ψ)-CFI operator ${}_{\mathrm{k}}^{\varpi}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}$, the following inequality holds:

$$\frac{{}_{\mathrm{k}}^{\varpi}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_\rho^{\zeta}(\mathrm{x})\right)}{{}_{\mathrm{k}}^{\varpi}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left(\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)}\ge\frac{{}_{\mathrm{k}}^{\varpi}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left(\hbar^{\varrho}(\mathrm{x})\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_\rho^{\zeta}(\mathrm{x})\right)}{{}_{\mathrm{k}}^{\varpi}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left(\hbar^{\varrho}(\mathrm{x})\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)}.$$

Proof. Under the given hypothesis, we have

$$[\hbar^{\varrho}(\vartheta)-\hbar^{\varrho}(\theta)]\left[\mathrm{g}_\rho^{\zeta-\delta_\rho}(\theta)-\mathrm{g}_\rho^{\zeta-\delta_\rho}(\vartheta)\right]\ge 0.$$

Consider the function

$$\begin{aligned}&{}_{\mathrm{k}}^{\varpi}\mathbb{N}_{\mathbb{b}}^{\lambda,\Psi}(\Psi(\mathrm{x}),\Psi(\vartheta),\Psi(\theta))\\&=\frac{1}{\mathrm{k}\Gamma_{\mathrm{k}}(\varpi)}\left(\frac{(\Psi(\mathbb{b})-\Psi(\mathrm{x}))^{\lambda}-(\Psi(\mathbb{b})-\Psi(\theta))^{\lambda}}{\lambda}\right)^{\frac{\varpi}{\mathrm{k}}-1}\frac{\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\theta)}{(\Psi(\mathbb{b})-\Psi(\theta))^{1-\lambda}}\\&[\hbar^{\varrho}(\vartheta)-\hbar^{\varrho}(\theta)]\left[\mathrm{g}_\rho^{\zeta-\delta_\rho}(\theta)-\mathrm{g}_\rho^{\zeta-\delta_\rho}(\vartheta)\right]\Psi'(\theta).\end{aligned}\tag{18}$$

In fact, the function ${}_{\mathrm{k}}^{\varpi}\mathbb{N}_{\mathbb{b}}^{\lambda,\Psi}(\Psi(\mathrm{x}),\Psi(\vartheta),\Psi(\theta))$ is positive $\forall\theta\in(\mathbb{a},\mathbb{b}]$. Integrating (18) with regard to θ from x to $\mathbb{b}$ yields

$$\begin{aligned}0&\le\int_{\mathrm{x}}^{\mathbb{b}}{}_{\mathrm{k}}^{\varpi}\mathbb{N}_{\mathbb{b}}^{\lambda,\Psi}(\Psi(\mathrm{x}),\Psi(\vartheta),\Psi(\theta))d\theta\\&=\frac{1}{\mathrm{k}\Gamma_{\mathrm{k}}(\varpi)}\int_{\mathrm{x}}^{\mathbb{b}}\left(\frac{(\Psi(\mathbb{b})-\Psi(\mathrm{x}))^{\lambda}-(\Psi(\mathbb{b})-\Psi(\theta))^{\lambda}}{\lambda}\right)^{\frac{\varpi}{\mathrm{k}}-1}\end{aligned}$$

$$\times \prod_{j=1}^{m} \mathrm{g}_j^{\delta_j}(\theta)\left[\hbar^{\varrho}(\vartheta)-\hbar^{\varrho}(\theta)\right]\left[\mathrm{g}_{\rho}^{\zeta-\delta_\rho}(\theta)-\mathrm{g}_{\rho}^{\zeta-\delta_\rho}(\vartheta)\right]\frac{\Psi'(\theta)d\theta}{\left(\Psi(\mathbb{b})-\Psi(\theta)\right)^{1-\lambda}}$$

$$=\hbar^{\varrho}(\vartheta)\left[{}_{\mathrm{k}}^{\varpi}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_{\rho}^{\zeta}(\mathrm{x})\right)\right]+\mathrm{g}_{\rho}^{\zeta-\delta_\rho}(\vartheta)\left[{}_{\mathrm{k}}^{\varpi}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left(\hbar^{\varrho}(\mathrm{x})\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right]$$

$$-\hbar^{\varrho}(\vartheta)\mathrm{g}_{\rho}^{\zeta-\delta_\rho}(\vartheta)\left[{}_{\mathrm{k}}^{\varpi}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left(\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right]-\left[{}_{\mathrm{k}}^{\varpi}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left(\hbar^{\varrho}(\mathrm{x})\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right]. \tag{19}$$

By conducting the product of (19) with (14) and integrating both sides about ϑ from x to $\mathbb{b}$ yields

$$0\leq\left[{}_{\mathrm{k}}^{\varpi}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_{\rho}^{\zeta}(\mathrm{x})\right)\right]\left[{}_{\mathrm{k}}^{\varpi}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left(\hbar^{\varrho}(\mathrm{x})\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right]$$

$$-\left[{}_{\mathrm{k}}^{\varpi}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left(\hbar^{\varrho}(\mathrm{x})\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_{\rho}^{\zeta}(\mathrm{x})\right)\right]\left[{}_{\mathrm{k}}^{\varpi}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left(\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right]. \tag{20}$$

Dividing both sides of (23) by

$${}_{\mathrm{k}}^{\varpi}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left(\hbar^{\varrho}(\mathrm{x})\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\left[{}_{\mathrm{k}}^{\varpi}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left(\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right],$$

leads to the proof of Theorem 3.3.

Corollary 3.5 Let continuous increasing function be $\hbar(\mathrm{x})$ and let the sequence of continuous positive decreasing functions be $\left\{\mathrm{g}_j,\, 1\leq j\leq m\right\}$ on the interval $[\mathbb{a},\,\mathbb{b}]$. If $\mathbb{a}<\mathrm{x}\leq\mathbb{b}$, $\varrho>0$, $\zeta\geq\delta_\rho>0$ for $1\leq\rho\leq m$. Then for the right $(k,\,\Psi)$-CFI operator ${}_{\mathrm{k}}^{\varpi}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}$, the following inequality holds:

$$\left[{}_{\mathrm{k}}^{\omega}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left(\hbar^{\varrho}(\mathrm{x})\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right]\left[{}_{\mathrm{k}}^{\varpi}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_{\rho}^{\zeta}(\mathrm{x})\right)\right]$$

$$+\left[{}_{\mathrm{k}}^{\omega}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_{\rho}^{\zeta}(\mathrm{x})\right)\right]\left[{}_{\mathrm{k}}^{\varpi}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left(\hbar^{\varrho}(\mathrm{x})\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right]$$

$$\geq\left[{}_{\mathrm{k}}^{\omega}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left(\hbar^{\varrho}(\mathrm{x})\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_{\rho}^{\zeta}(\mathrm{x})\right)\right]\left[{}_{\mathrm{k}}^{\varpi}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_{\rho}^{\zeta}(\mathrm{x})\right)\right]$$

$$+\left[{}_{\mathrm{k}}^{\omega}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left(\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right]\left[{}_{\mathrm{k}}^{\varpi}\mathcal{M}_{\mathbb{b}}^{\lambda,\Psi}\left(\hbar^{\varrho}(\mathrm{x})\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right].$$

Proof. By multiplying (19) with (16) and then integrating the resultant relation with respect to $\wp$ from x to $\mathbb{b}$ yields

$$
\begin{aligned}
0 \leq & \left[{}^{\omega}_{\mathrm{k}}\mathcal{M}^{\lambda,\Psi}_{\mathbb{b}}\left(\hbar^{\varrho}(\mathrm{x})\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right]\left[{}^{\varpi}_{\mathrm{k}}\mathcal{M}^{\lambda,\Psi}_{\mathbb{b}}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_{\rho}^{\zeta}(\mathrm{x})\right)\right] \\
& + \left[{}^{\omega}_{\mathrm{k}}\mathcal{M}^{\lambda,\Psi}_{\mathbb{b}}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_{\rho}^{\zeta}(\mathrm{x})\right)\right]\left[{}^{\varpi}_{\mathrm{k}}\mathcal{M}^{\lambda,\Psi}_{\mathbb{b}}\left(\hbar^{\varrho}(\mathrm{x})\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right] \\
& - \left[{}^{\omega}_{\mathrm{k}}\mathcal{M}^{\lambda,\Psi}_{\mathbb{b}}\left(\hbar^{\varrho}(\mathrm{x})\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_{\rho}^{\zeta}(\mathrm{x})\right)\right]\left[{}^{\varpi}_{\mathrm{k}}\mathcal{M}^{\lambda,\Psi}_{\mathbb{b}}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_{\rho}^{\zeta}(\mathrm{x})\right)\right] \\
& - \left[{}^{\omega}_{\mathrm{k}}\mathcal{M}^{\lambda,\Psi}_{\mathbb{b}}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_{\rho}^{\zeta}(\mathrm{x})\right)\right]\left[{}^{\varpi}_{\mathrm{k}}\mathcal{M}^{\lambda,\Psi}_{\mathbb{b}}\left(\hbar^{\varrho}(\mathrm{x})\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right].
\end{aligned}
\tag{21}
$$

Dividing both sides of (21) by the following relation

$$
\begin{aligned}
& \left[{}^{\omega}_{\mathrm{k}}\mathcal{M}^{\lambda,\Psi}_{\mathbb{b}}\left(\hbar^{\varrho}(\mathrm{x})\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_{\rho}^{\zeta}(\mathrm{x})\right)\right]\left[{}^{\varpi}_{\mathrm{k}}\mathcal{M}^{\lambda,\Psi}_{\mathbb{b}}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_{\rho}^{\zeta}(\mathrm{x})\right)\right] \\
& + \left[{}^{\omega}_{\mathrm{k}}\mathcal{M}^{\lambda,\Psi}_{\mathbb{b}}\left(\prod_{j\neq\rho}^{m}\mathrm{g}_j^{\delta_j}\mathrm{g}_{\rho}^{\zeta}(\mathrm{x})\right)\right]\left[{}^{\varpi}_{\mathrm{k}}\mathcal{M}^{\lambda,\Psi}_{\mathbb{b}}\left(\hbar^{\varrho}(\mathrm{x})\prod_{j=1}^{m}\mathrm{g}_j^{\delta_j}(\mathrm{x})\right)\right].
\end{aligned}
$$

Completes the proof of Corollary 3.5.

Remark 2

i. If we choose for $\Psi(\mathrm{x}) = \mathrm{x}$, we get the inequalities presented by [41].
ii. If we choose $k = 1$, we get new inequalities for the operators presented by [42].
iii. If we choose for $\Psi(\mathrm{x}) = \mathrm{x}$ and $\lambda = 1$, then we get inequalities for the k-fractional conformable operators.
iv. If we choose for $\Psi(\mathrm{x}) = \mathrm{x}$ and $\lambda = k = 1$, then we get inequalities presented by [45].

4. Conclusions

The left and right (k, Ψ)-CFI's have been established along with several significant integral inequalities associated with a finite series of positive and decreasing functions. The operators (k, Ψ)-CFIs presented in this work are the generalized form of the operators presented by [4, 28, 29]. The work presented in this paper can easily be reduced to the existing operators cited in the literature by considering Remark 1. If we consider $\Psi(\tau) = \tau$, then one can get the work presented earlier by Qi et al. [28]. If we take $k = 1$, then one can get the inequalities for the operators presented by [29]. Similarly, one can get the inequalities for the other existing operators by applying specific conditions given in Remark 1. One can obtain several other types of fractional integral inequalities and fractional differential equations by using the proposed (k, Ψ)-CFI's.

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Authors' contributions

The authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

Conflict of interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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