Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

The Generalized (k, ϑ)-Fractional Integral and Derivative Operators: Advances in Fractional Calculus

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Received: 29 November 2025; **Revised:** 26 February 2026; **Accepted:** 3 March 2026

Abstract: In this paper, we introduce the modified conformable fractional derivative and fractional integral operators, the so called (k, ϑ)-conformable fractional derivative (CFD) and (k, ϑ)-conformable fractional integral (CFI) operators. We examine their properties, including semi-group properties and the composition of CFD and CFI operators. Furthermore, we investigate the (k, ϑ)- Conformable Caputo Fractional Derivatives. Some properties are discussed graphically. The operators defined in this paper are the generalization of distinct operators cited therein literature. These generalized operators can easily be reduced to classical operators by giving certain specific values to the parameters.

***Keywords*:** conformable fractional derivative (CFD), conformable fractional integral (CFI), Caputo fractional derivative

MSC: 26A33, 26A51

1. Introduction

Fractional calculus (FC), which has applications in many scientific and technical fields, has made extensive use of fractional derivatives. Remarkably, the Riemann-Liouville derivative (RLD) and the Caputo derivative (CD) are the most often utilized derivatives. These derivatives have been shown to be highly effective in precisely characterizing complex dynamics found in physics, biology, engineering, and other domains [1–3].

Because memory effects are frequently observed in real-world scenarios, choosing the right non-local model is essential when working with a range of data types.

Thus, there has been a lot of research on the investigation of novel fractional operators with various properties, such as singular, non-singular, local, and non-local kernels [4–6].

Recent advances in FC have greatly simplified the use of different fractional operators to solve nonlinear functional integral problems. An investigation into the solvability of the Erdelyi-Kober fractional operator by Pathak et al. [7] provided additional information about the mathematical approaches that are effective in these circumstances. Paul et al. [8] presented an efficient method for solving nonlinear fractional integral problems. Additionally, Paul et al. [9] examined mixed-type nonlinear Volterra-Fredholm Erdelyi-Kober fractional integral equations. Bhat et al. [10] claimed that computer methods for heat and fluid flow issues were enhanced by the precision and effectiveness of an interpolation

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methodology to weakly singular integral equations. Farid et al. [11] proposed fractional Hadamard and Fej'er-Hadamard inequalities related to exponential $(\alpha, h\text{-}m)$-convexity and Riemann-Liouville fractional versions of Hadamard inequality for strongly m-convex functions, respectively, in an attempt to offer new tools for mathematical exploration. Rathour et al. have used fractional calculus to analyze convex functions [12]. For highly exponentially $(\alpha, h\text{-}m)$-convex functions, they have specifically included k-fractional integral inequalities of Hadamard type. Sadek [13] investigated the application of the fractional backward differentiation formulas (FBDF) approach to fractional differential matrix equations to address these difficult problems. For conformable fractional partial differential equations, Sadek et al. [14] devised a specific conformable finite element method. As a result, there was a significant increase in processing capacity available for these tasks. The work of Sadek on the stability of conformable linear infinite-dimensional systems provided important insights into their behavior and control [15].

In an attempt to expand the range of approaches for solving fractional differential equations, Sadek [16] looked at two-term fractional differential Sylvester matrix issues. It was discovered that fractional BDF equations were a helpful strategy. Sadek [17], working with Sami Bataineh, employed mathematical functions in fractional calculus in an innovative way to solve ϑ-fractional differential equations using the general Bernstein function. Last but certainly not least, Sadek et al. [18] developed the Galerkin-Bell technique to address fractional optimum control problems, demonstrating its efficacy and adaptability.

Collectively, these works improve our knowledge of fractional calculus and our ability to use it to solve a variety of challenging mathematical and control problems. The conventional FC method is probably not going to be able to provide the foundation needed to make intelligible inferences from such systems. An essential problems then arises: Is it possible to extend the fractional Riemann-Liouville integral (RLI) to incorporate RLD, Hadamard derivatives (HD), and other fractional derivatives [19, 20]? Determining the differentiation operator to use as the foundation for the iterative process is the work's challenge. The Cauchy formula, which allows us to substitute any integer value $w \in C$ for the standard integral, is used in FC to produce the standard integral of a function. In [21], it was suggested that the CFI be fractionalized correctly. Interestingly, an integral resemblance was originally utilized in [22] to illustrate a propensity for discounting in economic settings using mathematical economics. Furthermore, these integrals capture the behavior of nonlinear dissipative systems [23]. Classical fractional and integral operators, such as the Riemann-Liouville and Caputo operators, are widely used in various fields, including physics, engineering, and finance, to model phenomena characterized by non-local behavior and memory effects. Their applications range from control theory to viscoelasticity and anomalous diffusion. To solve fractional differential equations (FDEs) effectively, several numerical methods have been developed, including finite difference methods, fractional Adams-Bashforth and Adams-Moulton methods, and spectral methods. The introduction of new operators, like the (k, ϑ)-CFD and (k, ϑ)-CFI, offers a fresh perspective and potential advantages over classical methods. These new operators may facilitate more accurate modeling of complex systems by providing improved convergence properties and stability in numerical solutions. Additionally, they can lead to more efficient algorithms, enabling the handling of fractional equations in broader contexts and enhancing the applicability of fractional calculus in real-world problems.

The main objective of this paper is to define the generalized (k, ϑ)-conformable fractional derivative (CFD), conformable fractional integral (CFI) and Caputo fractional conformable derivative operators. Also, we study their various properties.

1.1 *Preliminaries*

In this section, we consider the following some known results.

In [24], the left and right CFDs are defined by:

Let $0 < \mathbb{w} \leq 1$, then

$$ {}_a\mathbb{T}^{\mathbb{w}} f(t) = \frac{df(t)}{d(t)}(t-a)^{1-\mathbb{w}} \quad \text{and} \quad \mathbb{T}_b^{\mathbb{w}} f(t) = \frac{df(t)}{d(t)}(b-t)^{1-\mathbb{w}}. $$

The left and right fractional integrals can be expressed as follows:

$$({}_a\mathbb{I}^{\wp} f)(t) = \int_a^t f(v)(v-a)^{\wp-1} dv \quad \text{and} \quad (\mathbb{I}_b^{\wp} f)(t) = \int_t^b f(v)(b-v)^{\wp-1} dv.$$

Let $\varkappa \in \mathfrak{R}_+^*$ and $m = [\varkappa] + 1$, the left RLI of order $\varkappa > 0$ is defined by

$$({}_a\mathbb{J}^{\varkappa} f)(t) = \frac{1}{\Gamma(\varkappa)} \int_a^t (t-v)^{\varkappa-1} f(v) dv. \tag{1}$$

The right RLI for order $\varkappa$ is defined by

$$(\mathbb{J}_b^{\varkappa} f)(t) = \frac{1}{\Gamma(\varkappa)} \int_t^b (v-t)^{\varkappa-1} f(v) dv. \tag{2}$$

The equation for the left RLD of order r is given below:

$$({}_a\mathbb{D}^{\varkappa} f)(t) = \left(\frac{d}{dt}\right)^m ({}_a\mathbb{J}^{m-\varkappa} f)(t).$$

The right RLD of order $\varkappa$ is defined by

$$(\mathbb{D}_b^{\varkappa} f)(t) = \left(-\frac{d}{dt}\right)^m (\mathbb{J}_b^{m-\varkappa} f)(t).$$

The form of left CD of order $\varkappa > 0$ is given by

$$({}_a^C\mathbb{D}^{\varkappa} f)(t) = ({}_a\mathbb{J}^{m-\varkappa} f^m)(t). \tag{3}$$

The right CD is:

$$({}^C\mathbb{D}_b^{\varkappa} f)(t) = \left(\mathbb{J}_b^{m-\varkappa} (-1)^m f^{(m)}\right)(t). \tag{4}$$

Take into consideration the increasing function $\vartheta \in C^1([a,b])$ such that for every $t \in [a,b]$, $\vartheta(t) \neq 0$. As per [7], the left ϑ-RLI of order $\varkappa > 0$, is as follows:

$$({}_a\mathbb{J}^{\varkappa,\vartheta} f)(t) = \frac{1}{\Gamma(\varkappa)} \int_a^t (\vartheta(t) - \vartheta(\mathrm{y}))^{\varkappa-1} \vartheta'(\mathrm{y}) f(\mathrm{y}) d\mathrm{y} \tag{5}$$

and the right ϑ-RLI of order $\varkappa > 0$ is defined by

$$(\mathbb{J}_b^{\varkappa,\vartheta}\ f)(t) = \frac{1}{\Gamma(\varkappa)} \int_t^b (\ \vartheta(\mathrm{y}) - \ \vartheta(t))^{\varkappa-1}\ \vartheta'(\mathrm{y})\ f(\mathrm{y})dy. \tag{6}$$

The equation for the left ϑ-RLD of order $\varkappa > 0$, is given below:

$$({}_a\mathbb{D}^{\varkappa,\ \vartheta} f(t) = \left(\frac{1}{\vartheta'(t)}\frac{d}{dt}\right)^m ({}_a\mathbb{J}^{m-\varkappa,\vartheta} f)(t). \tag{7}$$

The right ϑ-RLD of order $\varkappa$, is follows:

$$(\mathbb{D}_b^{\varkappa,\vartheta} f)(t) = \left(-\frac{1}{\vartheta'(t)}\frac{d}{dt}\right)^m (\mathbb{J}_b^{m-\varkappa,\ \vartheta} f)(t). \tag{8}$$

The left ϑ-CD is

$$({}_a^C\mathbb{D}^{\varkappa,\vartheta} f)(t) = {}_a\mathbb{J}^{m-\varkappa}\left(\frac{1}{\vartheta'(t)}\frac{d}{dt}\right)^m (t). \tag{9}$$

The right ϑ-CD of order $\varkappa$ is given by:

$$({}^c\mathbb{D}_b^{\varkappa,\vartheta} f)(t) = (\mathbb{J}_b^{m-\varkappa}\left(\left(\frac{-1}{\vartheta'(t)}\frac{d}{dt}\right)^m f\right)(t). \tag{10}$$

The left Hadamard fractional integral (HI) [25] is defined as

$$({}_a\mathcal{T}^{\varkappa} f)(t) = \frac{1}{\Gamma(\varkappa)} \int_a^t ((ln(t) - ln(y))^{\varkappa-1} f(y)\frac{dy}{y} \tag{11}$$

and the right HI is:

$$(\mathcal{T}_b^{\varkappa} f)(t) = \frac{1}{\Gamma(\varkappa)} \int_t^b ((\ln(\mathrm{y}) - \ln(t))^{\varkappa-1}\ f(\mathrm{y})\frac{d\mathrm{y}}{\mathrm{y}}.$$

The Left HD of order $\varkappa$ is:

$$({}_a\mathfrak{D}^{\varkappa}(t) = \left((t\frac{d}{dt})^m\right) ({}_a\mathbb{I}^{m-\varkappa} f)(t). \tag{12}$$

The right HD of order $\varkappa$ is:

$$(\mathfrak{D}_b^{\varkappa} f)(t) = \left((-t\frac{d}{dt})^{m}\right)(\mathbb{I}_b^{m-\varkappa} f)(t). \tag{13}$$

The left and right Caputo-HD [27–29] are defined respectively by

$$({}_a^C\mathfrak{D}^{\varkappa} f)(t) = {}_a\mathfrak{D}^{\varkappa}\left[f(t) - \sum_{\ell=0}^{m-1} \frac{(t\frac{d}{dt})^{\ell} f(a)}{\ell!}((\ln(t) - \ln(a))^{\ell}\right](t).$$

Let the space $C^m[a,b] = \left\{f : [a,b] \to \mathbb{C} : \left((-t\frac{d}{dt})^{m-1}[f(t)] \in C[a,b]\right\}\right.$, then the left and right Caputo-HD, respectively, are

$$({}_a^C\mathfrak{D}^{\varkappa} f)(t) = \left({}_a\mathcal{T}^{m-\varkappa}\left(t\frac{d}{dt}\right)^{m} f\right)(t). \tag{14}$$

and

$$({}^c\mathfrak{D}_b^{\varkappa} f)(t) = \mathfrak{D}_b^{\varkappa}\left[f(t) - \sum_{\ell=0}^{m-1} \frac{(-1)^{\ell}(t\frac{d}{dt})^{\ell} f(b)}{\ell!}\ (\ln(b) - \ln(t))^{k}\right](t)$$

in $C^m[a,b]$ equivalentaly by

$$({}^c\mathfrak{D}_b^{\varkappa} f)(t) = \left(\mathcal{T}_b^{m-\varkappa}(-t\frac{d}{dt})^{m} f\right)(t). \tag{15}$$

Let the function space be defined by

$$Y_c^q(a,b) = \left\{f : [a,b] \to \mathbb{R} : \| f \|_{X_c^q} = \left(\int_a^b |t^c f(t)|^q\right)^{\frac{1}{q}} < \infty\right\},$$

where $1 \leq q < \infty$, $a < b$, and $c \in \Re$. The norm $|f|_{Y_c^q}$ in the case of $q = \infty$ is defined as the essential supremum over $a \leq l \leq b$ of the equation $[t^c\ |f(t)|]$.

The left and right Katugampola Fractional Integrals (KFI) [19] can be represented in the following way within the context of the function space mentioned:

$$({}_a\mathbb{I}^{\varkappa,\mathbb{w}} f)(t) = \frac{1}{\Gamma(\varkappa)} \int_a^t \left(\frac{t^{\mathbb{w}} - y^{\mathbb{w}}}{\mathbb{w}}\right)^{\varkappa-1} f(y)\frac{dy}{y^{1-\mathbb{w}}}$$

and

$$(\mathbb{I}_b^{\varkappa,\mathbb{w}} f)(t) = \frac{1}{\Gamma(\varkappa)} \int_t^b \left(\frac{y^{\mathbb{w}} - t^{\mathbb{w}}}{\mathbb{w}} \right)^{\varkappa-1} f(y) \frac{dy}{y^{1-\mathbb{w}}} \tag{16}$$

respectively.

The fractional derivatives of left and right Katugampola [20] are defined by

$$({}_a\mathbb{D}^{\varkappa,\mathbb{w}} f)(t) = \frac{\left(t^{1-\mathbb{w}} \frac{d}{dt}\right)^m}{\Gamma(m-\varkappa)} \int_a^t \left(\frac{x^{\mathbb{w}} - y^{\mathbb{w}}}{\mathbb{w}} \right)^{m-\varkappa-1} f(y) \frac{dy}{y^{1-\mathbb{w}}}$$

and

$$(\mathbb{D}_b^{\varkappa,\mathbb{w}} f)(t) = \frac{\left(-t^{1-\mathbb{w}} \frac{d}{dt}\right)^m}{\Gamma(m-\varkappa)} \int_t^b \left(\frac{y^{\mathbb{w}} - x^{\mathbb{w}}}{\mathbb{w}} \right)^{m-\varkappa-1} f(y) \frac{dy}{y^{1-\mathbb{w}}}$$

respectively.

Based on the framework established by Katugampola [20], Jarad et al. [30] proposed modified Caputo-type versions of the left and right generalized fractional derivatives. These definitions are given by:

$$({}_a^C\mathbb{D}^{\varkappa,\mathbb{w}} f)(t) = \frac{1}{\Gamma(m-\varkappa)} \int_a^t \left(\frac{t^{\mathbb{w}} - u^{\mathbb{w}}}{\mathbb{w}} \right)^{m-\varkappa-1} \left(t^{1-\mathbb{w}} \frac{d}{dt} \right)^m f(y) \frac{dy}{y^{1-\mathbb{w}}}$$

and

$$({}^C\mathbb{D}_b^{\varkappa,\mathbb{w}} f)(t) = \frac{1}{\Gamma(m-\varkappa)} \int_t^b \left(\frac{y^{\mathbb{w}} - t^{\mathbb{w}}}{\mathbb{w}} \right)^{m-\varkappa-1} \left(-t^{1-\mathbb{w}} \frac{d}{dt} \right)^m f(y) \frac{dy}{y^{1-\mathbb{w}}}.$$

The left and right FI were presented by the authors of [25] by using the concept of [21]. The following are the formulations for these integrals:

$${}_a^{\varkappa}\mathcal{T}^{\mathbb{w}} f(t) = \frac{1}{\Gamma(\varkappa)} \int_a^t \left(\frac{(t-a)^{\mathbb{w}} - (\varsigma-a)^{\mathbb{w}}}{\mathbb{w}} \right)^{\varkappa-1} \frac{f(\varsigma) d\varsigma}{(\varsigma-a)^{1-\mathbb{w}}} \tag{17}$$

and

$$({}^{\varkappa}\mathcal{T}_b^{\mathbb{w}} f)(t) = \frac{1}{\Gamma(\varkappa)} \int_t^b \left(\frac{(b-t)^{\mathbb{w}} - (b-\varsigma)^{\mathbb{w}}}{\mathbb{w}} \right)^{\varkappa-1} \frac{f(\varsigma) d\varsigma}{(b-\varsigma)^{1-\mathbb{w}}}. \tag{18}$$

By using the idea of [21] , the authors in [25] introduced the FD as follows:

$$ {}_a^r\mathfrak{D}^{\mathbb{w}} f(t) = \frac{{}_a^m\mathbb{T}^w}{\Gamma(m-r)} \int_a^t \left(\frac{(t-a)^{\mathbb{w}} - (\varsigma-a)^{\mathbb{w}}}{\mathbb{w}} \right)^{r-1} \frac{f(\varsigma)d\varsigma}{(\varsigma-a)^{1-\mathbb{w}}} \tag{19} $$

and

$$ {}^r\mathfrak{D}_b^{\mathbb{w}} f(t) = \frac{(-1)^m\ {}^m\mathbb{T}_b^w}{\Gamma(m-r)} \int_t^b \left(\frac{(b-t)^{\mathbb{w}} - (b-\varsigma)^{\mathbb{w}}}{\mathbb{w}} \right)^{r-1} \frac{f(\varsigma)d\varsigma}{(b-\varsigma)^{1-\mathbb{w}}} \tag{20} $$

where

$$ {}_a^m\mathbb{T}^{\mathbb{w}} = {}_a\mathbb{T}^{\mathbb{w}}\ {}_a\mathbb{T}^{\mathbb{w}} \ldots {}_a\mathbb{T}^{\mathbb{w}}\ (m \text{ times}) \text{ and } {}^m\mathbb{T}_b^{\mathbb{w}} = \mathbb{T}_b^{\mathbb{w}}\ \mathbb{T}_b^{\mathbb{w}} \ldots \mathbb{T}_b^{\mathbb{w}} (m \text{ times}) $$

With ${}_a\mathbb{T}^{\mathbb{w}}$ and $\mathbb{T}_b^{\mathbb{w}}$ are the left and right CFD.
The CFD in Caputo type are;

$$ {}_a^C\mathfrak{D}^{r,\mathbb{w}} f(t) = \frac{1}{\Gamma(m-r)} \int_a^t \left(\frac{(t-a)^{\mathbb{w}} - (\varsigma-a)^{\mathbb{w}}}{\mathbb{w}} \right)^{m-r-1} \frac{{}_a^m\mathbb{T}^{\mathbb{w}} f(\varsigma)d\varsigma}{(\varsigma-a)^{1-\mathbb{w}}} \tag{21} $$

and

$$ {}^C\mathfrak{D}_b^{r,\mathbb{w}} f(t) = \frac{(-1)^m}{\Gamma(m-r)} \int_t^b \left(\frac{(b-t)^{\mathbb{w}} - (b-\varsigma)^{\mathbb{w}}}{\mathbb{w}} \right)^{m-r-1} \frac{{}^m\mathbb{T}_b^{\mathbb{w}} f(\varsigma)d\varsigma}{(b-\varsigma)^{1-\mathbb{w}}}, \tag{22} $$

left and right respectively.

Definition 1 [26] Let $r \in \mathbb{C}, \Re(r) > 0$. *The* left ϑ-CFI is

$$ {}_a^r\mathcal{T}_k^{\mathbb{w}} f(t) = \frac{1}{k\Gamma_k(r)} \int_a^t \left(\frac{(t-a)^{\mathbb{w}} - (\varsigma-a)^{\mathbb{w}}}{\mathbb{w}} \right)^{\frac{r}{k}-1} \frac{f(\varsigma)d\varsigma}{(\varsigma-a)^{1-\mathbb{w}}} $$

and

$$ ({}^r\mathcal{T}_{b,k}^{\mathbb{w}} f)(t) = \frac{1}{k\Gamma_k(r)} \int_t^b \left(\frac{(b-t)^{\mathbb{w}} - (b-\varsigma)^{\mathbb{w}}}{\mathbb{w}} \right)^{\frac{r}{k}-1} \frac{f(\varsigma)d\varsigma}{(b-\varsigma)^{1-\mathbb{w}}}. $$

Definition 2 [27] Let ϑ an increasing function and $\vartheta \in \mathbb{C}^1[a,b]$ such that $\vartheta \neq 0$.
The left ϑ-CFD is given by:

$$ {}_{a}^{m}\mathbb{T}^{\mathbb{w},\vartheta} f(t) = \left(\frac{\frac{df(t)}{dt}}{(\vartheta(t)-\vartheta(a))^{\mathbb{w}-1}\frac{d\vartheta(t)}{d(t)}} \right)^{m} $$

and the right ϑ-CFD is given by

$$ {}^{m}\mathbb{T}_{b}^{\mathbb{w},\vartheta} f(t) = \left(\frac{\frac{df(t)}{dt}}{(\vartheta(b)-\vartheta(t))^{\mathbb{w}-1}\frac{d\vartheta(t)}{d(t)}} \right)^{m}. $$

If $\vartheta(t) = t$, we get the left and right CFD see [18].

Definition 3 [27] Let the function $\vartheta \in \mathbb{C}^1[a,b]$ be an increasing with $\vartheta(t) \neq 0$.
The left ϑ-CFI is defined by:

$$ ({}_{a}\mathbb{I}^{\mathbb{w},\vartheta} f)(t) = \int_{a}^{t} (\vartheta(u)-\vartheta(a))^{\mathbb{w}-1} f(u)\vartheta'(u)du $$

and the right ϑ-CFI is given by:

$$ (\mathbb{I}_{b}^{\mathbb{w},\vartheta} f)(t) = \int_{t}^{b} (\vartheta(b)-\vartheta(u))^{\mathbb{w}-1} f(u)\vartheta'(u)du. $$

If $\vartheta(u) = u$, we get the left and right CFI see [21]. From Definition 3, we have

$$ {}_{a}^{m}\mathbb{I}^{\mathbb{w},\vartheta} f(t) = \int_{a}^{t} (\vartheta(u_1)-\vartheta(a))^{\mathbb{w}-1}\vartheta'(u_1)du_1 \times \int_{a}^{u_1} (\vartheta(u_2)-\vartheta(a))^{\mathbb{w}-1}\vartheta'(u_2)du_2 $$

$$ \times \int_{a}^{u_2} (\vartheta(u_3)-\vartheta(a))^{\mathbb{w}-1}\vartheta'(u_3)\, u_3 \ldots \int_{a}^{u_{m-1}} (\vartheta(u_m)-\vartheta(a))^{\mathbb{w}-1}\vartheta'(u_m)du_m $$

$$ {}_{a}^{m}\mathbb{I}^{\mathbb{w},\vartheta} f(t) = \frac{1}{\Gamma(m)} \int_{a}^{t} \left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}} - (\vartheta(u)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}} \right)^{m-1} \frac{f(u)\vartheta'(u)du}{(\vartheta(u)-\vartheta(a))^{1-\mathbb{w}}}. $$

Definition 4 [28] Let $\varkappa \in \mathbb{C}, \Re(\varkappa) > 0$. The left ϑ-CFI is

$$ {}_{a}^{\varkappa}\mathcal{J}^{\mathbb{w},\vartheta} f(t) = \frac{1}{\Gamma(\varkappa)} \int_{a}^{t} \left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}} - (\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}} \right)^{\varkappa-1} \frac{f(\varsigma)\vartheta'(\varsigma)d\varsigma}{(\vartheta(\varsigma)-\vartheta(a))^{1-\mathbb{w}}}. \tag{23} $$

Definition 5 [28] Let $\varkappa \in \mathbb{C}$, $\Re(\varkappa) > 0$. The right ϑ-CFI is

$$ {}^{\varkappa}\mathcal{J}_{b}^{\mathbb{w},\vartheta} f(t) = \frac{1}{\Gamma(\varkappa)} \int_{t}^{b} \left(\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}} - (\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{\mathbb{w}} \right)^{\varkappa-1} \frac{f(\varsigma)\vartheta'(\varsigma)d\varsigma}{(\vartheta(b)-\vartheta(\varsigma))^{1-\mathbb{w}}}. \tag{24} $$

Recently, the development and application of conformable fractional integral and derivative operators have attracted a lot of attention. Salah et al. [29] used conformable fractional-order models to study the dynamics of Human Immunodeficiency Virus/Acquired Immunodeficiency Syndrome (HIV/AIDS) transmission, providing new insights into the disease's spread and prevention strategies. By employing extended CFI to give additional midpoint-type inequalities for coordinated convex functions, Kiriş et al. [31] expanded the mathematical tools available for handling these types of functions. The concept of fractional inequalities and their applications was expanded by Ying et al. [32], who provided conformable fractional Milne-type inequalities. Hezenci and Budak [33] refined classical inequalities by establishing Bullen-type inequalities for twice-differentiable functions using CFI. Wang and Yuan [34] explored the existence, uniqueness, and Ulam stability of solutions to the fractional conformable Langevin system on the ethane graph, demonstrating the sophisticated application of fractional calculus to complex systems analysis.

In the last few decades, the field of fractional calculus has gained more recognition due to wide application in diverse domains. Many researchers have developed distinct generalizations of fractional operators cited in literature. In this paper, our aim is to define further generalization of conformable fractional integral and derivative operators inspired by the work done by [25, 26, 28].

The structure of the paper is as follows: Section 1 and 2 provide a brief history and results of conformable fractional integrals and derivatives. In addition, we provide a few further findings in section 1. Definitions and notations for left and right (k,ϑ)-conformable fractional integrals (k,ϑ)-CFI are provided in Section 3. In this section, we also present (k,ϑ)-CFD for functions in certain spaces, as well as establish the properties of (k,ϑ)-CFD in these spaces and calculate (k,ϑ)-CFD of functions in provided spaces. The concept, characteristics, and Caputo-type (k,ϑ)-CFD are covered in section 4. The concluding remarks are given in section 5.

2. The (k,ϑ)-CFD and (k,ϑ)-CFI

In this section, we define the novel (k,ϑ)-CFI (conformable fractional integral) and (k,ϑ)-CFD (conformable fractional derivative) operators. We also address their different characteristics.

Definition 6 Let the function ϑ be an increasing and $\vartheta \in \mathbb{C}^1[a,b]$, such that $\vartheta \neq 0$.

The left ϑ-CFD is defined by

$$ {}_{a}^{m}\mathbb{T}_{k}^{\mathbb{w},\vartheta} f(t) = k^{m} \left(\frac{\frac{df(t)}{dt}}{(\vartheta(t)-\vartheta(a))^{\mathbb{w}-1}\frac{d\vartheta(t)}{d(t)}} \right)^{m} $$

and the right ϑ-CFD is defined by

$$ {}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta} f(t) = k^{m} \left(\frac{\frac{df(t)}{dt}}{(\vartheta(b)-\vartheta(t))^{\mathbb{w}-1}\frac{d\vartheta(t)}{d(t)}} \right)^{m}. $$

Next, we define (k,ϑ) conformable fractional integrals.

Definition 7 Let $\varkappa \in \mathbb{C}$, $\Re(\varkappa) > 0$ and $k > 0$. The left $(k,\ \vartheta)$-CFI is

$$ {}_{a}^{\rho}\mathcal{J}_{k}^{\mathbb{w},\vartheta} f(t) = \frac{1}{k\Gamma_k(\rho)} \int_a^t \left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}} - (\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}} \right)^{\frac{\rho}{k}-1} \frac{f(\varsigma)\vartheta'(\varsigma)d\varsigma}{(\vartheta(\varsigma)-\vartheta(a))^{1-\mathbb{w}}}. \tag{25} $$

Remark 1 We have,
• Applying (25) for $\vartheta(t) = t$, we get the operators defined by Mubeen et al. [26].
• Applying (25) for $k = 1$, we get left ϑ-CFI (23)
• Applying (25) for $\vartheta(t) = t$ and $\mathbb{w} = 1$, we obtain (1).
• Applying (25) for $\mathbb{w} = 1$, we obtain $(k,\ \vartheta)$-RLI (5).
• Applying (25) for $\vartheta(t) = t$, we have k-RL fractional integral (17).
• Applying (25) for $\vartheta(t) = \ln t$ and $\mathbb{w} \to 0$, we obtain (11).
Definition 8 Let $\rho \in \mathbb{C},\ \Re(\rho) > 0$ and $k > 0$. The right (k, ϑ)-CFI is

$$ {}^{\rho}\mathcal{J}_{b,k}^{\mathbb{w},\vartheta} f(t) = \frac{1}{k\Gamma_k(\rho)} \int_t^b \left(\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}} - (\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{\mathbb{w}} \right)^{\frac{\rho}{k}-1} \frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}. \tag{26} $$

Remark 2
• If we consider $k = 1$, then (26) reduces to the right ϑ-CFI (24).
• If we consider $\vartheta(t) = t$ and $\mathbb{w} = 1$, then (26) reduces to the k-RL fractional integral (2).
• If we take $\mathbb{w} = 1$ in (26), then we have $(k,\ \vartheta)$-RLI (6).
• If we take $\vartheta(t) = t$ in (26), then we obtain k-RL fractional integral (18).
• If we choose $\vartheta(t) = \ln t$ and $\mathbb{w} \to 0$ in (26), we obtain (16).
In Definition 8, we present the (k, ϑ)-CFD.
Definition 9 Let $\rho \in \mathbb{C} and \Re(\rho) > 0$ and $k > 0$. The left and right (k, ϑ)-CFD, are defined as follows:

$$ {}_{a}^{\rho}\mathcal{D}_{k}^{\mathbb{w},\vartheta} f(t) = {}_{a}^{m}\mathbb{T}_{k}^{\mathbb{w},\vartheta}\ {}_{a}^{mk-\rho}\mathcal{J}_{k}^{\mathbb{w},\vartheta} f(t) $$

$$ = \frac{{}_{a}^{m}\mathbb{T}_{k}^{\mathbb{w},\vartheta}}{k\Gamma_k(mk-\rho)} \int_a^t \left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}} - (\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}} \right)^{\frac{mk-\rho}{k}-1} \frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(\varsigma)-\vartheta(a)\ )^{1-\mathbb{w}}} \tag{27} $$

and

$$ {}^{\rho}\mathcal{D}_{b,k}^{\mathbb{w},\vartheta} f(t) = {}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}\ {}^{mk-\rho}\mathcal{J}_{b,k}^{\mathbb{w},\vartheta} f\ (t) $$

$$ = \frac{(-1)^m\ {}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}}{k\Gamma_k(mk-\rho)} \int_t^b \left(\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}} - (\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{\mathbb{w}} \right)^{\frac{mk-\rho}{k}-1} \frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}} \tag{28} $$

where $m = \Re(\rho) + 1$ and

$$\left.\begin{aligned} {}^{m}_{a}\mathbb{T}_{k}^{\mathbb{w},\vartheta} &= {}_{a}\mathbb{T}_{k}^{\mathbb{w},\vartheta}\,{}_{a}\mathbb{T}_{k}^{\mathbb{w},\vartheta}\cdots{}_{a}\mathbb{T}_{k}^{\mathbb{w},\vartheta} \\ {}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta} &= \mathbb{T}_{b,k}^{\mathbb{w},\vartheta}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}\ldots\mathbb{T}_{b,k}^{\mathbb{w},\vartheta} \end{aligned}\right\} m\text{-times}$$

with ${}^{m}_{a}\mathbb{T}_{k}^{\mathbb{w},\vartheta}$ and ${}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}$ are presented in Definition 5.

Remark 3 We have

• If we choose $k=1$, $\vartheta(t)=t$ and $\mathbb{w}=1$, then (27) becomes (7).
• If we choose $\mathbb{w}=1$, then (27) becomes (19).
• If we choose $\vartheta(t)=t$ and $\mathbb{w}\to 0$, then (27) reduces to (12).
• If we choose $\mathbb{w}=1$, then (27) reduces to (6).
• If we choose $\vartheta(t)=t$ and $\mathbb{w}=1$, we get (8).
• If we choose $\mathbb{w}=1$, then (27) reduces to (20).
• If we choose $\vartheta(t)=t$ and $\mathbb{w}\to 0$, then (28) reduces to (13).

Next, we present properties of the (k,ϑ)-CFD and (k,ϑ)-CFI.

Theorem 1 Let $\varkappa,\varkappa^{*}\in\mathscr{R}^{\circ}_{+}$, we have

$$\left.\begin{aligned} &(i)\,{}^{\varkappa}_{a}\mathscr{J}_{k}^{\mathbb{w},\vartheta}\left({}^{\varkappa^{*}}_{a}\mathscr{J}_{k}^{\mathbb{w},\vartheta}\right)f(t)={}^{\varkappa+\varkappa^{*}}_{a}\mathscr{J}_{k}^{\mathbb{w},\vartheta}\,f(t) \\ &(ii)\,{}^{\varkappa}\mathscr{J}_{b,k}^{\mathbb{w},\vartheta}\left({}^{\varkappa^{*}}\mathscr{J}_{b,k}^{\mathbb{w},\vartheta}\right)f(t)={}^{\varkappa+\varkappa^{*}}\mathscr{J}_{b,k}^{\mathbb{w},\vartheta}\,f(t) \end{aligned}\right\}. \tag{29}$$

Proof

i) Let us consider

$${}^{\varkappa}_{a}\mathscr{J}_{k}^{\mathbb{w},\vartheta}\left({}^{\varkappa^{*}}_{a}\mathscr{J}_{k}^{\mathbb{w},\vartheta}\right)f(t)$$

$$=\frac{1}{k\Gamma_{k}(\varkappa)k\Gamma_{k}(\varkappa^{*})}\int_{a}^{t}\int_{a}^{\nu}\left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\nu)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{\varkappa}{k}-1}\left(\frac{(\vartheta(\nu)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{\varkappa^{*}}{k}-1}$$

$$\times\frac{f(\varsigma)\vartheta'(\varsigma)d\varsigma}{(\vartheta(\varsigma)-\vartheta(a))^{1-\mathbb{w}}}\frac{\vartheta'(\nu)d\nu}{(\vartheta(\nu)-\vartheta(a))^{1-\mathbb{w}}}$$

$$=\frac{1}{k\Gamma_{k}(\varkappa)k\Gamma_{k}(\varkappa^{*})\mathbb{w}^{\frac{\varkappa}{k}+\frac{\varkappa^{*}}{k}-2}}\int_{a}^{t}\int_{\varsigma}^{t}((\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\nu)-\vartheta(a))^{\mathbb{w}})^{\frac{\varkappa}{k}-1}$$

$$\times((\vartheta(\nu)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}})^{\frac{\varkappa^{*}}{k}-1}\frac{f(\varsigma)\,\vartheta'(\varsigma)d\varsigma}{(\vartheta(\varsigma)-\vartheta(a))^{1-\mathbb{w}}}\frac{\vartheta'(\nu)d\nu}{(\vartheta(\nu)-\vartheta(a))^{1-\mathbb{w}}}.$$

Substituting

$$y=\frac{(\vartheta(\nu)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}},\ \ dy=\frac{\mathbb{w}(\vartheta(\nu)-\vartheta(a))^{\mathbb{w}-1}\ \vartheta'(\nu)d\nu}{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}$$

and

$$1-y=1-\frac{(\vartheta(\nu)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}=\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\nu)-\vartheta(a))^{\mathbb{w}}}{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}$$

in above integral, we get

$$\begin{aligned}
&{}_{a}^{\varkappa}\mathcal{J}_{k}^{\mathbb{w},\vartheta}\left({}_{a}^{\varkappa^*}\mathcal{J}_{k}^{\mathbb{w},\vartheta}\right)f(t)\\
&=\frac{1}{k\Gamma_k(\varkappa)k\Gamma_k(\varkappa^*)\,\mathbb{w}^{\frac{\varkappa}{k}+\frac{\varkappa^*}{k}-2+1}}\int_a^t((\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}})^{\frac{\varkappa}{k}+\frac{\varkappa^*}{k}-1}\frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(\varsigma)-\vartheta(a))^{1-\mathbb{w}}}\\
&\quad\times\int_0^1(1-y)^{\frac{\varkappa}{k}-1}(y)^{\frac{\varkappa^*}{k}-1}dy\\
&=\frac{1}{k(\Gamma_k(\varkappa)\Gamma_k(\varkappa^*))}\int_a^t\left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{\varkappa}{k}+\frac{\varkappa^*}{k}-1}\frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(\varsigma)-\vartheta(a)\ )^{1-\mathbb{w}}}\\
&\quad\times\frac{1}{k}\int_0^1(1-y)^{\frac{\varkappa}{k}-1}(y)^{\frac{\varkappa^*}{k}-1}dy.
\end{aligned}$$

By using k-Beta function, we get

$$\begin{aligned}
{}_{a}^{\varkappa}\mathcal{J}_{k}^{\mathbb{w},\vartheta}\left({}_{a}^{\varkappa^*}\mathcal{J}_{k}^{\mathbb{w},\vartheta}\right)f(t)&=\frac{1}{k\Gamma_k(\varkappa+\varkappa^*)}\int_a^t\left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{\varkappa}{k}+\frac{\varkappa^*}{k}-1}\frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(\varsigma)-\vartheta(a)\ )^{1-\mathbb{w}}}\\
&={}_{a}^{\varkappa+\varkappa^*}\mathcal{J}_{k}^{\mathbb{w},\vartheta}\ f(t),
\end{aligned}$$

which completes the proof of (i).

ii) Now, consider

$$\begin{aligned}
&{}^{\varkappa}\mathcal{J}_{b,k}^{\mathbb{w},\vartheta}\left({}^{\varkappa^*}\mathcal{J}_{b,k}^{\mathbb{w},\vartheta}\right)\ f(t)\\
&=\frac{1}{k\Gamma_k(\varkappa)k\Gamma_k(\varkappa^*)}\int_t^b\int_\nu^b\left(\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\nu))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{\varkappa}{k}-1}\left(\frac{(\vartheta(b)-\vartheta(\nu))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{\varkappa^*}{k}-1}
\end{aligned}$$

$$\times \frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}\ \frac{\vartheta'(\nu)d\nu}{(\ \vartheta(b)-\vartheta(\nu)\ )^{1-\mathbb{w}}}$$

$$= \frac{1}{k\Gamma_k(\mathscr{r})k\Gamma_k(\mathscr{r}^*)\mathbb{w}^{\frac{\mathscr{r}}{k}+\frac{\mathscr{r}^*}{k}-2}} \int_t^b \int_\varsigma^b \left((\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\nu))^{\mathbb{w}}\right)^{\frac{\mathscr{r}}{k}-1}$$

$$\times \left((\vartheta(b)-\vartheta(\nu))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma)^{\mathbb{w}}\right)^{\frac{\mathscr{r}^*}{k}-1} \frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}\ \frac{\vartheta'(\nu)d\nu}{(\ \vartheta(b)-\vartheta(\nu)\ )^{1-\mathbb{w}}}.$$

Substituting

$$y = \frac{(\vartheta(b)-\vartheta(\nu))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}$$

$$-dy = \frac{\mathbb{w}(\vartheta(b)-\vartheta(\nu))^{\mathbb{w}-1}\ \vartheta'(\nu)d\nu}{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(a)-\vartheta(\varsigma))^{\mathbb{w}}}$$

and

$$1-y = \frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\nu))^{\mathbb{w}}}{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}$$

in above integral, we get

$${}^{\mathscr{r}}\mathscr{I}_{b,k}^{\mathbb{w},\vartheta}\left({}^{\mathscr{r}^*}\mathscr{I}_{b,k}^{\mathbb{w},\vartheta}\right)\ f(t)$$

$$= \frac{1}{k\Gamma_k(\mathscr{r})k\Gamma_k(\mathscr{r}^*)\mathbb{w}^{\frac{\mathscr{r}}{k}+\frac{\mathscr{r}^*}{k}-2+1}} \int_t^b \left((\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}\right)^{\frac{\mathscr{r}}{k}+\frac{\mathscr{r}^*}{k}-1}$$

$$\frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}} \int_0^1 (1-y)^{\frac{\mathscr{r}}{k}-1}(y)^{\frac{\mathscr{r}^*}{k}-1}dy$$

$$= \frac{1}{k(\Gamma_k(\mathscr{r})\Gamma_k(\mathscr{r}^*))} \int_t^b \left(\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{\mathscr{r}}{k}+\frac{\mathscr{r}^*}{k}-1} \frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}$$

$$\times \frac{1}{k}\int_0^1 (1-y)^{\frac{\mathscr{r}}{k}-1}(y)^{\frac{\mathscr{r}^*}{k}-1}dy.$$

By using k-Beta function, we get

$$
{}^{\varkappa}\mathcal{J}_{b,k}^{\mathbb{w},\vartheta}\left({}^{\varkappa^*}\mathcal{J}_{b,k}^{\mathbb{w},\vartheta}\right) f(t) = \frac{1}{(-1)k\Gamma_k(\varkappa+\varkappa^*)} \int_t^b \left(\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}} - (\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{\varkappa}{k}+\frac{\varkappa^*}{k}-1} \frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}
$$

$$
= {}^{\varkappa+\varkappa^*}\mathcal{J}_{b,k}^{\mathbb{w},\vartheta}\ f(t),
$$

which proves the desired proof. □

Lemma 1 We have

$$
(1) \qquad {}_a^{\varkappa}\mathcal{J}_k^{\mathbb{w},\vartheta}\left(\vartheta(t)-\vartheta(a)\right)^{\frac{\mathbb{w}\varkappa^*}{k}-\mathbb{w}} = \frac{1}{\mathbb{w}^{\frac{\varkappa}{k}}} \frac{k\Gamma_k(\varkappa^*)}{k\Gamma_k(\varkappa+\varkappa^*)} \left(\vartheta(t)-\vartheta(a)\right)^{\mathbb{w}\left(\frac{\varkappa}{k}+\frac{\varkappa^*}{k}-1\right)} \tag{30}
$$

$$
(2) \qquad {}^{\varkappa}\mathcal{J}_{b,k}^{\mathbb{w},\vartheta}\left(\vartheta(b)-\vartheta(t)\right)^{\frac{\mathbb{w}\varkappa^*}{k}-\mathbb{w}} = \frac{1}{\mathbb{w}^{\frac{\varkappa}{k}}} \frac{k\Gamma_k(\varkappa^*)}{k\Gamma_k(\varkappa+\varkappa^*)} \left(\vartheta(b)-\vartheta(t)\right)^{\mathbb{w}\left(\frac{\varkappa}{k}+\frac{\varkappa^*}{k}-1\right)} \tag{31}
$$

with $\varkappa^* \varepsilon \mathcal{R}_+^{\circ}$.

Proof (1). Consider

$$
{}_a^{\varkappa}\mathcal{J}_k^{\mathbb{w},\vartheta}\left(\vartheta(t)-\vartheta(a)\right)^{\frac{\mathbb{w}\varkappa^*}{k}-\mathbb{w}} = \frac{1}{k\Gamma_k(\varkappa)} \int_a^t \left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}} - (\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{\varkappa}{k}-1} (\vartheta(\varsigma)-\vartheta(a))^{\frac{\mathbb{w}\varkappa^*}{k}-\mathbb{w}}
$$

$$
\times \frac{\vartheta'(\varsigma)d\varsigma}{(\ \vartheta(\varsigma)-\vartheta(a)\ )^{1-\mathbb{w}}}.
$$

Substituting

$$
y = \frac{(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}},\ \ dy = \frac{\mathbb{w}(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}-1}\ \vartheta'(\varsigma)d\varsigma}{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}}
$$

and

$$
1-y = \frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}} - (\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}}
$$

in above integral, we get

$$
{}_a^{\varkappa}\mathcal{J}_k^{\mathbb{w},\vartheta}\left(\vartheta(t)-\vartheta(a)\right)^{\frac{\mathbb{w}\varkappa^*}{k}-\mathbb{w}} = \frac{1}{\Gamma_k(\varkappa)\mathbb{w}^{\frac{\varkappa}{k}-1+1}}\ \left(\vartheta(t)-\vartheta(a)\right)^{\mathbb{w}\left(\frac{\varkappa}{k}+\frac{\varkappa^*}{k}-1\right)} \frac{1}{k}\int_0^1 (1-y)^{\frac{\varkappa}{k}-1}(y)^{\frac{\varkappa^*}{k}-1}dy.
$$

Using k-Beta function, we get

$$ {}^{\varkappa}_{a}\mathcal{J}^{\mathbb{w},\vartheta}_{k}\left(\vartheta\left(t\right)-\vartheta(a)\right)^{\frac{\mathbb{w}\varkappa^{*}}{k}-\mathbb{w}}=\frac{1}{\mathbb{w}^{\frac{\varkappa}{k}}}\frac{\Gamma_{k}\left(\varkappa^{*}\right)}{\Gamma_{k}\left(\varkappa+\varkappa^{*}\right)}\left(\vartheta\left(t\right)-\vartheta(a)\right)^{\mathbb{w}\left(\frac{\varkappa}{k}+\frac{\varkappa^{*}}{k}-1\right)}, $$

which completes the proof of (1). □

Proof (2). Consider

$$ {}^{\varkappa}\mathcal{J}^{\mathbb{w},\vartheta}_{b,k}\left(\vartheta(b)-\vartheta\left(t\right)\right)^{\frac{\mathbb{w}\varkappa^{*}}{k}-\mathbb{w}}=\frac{1}{k\Gamma_{k}(\varkappa)}\int_{t}^{b}\left(\frac{\left(\vartheta(b)-\vartheta\left(t\right)\right)^{\mathbb{w}}-\left(\vartheta(b)-\vartheta(\varsigma)\right)^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{\varkappa}{k}-1}\left(\vartheta(b)-\vartheta(\varsigma)\right)^{\mathbb{w}\frac{\varkappa^{*}}{k}-\mathbb{w}} $$

$$ \times\frac{\vartheta'(\varsigma)d\varsigma}{\left(\vartheta(b)-\vartheta(\varsigma)\right)^{1-\mathbb{w}}}. $$

Substituting

$$ y=\frac{\left(\vartheta(b)-\vartheta(\varsigma)\right)^{\mathbb{w}}}{\left(\vartheta(b)-\vartheta(t)\right)^{\mathbb{w}}},\ \ -dy=\frac{\mathbb{w}(\vartheta(b)-\vartheta(\varsigma)^{\mathbb{w}-1}\ \vartheta'(\varsigma)d\varsigma}{\left(\vartheta(b)-\vartheta(t)\right)^{\mathbb{w}}} $$

and

$$ 1-y=\frac{\left(\vartheta(b)-\vartheta(t)\right)^{\mathbb{w}}-\left(\vartheta(b)-\vartheta(\varsigma)\right)^{\mathbb{w}}}{\left(\vartheta(b)-\vartheta(t)\right)^{\mathbb{w}}} $$

in above integral, we get

$$ {}^{\varkappa}\mathcal{J}^{\mathbb{w},\vartheta}_{b,k}\left(\vartheta(b)-\vartheta\left(t\right)\right)^{\frac{\mathbb{w}\varkappa^{*}}{k}-\mathbb{w}}=\frac{1}{\Gamma_{k}\left(\varkappa\right)\mathbb{w}^{\frac{\varkappa}{k}-1+1}}\left(\vartheta(b)-\vartheta\left(t\right)\right)^{\mathbb{w}\left(\frac{\varkappa}{k}+\frac{\varkappa^{*}}{k}-1\right)}\frac{1}{k}\int_{0}^{1}(1-y)^{\frac{\varkappa}{k}-1}(y)^{\frac{\varkappa^{*}}{k}-1}dy. $$

Using k-Beta function, we get the desired proof as follows

$$ {}^{\varkappa}\mathcal{J}^{\mathbb{w},\vartheta}_{b,k}\left(\vartheta(b)-\vartheta\left(t\right)\right)^{\frac{\mathbb{w}\varkappa^{*}}{k}-\mathbb{w}}=\frac{1}{\mathbb{w}^{\frac{\varkappa}{k}}}\frac{k\Gamma_{k}\left(\varkappa^{*}\right)}{k\Gamma_{k}\left(\varkappa+\varkappa^{*}\right)}\left(\vartheta(b)-\vartheta(t)\right)^{\mathbb{w}\left(\frac{\varkappa}{k}+\frac{\varkappa^{*}}{k}-1\right)}. $$

□

Example 1 For visualization we choose $\vartheta\left(t\right)=t^{2},\ 0<t<100,\ k=0.5,\ 1.5,\ \mathbb{w}=1,\ \varkappa=1,\ \varkappa^{*}=0.5$ in (30), we have

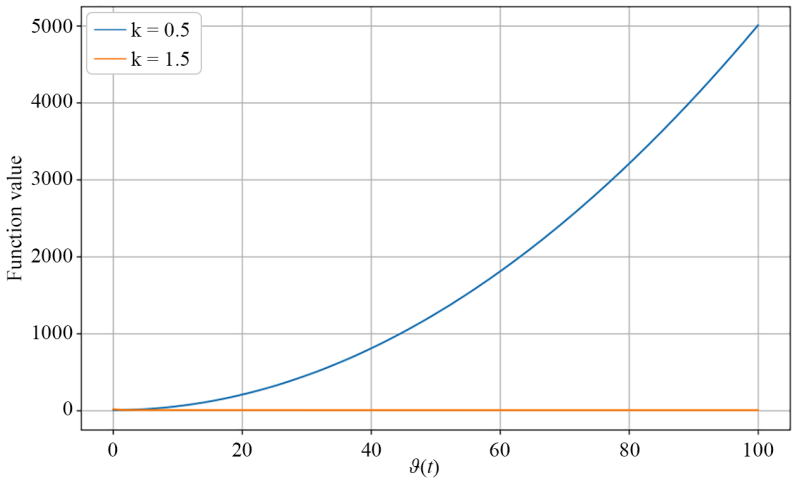


Figure 1. Graphical representation of equation (30) for different values of $k = 0.5, 1.5$

Example 2 For visualization we choose $\vartheta(t) = t^2,\ 0 < t < 100,\ k = 0.5,,\ \mathbb{w} = 1,\ \varkappa = 1,\ \varkappa^* = 0.2,\ 0.8$ in (30)

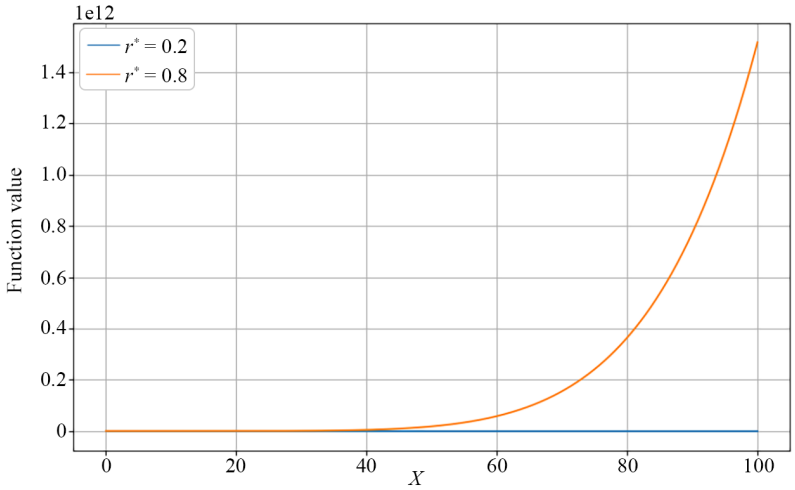


Figure 2. Graphical representation of equation (30) for different values of $\varkappa^* = 0.2, 0.8$

Example 3 For visualization we choose $\vartheta(t) = t^2,\ 0 < t < 100,\ k = 0.5,\ 1.5,\ \mathbb{w} = 1,\ \varkappa = 1,\ \varkappa^* = 0.5$ in (31)

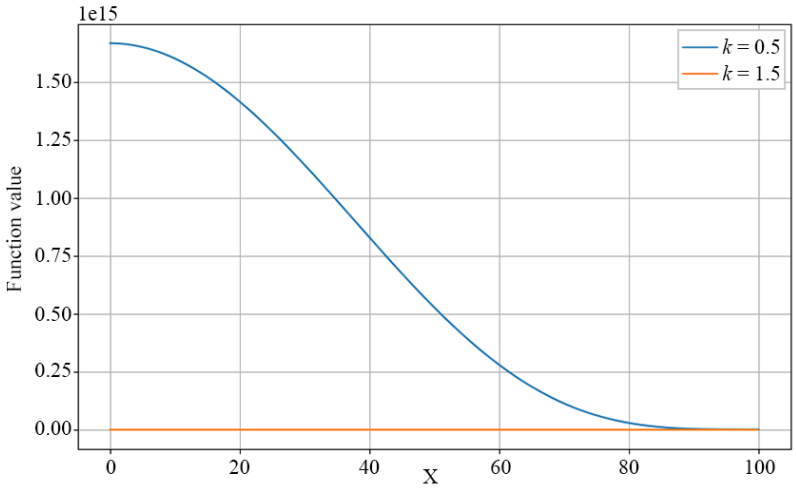


Figure 3. Graphical representation of equation (31) for different values of $k = 0.5, 1.5$

Example 4 For visualization we choose $\vartheta(t) = t^2,\ 0 < t < 100,\ k = 0.5,\ \mathbb{w} = 1,\ \varkappa = 1,\ \varkappa^* = 0.2,\ 0.8$ in (31)

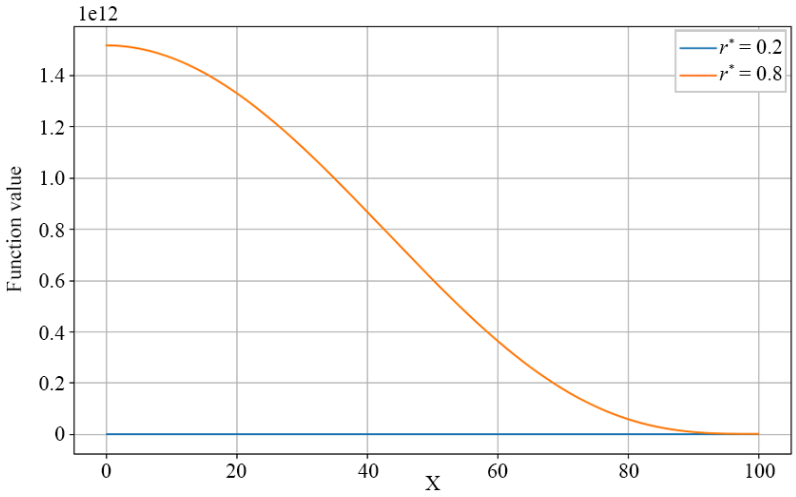


Figure 4. Graphical representation of equation (30) for different values of $\varkappa^* = 0.2, 0.8$

Lemma 2 Let $-\mathbb{w}\varepsilon\ \Re_+^\circ$, we have

$$\left[{}_a^{\varkappa}\mathfrak{D}_k^{\mathbb{w},\vartheta}\left(\vartheta(t)-\vartheta(a)\right)^{\frac{\mathbb{w}\varkappa^*}{k}-\mathbb{w}}\right](t) = \mathbb{w}^{\frac{\varkappa}{k}}\frac{\Gamma_k(\varkappa^*)}{\Gamma_k(\varkappa^*-\varkappa)}\left(\vartheta(t)-\vartheta(a)\right)^{\mathbb{w}\left(\frac{\varkappa^*}{k}-\frac{\varkappa}{k}-1\right)} \tag{32}$$

and

$$\left[{}^{\varkappa}\mathfrak{D}_{b,k}^{\mathbb{w},\vartheta}\left(\vartheta(t)-\vartheta(a)\right)^{\frac{\mathbb{w}\varkappa^*}{k}-\mathbb{w}}\right](t) = \mathbb{w}^{\frac{\varkappa}{k}}\frac{\Gamma_k(\varkappa^*)}{\Gamma_k((\varkappa^*-\varkappa))}\left(\vartheta(b)-\vartheta(t)\right)^{\mathbb{w}\left(\frac{\varkappa^*}{k}-\frac{\varkappa}{k}-1\right)}. \tag{33}$$

Proof. To prove (32), we have

$$\left[{}_a^r\mathfrak{D}_k^{\mathbb{w},\vartheta}\left(\vartheta(t)-\vartheta(a)\right)^{\frac{\mathbb{w}r^*}{k}-\mathbb{w}}\right](t)=\frac{{}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta}}{k\Gamma_k(mk-r)}\int_a^t\left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{mk-r}{k}-1}$$

$$(\vartheta(\varsigma)-\vartheta(a)^{\mathbb{w}\left(\frac{r^*}{k}-1\right)}\frac{\vartheta'(\varsigma)d\varsigma}{\left(\vartheta(\varsigma)-\vartheta(a)\right)^{1-\mathbb{w}}},$$

where

$${}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta}=\underline{{}_a\mathbb{T}_k^{\mathbb{w},\vartheta}\,{}_a\mathbb{T}_k^{\mathbb{w},\vartheta}\cdots{}_a\mathbb{T}_k^{\mathbb{w},\vartheta}}(m\text{-times})$$

and

$${}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta}f(t)=\left[(\vartheta(t)-\vartheta(a))^{1-\mathbb{w}}\frac{k}{\vartheta'(t)}\frac{df(t)}{dt}\right]^m.$$

Substituting

$$y=\frac{(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}},\;\; dy=\frac{\mathbb{w}(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}-1}\vartheta'(\varsigma)d\varsigma}{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}}$$

and

$$1-y=\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}}$$

in above integral, we have

$$\left[{}_a^r\mathfrak{D}_k^{\mathbb{w},\vartheta}\left(\vartheta(t)-\vartheta(a)\right)^{\frac{\mathbb{w}r^*}{k}-\mathbb{w}}\right](t)=\frac{{}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta}}{\Gamma_k(mk-r)\,\mathbb{w}^{\frac{mk-r}{k}-1+1}}(\vartheta(t)-\vartheta(a))^{\mathbb{w}\left(\frac{mk-r}{k}+\frac{r^*}{k}-1\right)}$$

$$\frac{1}{k}\int_0^1(1-y)^{m-\frac{r}{k}-1}(y)^{\frac{r^*}{k}-1}\,dy.$$

Using k-Beta function, we have

$$\left[{}_a^r\mathfrak{D}_k^{\mathbb{w},\vartheta}\left(\vartheta(t)-\vartheta(a)\right)^{\frac{\mathbb{w}r^*}{k}-\mathbb{w}}\right](t)$$

$$= \frac{{}^{m}_{a}\mathbb{T}_{k}^{\mathbb{w},\vartheta}}{\mathbb{w}^{m-\frac{r}{k}}} \left(\vartheta(t)-\vartheta(a)\right)^{\mathbb{w}\left(\frac{mk-r}{k}+\frac{r^*}{k}-1\right)} \frac{\Gamma_k(r^*)}{\Gamma_k(mk-r+r^*)}$$

$$= \frac{\Gamma_k(r^*)}{\mathbb{w}^{m-\frac{r}{k}}\,\Gamma_k(mk-r+r^*)} {}^{m}_{a}\mathbb{T}_{k}^{\mathbb{w},\vartheta}\left(\vartheta(t)-\vartheta(a)\right)^{\mathbb{w}\left(\frac{mk-r}{k}+\frac{r^*}{k}-1\right)}$$

$$= \frac{\Gamma_k(r^*)}{\mathbb{w}^{m-\frac{r}{k}}\,\Gamma_k(mk-r+r^*)} \left[\left(\vartheta(t)-\vartheta(a)\right)^{1-\mathbb{w}} \frac{k}{\vartheta'(t)}\frac{d}{dt}\right]^{m} \left(\vartheta(t)-\vartheta(a)\right)^{\mathbb{w}\left(\frac{mk-r}{k}+\frac{r^*}{k}-1\right)}$$

$$= \frac{\Gamma_k(r^*)}{\mathbb{w}^{m-\frac{r}{k}}\,\Gamma_k(mk-r+r^*)} \left(\vartheta(t)-\vartheta(a)\right)^{m-m\mathbb{w}} \left[\frac{k}{\vartheta'(t)}\frac{d}{dt}\right]^{m} \left(\vartheta(t)-\vartheta(a)\right)^{\mathbb{w}\left(\frac{mk-r}{k}+\frac{r^*}{k}-1\right)}.$$

Differentiating, we have

$$\left[{}^{r}_{a}\mathfrak{D}_{k}^{\mathbb{w},\vartheta}\left(\vartheta(t)-\vartheta(a)\right)^{\frac{\mathbb{w}r^*}{k}-\mathbb{w}}\right](t) = \frac{\Gamma_k(r^*)}{\mathbb{w}^{m-\frac{r}{k}}\,\Gamma_k(mk-r+r^*)} \left(\vartheta(t)-\vartheta(a)\right)^{1-\mathbb{w}} \frac{1}{\vartheta'(t)}$$

$$\frac{d}{dt}\left(\vartheta(t)-\vartheta(a)\right)^{\mathbb{w}\left(\frac{mk-r}{k}+\frac{r^*}{k}-1\right)}$$

$$= \frac{\Gamma_k(r^*)}{\mathbb{w}^{m-\frac{r}{k}}\,\Gamma_k(mk-r+r^*)} \mathbb{w}\left(m-\frac{r}{k}+\frac{r^*}{k}-1\right)\left(\vartheta(t)-\vartheta(a)\right)^{\mathbb{w}\left(\frac{mk-r}{k}+\frac{r^*}{k}-2\right)}.$$

Again differentiating, we obtain

$$\left[{}^{r}_{a}\mathfrak{D}_{k}^{\mathbb{w},\vartheta}\left(\vartheta(t)-\vartheta(a)\right)^{\frac{\mathbb{w}r^*}{k}-\mathbb{w}}\right](t) = \frac{\Gamma_k(r^*)}{\mathbb{w}^{m-\frac{r}{k}}\,\Gamma_k(mk-r+r^*)} \mathbb{w}^2\left(\frac{mk}{k}-\frac{r}{k}+\frac{r^*}{k}-1\right)$$

$$\left(\frac{mk}{k}-\frac{r}{k}+\frac{r^*}{k}-2\right)\left(\vartheta(t)-\vartheta(a)\right)^{\mathbb{w}\left(m-\frac{r}{k}+\frac{r^*}{k}-3\right)}.$$

Continuing up to m-times and using $\Gamma_k(x) = k^{\frac{x}{k}}\Gamma(\frac{x}{k})$, we get

$$\left[{}^{r}_{a}\mathfrak{D}_{k}^{\mathbb{w},\vartheta}\left(\vartheta(t)-\vartheta(a)\right)^{\frac{\mathbb{w}r^*}{k}-\mathbb{w}}\right](t) = \frac{\Gamma_k(r^*)}{\mathbb{w}^{m-\frac{r}{k}}\,\Gamma_k(mk-r+r^*)} \mathbb{w}^{m}\left(\frac{mk}{k}-\frac{r}{k}+\frac{r^*}{k}-1\right)\ldots\left(\frac{mk}{k}-\frac{r}{k}+\frac{r^*}{k}-m-1\right)$$

$$\left(\vartheta(t)-\vartheta(a)\right)^{\mathbb{w}\left(m-\frac{r}{k}+\frac{r^*}{k}-m-1\right)}.$$

Thus, we obtain

$$\left[{}_{a}^{r}\mathfrak{D}_{k}^{\mathbb{w},\vartheta}\left(\vartheta\left(t\right)-\vartheta(a)\right)^{\frac{\mathbb{w}r^{*}}{k}-\mathbb{w}}\right](t)=\frac{\Gamma_{k}\left(r^{*}\right)}{\mathbb{w}^{m-\frac{r}{k}}\,\Gamma_{k}\left(mk-r+r^{*}\right)}\mathbb{w}^{m}\frac{\Gamma_{k}\left(mk-r+r^{*}\right)}{\Gamma_{k}\left(r^{*}-r\right)}\left(\vartheta(t)-\vartheta(a)\right)^{\mathbb{w}\left(m-\frac{r}{k}+\frac{r^{*}}{k}-m-1\right)}$$

$$=\mathbb{w}^{\frac{r}{k}}\frac{k\Gamma_{k}\left(r^{*}\right)}{k\Gamma_{k}\left(r^{*}-r\right)}\left(\vartheta(t)-\vartheta(a)\right)^{\mathbb{w}\left(-\frac{r}{k}+\frac{r^{*}}{k}-1\right)},$$

which completes the desired result (32). □

Now, to prove (33), we consider

$$\left[{}^{r}\mathfrak{D}_{b,k}^{\mathbb{w},\vartheta}\left(\vartheta\left(t\right)-\vartheta(a)\right)^{\frac{\mathbb{w}r^{*}}{k}-\mathbb{w}}\right](t)=\frac{(-1)^{m}\;{}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}}{k\Gamma_{k}(mk-r)}\int_{t}^{b}\left(\frac{\left(\vartheta(b)-\vartheta(t)\right)^{\mathbb{w}}-\left(\vartheta(b)-\vartheta(\varsigma)\right)^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{mk-r}{k}-1}$$

$$\left(\vartheta(b)-\vartheta\left(t\right)\right)^{\mathbb{w}\left(\frac{r^{*}}{k}-1\right)}\frac{f(\varsigma)\;\vartheta'(\varsigma)d\varsigma}{\left(\;\vartheta(b)-\vartheta(\varsigma)\;\right)^{1-\mathbb{w}}}.$$

Substituting

$$y=\frac{\left(\vartheta(b)-\vartheta(\varsigma)^{\mathbb{w}}\right.}{\left(\vartheta(b)-\vartheta(t)\right)^{\mathbb{w}}},\;\;-dy=\frac{\mathbb{w}(\vartheta(b)-\vartheta(\varsigma)^{\mathbb{w}-1}\left(\vartheta'(\varsigma)d\varsigma\right)}{\left(\vartheta(b)-\vartheta(t)\right)^{\mathbb{w}}}$$

and

$$1-y=\frac{\left(\vartheta(b)-\vartheta(t)\right)^{\mathbb{w}}-\left(\vartheta(b)-\vartheta(\varsigma)\right)^{\mathbb{w}}}{\left(\vartheta(b)-\vartheta(t)\right)^{\mathbb{w}}}$$

in above integral, we obtain

$$\left[{}^{r}\mathfrak{D}_{b,k}^{\mathbb{w},\vartheta}\left(\vartheta\left(t\right)-\vartheta(a)\right)^{\frac{\mathbb{w}r^{*}}{k}-\mathbb{w}}\right](t)=\frac{(-1)^{m}\;{}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}}{\Gamma_{k}\left(mk-r\right)\mathbb{w}^{m-\frac{r}{k}-1+1}}\;\left(\vartheta(b)-\vartheta(t)\right)^{\mathbb{w}\left(\frac{mk-r}{k}+\frac{r^{*}}{k}-1\right)}$$

$$\frac{1}{k}\int_{0}^{1}(1-y)^{\frac{mk-r}{k}-1}(y)^{\frac{r^{*}}{k}-1}(-dy).$$

Using k-Beta function, we have

$$\left[{}^{r}\mathfrak{D}_{b,k}^{\mathbb{w},\vartheta}\left(\vartheta\left(t\right)-\vartheta(a)\right)^{\mathbb{w}r^{*}-\mathbb{w}}\right](t)$$

$$=\frac{(-1)^{m}\,{}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}}{\mathbb{w}^{m-\frac{r}{k}}}\;\left(\vartheta(b)-\vartheta(t)\right)^{\mathbb{w}\left(\frac{mk-r}{k}+\frac{r^{*}}{k}-1\right)}(-1)\frac{\Gamma_{k}\left(r^{*}\right)}{\Gamma_{k}\left(mk-r+r^{*}\right)}$$

$$= \frac{\Gamma_k(\varkappa^*)}{\mathbb{w}^{m-\frac{\varkappa}{k}}\,\Gamma_k(mk-\varkappa+\varkappa^*)}(-1)^{m+1}\ {}^{m}\mathbb{T}^{\mathbb{w},\vartheta}_{b,k}(\vartheta(b)-\vartheta(t))^{\mathbb{w}\left(\frac{mk-\varkappa}{k}+\frac{\varkappa^*}{k}-1\right)}$$

$$= \frac{\Gamma_k(\varkappa^*)}{\mathbb{w}^{m-\frac{\varkappa}{k}}\,\Gamma_k(mk-\varkappa+\varkappa^*)}(-1)^{m+1}\ \left[(\vartheta(b)-\vartheta(t))^{1-\mathbb{w}}\frac{k}{\vartheta'(t)}\frac{d}{dt}\right]^m\ (\vartheta(b)-\vartheta(t))^{\mathbb{w}\left(\frac{mk-\varkappa}{k}+\frac{\varkappa^*}{k}-1\right)}$$

$$= \frac{\Gamma_k(\varkappa^*)}{\mathbb{w}^{m-\frac{\varkappa}{k}}\,\Gamma_k(mk-\varkappa+\varkappa^*)}(-1)^{m+1}\ (\vartheta(b)-\vartheta(t))^{m-m\mathbb{w}}\ \left[\frac{k}{\vartheta'(t)}\frac{d}{dt}\right]^m\ (\vartheta(b)-\vartheta(t))^{\mathbb{w}\left(\frac{mk-\varkappa}{k}+\frac{\varkappa^*}{k}-1\right)}.$$

Differentiating m-times, we get

$$\left[{}^{\varkappa}\mathfrak{D}^{\mathbb{w},\vartheta}_{b,k}\left(\vartheta(t)-\vartheta(a)\right)^{\frac{\mathbb{w}\varkappa^*}{k}-\mathbb{w}}\right](t) = \frac{\Gamma_k(\varkappa^*)}{\mathbb{w}^{m-\frac{\varkappa}{k}}\,\Gamma_k(mk-\varkappa+\varkappa^*)}k^m(-1)^{m-1}\mathbb{w}^m\left(\frac{mk}{k}-\frac{\varkappa}{k}+\frac{\varkappa^*}{k}-1\right)$$

$$\ldots\left(\frac{mk}{k}-\frac{\varkappa}{k}+\frac{\varkappa^*}{k}-m-1\right)(\vartheta(b)-\vartheta(t))^{\mathbb{w}\left(\frac{mk-\varkappa}{k}+\frac{\varkappa^*}{k}-m-1\right)}.$$

Thus, we obtain

$$\left[{}^{\varkappa}\mathfrak{D}^{\mathbb{w},\vartheta}_{b,k}\left(\vartheta(t)-\vartheta(a)\right)^{\frac{\mathbb{w}\varkappa^*}{k}-\mathbb{w}}\right](t) = \frac{\Gamma_k(\varkappa^*)}{\mathbb{w}^{m-\frac{\varkappa}{k}}\,\Gamma_k(m-\varkappa+\varkappa^*)}\mathbb{w}^m\frac{\Gamma_k(m-\varkappa+\varkappa^*)}{\Gamma_k(\varkappa^*-\varkappa)}\ (\vartheta(b)-\vartheta(t))^{\mathbb{w}\left(m-\frac{\varkappa}{k}+\frac{\varkappa^*}{k}-m-1\right)}$$

$$= \mathbb{w}^{\frac{\varkappa}{k}}\frac{\Gamma_k(\varkappa^*)}{\Gamma_k(\varkappa^*-\varkappa)}\ (\vartheta(b)-\vartheta(t))^{\mathbb{w}\left(\frac{\varkappa^*}{k}-\frac{\varkappa}{k}-1\right)},$$

which proves the desired result given in (33).

Example 5 For visualization we choose $\vartheta(t)=t^2$, $0<t<100$, $k=0.5,\ 1.5,\ \mathbb{w}=1,\ \varkappa=1,\ \varkappa^*=0.5$ in (32)

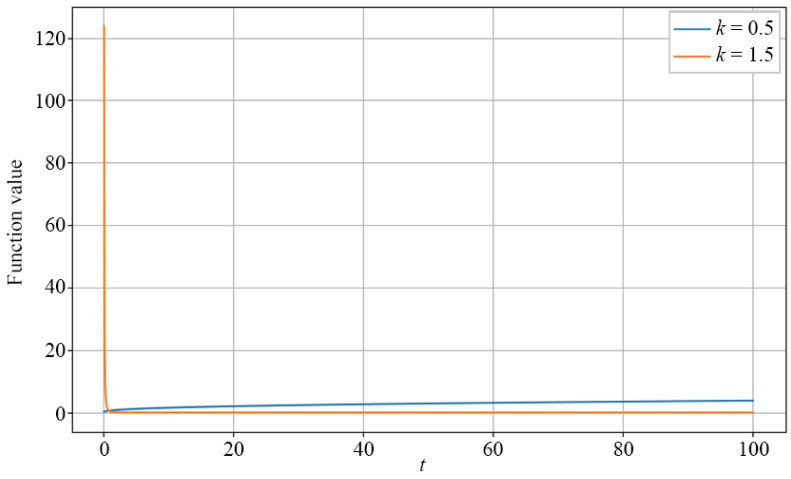


Figure 5. Graphical representation of equation (32) for different values of k = 0.5, 1.5

Example 6 For visualization we choose $\vartheta(t) = t^2,\ 0 < t < 100,\ k = 0.5,\ \mathbb{w} = 1,\ \varkappa = 1,\ \varkappa^* = 0.2,\ 0.8$ in (32)

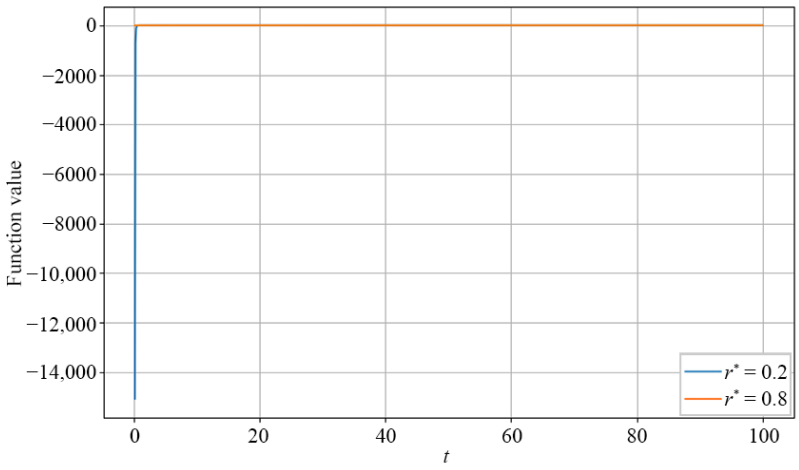


Figure 6. Graphical representation of equation (32) for different values of $\varkappa^* = 0.2, 0.8$

Example 7 For visualization we choose $\vartheta(t) = t^2,\ 0 < t < 100,\ k = 0.5,,\ \mathbb{w} = 1,\ \varkappa = 1,\ \varkappa^* = 0.2,\ 0.8$ in equation (33)

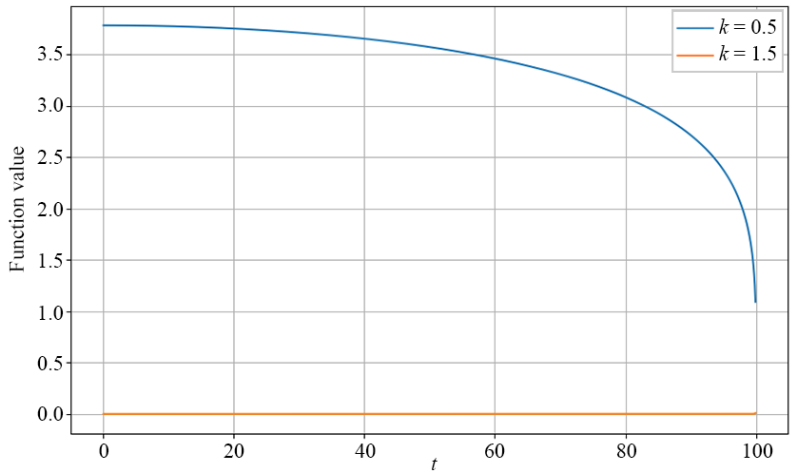


Figure 7. Graphical representation of equation (33) for different values of $k = 0.5, 1.5$

Example 8 For visualization we choose $\vartheta(t) = t^2,\ 0 < t < 100,\ k = 0.5,\ 1.5,\ \mathbb{w} = 1,\ \varkappa = 1,\ \varkappa^* = 0.5$ in (33)

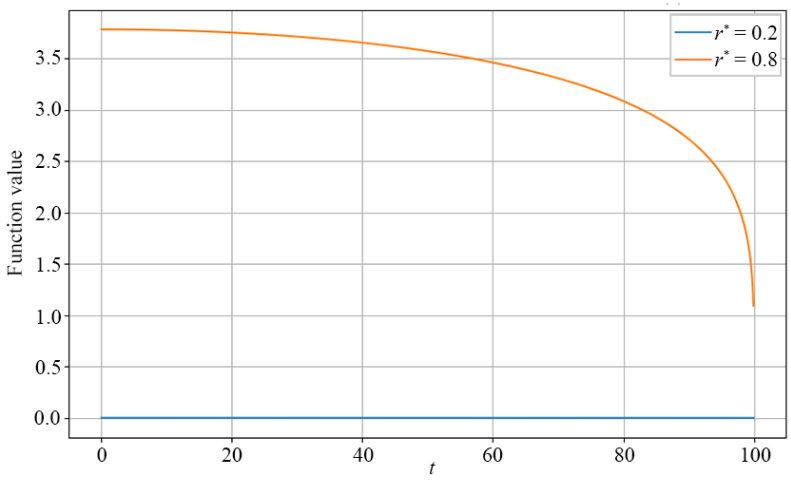


Figure 8. Graphical representation of equation (30) for different values of $r^* = 0.2, 0.8$

Lemma 3 The following results hold true:

$$ {}_a^r\mathfrak{D}_k^{\mathbb{w},\vartheta} f(t) = {}_a^{-r}\mathcal{I}_k^{\mathbb{w},\vartheta} f(t) \text{ and } {}^r\mathfrak{D}_{b,k}^{\mathbb{w},\vartheta} f(t) = {}^{-r}\mathcal{I}_{b,k}^{\mathbb{w},\vartheta} f(t). \tag{34} $$

Proof. Consider

$$ {}_a^r\mathfrak{D}_k^{\mathbb{w},\vartheta} f(t) = {}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta}\, {}_a^{m-r}\mathcal{I}_k^{\mathbb{w},\vartheta} f(t) $$

$$ = \frac{{}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta}}{k\Gamma_k(mk-r)} \int_a^t \left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}} - (\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}} \right)^{\frac{mk-r}{k}-1} \frac{f(\varsigma)\,\vartheta'(\varsigma)d\varsigma}{(\,\vartheta(\varsigma)-\vartheta(a)\,)^{1-\mathbb{w}}} $$

$$ = \frac{1}{k\Gamma_k(mk-r)}\, {}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta} \int_a^t \left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}} - (\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}} \right)^{\frac{mk-r}{k}-1} \frac{f(\varsigma)\,\vartheta'(\varsigma)d\varsigma}{(\,\vartheta(\varsigma)-\vartheta(a)\,)^{1-\mathbb{w}}} $$

$$ = \frac{1}{k\Gamma_k(mk-r)\mathbb{w}^{m-\frac{r}{k}-1}} (\vartheta(t)-\vartheta(a))^{m-m\mathbb{w}} \left[\frac{k}{\vartheta'(t)} \frac{d}{dt} \right]^m $$

$$ \int_a^t ((\vartheta(t)-\vartheta(a))^{\mathbb{w}} - (\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}})^{\frac{mk-r}{k}-1} \frac{f(\varsigma)\,\vartheta'(\varsigma)d\varsigma}{(\,\vartheta(\varsigma)-\vartheta(a)\,)^{1-\mathbb{w}}} $$

$$ = \frac{1}{k\Gamma_k(mk-r)\mathbb{w}^{m-\frac{r}{k}-1}} \mathbb{w}^m \frac{\Gamma_k(mk-r)}{\Gamma_k(-r)} \int_a^t ((\vartheta(t)-\vartheta(a))^{\mathbb{w}} - (\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}})^{-\frac{r}{k}-1} \frac{f(\varsigma)\,\vartheta'(\varsigma)d\varsigma}{(\,\vartheta(\varsigma)-\vartheta(a)\,)^{1-\mathbb{w}}} $$

$$ = \frac{1}{k\Gamma_k(-r)\mathbb{w}^{-\frac{r}{k}-1}} \int_a^t ((\vartheta(t)-\vartheta(a))^{\mathbb{w}} - (\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}})^{-\frac{r}{k}-1} \frac{f(\varsigma)\,\vartheta'(\varsigma)d\varsigma}{(\,\vartheta(\varsigma)-\vartheta(a)\,)^{1-\mathbb{w}}} $$

$$= \frac{1}{k\Gamma_k(-\varkappa)} \int_a^t \left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}} - (\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}} \right)^{-\frac{\varkappa}{k}-1} \frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(\varsigma)-\vartheta(a)\ )^{1-\mathbb{w}}}$$

$$= {}_a^{-\varkappa}\mathcal{J}_k^{\mathbb{w},\vartheta} f(t),$$

which completes the first part of Lemma 3.

To prove the second part of Lemma 3, we have

$${}^{\varkappa}\mathfrak{D}_{b,k}^{\mathbb{w},\vartheta} f(t) = {}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}\, {}^{m-\varkappa}\mathcal{J}_{b,k}^{\mathbb{w},\vartheta} f(t)$$

$$= \frac{(-1)^m\, {}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}}{k\Gamma_k(mk-\varkappa)} \int_t^b \left(\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}} - (\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{\mathbb{w}} \right)^{\frac{mk-\varkappa}{k}-1} \frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}$$

$$= \frac{1}{k\Gamma_k(mk-\varkappa)} (-1)^m\ {}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta} \int_t^b \left(\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}} - (\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{\mathbb{w}} \right)^{\frac{mk-\varkappa}{k}-1} \frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}$$

$$= \frac{1}{k\Gamma_k(mk-\varkappa)\mathbb{w}^{m-\frac{\varkappa}{k}-1}} (-1)^m\ (\vartheta(b)-\vartheta(t))^{m-m\mathbb{w}} \left[\frac{k}{\vartheta'(t)} \frac{d}{dt} \right]^m$$

$$\times \int_t^b \left((\vartheta(b)-\vartheta(t))^{\mathbb{w}} - (\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}} \right)^{m-\frac{\varkappa}{k}-1} \frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}$$

$$= \frac{1}{k\Gamma_k(mk-\varkappa)\mathbb{w}^{m-\frac{\varkappa}{k}-1}} (-1)^{m-1}\mathbb{w}^m\ \frac{\Gamma_k(m-\varkappa)}{\Gamma_k(-\varkappa)} \int_t^b \left((\vartheta(b)-\vartheta(t))^{\mathbb{w}} - (\vartheta(b)-\ \vartheta(\varsigma))^{\mathbb{w}} \right)^{-\frac{\varkappa}{k}-1}$$

$$\frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}$$

$$= \frac{1}{k\Gamma_k(-\varkappa)} (-1)^{m-1} \int_t^b \left(\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}} - (\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{\mathbb{w}} \right)^{-\frac{\varkappa}{k}-1} \frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}$$

$$= {}^{-\varkappa}\mathcal{J}_{b,k}^{\mathbb{w},\vartheta} f(t),$$

which proves the desired result. □

3. (k, ϑ)-CFD on the spaces $\mathbb{C}_{\mathbb{w},\vartheta,a,k}^{m}$ and $\mathbb{C}_{\mathbb{w},\vartheta,b,k}^{m}$

In this section, $(k,\ \vartheta)$ -CFD of functions inside the spaces defined by the following definitions are examined.

Definition 10 Let $\mathbb{w} > 0$, $n \in \mathcal{N}$ and an interval $[a,b]$ define

$$\mathbb{C}^{m}_{\mathbb{w},\vartheta,a,k}([a,b]) = \left\{ \frac{f;[a,b]\to\ \Re}{{}^{m-1}_{a}\mathbb{T}^{\mathbb{w},\vartheta} f\ \in\ \mathbb{I}_{\mathbb{w},\vartheta,k}([a,b])} \right\} \tag{35}$$

and

$$\mathbb{C}^{m}_{\mathbb{w},\vartheta,b,k}([a,b]) = \left\{ \frac{f;[a,b]\to\ \Re}{{}^{m-1}\mathbb{T}^{\mathbb{w},\vartheta}_{b}\ f\ \in\ \mathbb{I}^{\mathbb{w},\vartheta,k}([a,b])} \right\} \tag{36}$$

with

$$\mathbb{I}_{\mathbb{w},\vartheta}([a,b]) = \left\{ f;[a,b]\to\ \Re : f(t) = \left({}_{a}\mathbb{I}^{\mathbb{w},\vartheta} g\right)(t) + f(a),\ \text{for some}\ g = \mathscr{L}_{\mathbb{w},\vartheta}(a) \right\}$$

and the right ϑ-CFI of order $\mathbb{w}$ is:

$$\mathbb{I}^{\mathbb{w},\vartheta}([a,b]) = \left\{ f;[a,b]\to\ \Re : f(t) = \left((\mathbb{I}^{\mathbb{w},\vartheta}_{b} g(t))\right)(t) + f(b),\ \text{for some}\ g = \mathscr{L}_{\mathbb{w},\vartheta}(b) \right\}$$

$$\mathscr{L}_{\mathbb{w},\vartheta}(a) = \left\{ f : [a,b]\to\ \Re\ :\ \left({}_{a}\mathbb{I}^{\mathbb{w},\vartheta} f\right)(t)\ \text{exists for all}\ f\ \in [a,b] \right\}$$

and

$$\mathscr{L}_{\mathbb{w},\vartheta}(b) = \left\{ f;[a,b]\to\ \Re\ :\left((\mathbb{I}^{\mathbb{w},\vartheta}_{b} f\right)(t), exists\ for\ all\ f\ \in [a,b] \right\}$$

Lemma 4 For $\mathbb{w} > 0$, $f\ \in \mathbb{C}^{m}_{\mathbb{w},\vartheta,a}([a,b])$ If and only if f may be stated as follows:

$$f(t) = \frac{1}{(m-1)!}\int_{a}^{t} \left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}} - (\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}} \right)^{\frac{mk}{k}-1} \frac{g(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(\varsigma)-\vartheta(a)\ )^{1-\mathbb{w}}}$$

$$+ \sum_{i=0}^{m-1} \frac{{}^{i}_{a}\mathbb{T}_{k}{}^{\mathbb{w},\vartheta} f(a)}{\mathbb{w}^{\frac{i}{k}}\, i!} (\vartheta(t)-\vartheta(a))^{\frac{\mathbb{w}i}{k}}, \tag{37}$$

where $g(u) = {}^{m}_{a}\mathbb{T}_{k}{}^{\mathbb{w},\vartheta} f(u)$.

Lemma 5 Let $f\ \in \mathbb{C}^{m}_{\mathbb{w},\vartheta,b,k}([a,b])$ if and only if

$$f(t) = \frac{1}{(m-1)!}\int_{t}^{b} \left(\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}} - (\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{\mathbb{w}} \right)^{\frac{mk}{k}-1} \frac{\vartheta'(\varsigma)({}^{m}\mathbb{T}^{\mathbb{w},\vartheta}_{b,k} f)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}$$

$$+\sum_{i=0}^{m-1}\frac{(-1)^{i^i}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}f(b)}{\mathbb{w}^{\frac{i}{k}}i!}(\vartheta(b)-\vartheta(t))^{\frac{\mathbb{w}i}{k}}. \tag{38}$$

The proof of Lemma's 4 and 5 are similar to the proof of Lemma's 3 and 4 given in [25].
Next, we prove the following results based on Lemma 4 and 5.
Theorem 2 Let $\in \mathfrak{R}_+^{\circ}$,$m=[r]+1$ and $k>0$. We have
1. Let $f\in\mathbb{C}_{\mathbb{w},\vartheta,a,k}^{m}([a,b])$, the left (k,ϑ)-CFD is given by

$${}_a^r\mathfrak{D}_k^{\mathbb{w},\vartheta}f(t)=\frac{\Gamma_k(mk)}{k\ (m-1)!\Gamma_k(m-r)}\int_a^t\left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{mk-r}{k}-1}\frac{({}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta}f(\varsigma))\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(\varsigma)-\vartheta(a)\ )^{1-\mathbb{w}}}$$

$$+\sum_{i=0}^{m-1}\frac{\left({}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta}f(a)\right)\Gamma_k(i+k)}{\mathbb{w}^{\frac{i}{k}-\frac{r}{k}}\Gamma_k(mk+i+k-r)\,i!}(\vartheta(t)-\vartheta(a))^{\frac{\mathbb{w}(i-r)}{k}}. \tag{39}$$

2. Let $f\ \in\mathbb{C}_{\mathbb{w},\vartheta,a}^{m}([a,b])$ and $k>0$, the right k,ϑ-CFD is given by

$${}^r\mathfrak{D}_{b,k}^{\mathbb{w},\vartheta}f(t)=\frac{\Gamma_k(mk)\,{}^m\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}}{k\ (m-1)!\Gamma_k(m-r)}\int_t^b\left(\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{mk-r}{k}-1}\frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}$$

$$+\sum_{i=0}^{m-1}\mathbb{w}^{\frac{r}{k}}\frac{\Gamma_k(i+k)}{\Gamma_k(mk+i+k-r)}\frac{(-1)^{i^i}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}f(b)}{\mathbb{w}^{\frac{i}{k}}i!}(\vartheta(b)-\vartheta(t))^{\frac{\mathbb{w}(i-r)}{k}}. \tag{40}$$

Proof. To prove 1, we have
Since $f\ \in\mathbb{C}_{\mathbb{w},\vartheta,a,k}^{m}([a,b])$, from Lemma 4, we have

$$f(t)=\frac{1}{(m-1)!}\int_a^t\left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{mk}{k}-1}\frac{({}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta}f)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(\varsigma)-\vartheta(a)\ )^{1-\mathbb{w}}}$$

$$+\sum_{i=0}^{m-1}\frac{{}_a^i\mathbb{T}_k^{\mathbb{w},\vartheta}f(a)}{\mathbb{w}^{\frac{i}{k}}i!}(\vartheta(t)-\vartheta(a))^{\frac{\mathbb{w}i}{k}}.$$

Therefore, we have

$${}_a^r\mathfrak{D}_k^{\mathbb{w},\vartheta}f(t)=\frac{{}_a^m\mathbb{T}^{\mathbb{w},\vartheta}}{(m-1)!k\Gamma_k(mk-r)}\int_a^t\int_a^{\nu}\left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\nu)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{mk-r}{k}-1}$$

$$\left(\frac{(\vartheta(\nu)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{mk}{k}-1}$$

$$\times \frac{({}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta} f(\varsigma))\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(\varsigma)-\vartheta(a)\ )^{1-\mathbb{w}}}\ \frac{\vartheta'(\nu)d\nu}{(\ \vartheta(\nu)-\vartheta(a)\ )^{1-\mathbb{w}}}$$

$$+\sum_{i=0}^{m-1}\frac{\left({}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta} f(a)\right)}{\mathbb{w}^{\frac{i}{k}}\, i!}\mathbb{w}^{\frac{r}{k}}\frac{\Gamma_k(i+k)}{\Gamma_k(mk+i+k-r)}(\vartheta(t)-\vartheta(a))^{\frac{\mathbb{w}(i-r)}{k}}$$

$$=\frac{{}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta}}{(m-1)!k\Gamma_k(mk-r)\mathbb{w}^{m-\frac{r}{k}+m-2}}\int_a^t\int_\varsigma^t\left((\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\nu)-\vartheta(a))^{\mathbb{w}}\right)^{\frac{mk-r}{k}-1}$$

$$\left((\vartheta(\nu)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}\right)^{m-1}$$

$$\times \frac{({}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta} f(\varsigma))\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(\varsigma)-\vartheta(a)\ )^{1-\mathbb{w}}}\ \frac{\vartheta'(\nu)d\nu}{(\ \vartheta(\nu)-\vartheta(a)\ )^{1-\mathbb{w}}}$$

$$\sum_{i=0}^{m-1}\frac{\left({}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta} f(a)\right)}{i!\ \mathbb{w}^{\frac{i}{k}}}\mathbb{w}^{\frac{r}{k}}\frac{\Gamma_k(i+k)}{\Gamma_k(mk+i+k-r)}(\vartheta(t)-\vartheta(a))^{\frac{\mathbb{w}(i-r)}{k}}.$$

Substituting

$$y=\frac{(\vartheta(\nu)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}},\ \ dy=\frac{\mathbb{w}(\vartheta(\nu)-\vartheta(a))^{\mathbb{w}-1}\ \vartheta'(\nu)d\nu}{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}$$

and

$$1-y=1-\frac{(\vartheta(\nu)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta\ (a))^{\mathbb{w}}}{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}=\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\nu)-\vartheta(a))^{\mathbb{w}}}{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}$$

in above integral, we get

$${}_a^r\mathfrak{D}_k^{\mathbb{w},\vartheta} f(t)=\frac{{}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta}}{(m-1)!\Gamma_k(mk-r)}\int_a^t\left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{2mk-r}{k}-1}\ \frac{({}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta} f(\varsigma))\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(\varsigma)-\vartheta(a)\ )^{1-\mathbb{w}}}$$

$$\frac{1}{k}\int_0^1(1-y\ )^{m-\frac{r}{k}-1}(y)^{\frac{mk}{k}-1}dy+\sum_{i=0}^{m-1}\frac{\left({}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta} f(a)\right)\ \Gamma_k(i+k)}{i!\mathbb{w}^{\frac{i}{k}-\frac{r}{k}}\Gamma_k(mk+i+k-r)}(\vartheta(t)-\vartheta(a))^{\frac{\mathbb{w}(i-r)}{k}}.$$

By using k-Beta function, we get

$${}_a^r\mathfrak{D}_k^{\mathbb{w},\vartheta} f(t)=\frac{{}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta}}{(m-1)!\ \Gamma_k(mk-r)}\int_a^t\left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{2mk-r}{k}-1}\ \frac{({}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta} f(\varsigma))\vartheta'(\varsigma)d\varsigma}{(\vartheta(\varsigma)-\vartheta(a))^{1-\mathbb{w}}}$$

$$\frac{\Gamma_k(mk-r)\Gamma_k(mk)}{\Gamma_k(2mk-r)}+\sum_{i=0}^{m-1}\frac{\left({}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta}f(a)\right)\ \Gamma_k(i+k)}{i!\mathbb{w}^{\frac{i}{k}-\frac{r}{k}}\Gamma_k(mk+i+k-r)}(\vartheta(t)-\vartheta(a))^{\frac{\mathbb{w}(i-r)}{k}}$$

$$=\frac{{}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta}\ \Gamma_k(mk)}{(m-1)!\Gamma_k(2mk-r)}\ \int_a^t\left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{2mk}{k}-\frac{r}{k}-1}$$

$$\frac{({}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta}f(\varsigma))\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(\varsigma)-\vartheta(a)\ )^{1-\mathbb{w}}}+\sum_{i=0}^{m-1}\frac{\left({}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta}f(a)\right)\ \Gamma_k(i+k)}{i!\mathbb{w}^{\frac{i}{k}-\frac{r}{k}}\Gamma_k(mk+i+k-r)}(\vartheta(t)-\vartheta(a))^{\frac{\mathbb{w}(i-r)}{k}}$$

$$=\frac{\Gamma_k(mk)\,{}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta}}{(m-1)!\,\Gamma_k(2mk-r)}\ \int_a^t\left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}}\right)^{2m-\frac{r}{k}-1}$$

$$\frac{({}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta}f(\varsigma))\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(\varsigma)-\vartheta(a)\ )^{1-\mathbb{w}}}+\sum_{i=0}^{m-1}\frac{\left({}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta}f(a)\right)\ \Gamma_k(i+k)}{i!\mathbb{w}^{\frac{i}{k}-\frac{r}{k}}\Gamma_k(mk+i+k-r)}(\vartheta(t)-\vartheta(a))^{\frac{\mathbb{w}(i-r)}{k}}$$

$$=\frac{\Gamma_k(mk)}{(m-1)!\Gamma_k(2mk-r)\ \mathbb{w}^{2m-\frac{r}{k}-1}}\ \left[(\vartheta(t)-\vartheta(a))^{1-\mathbb{w}}\frac{k}{\vartheta'(t)}\frac{d}{dt}\right]^m$$

$$\int_a^t((\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}})^{\frac{2mk}{k}-\frac{r}{k}-1}\frac{({}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta}f(\varsigma))\vartheta'(\varsigma)d\varsigma}{(\ \vartheta(\varsigma)-\vartheta(a)\ )^{1-\mathbb{w}}}$$

$$+\sum_{i=0}^{m-1}\frac{\left({}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta}f(a)\right)\ \Gamma_k(i+k)}{i!\mathbb{w}^{\frac{i}{k}-\frac{r}{k}}\Gamma_k(mk+i+k-r)}(\vartheta(t)-\vartheta(a))^{\frac{\mathbb{w}(i-r)}{k}}. \tag{41}$$

Differentiating the first part of (41) m-times, we get

$${}_a^r\mathfrak{D}_k^{\mathbb{w},\vartheta}f(t)$$

$$=\frac{\Gamma_k(mk)}{(m-1)!k\Gamma_k(2mk-r)\ \mathbb{w}^{2m-\frac{r}{k}-1}}\frac{\Gamma_k(2mk-r)}{\Gamma_k(mk-r)}\ \mathbb{w}^m\int_a^t((\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}})^{\frac{mk}{k}-\frac{r}{k}-1}$$

$$\times\frac{({}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta}f(\varsigma))\vartheta'(\varsigma)d\varsigma}{(\vartheta(\varsigma)-\vartheta(a)\ )^{1-\mathbb{w}}}+\sum_{i=0}^{m-1}\frac{\left({}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta}f(a)\right)}{\mathbb{w}^{i-r}\Gamma(i+1-r)}(\vartheta(t)-\vartheta(a))^{\mathbb{w}(i-r)}$$

$$+\sum_{i=0}^{m-1}\frac{\left({}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta}f(a)\right)\ \Gamma_k(i+k)}{i!\mathbb{w}^{\frac{i}{k}-\frac{r}{k}}\Gamma_k(mk+i+k-r)}(\vartheta(t)-\vartheta(a))^{\frac{\mathbb{w}(i-r)}{k}}$$

$$=\frac{\Gamma_k(mk)}{k\ (m-1)!\Gamma_k(m-r)\ \mathbb{w}^{m-\frac{r}{k}-1}}\ \int_a^t((\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}})^{m-\frac{r}{k}-1}\frac{({}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta}f(\varsigma))\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(\varsigma)-\vartheta(a)\ )^{1-\mathbb{w}}}$$

$$+\sum_{i=0}^{m-1}\frac{\left({}_a^{m}\mathbb{T}_k^{\mathbb{w},\vartheta}f(a)\right)\ \Gamma_k(i+k)}{i!\mathbb{w}^{\frac{i}{k}-\frac{r}{k}}\Gamma_k(mk+i+k-r)}(\vartheta(t)-\vartheta(a))^{\frac{\mathbb{w}(i-r)}{k}}$$

$$=\frac{\Gamma_k(mk)}{k\ (m-1)!\Gamma_k(m-r)}\int_a^t\left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}}\right)^{m-\frac{r}{k}-1}\frac{({}_a^{m}\mathbb{T}_k^{\mathbb{w},\vartheta}f(\varsigma))\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(\varsigma)-\vartheta(a)\ )^{1-\mathbb{w}}}$$

$$+\sum_{i=0}^{m-1}\frac{\left({}_a^{m}\mathbb{T}_k^{\mathbb{w},\vartheta}f(a)\right)\ \Gamma_k(i+k)}{i!\mathbb{w}^{\frac{i}{k}-\frac{r}{k}}\Gamma_k(mk+i+k-r)}(\vartheta(t)-\vartheta(a))^{\frac{\mathbb{w}(i-r)}{k}},$$

which is the required proof.

Now, to prove the second part of the theorem, we have

Since $f\ \in\mathbb{C}^{m}_{\mathbb{w},\vartheta,a,k}([a,b])$, from Lemma 5, we have

$$f(t)=\frac{1}{(m-1)!}\int_t^b\left(\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{mk}{k}-1}\frac{\vartheta'(\varsigma)({}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}f)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}$$

$$+\sum_{i=0}^{m-1}\frac{(-1)^{ii}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}f(b)}{\mathbb{w}^{\frac{i}{k}}\ i!}(\vartheta(b)-\vartheta(t))^{\frac{\mathbb{w}i}{k}}.$$

Therefore, we have

$${}^{r}\mathfrak{D}_{b,k}^{\mathbb{w},\vartheta}f(t)=\frac{(-1)^{m}\ {}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}}{(m-1)!k\Gamma_k(mk-r)}\int_t^b\int_v^b\left(\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(v))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{mk-r}{k}-1}$$

$$\left(\frac{(\vartheta(b)-\vartheta(v))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{\mathbb{w}}\right)^{m-1}\times\frac{\vartheta'(\varsigma)({}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}f)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}\ \frac{\vartheta'(v)dv}{(\ \vartheta(b)-\vartheta(v)\ )^{1-\mathbb{w}}}$$

$$+\sum_{i=0}^{m-1}\frac{\left({}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}f(a)\right)\ \Gamma_k(i+k)}{i!\mathbb{w}^{\frac{i}{k}-\frac{r}{k}}\Gamma_k(mk+i+k-r)}(\vartheta(b)-\vartheta(t))^{\frac{\mathbb{w}(i-r)}{k}}$$

$${}^{r}\mathfrak{D}_{b,k}^{\mathbb{w},\vartheta}f(t)=\frac{(-1)^{m}{}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}}{(m-1)!k\Gamma_k(mk-r)\ \mathbb{w}^{m-\frac{r}{k}-1+m-1}}\int_t^b\int_\varsigma^b\left((\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(v))^{\mathbb{w}}\right)^{\frac{mk-r}{k}-1}$$

$$\times\left((\vartheta(b)-\vartheta(v))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}\right)^{m-1}\frac{\vartheta'(\varsigma)({}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}f)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}\ \frac{\vartheta'(v)dv}{(\ \vartheta(b)-\vartheta(v)\ )^{1-\mathbb{w}}}$$

$$+\sum_{i=0}^{m-1}\frac{\left({}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}f(a)\right)\ \Gamma_k(i+k)}{i!\mathbb{w}^{\frac{i}{k}-\frac{r}{k}}\Gamma_k(mk+i+k-r)}(\vartheta(b)-\vartheta(t))^{\frac{\mathbb{w}(i-r)}{k}}.$$

Substituting

$$y=\frac{(\vartheta(b)-\vartheta(\nu))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}},\quad -dy=\frac{\mathbb{w}(\vartheta(b)-\vartheta(\nu))^{\mathbb{w}-1}\,\vartheta'(\nu)d\nu}{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(a)-\vartheta(\varsigma))^{\mathbb{w}}}$$

and

$$1-y=1-\frac{(\vartheta(b)-\vartheta(\nu))^{\mathbb{w}}-(\vartheta(b)-\vartheta\,(\varsigma))^{\mathbb{w}}}{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}=\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\nu))^{\mathbb{w}}}{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}$$

in above integral, we obtain

$$
\begin{aligned}
{}^{r}\mathfrak{D}_{b,k}^{\mathbb{w},\vartheta}f(t)&=\frac{(-1)^{m+1}\;{}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}}{(m-1)!\;k\Gamma_k(m-r)\mathbb{w}^{2m-\frac{r}{k}-2+1}}\int_t^b((\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}})^{\frac{2mk-r}{k}-1}\\
&\quad\frac{\vartheta'(\varsigma)({}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}f)d\varsigma}{(\,\vartheta(b)-\vartheta(\varsigma))^{1-\mathbb{w}}}\left(\int_0^1(1-y\,)^{\frac{mk-r}{k}-1}(y)^{\frac{mk}{k}-1}dy\right)\\
&\quad+\sum_{i=0}^{m-1}\frac{\left({}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}f(a)\right)\;\Gamma_k(i+k)}{i!\mathbb{w}^{\frac{i}{k}-\frac{r}{k}}\Gamma_k(mk+i+k-r)}(\vartheta(b)-\vartheta(t))^{\frac{\mathbb{w}(i-r)}{k}}\\
&=\frac{(-1)^{m+1}\;{}^{m}\mathbb{T}_{b}^{\mathbb{w},\vartheta}}{(m-1)!\Gamma_k(m-r)\mathbb{w}^{2m-\frac{r}{k}-1}}\int_t^b((\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}})^{2m-\frac{r}{k}-1}\\
&\quad\frac{\vartheta'(\varsigma)({}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}f)d\varsigma}{(\,\vartheta(b)-\vartheta(\varsigma))^{1-\mathbb{w}}}\left(\frac{1}{k}\int_0^1(1-y\,)^{m-\frac{r}{k}-1}(y)^{\frac{mk}{k}-1}dy\right)\\
&\quad+\sum_{i=0}^{m-1}\frac{\left({}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}f(a)\right)\;\Gamma_k(i+k)}{i!\mathbb{w}^{\frac{i}{k}-\frac{r}{k}}\Gamma_k(mk+i+k-r)}(\vartheta(b)-\vartheta(t))^{\frac{\mathbb{w}(i-r)}{k}}.
\end{aligned}
$$

By using k-Beta function, we have

$$
\begin{aligned}
&=\frac{(-1)^{m+1}\;\Gamma_k(mk)\,{}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}}{(m-1)!\Gamma_k(2mk-r)\mathbb{w}^{2m-\frac{r}{k}-1}}\int_t^b\left((\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}\right)^{\frac{2mk-r}{k}-1}\frac{\vartheta'(\varsigma)({}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}f)d\varsigma}{(\,\vartheta(b)-\vartheta(\varsigma)\,)^{1-\mathbb{w}}}\\
&\quad+\sum_{i=0}^{m-1}\frac{\left({}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}f(a)\right)\;\Gamma_k(i+k)}{i!\mathbb{w}^{\frac{i}{k}-\frac{r}{k}}\Gamma_k(mk+i+k-r)}(\vartheta(b)-\vartheta(t))^{\frac{\mathbb{w}(i-r)}{k}}
\end{aligned}
$$

$$= \frac{\Gamma_k(mk)}{(m-1)!\Gamma_k(2mk-r)\,\mathbb{w}^{2m-\frac{r}{k}-1}}(-1)^{m+1}\ {}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}$$

$$\int_t^b \left((\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}\right)^{\frac{2mk}{k}-\frac{r}{k}-1}\frac{\vartheta'(\varsigma)({}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}f)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma))^{1-\mathbb{w}}}$$

$$+\sum_{i=0}^{m-1}\frac{\left({}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}f(a)\right)\ \Gamma_k(i+k)}{i!\mathbb{w}^{\frac{i}{k}-\frac{r}{k}}\Gamma_k(mk+i+k-r)}(\vartheta(b)-\vartheta(t))^{\frac{\mathbb{w}(i-r)}{k}}$$

$$= \frac{\Gamma_k(mk)}{(m-1)!\Gamma_k(2mk-r)\mathbb{w}^{2m-\frac{r}{k}-1}}(-1)^{m+1}\ (\vartheta(b)-\vartheta(t))^{m-m\mathbb{w}}\ \left[\frac{k}{\vartheta'(t)}\frac{d}{dt}\right]^{m}$$

$$\int_t^b \left((\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}\right)^{2m-\frac{r}{k}-1}\frac{\vartheta'(\varsigma)({}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}f)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}$$

$$+\sum_{i=0}^{m-1}\frac{\left({}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}f(a)\right)\ \Gamma_k(i+k)}{i!\mathbb{w}^{\frac{i}{k}-\frac{r}{k}}\Gamma_k(mk+i+k-r)}(\vartheta(b)-\vartheta(t))^{\frac{\mathbb{w}(i-r)}{k}}$$

$$= \frac{\Gamma_k(mk)}{k(m-1)!\Gamma_k(2mk-r)\mathbb{w}^{2m-\frac{r}{k}-1}}\mathbb{w}^{m}\frac{\Gamma_k(2mk-r)}{\Gamma_k(mk-r)}$$

$$\int_t^b \left((\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}\right)^{m-\frac{r}{k}-1}\frac{\vartheta'(\varsigma)({}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}f)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}$$

$$+\sum_{i=0}^{m-1}\frac{\left({}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}f(a)\right)\ \Gamma_k(i+k)}{i!\mathbb{w}^{\frac{i}{k}-\frac{r}{k}}\Gamma_k(mk+i+k-r)}(\vartheta(b)-\vartheta(t))^{\frac{\mathbb{w}(i-r)}{k}}$$

$$= \frac{\Gamma_k(mk)}{k\ (m-1)!\Gamma_k(mk-r)\ \mathbb{w}^{2m-\frac{r}{k}-1-m}}\int_t^b \left((\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}\right)^{m-\frac{r}{k}-1}\frac{\vartheta'(\varsigma)({}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}f)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}$$

$$+\sum_{i=0}^{m-1}\frac{\left({}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}f(a)\right)\ \Gamma_k(i+k)}{i!\mathbb{w}^{\frac{i}{k}-\frac{r}{k}}\Gamma_k(mk+i+k-r)}(\vartheta(b)-\vartheta(t))^{\frac{\mathbb{w}(i-r)}{k}}$$

$$= \frac{\Gamma_k(mk)\,{}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}}{k\ (m-1)!\Gamma_k(m-r)}\int_t^b \left(\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{mk-r}{k}-1}\frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}$$

$$+\sum_{i=0}^{m-1}\frac{\left({}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}f(a)\right)\ \Gamma_k(i+k)}{i!\mathbb{w}^{\frac{i}{k}-\frac{r}{k}}\Gamma_k(mk+i+k-r)}(\vartheta(b)-\vartheta(t))^{\frac{\mathbb{w}(i-r)}{k}},$$

which completes the desired proof. □

Theorem 3 Let $\Re(\varkappa) > m > 0$ with $m \in \mathcal{N}$. We have

$$ {}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta}\,{}_a^\varkappa\mathcal{J}_k^{\mathbb{w},\vartheta} f(t) = {}_a^{\varkappa-mk}\mathcal{J}_k^{\mathbb{w},\vartheta} f(t) \text{ and } {}^m\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}\,{}^\varkappa\mathcal{J}_{b,k}^{\mathbb{w},\vartheta} f(t) = {}^{\varkappa-mk}\mathcal{J}_{b,k}^{\mathbb{w},\vartheta} f(t). \tag{42} $$

Proof. Consider

$$ {}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta}\,{}_a^\varkappa\mathcal{J}_k^{\mathbb{w},\vartheta} f(t) $$

$$ = {}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta}\left[\frac{1}{k\Gamma_k(\varkappa)}\int_a^t\left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{\varkappa}{k}-1}\frac{f(\varsigma)\,\vartheta'(\varsigma)d\varsigma}{(\,\vartheta(\varsigma)-\vartheta(a)\,)^{1-\mathbb{w}}}\right] $$

$$ = {}_a^{m-1}\mathbb{T}_k^{\mathbb{w},\vartheta}\left[\frac{1}{k\Gamma_k(\varkappa-k)}\int_a^t\left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{\varkappa-k}{k}-1}\frac{f(\varsigma)\,\vartheta'(\varsigma)d\varsigma}{(\,\vartheta(\varsigma)-\vartheta(a)\,)^{1-\mathbb{w}}}\right] $$

$$ = {}_a^{m-2}\mathbb{T}_k^{\mathbb{w},\vartheta}\left[\frac{1}{k\Gamma_k(\varkappa-2k)}\int_a^t\left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{\varkappa-2k}{k}-1}\frac{f(\varsigma)\,\vartheta'(\varsigma)d\varsigma}{(\,\vartheta(\varsigma)-\vartheta(a)\,)^{1-\mathbb{w}}}\right] $$

$$ = {}_a^{m-3}\mathbb{T}_k^{\mathbb{w},\vartheta}\left[\frac{1}{k\Gamma_k(\varkappa-3k)}\int_a^t\left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{\varkappa-3k}{k}-1}\frac{f(\varsigma)\,\vartheta'(\varsigma)d\varsigma}{(\,\vartheta(\varsigma)-\vartheta(a)\,)^{1-\mathbb{w}}}\right]. $$

Continuing in this way, we get

$$ {}_a^m\mathbb{T}_k^{\mathbb{w},\vartheta}\,{}_a^\varkappa\mathcal{J}_k^{\mathbb{w},\vartheta} f(t) $$

$$ = {}_a^{m-m}\mathbb{T}_k^{\mathbb{w},\vartheta}\left[\frac{1}{k\Gamma_k(\varkappa-mk)}\int_a^t\left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{\varkappa-mk}{k}-1}\frac{f(\varsigma)\,\vartheta'(\varsigma)d\varsigma}{(\,\vartheta(\varsigma)-\vartheta(a)\,)^{1-\mathbb{w}}}\right] $$

$$ = \left[\frac{1}{k\Gamma_k(\varkappa-mk)}\int_a^t\left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{\varkappa}{k}-m-1}\frac{f(\varsigma)\,\vartheta'(\varsigma)d\varsigma}{(\,\vartheta(\varsigma)-\vartheta(a)\,)^{1-\mathbb{w}}}\right] $$

$$ = {}_a^{\varkappa-mk}\mathcal{J}_k^{\mathbb{w},\vartheta} f(t), $$

which proves the first part of Theorem 3.

Now, to prove the second part of Theorem 3, we have

$$ {}^m\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}\,{}^\varkappa\mathcal{J}_{b,k}^{\mathbb{w},\vartheta} f(t) $$

$$= {}^{m}\mathbb{T}^{\mathbb{w},\vartheta}_{b,k}\left[\frac{1}{k\Gamma_k(\varkappa)}\int_t^b\left(\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{\varkappa}{k}-1}\frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}\right]$$

$$= {}^{m-1}\mathbb{T}^{\mathbb{w},\vartheta}_{b,k}\left[\frac{1}{k\Gamma_k(\varkappa-k)}\int_t^b\left(\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{\varkappa-k}{k}-1}\frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}\right]$$

$$= {}^{m-2}\mathbb{T}^{\mathbb{w},\vartheta}_{b,k}\left[\frac{1}{k\Gamma_k(\varkappa-2k)}\int_t^b\left(\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{\varkappa-2k}{k}-1}\frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}\right]$$

$$= {}^{m-3}\mathbb{T}^{\mathbb{w},\vartheta}_{b,k}\left[\frac{1}{k\Gamma_k(\varkappa-3k)}\int_t^b\left(\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{\varkappa-3k}{k}-1}\frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}\right]$$

$$\vdots$$

$${}^{m}\mathbb{T}^{\mathbb{w},\vartheta}_{b,k}\ {}^{\varkappa}\mathcal{J}^{\mathbb{w},\vartheta}_{b,k}f(t)$$

$$= {}^{m-m}\mathbb{T}^{\mathbb{w},\vartheta}_{b,k}\left[\frac{1}{k\Gamma_k(\varkappa-mk)}\int_t^b\left(\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{\varkappa-mk}{k}-1}\frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}\right]$$

$$= \left[\frac{1}{k\Gamma_k(\varkappa-mk)}\int_t^b\left(\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{\varkappa-mk}{k}-1}\frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}\right]$$

$$= {}^{\varkappa-mk}\mathcal{J}^{\mathbb{w},\vartheta}_{b,k}f(t),$$

which proves the second part of Theorem 3. □

Corollary 1 Let $\Re(\varkappa^*) < \Re(\varkappa)$, we have

$${}^{\varkappa^*}_{a}\mathfrak{D}^{\mathbb{w},\vartheta}_{k}\left({}^{\varkappa}_{a}\mathcal{J}^{\mathbb{w},\vartheta}_{k}\ f(t)\right) = {}^{\varkappa-\varkappa^*}_{a}\mathcal{J}^{\mathbb{w},\vartheta}_{k}\ f(t) \text{ and } {}^{\varkappa^*}\mathfrak{D}^{\mathbb{w},\vartheta}_{b,k}\left({}^{\varkappa}\mathcal{J}^{\mathbb{w},\vartheta}_{b,k}f(t)\right) = {}^{\varkappa-\varkappa^*}\mathcal{J}^{\mathbb{w},\vartheta}_{b,k}f(t). \tag{43}$$

Proof. By utilizing Theorems 1 and 3, we obtain

$${}^{\varkappa^*}_{a}\mathfrak{D}^{\mathbb{w},\vartheta}_{k}\left({}^{\varkappa}_{a}\mathcal{J}^{\mathbb{w},\vartheta}_{k}\ f(t)\right) = {}^{m}_{a}\mathbb{T}_{k}{}^{\mathbb{w},\vartheta}\left({}^{mk-\varkappa^*}_{a}\mathcal{J}^{\mathbb{w},\vartheta}_{k}\ \left({}^{\varkappa}_{a}\mathcal{J}^{\mathbb{w},\vartheta}_{k}\ f(t)\right)\right)$$

$$= {}^{m}_{a}\mathbb{T}_{k}{}^{\mathbb{w},\vartheta}\left({}^{\varkappa+mk-\varkappa^*}_{a}\quad \mathcal{J}^{\mathbb{w},\vartheta}_{k}\ f(t)\right)$$

$$= {}^{\varkappa+mk-\varkappa^*-mk}_{a}\mathcal{J}^{\mathbb{w},\vartheta}_{k}\ f(t)$$

$$= {}_a^{\varkappa-\varkappa^*}\mathcal{J}_k^{\mathbb{w},\vartheta} f(t),$$

which proves the first part of Corollary.

To prove the second part, we have

$$\begin{aligned}
{}^{\varkappa^*}\mathfrak{D}_{b,k}^{\mathbb{w},\vartheta}\left({}^{\varkappa}\mathcal{J}_{b,k}^{\mathbb{w},\vartheta} f(t)\right) &= (-1)^m\, {}^m\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}\left({}^{mk-\varkappa^*}\mathcal{J}_{b,k}^{\mathbb{w},\vartheta}\left({}^{\varkappa}\mathcal{J}_{b,k}^{\mathbb{w},\vartheta} f(t)\right)\right)\\
&= (-1)^m\, {}^m\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}\left({}^{\varkappa+mk-\varkappa^*}\mathcal{J}_{b,k}^{\mathbb{w},\vartheta} f(t)\right)\\
&= (-1)^m\, {}^m\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}\left({}^{\varkappa+mk-\varkappa^*}\mathcal{J}_{b,k}^{\mathbb{w},\vartheta} f(t)\right)\\
&= {}^{\varkappa+mk-\varkappa^*-mk}\mathcal{J}_{b,k}^{\mathbb{w},\vartheta} f(t)\\
&= {}^{\varkappa-\varkappa^*}\mathcal{J}_{b,k}^{\mathbb{w},\vartheta} f(t),
\end{aligned}$$

which completes the second part of Corollary 1. □

Theorem 4 Let $\varkappa > 0$ and $f \in \mathbb{C}_{\mathbb{w},\vartheta,a}^m([a,b])$, $\left(f \in \mathbb{C}_{\mathbb{w},\vartheta,b}^m([a,b])\right)$, we have

$${}_a^{\varkappa}\mathfrak{D}_k^{\mathbb{w},\vartheta}\left({}_a^{\varkappa}\mathcal{J}_k^{\mathbb{w},\vartheta} f(t)\right) = f(t) \text{ and } {}^{\varkappa}\mathfrak{D}_{b,k}^{\mathbb{w},\vartheta}\left({}^{\varkappa}\mathcal{J}_{b,k}^{\mathbb{w},\vartheta} f(t)\right) = f(t). \tag{44}$$

Proof Consider

$$\begin{aligned}
&{}_a^{\varkappa}\mathfrak{D}_k^{\mathbb{w},\vartheta}\left({}^{\varkappa}\mathcal{J}_{a,k}^{\mathbb{w},\vartheta} f(t)\right)\\
&= \frac{{}_a^m\mathbb{T}_k{}^{\mathbb{w},\vartheta}}{k\Gamma_k(mk-\varkappa)k\Gamma_k(\varkappa)}\int_a^t\int_a^{\nu}\left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\nu)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{mk-\varkappa}{k}-1}\\
&\quad\left(\frac{(\vartheta(\nu)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{\varkappa}{k}-1}\times\frac{f(\varsigma)\,\vartheta'(\varsigma)d\varsigma}{(\,\vartheta(\varsigma)-\vartheta(a)\,)^{1-\mathbb{w}}}\,\frac{\vartheta'(\nu)d\nu}{(\,\vartheta(\nu)-\vartheta(a)\,)^{1-\mathbb{w}}}\\
&= \frac{{}_a^m\mathbb{T}_k{}^{\mathbb{w},\vartheta}}{k\Gamma_k(mk-\varkappa)k\Gamma_k(\varkappa)\mathbb{w}^{m-\frac{\varkappa}{k}+\frac{\varkappa}{k}-2}}\int_a^t\int_{\varsigma}^t\left((\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\nu)-\vartheta(a))^{\mathbb{w}}\right)^{\frac{mk-\varkappa}{k}-1}\\
&\quad\left((\vartheta(\nu)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}\right)^{\frac{\varkappa}{k}-1}\frac{f(\varsigma)\,\vartheta'(\varsigma)d\varsigma}{(\,\vartheta(\varsigma)-\vartheta(a)\,)^{1-\mathbb{w}}}\,\frac{\vartheta'(\nu)d\nu}{(\,\vartheta(\nu)-\vartheta(a)\,)^{1-\mathbb{w}}}.
\end{aligned}$$

Substituting

$$y=\frac{(\vartheta(\nu)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}},\ \ dy=\frac{\mathbb{w}(\vartheta(\nu)-\vartheta(a))^{\mathbb{w}-1}\ \vartheta'(\nu)d\nu}{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}$$

and

$$1-y=1-\frac{(\vartheta(\nu)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta\ (a))^{\mathbb{w}}}{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}=\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\nu)-\vartheta(a))^{\mathbb{w}}}{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}$$

in above integral, we get

$$\begin{aligned}
&{}^{r}_{a}\mathfrak{D}^{\mathbb{w},\vartheta}_{k}\left({}^{r}\mathcal{J}^{\mathbb{w},\vartheta}_{a,k}f(t)\right)\\
&=\frac{{}^{m}_{a}\mathbb{T}_{k}{}^{\mathbb{w},\vartheta}}{k\Gamma_{k}(mk-r)k\Gamma_{k}(r)\mathbb{w}^{m-1}}\int_{a}^{t}\left((\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}\right)^{\frac{mk-r}{k}+\frac{r}{k}-1}\\
&\quad\frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(\varsigma)-\vartheta(a))^{1-\mathbb{w}}}\left(\int_{0}^{1}(1-y\ )^{\frac{mk-r}{k}-1}(y)^{\frac{r}{k}-1}dy\right)\\
&=\frac{{}^{m}_{a}\mathbb{T}_{k}{}^{\mathbb{w},\vartheta}}{k(\Gamma_{k}(mk-r)\Gamma_{k}(r))}\int_{a}^{t}\left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}}\right)^{m-1}\\
&\quad\frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(\varsigma)-\vartheta(a)\ )^{1-\mathbb{w}}}\times\frac{1}{k}\int_{0}^{1}(1-y\ )^{\frac{mk-r}{k}-1}(y)^{\frac{r}{k}-1}dy.
\end{aligned}$$

By using k-Beta function, we have

$${}^{r}_{a}\mathfrak{D}^{\mathbb{w},\vartheta}_{k}\left({}^{r}\mathcal{J}^{\mathbb{w},\vartheta}_{a,k}f(t)\right)=\frac{{}^{m}_{a}\mathbb{T}_{k}{}^{\mathbb{w},\vartheta}}{k\Gamma_{k}(mk-r+r)}\int_{a}^{t}\left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}}\right)^{m-1}\frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(\varsigma)-\vartheta(a)\ )^{1-\mathbb{w}}}$$

$$=\frac{{}^{m}_{a}\mathbb{T}^{\mathbb{w},\vartheta}}{k^{m}\Gamma(m)}\int_{a}^{t}\left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}}\right)^{m-1}\frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(\varsigma)-\vartheta(a)\ )^{1-\mathbb{w}}}$$

$${}^{r}_{a}\mathfrak{D}^{\mathbb{w},\vartheta}_{k}\left({}^{r}\mathcal{J}^{\mathbb{w},\vartheta}_{a,k}f(t)\right)=\frac{1}{k^{m}}\ {}^{m}_{a}\mathbb{T}_{k}{}^{\mathbb{w},\vartheta}\left({}^{m}_{a}\mathbb{I}^{\mathbb{w},\vartheta}f(t)\right)=f(t),\ (\text{using}\ \Gamma_{k}(x)=k^{\frac{x}{k}-1}\Gamma(\frac{x}{k})),$$

this completes the first part of Theorem 4.

To prove the second part of Theorem 4, we have

$${}^{r}\mathfrak{D}^{\mathbb{w},\vartheta}_{b,k}\left({}^{r}\mathcal{J}^{\mathbb{w},\vartheta}_{b,k}f(t)\right)$$

$$= \frac{(-1)^m\ {}^m\mathbb{T}^{\mathbb{w},\vartheta}_{b,k}}{k\Gamma_k(mk-\mathscr{r})k\Gamma_k(\mathscr{r})} \int_t^b \int_\nu^b \left(\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\nu))^{\mathbb{w}}}{\mathbb{w}}\right)^{m-\frac{\mathscr{r}}{k}-1}$$

$$\left(\frac{(\vartheta(b)-\vartheta(\nu))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{\mathscr{r}}{k}-1} \times \frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}\ \frac{\vartheta'(\nu)d\nu}{(\ \vartheta(b)-\vartheta(\nu)\ )^{1-\mathbb{w}}}$$

$$= \frac{(-1)^m\ {}^m\mathbb{T}^{\mathbb{w},\vartheta}_{b,k}}{k\Gamma_k(mk-\mathscr{r})k\Gamma_k(\mathscr{r})\mathbb{w}^{m-\frac{\mathscr{r}}{k}+\frac{\mathscr{r}}{k}-2}} \int_t^b \int_\varsigma^b \left((\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\nu))^{\mathbb{w}}\right)^{m-\frac{\mathscr{r}}{k}-1}$$

$$\times \left((\vartheta(b)-\vartheta(\nu))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma)^{\mathbb{w}}\right)^{\frac{\mathscr{r}}{k}-1} \frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}\ \frac{\vartheta'(\nu)d\nu}{(\ \vartheta(b)-\vartheta(\nu)\ )^{1-\mathbb{w}}}.$$

Substituting

$$y = \frac{(\vartheta(b)-\vartheta(\nu))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}},\quad -dy = \frac{\mathbb{w}(\vartheta(b)-\vartheta(\nu))^{\mathbb{w}-1}\ \vartheta'(\nu)d\nu}{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(a)-\vartheta(\varsigma))^{\mathbb{w}}}$$

and

$$1-y = 1-\frac{(\vartheta(b)-\vartheta(\nu))^{\mathbb{w}}-(\vartheta(b)-\vartheta\ (\varsigma))^{\mathbb{w}}}{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}} = \frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\nu))^{\mathbb{w}}}{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}$$

in above integral, we get

$${}^{\mathscr{r}}\mathfrak{D}^{\mathbb{w},\vartheta}_{b,k}\left({}^{\mathscr{r}}\mathscr{J}^{\mathbb{w},\vartheta}_{b,k}f(t)\right) = \frac{(-1)^{m+1}\ {}^m\mathbb{T}^{\mathbb{w},\vartheta}_{b,k}}{k\Gamma_k(mk-\mathscr{r})k\Gamma_k(\mathscr{r})\mathbb{w}^{m-1}} \int_t^b \left((\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}\right)^{m-\frac{\mathscr{r}}{k}+\frac{\mathscr{r}}{k}-1}$$

$$\frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma))^{1-\mathbb{w}}} \times \int_0^1 (1-y\ )^{m-\frac{\mathscr{r}}{k}-1}(y)^{\frac{\mathscr{r}}{k}-1}dy$$

$$= \frac{(-1)^{m+1}\ {}^m\mathbb{T}^{\mathbb{w},\vartheta}_{b,k}}{k(\Gamma_k(mk-\mathscr{r})\Gamma_k(\mathscr{r}))} \int_t^b \left(\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{\mathbb{w}}\right)^{m-1}$$

$$\frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma))^{1-\mathbb{w}}} \left(\frac{1}{k}\int_0^1 (1-y\ )^{m-\frac{\mathscr{r}}{k}-1}(y)^{\frac{\mathscr{r}}{k}-1}dy\right).$$

By using k-Beta function, we get

$${}^{\mathscr{r}}\mathfrak{D}^{\mathbb{w},\vartheta}_{b,k}\left({}^{\mathscr{r}}\mathscr{J}^{\mathbb{w},\vartheta}_{b,k}f(t)\right) = \frac{(-1)^{m+1}\ {}^m\mathbb{T}^{\mathbb{w},\vartheta}_{b,k}}{k\Gamma_k(mk-\mathscr{r}+\mathscr{r})} \int_t^b \left(\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{\mathbb{w}}\right)^{m-1} \frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}$$

$$= \frac{(-1)^{m+1}\ {}_{m}\mathbb{T}^{\mathbb{w},\vartheta}_{b,k}}{k\Gamma_k(mk)} \int_t^b \left(\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}} - (\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{\mathbb{w}} \right)^{m-1} \frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}$$

$$= \frac{(-1)^{m+1}\ {}_{m}\mathbb{T}^{\mathbb{w},\vartheta}_{b,k}}{k^m\Gamma(m)} \int_t^b \left(\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}} - (\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{\mathbb{w}} \right)^{m-1} \frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}$$

$$= (-1)^{m+1} \frac{1}{k^m}\, {}_{m}\mathbb{T}^{\mathbb{w},\vartheta}_{b,k} \left({}_{m}\mathbb{I}^{\mathbb{w},\vartheta}_{b} f(t) \right) = f(t),$$

which completes the proof. □

Theorem 5 Let $\Re(r) > 0, m = -[-\Re(r)]$, $f \in \mathscr{L}(a,b)$ and ${}^{r}_{a}\mathscr{I}^{\mathbb{w},\vartheta}_{k} f \in \mathbb{C}^{m}_{\mathbb{w},\vartheta,a,k}[a,b]$ and ${}^{r}\mathscr{I}^{\mathbb{w},\vartheta}_{b,k} f \in \mathbb{C}^{m}_{\mathbb{w},\vartheta,b,k}[a,b]$. Then

$$ {}^{r}_{a}\mathscr{I}^{\mathbb{w},\vartheta}_{k} \left({}^{r}_{a}\mathfrak{D}^{\mathbb{w},\vartheta}_{k}\ f(t) \right) = f(t) - \sum_{i=0}^{m} \frac{{}^{r-i}_{a}\mathfrak{D}^{\mathbb{w},\vartheta}_{k}\ f(a)}{\mathbb{w}^{\frac{r}{k}-\frac{i}{k}}\ k\Gamma_k(r-ik+k)} (\vartheta(t)-\vartheta(a)^{\mathbb{w}\left(\frac{r}{k}-\frac{i}{k}\right)} \tag{45}$$

and

$$ {}^{r}\mathscr{I}^{\mathbb{w},\vartheta}_{b,k} \left({}^{r}\mathfrak{D}^{\mathbb{w},\vartheta}_{b,k} f(t) \right) = f(t) - \sum_{i=1}^{m} \frac{{}^{r-i}\mathfrak{D}^{\mathbb{w},\vartheta}_{b,k} f(b)}{\mathbb{w}^{\frac{r}{k}-\frac{i}{k}}\ k\Gamma_k(r-ik+k)} (\vartheta(b)-\vartheta(t))^{\mathbb{w}\left(\frac{r}{k}-\frac{i}{k}\right)}. \tag{46}$$

Proof. To prove (45), we have

$${}^{r}_{a}\mathscr{I}^{\mathbb{w},\vartheta}_{k} \left({}^{r}_{a}\mathfrak{D}^{\mathbb{w},\vartheta}_{k}\ f(t) \right)$$

$$= \frac{1}{k\Gamma_k(r)} \int_a^t \left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}} - (\vartheta(\varsigma)-\vartheta(a)^{\mathbb{w}}}{\mathbb{w}} \right)^{\frac{r}{k}-1} \left({}^{m}_{a}\mathbb{T}_{k}{}^{\mathbb{w},\vartheta} \left({}^{mk-r}_{a}\mathscr{I}^{\mathbb{w},\vartheta}_{k} f(\varsigma) \right) \right) \frac{\vartheta'(\varsigma)d\varsigma}{(\ \vartheta(\varsigma)-\vartheta(a)\ )^{1-\mathbb{w}}}$$

$$= \frac{{}^{1}_{a}\mathbb{T}_{k}{}^{\mathbb{w},\vartheta}}{k\Gamma_k(r+k)} \left[\int_a^t \left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}} - (\vartheta(\varsigma)-\vartheta(a)^{\mathbb{w}}}{\mathbb{w}} \right)^{\frac{r}{k}} \left({}^{m}_{a}\mathbb{T}_{k}{}^{\mathbb{w},\vartheta} \left({}^{mk-r}_{a}\mathscr{I}^{\mathbb{w},\vartheta}_{k} f(\varsigma) \right) \right) \frac{\vartheta'(\varsigma)d\varsigma}{(\ \vartheta(\varsigma)-\vartheta(a)\ )^{1-\mathbb{w}}} \right].$$

Using integration by parts m times, we get

$${}^{r}_{a}\mathscr{I}^{\mathbb{w},\vartheta}_{k} \left({}^{r}_{a}\mathfrak{D}^{\mathbb{w},\vartheta}_{k}\ f(t) \right)$$

$$= {}^{1}_{a}\mathbb{T}_{k}{}^{\mathbb{w},\vartheta} \left[\frac{1}{k\Gamma_k(r-mk+k)} \int_a^t \left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}} - (\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}} \right)^{\frac{r}{k}-m} \left({}^{mk-r}_{a}\mathscr{I}^{\mathbb{w},\vartheta}_{k} f(\varsigma) \right) \right.$$

$$\frac{\vartheta'(\varsigma)d\varsigma}{(\ \vartheta(\varsigma)-\vartheta(a)\ )^{1-\mathbb{w}}}-\sum_{i=1}^{m}\frac{\left(\left({}_a^{m-i}\mathbb{T}_k^{\ \mathbb{w},\vartheta}f(a)\right)\left({}_a^{mk-\mathscr{r}}\mathscr{J}_k^{\mathbb{w},\vartheta}f(\varsigma)\right)\right)}{k\Gamma_k(\mathscr{r}+2k-ik)\,\mathbb{w}^{\frac{\mathscr{r}}{k}-i+1}}(\vartheta(t)-\vartheta(a))^{\mathbb{w}\left(\frac{\mathscr{r}}{k}-\frac{i}{k}+1\right)}\Bigg]$$

$$={}_a^1\mathbb{T}_k^{\ \mathbb{w},\vartheta}\left[{}_a^{\mathscr{r}-mk+k}\mathscr{J}_k^{\mathbb{w},\vartheta}\left({}_a^{mk-\mathscr{r}}\mathscr{J}_k^{\mathbb{w},\vartheta}f(\varsigma)\right)-\sum_{i=1}^{m}\frac{\left(\left({}_a^{m-i}\mathbb{T}_k^{\ \mathbb{w},\vartheta}f(a)\right)\left({}_a^{mk-\mathscr{r}}\mathscr{J}_k^{\mathbb{w},\vartheta}f(\varsigma)\right)\right)}{k\Gamma_k(\mathscr{r}+2k-ik)\,\mathbb{w}^{\frac{\mathscr{r}-i}{k}+1}}(\vartheta(t)-\vartheta(a))^{\mathbb{w}\left(\frac{\mathscr{r}-i}{k}+1\right)}\right].$$

Now by using Theorem 1, we get

$$={}_a^1\mathbb{T}_k^{\ \mathbb{w},\vartheta}\left[{}_a^k\mathscr{J}_k^{\mathbb{w},\vartheta}f(t)-\sum_{i=1}^{m}\frac{\left(\left({}_a^{m-i}\mathbb{T}_k^{\ \mathbb{w},\vartheta}f(a)\right)\left({}_a^{mk-\mathscr{r}}\mathscr{J}_k^{\mathbb{w},\vartheta}f(\varsigma)\right)\right)}{k\Gamma_k(\mathscr{r}+2k-ik)\,\mathbb{w}^{\frac{\mathscr{r}}{k}-i+1}}(\vartheta(t)-\vartheta(a))^{\mathbb{w}\left(\frac{\mathscr{r}-i}{k}+1\right)}\right]$$

$$=f(t)-\sum_{i=0}^{m}\frac{{}_a^{\mathscr{r}-i}\mathfrak{D}_k^{\mathbb{w},\vartheta}f(a)}{\mathbb{w}^{\frac{\mathscr{r}}{k}-i}\,k\Gamma_k(\mathscr{r}-ik+k)}(\vartheta(t)-\vartheta(a))^{\mathbb{w}\left(\frac{\mathscr{r}-i}{k}\right)},$$

which completes the first part.

To prove (46), we have

$${}^{\mathscr{r}}\mathscr{J}_{b,k}^{\mathbb{w},\vartheta}\left({}^{\mathscr{r}}\mathfrak{D}_{b,k}^{\mathbb{w},\vartheta}f(t)\right)$$

$$=\frac{1}{k\Gamma_k(\mathscr{r})}\int_t^b\left(\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{\mathscr{r}}{k}-1}\left((-1)^m\ {}^m\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}\ \left({}^{mk-\mathscr{r}}\mathscr{J}_{b,k}^{\mathbb{w},\vartheta}\right)\right)\frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}$$

$$=\frac{{}^1\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}}{k\Gamma_k(\mathscr{r}+1)}\int_t^b\left(\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{\mathscr{r}}{k}}\left(\ (-1)^m\ {}^m\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}\ \left({}^{mk-\mathscr{r}}\mathscr{J}_{b,k}^{\mathbb{w},\vartheta}\right)\right)\frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}.$$

Using integration by parts m times, we get

$${}^{\mathscr{r}}\mathscr{J}_{b,k}^{\mathbb{w},\vartheta}\left({}^{\mathscr{r}}\mathfrak{D}_{b,k}^{\mathbb{w},\vartheta}f(t)\right)$$

$$={}^1\mathbb{T}_b^{\mathbb{w},\vartheta}\left[\frac{1}{k\Gamma_k(\mathscr{r}-mk+k)}\int_t^b\left(\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{\mathscr{r}-mk}{k}}\left(\ (-1)^m\left({}^{m-\mathscr{r}}\mathscr{J}_{b,k}^{\mathbb{w},\vartheta}f(\varsigma)\right)\right)\right.$$

$$\left.\frac{f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma))^{1-\mathbb{w}}}-\sum_{i=1}^{m}\frac{\left((-1)^{m-i}{}^{m-i}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}f(b)\right)\left({}^{mk-\mathscr{r}}\mathscr{J}_{b,k}^{\mathbb{w},\vartheta}f(\varsigma)\right)}{k\Gamma_k(\mathscr{r}+2k-ik)\,\mathbb{w}^{\frac{\mathscr{r}-i}{k}+1}}(\vartheta(b)-\vartheta(t))^{\mathbb{w}\left(\frac{\mathscr{r}-i}{k}+1\right)}\right]$$

$$= {}^{1}\mathbb{T}_{b}^{\mathbb{w},\vartheta}\left[{}^{r-m+1}\mathcal{J}_{b,k}^{\mathbb{w},\vartheta}\left({}^{m-r}\mathcal{J}_{b,k}^{\mathbb{w},\vartheta}f(\varsigma)\right)-\sum_{i=1}^{m}\frac{\left((-1)^{m-i}\ {}^{m-i}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}f(b)\right)\left({}^{mk-r}\mathcal{J}_{b,k}^{\mathbb{w},\vartheta}f(\varsigma)\right)}{k\Gamma_{k}(r+2k-ik)\,\mathbb{w}^{\frac{r-i}{k}+1}}(\vartheta(b)-\vartheta(t))^{\mathbb{w}\left(\frac{r-i}{k}+1\right)}\right].$$

Now by using Theorem 1, we get

$${}^{r}\mathcal{J}_{b,k}^{\mathbb{w},\vartheta}\left({}^{r}\mathfrak{D}_{b,k}^{\mathbb{w},\vartheta}f(t)\right)$$

$$= {}^{1}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}\left[{}^{k}\mathcal{J}_{b,k}^{\mathbb{w},\vartheta}f(\varsigma)-\sum_{i=1}^{m}\frac{\left((-1)^{m-i}\ \ {}^{m-i}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}\left({}^{mk-r}\mathcal{J}_{b,k}^{\mathbb{w},\vartheta}f(\varsigma)\right)\right)}{k\Gamma_{k}(r+2k-ik)\,\mathbb{w}^{\frac{r-i}{k}+1}}(\vartheta(b)-\vartheta(t))^{\mathbb{w}\left(\frac{r-i}{k}+1\right)}\right]$$

$$= f(t)-\sum_{i=1}^{m}\frac{{}^{r-i}\mathfrak{D}_{b,k}^{\mathbb{w},\vartheta}f(b)}{\mathbb{w}^{\frac{r-i}{k}}\ k\Gamma_{k}(r-ik+k)}(\vartheta(b)-\vartheta(t))^{\mathbb{w}\left(\frac{r-i}{k}\right)},$$

which completes the proof. □

4. The (k,ϑ)-CFD with the Caputo type

In this section, we present (k,ϑ)-CFD (Caputo Fractional Derivative) and its properties.

Definition 11 Let $\mathbb{w}>0$, $\Re(r)\geq 0$ and $m=[\Re(r)]+1$

a. If $\in \mathbb{C}_{\mathbb{w},\vartheta,a}^{m}([a,b])$. The left Caputo (k,ϑ)-CFD of f is

$${}_{a}^{\mathbb{C}}\mathfrak{D}_{k}^{r,\mathbb{w},\vartheta}\ f(t)={}_{a}^{r}\mathfrak{D}_{k}^{\mathbb{w},\vartheta}\left[f(t)-\sum_{i=0}^{m-1}\frac{{}_{a}^{i}\mathbb{T}_{k}^{\mathbb{w},\vartheta}\ f(a)}{\mathbb{w}^{\frac{i}{k}}\ i!}(\vartheta(t)-\vartheta(a)^{\frac{\mathbb{w}i}{k}}\right](t). \tag{47}$$

b. If $f\in\mathbb{C}_{\mathbb{w},\vartheta,b}^{m}([a,b])$. The left Caputo (k,ϑ)-CFD of f is

$${}^{\mathbb{C}}\mathfrak{D}_{b,k}^{r,\mathbb{w},\vartheta}\ f(t)={}^{r}\mathfrak{D}_{b,k}^{\mathbb{w},\vartheta}\left[f(t)-\sum_{i=0}^{m-1}\frac{(-1)^{i\,i}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}f(b)}{\mathbb{w}^{\frac{i}{k}}\ i!}\ (\vartheta(b)-\vartheta(t))^{\frac{\mathbb{w}i}{k}}\right](t). \tag{48}$$

Theorem 6 Let $\Re(r)\geq 0$, $m=[\Re(r)]+1$, $f\in\mathbb{C}_{\mathbb{w},\vartheta,a,}^{m}[a,b]$, $\left(f\in\ \mathbb{C}_{\mathbb{w},\vartheta,b}^{m}([a,b])\right)$. Then we have the left and right k,ϑ-CFD in the Caputo type are

$${}_{a}^{\mathbb{C}}\mathfrak{D}_{k}^{r,\mathbb{w},\vartheta}\ f(t)=\frac{1}{k\Gamma_{k}(mk-r)}\int_{a}^{t}\left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{mk-r}{k}-1}\frac{{}_{a}^{m}\mathbb{T}_{k}^{\mathbb{w},\vartheta}f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(\varsigma)-\vartheta(a)\ )^{1-\mathbb{w}}}$$

$${}_{a}^{\mathbb{C}}\mathfrak{D}_{k}^{r,\mathbb{w},\vartheta}f(t)={}_{a}^{mk-r}\mathcal{J}_{k}^{\mathbb{w},\vartheta}\left({}_{a}^{m}\mathbb{T}_{k}^{\mathbb{w},\vartheta}\ f(t)\right) \tag{49}$$

and

$$ {}^{\mathbb{C}}\mathfrak{D}_{b,k}^{\varkappa,\mathbb{w},\vartheta} f(t) = \frac{(-1)^m}{k\Gamma_k(mk-\varkappa)} \int_t^b \left(\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}} - (\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{\mathbb{w}} \right)^{\frac{mk-\varkappa}{k}-1} \frac{{}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta} f(\varsigma)\vartheta'(\varsigma)d\varsigma}{(\vartheta(b)-\vartheta(\varsigma))^{1-\mathbb{w}}} $$

$$ {}^{\mathbb{C}}\mathfrak{D}_{b,k}^{\varkappa,\mathbb{w},\vartheta} f(t) = {}^{mk-\varkappa}\mathscr{I}_{b,k}^{\mathbb{w},\vartheta} \left({}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta} f(t) \right) \tag{50} $$

respectively.

Proof. By using (47), we have

$$ \left[{}_{a}^{\mathbb{C}}\mathfrak{D}_{k}^{\varkappa,\mathbb{w},\vartheta} f(t) = {}_{a}^{\varkappa}\mathfrak{D}_{k}^{\mathbb{w},\vartheta} f(t) - \sum_{i=0}^{m-1} \frac{{}_{a}^{i}\mathbb{T}_{k}^{\mathbb{w},\vartheta} f(a)}{\mathbb{w}^{\frac{i}{k}}\, i!} (\vartheta(t)-\vartheta(a))^{\frac{\mathbb{w}i}{k}} \right](t) $$

$$ = {}_{a}^{\varkappa}\mathfrak{D}_{k}^{\mathbb{w},\vartheta} f(t) - \sum_{i=0}^{m-1} \frac{{}_{a}^{i}\mathbb{T}_{k}^{\mathbb{w},\vartheta} f(a)}{\mathbb{w}^{\frac{i}{k}}\, i!} {}_{a}^{\varkappa}\mathfrak{D}_{k}^{\mathbb{w},\vartheta} (\vartheta(t)-\vartheta(a))^{\frac{\mathbb{w}i}{k}}(t). $$

Using Lemma 2 and Theorem 2, we get

$$ = {}_{a}^{\varkappa}\mathfrak{D}_{k}^{\mathbb{w},\vartheta} f(t) - \sum_{i=0}^{m-1} \frac{{}_{a}^{i}\mathbb{T}_{k}^{\mathbb{w},\vartheta} f(a)}{\mathbb{w}^{\frac{i-\varkappa}{k}}\, i!} \frac{\Gamma_k(i+k)}{\Gamma_k(mk+i-\varkappa+k)} (\vartheta(t)-\vartheta(a))^{\mathbb{w}\left(\frac{i}{k}-\frac{\varkappa}{k}\right)} $$

$$ = {}_{a}^{\varkappa}\mathfrak{D}_{k}^{\mathbb{w},\vartheta} f(t) - \sum_{i=0}^{m-1} \frac{{}_{a}^{i}\mathbb{T}_{k}^{\mathbb{w},\vartheta} f(a)}{\mathbb{w}^{\frac{i-\varkappa}{k}}\, i!} \frac{\Gamma_k(i+k)}{\Gamma_k(mk+i-\varkappa+k)} (\vartheta(t)-\vartheta(a))^{\mathbb{w}\left(\frac{i}{k}-\frac{\varkappa}{k}\right)} $$

$$ = \frac{1}{k\Gamma_k(mk-\varkappa)} \int_a^t \left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}} - (\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}} \right)^{\frac{mk-\varkappa}{k}-1} \frac{{}_{a}^{m}\mathbb{T}_{k}^{\mathbb{w},\vartheta} f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(\varsigma)-\vartheta(a)\ )^{1-\mathbb{w}}} $$

$$ = {}_{a}^{mk-\varkappa}\mathscr{I}_{k}^{\mathbb{w},\vartheta} \left({}_{a}^{m}\mathbb{T}_{k}^{\mathbb{w},\vartheta} f(t) \right), $$

which proves the first part of Theorem 6.

Next, to prove (50), we employ (48)

$$ = {}^{\varkappa}\mathfrak{D}_{b,k}^{\mathbb{w},\vartheta} \left[f(t) - \sum_{i=0}^{m-1} \frac{(-1)^{ii}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta} f(b)}{\mathbb{w}^{\frac{i}{k}} i!} (\vartheta(b)-\vartheta(t))^{\frac{\mathbb{w}i}{k}} \right](t). $$

By applying Theorem 2 and Lemma 2, we obtain

$$
\begin{aligned}
{}^{\mathbb{C}}\mathfrak{D}_{b,k}^{\varkappa,\mathbb{w},\vartheta}\, f(t) &= {}^{\varkappa}\mathfrak{D}_{b,k}^{\mathbb{w},\vartheta}\, f(t) - \sum_{i=0}^{m-1} \frac{(-1)^{i^i}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta} f(b)}{\mathbb{w}^{\frac{i}{k}} i!}\, {}^{\varkappa}\mathfrak{D}_{b,k}^{\mathbb{w},\vartheta}(\vartheta(b)-\vartheta(t))^{\frac{\mathbb{w}i}{k}}(t) \\
&= {}^{\varkappa}\mathfrak{D}_{b,k}^{\mathbb{w},\vartheta}\, f(t) - \sum_{i=0}^{m-1} \frac{(-1)^{i^i}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta} f(b)}{\mathbb{w}^{\frac{i-\varkappa}{k}}\ i!}\, \frac{\Gamma_k(i+k)}{\Gamma_k(mk+i-\varkappa+k)}\,(\vartheta(b)-\vartheta(t))^{\mathbb{w}\left(\frac{i}{k}-\frac{\varkappa}{k}\right)} \\
&= {}^{\varkappa}\mathfrak{D}_{b,k}^{\mathbb{w},\vartheta}\, f(t) - \sum_{i=0}^{m-1} \frac{(-1)^{i^i}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta} f(b)}{\mathbb{w}^{\frac{i-\varkappa}{k}} i!}\ (\vartheta(b)-\vartheta(t))^{\mathbb{w}\left(\frac{i}{k}-\frac{\varkappa}{k}\right)} \\
{}^{\mathbb{C}}\mathfrak{D}_{b,k}^{\varkappa,\mathbb{w},\vartheta}\, f(t) &= \frac{(-1)^m}{k\Gamma_k(mk-\varkappa)} \int_t^b \left(\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{mk-\varkappa}{k}-1} \frac{m\mathbb{T}_{b,k}^{\mathbb{w},\vartheta} f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}} \\
&= {}^{mk-\varkappa}\mathfrak{I}_{b,k}^{\mathbb{w},\vartheta}\left({}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta} f(t)\right),
\end{aligned}
$$

which is the desired proof. □

Remark 5

We have

• If we take k=1, $\vartheta(t)=t$ and $\mathbb{w}=1$, then (49) reduces to (3).
• If we take $\mathbb{w}=1$, then (49) reduces to (9).
• If we take $\vartheta(t)=t$ and $\mathbb{w}\to 0$, then (49) reduces to (14).
• If we consider $\vartheta(t)=t$, then (49) becomes (21).
• If we take $\vartheta(t)=t$ and $\mathbb{w}=1$, then (50) reduces to (4).
• If we consider $\mathbb{w}=1$, then (50) becomes (10).
• If we consider $\vartheta(t)=t$ and $\mathbb{w}=1$ in (50), we get (8).
• If we take $\vartheta(t)=\ln t$ and $\mathbb{w}\to 0$, then (50) reduces to (15).
• If we consider $\vartheta(t)=t$, then (50) becomes (22).

Lemma 5 Let $\Re(\varkappa)>0$ with $m=[\Re(\varkappa)]+1, \Re(\varkappa)\notin \mathcal{N}$ and $\in \mathbb{C}[a,b]$. We have

$$
{}_{a}^{\varkappa-\ell k}\mathfrak{I}_{k}^{\mathbb{w},\vartheta}\, f(a)=0,
$$

and

$$
{}^{\varkappa-\ell k}\mathfrak{I}_{b,k}^{\mathbb{w},\vartheta} f(b)=0,
$$

for $\ell=0,\ 1,\ 2,\cdots,m-1$.

Proof. We have

$$\left|{}_{a}^{r-\ell k}\mathcal{J}_{k}^{\mathbb{w},\vartheta}\ f(t)\right| \leq \frac{\|f\|_{\mathbb{C}}}{\left|\begin{array}{l} k\Gamma_{k}(r-\ell k) \\ \end{array}\right|\left(\Re\left(\frac{r}{k}\right)-\ell\right)} \frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}((\Re(\frac{r}{k})-\ell))}}{\mathbb{w}^{\Re(\frac{r}{k})-\ell}}.$$

Replacing t by a

$$\left|{}_{a}^{r-\ell k}\mathcal{J}_{k}^{\mathbb{w},\vartheta}\ f(t)\right| \leq \frac{\|f\|_{\mathbb{C}}}{\left|\begin{array}{l} k\Gamma_{k}(r-\ell k) \\ \end{array}\right|\left(\Re\left(\frac{r}{k}\right)-\ell\right)} \frac{(\vartheta(a)-\vartheta(a))^{\mathbb{w}((\Re(\frac{r}{k})-\ell))}}{\mathbb{w}^{\Re(\frac{r}{k})-\ell}}$$

$$= 0.$$

To prove, the second part, we have

$$\left|{}^{r-\ell k}\mathcal{J}_{b,k}^{\mathbb{w},\vartheta}\ f(t)\right| \leq \frac{\|f\|_{\mathbb{C}}}{\left|\begin{array}{l} k\Gamma_{k}(r-\ell k) \\ \end{array}\right|\left(\Re\left(\frac{r}{k}\right)-\ell\right)} \frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}((\Re(\frac{r}{k})-\ell))}}{\mathbb{w}^{\Re(\frac{r}{k})-\ell}}.$$

Replacing t by b

$$\left|{}^{r-\ell k}\mathcal{J}_{b,k}^{\mathbb{w},\vartheta}\ f(t)\right| \leq \frac{\|f\|_{\mathbb{C}}}{\left|\begin{array}{l} k\Gamma_{k}(r-\ell k) \\ \end{array}\right|\left(\Re\left(\frac{r}{k}\right)-\ell\right)} \frac{(\vartheta(b)-\vartheta(b))^{\mathbb{w}((\Re(\frac{r}{k})-\ell))}}{\mathbb{w}^{\Re(\frac{r}{k})-\ell}}$$

$$= 0.$$

□

Lemma 6 Let $\Re(r) \geq 0$, $m = [\Re(r)]+1$, and ${}_{a}^{m}\mathbb{T}_{k}^{\mathbb{w},\vartheta} \in \mathbb{C}[a,b]$ $\left({}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta} \in \mathbb{C}[a,b]\right)$. We have

$${}_{a}^{\mathbb{C}}\mathfrak{D}_{k}^{r,\mathbb{w},\vartheta} f(a) = 0$$

and

$${}^{\mathbb{C}}\mathfrak{D}_{b,k}^{r,\mathbb{w},\vartheta} f(b) = 0.$$

Proof. We have

$$\left|{}_{a}^{\mathbb{C}}\mathfrak{D}_{k}^{\varkappa,\mathbb{w},\vartheta}\, f(t)\right| \leq \frac{\left\|{}_{a}^{m}\mathbb{T}_{k}^{\mathbb{w},\vartheta}\, f\right\|_{\mathbb{C}}}{\left|\begin{array}{l} k\Gamma_{k}(mk-\varkappa) \end{array}\right|\left(m-\Re\left(\frac{\varkappa}{k}\right)\right)} \frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}\left(m-\Re\left(\frac{\varkappa}{k}\right)\right)}}{\mathbb{w}^{m-\Re\left(\frac{\varkappa}{k}\right)}}.$$

Replacing t by a

$$\left|{}_{a}^{\mathbb{C}}\mathfrak{D}_{k}^{\varkappa,\mathbb{w},\vartheta}\, f(t)\right| \leq \frac{\left\|{}_{a}^{m}\mathbb{T}_{k}^{\mathbb{w},\vartheta}\, f\right\|_{\mathbb{C}}}{\left|\begin{array}{l} k\Gamma_{k}(mk-\varkappa) \end{array}\right|\left(m-\Re\left(\frac{\varkappa}{k}\right)\right)} \frac{(\vartheta(a)-\vartheta(a))^{\mathbb{w}\left(m-\Re\left(\frac{\varkappa}{k}\right)\right)}}{\mathbb{w}^{m-\Re\left(\frac{\varkappa}{k}\right)}}$$

$$= 0.$$

For the second part, we have

$$\left|{}^{\mathbb{C}}\mathfrak{D}_{b,k}^{\varkappa,\mathbb{w},\vartheta} f(t)\right| \leq \frac{\left\|{}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta} f\right\|_{\mathbb{C}}}{\left|\begin{array}{l} k\Gamma_{k}(mk-\varkappa) \end{array}\right|\left(m-\Re\left(\frac{\varkappa}{k}\right)\right)} \frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}\left(m-\Re\left(\frac{\varkappa}{k}\right)\right)}}{\mathbb{w}^{m-\Re\left(\frac{\varkappa}{k}\right)}}.$$

Replacing t by b

$$\left|{}^{\mathbb{C}}\mathfrak{D}_{b,k}^{\varkappa,\mathbb{w},\vartheta} f(t)\right| \leq \frac{\left\|{}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta} f\right\|_{\mathbb{C}}}{\left|\begin{array}{l} k\Gamma_{k}(mk-\varkappa) \end{array}\right|\left(m-\Re\left(\frac{\varkappa}{k}\right)\right)} \frac{(\vartheta(b)-\vartheta(b))^{\mathbb{w}\left(m-\Re\left(\frac{\varkappa}{k}\right)\right)}}{\mathbb{w}^{m-\Re\left(\frac{\varkappa}{k}\right)}}$$

$$= 0.$$

□

Theorem 7 Let $\Re(\varkappa) \geq 0,\ \Re(\varkappa^{*}) \leq 0,\ m-1 < \Re(\varkappa) \leq n$ and $m-1 < \Re(\varkappa^{*}) \leq m,\ f \in \mathbb{C}_{\mathbb{w},\vartheta,a,k}^{m+n}[a,b]$, $\left(f \in \mathbb{C}_{\mathbb{w},\vartheta,b,k}^{m+n}[a,b]\right)$. We have

$${}_{a}^{\mathbb{C}}\mathfrak{D}_{k}^{\varkappa,\mathbb{w},\vartheta}\left({}_{a}^{\mathbb{C}}\mathfrak{D}_{k}^{\varkappa^{*},\mathbb{w},\vartheta}\, f(t)\right) = {}_{a}^{\mathbb{C}}\mathfrak{D}_{k}^{(\varkappa+\varkappa^{*}),\mathbb{w},\vartheta}\, f(t), \tag{51}$$

and

$${}^{\mathbb{C}}\mathfrak{D}_{b,k}^{\varkappa,\mathbb{w},\vartheta}\left({}^{\mathbb{C}}\mathfrak{D}_{b,k}^{\varkappa^{*},\mathbb{w},\vartheta}\, f(t)\right) = {}^{\mathbb{C}}\mathfrak{D}_{b,k}^{(\varkappa+\varkappa^{*}),\mathbb{w},\vartheta}\, f(t). \tag{52}$$

Proof. Consider

$$
{}_{a}^{\mathbb{C}}\mathfrak{D}_{k}^{\mathfrak{r},\mathbb{w},\vartheta}\left({}_{a}^{\mathbb{C}}\mathfrak{D}_{k}^{\mathfrak{r}^{*},\mathbb{w},\vartheta}\, f(t)\right)
$$

$$
=\frac{1}{k\Gamma_{k}(mk-\mathfrak{r})k\Gamma_{k}(nk-\mathfrak{r}^{*})}\int_{a}^{t}\int_{a}^{\nu}\left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\nu)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{mk-\mathfrak{r}}{k}-1}
$$

$$
\left(\frac{(\vartheta(\nu)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{nk-\mathfrak{r}^{*}}{k}-1}\times\frac{{}_{a}^{n}\mathbb{T}_{k}^{\mathbb{w},\vartheta}f(\varsigma)\;\vartheta'(\varsigma)d\varsigma}{(\;\vartheta(\varsigma)-\vartheta(a)\;)^{1-\mathbb{w}}}\;\frac{{}_{a}^{m}\mathbb{T}_{k}^{\mathbb{w},\vartheta}\;\vartheta'(\nu)d\nu}{(\;\vartheta(\nu)-\vartheta(a)\;)^{1-\mathbb{w}}}
$$

$$
=\frac{1}{k\Gamma_{k}(mk-\mathfrak{r})k\Gamma_{k}(nk-\mathfrak{r}^{*})\mathbb{w}^{m-\frac{\mathfrak{r}}{k}+n-\frac{\mathfrak{r}^{*}}{k}-2}}\int_{a}^{t}\int_{\varsigma}^{t}((\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\nu)-\vartheta(a))^{\mathbb{w}})^{\frac{mk-\mathfrak{r}}{k}-1}
$$

$$
\times((\vartheta(\nu)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}})^{\frac{nk-\mathfrak{r}^{*}}{k}-1}\frac{{}_{a}^{n}\mathbb{T}_{k}^{\mathbb{w},\vartheta}f(\varsigma)\;\vartheta'(\varsigma)d\varsigma}{(\;\vartheta(\varsigma)-\vartheta(a)\;)^{1-\mathbb{w}}}\;\frac{{}_{a}^{m}\mathbb{T}_{k}^{\mathbb{w},\vartheta}\;\vartheta'(\nu)d\nu}{(\;\vartheta(\nu)-\vartheta(a)\;)^{1-\mathbb{w}}}.
$$

Substituting

$$
y=\frac{(\vartheta(\nu)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}},\;\; dy=\frac{\mathbb{w}(\vartheta(\nu)-\vartheta(a))^{\mathbb{w}-1}\;\vartheta'(\nu)d\nu}{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}
$$

and

$$
1-y=1-\frac{(\vartheta(\nu)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta\;(a))^{\mathbb{w}}}{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}=\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\nu)-\vartheta(a))^{\mathbb{w}}}{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}
$$

in above integral, we have

$$
{}_{a}^{\mathbb{C}}\mathfrak{D}_{k}^{\mathfrak{r},\mathbb{w},\vartheta}\left({}_{a}^{\mathbb{C}}\mathfrak{D}_{k}^{\mathfrak{r}^{*},\mathbb{w},\vartheta}\, f(t)\right)
$$

$$
=\frac{1}{k\Gamma_{k}(mk-\mathfrak{r})k\Gamma_{k}(nk-\mathfrak{r}^{*})\mathbb{w}^{\frac{mk+nk-\mathfrak{r}}{k}-\frac{\mathfrak{r}^{*}}{k}-1}}\int_{a}^{t}\left((\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}\right)^{\frac{mk+nk-\mathfrak{r}}{k}-\frac{\mathfrak{r}^{*}}{k}-1}
$$

$$
\frac{{}_{a}^{m+n}\mathbb{T}_{k}^{\mathbb{w},\vartheta}f(\varsigma)\;\vartheta'(\varsigma)d\varsigma}{(\;\vartheta(\varsigma)-\vartheta(a))^{1-\mathbb{w}}}\left(\int_{0}^{1}(1-y)^{\frac{mk-\mathfrak{r}}{k}-1}(y)^{\frac{nk-\mathfrak{r}^{*}}{k}-1}dy\right)
$$

$$
=\frac{1}{k(\Gamma_{k}(m-\mathfrak{r})\Gamma_{k}(n-\mathfrak{r}^{*}))}\int_{a}^{t}\left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{mk+nk-\mathfrak{r}}{k}-\frac{\mathfrak{r}^{*}}{k}-1}
$$

$$\frac{{}^{m+n}_{a}\mathbb{T}_k^{\mathbb{w},\vartheta} f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(\varsigma)-\vartheta(a))^{1-\mathbb{w}}} \times (\frac{1}{k}\int_0^1 (1-y\ )^{\frac{mk-r}{k}-1} y^{\frac{nk-r^*}{k}-1} dy).$$

By using k-Beta function, we get

$$ {}^{\mathbb{C}}_{a}\mathfrak{D}_k^{r,\mathbb{w},\vartheta}\left({}^{\mathbb{C}}_{a}\mathfrak{D}_k^{r^*,\mathbb{w},\vartheta}\ f(t)\right)$$

$$= \frac{1}{k\Gamma_k(mk+nk-r-r^*)}\ \int_a^t \left(\frac{(\vartheta(t)-\vartheta(a))^{\mathbb{w}}-(\vartheta(\varsigma)-\vartheta(a))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{mk+nk-r}{k}-\frac{r^*}{k}-1} \frac{{}^{m+n}_{a}\mathbb{T}_k^{\mathbb{w},\vartheta} f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(\varsigma)-\vartheta(a)\ )^{1-\mathbb{w}}}$$

$$= {}^{mk+nk-r-r^*}_{a}\mathcal{I}_k^{\mathbb{w},\vartheta}\ \left({}^{m+n}_{a}\mathbb{T}_k^{\mathbb{w},\vartheta}\ f(t)\right),$$

this completes the proof of the first part.

To prove ${}^{\mathbb{C}}\mathfrak{D}_{b,k}^{r,\mathbb{w},\vartheta}\left({}^{\mathbb{C}}\mathfrak{D}_{b,k}^{r^*,\mathbb{w},\vartheta}\ f(t)\right) = {}^{\mathbb{C}}\mathfrak{D}_{b,k}^{(r+r^*),\mathbb{w},\vartheta}\ f(t)$. As we know that

$${}^{\mathbb{C}}\mathfrak{D}_{b,k}^{r,\mathbb{w},\vartheta}\ f(t) = \frac{(-1)^m}{k\Gamma_k(mk-r)}\ \int_t^b \left(\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{mk-r}{k}-1} \frac{{}^{m}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta} f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}.$$

Therefore, we have

$${}^{\mathbb{C}}\mathfrak{D}_{b,k}^{r,\mathbb{w},\vartheta}\left({}^{\mathbb{C}}\mathfrak{D}_{b,k}^{r^*,\mathbb{w},\vartheta}\ f(t)\right)$$

$$= \frac{1}{k\Gamma_k(mk-r)k\Gamma_k(nk-r^*)}\ \int_t^b \int_v^b \left(\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(v))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{mk-r}{k}-1}$$

$$\left(\frac{(\vartheta(b)-\vartheta(v))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma)^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{nk-r^*}{k}-1} \times \frac{(-1)^{nn}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta} f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}\ \frac{(-1)^{mm}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}\ \vartheta'(v)dv}{(\ \vartheta(b)-\vartheta(v)\ )^{1-\mathbb{w}}}$$

$$= \frac{1}{k\Gamma_k(mk-r)k\Gamma_k(nk-r^*)\mathbb{w}^{m-\frac{r}{k}+n-\frac{r^*}{k}-2}} \int_t^b \int_\varsigma^b \left((\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(v))^{\mathbb{w}}\right)^{\frac{mk-r}{k}-1}$$

$$\times \left((\vartheta(b)-\vartheta(v))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}\right)^{\frac{nk-r^*}{k}-1} \frac{(-1)^{nn}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta} f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma))^{1-\mathbb{w}}}\ \frac{(-1)^{mm}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}\ \vartheta'(v)dv}{(\ \vartheta(b)-\vartheta(v))^{1-\mathbb{w}}}.$$

Substituting

$$y=\frac{(\vartheta(b)-\vartheta(\nu))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}},\ \ dy=\frac{\mathbb{w}(\vartheta(b)-\vartheta(\nu))^{\mathbb{w}-1}\ \vartheta'(\nu)d\nu}{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(a)-\vartheta(\varsigma))^{\mathbb{w}}}$$

and

$$1-y=1-\frac{(\vartheta(b)-\vartheta(\nu))^{\mathbb{w}}-(\vartheta(b)-\vartheta\ (\varsigma))^{\mathbb{w}}}{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}=\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\nu))^{\mathbb{w}}}{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}$$

in above integral, we get

$$\begin{aligned}
&{}^{\mathbb{C}}\mathfrak{D}_{b,k}^{\mathscr{r},\mathbb{w},\vartheta}\left({}^{\mathbb{C}}\mathfrak{D}_{b,k}^{\mathscr{r}^*,\mathbb{w},\vartheta}\ f(t)\right)\\
&=\frac{1}{k\Gamma_k(\mathscr{m}k-\mathscr{r})k\Gamma_k(\mathscr{n}k-\mathscr{r}^*)\,\mathbb{w}^{\mathscr{m}-\frac{\mathscr{r}}{k}+\mathscr{n}-\frac{\mathscr{r}^*}{k}-1}}\int_t^b\left((\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}\right)^{\mathscr{m}-\frac{\mathscr{r}}{k}+\mathscr{n}-\frac{\mathscr{r}^*}{k}-1}\\
&\quad\times\frac{(-1)^{\mathscr{m}+\mathscr{n}}\ {}^{\mathscr{m}+\mathscr{n}}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}\ f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}\int_0^1(1-y\ )^{\frac{\mathscr{m}k-\mathscr{r}}{k}-1}(y)^{\frac{\mathscr{n}k-\mathscr{r}^*}{k}-1}dy\\
&=\frac{1}{k\Gamma_k(\mathscr{m}k-\mathscr{r})\Gamma_k(\mathscr{n}k-\mathscr{r}^*))}\int_t^b\left(\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{\mathscr{m}k+\mathscr{n}k-\mathscr{r}}{k}-\frac{\mathscr{r}^*}{k}-1}\\
&\quad\frac{(-1)^{\mathscr{m}+\mathscr{n}}\ {}^{\mathscr{m}+\mathscr{n}}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}\times\frac{1}{k}\int_0^1(1-y\ )^{\frac{\mathscr{m}k-\mathscr{r}}{k}-1}(y)^{\frac{\mathscr{n}k-\mathscr{r}^*}{k}-1}dy.
\end{aligned}$$

By using k-Beta function, we get

$$\begin{aligned}
&=\frac{1}{k\Gamma_k(\mathscr{m}k+\mathscr{n}k-\mathscr{r}-\mathscr{r}^*)}\int_t^b\left(\frac{(\vartheta(b)-\vartheta(t))^{\mathbb{w}}-(\vartheta(b)-\vartheta(\varsigma))^{\mathbb{w}}}{\mathbb{w}}\right)^{\frac{\mathscr{m}k+\mathscr{n}k-\mathscr{r}}{k}-\frac{\mathscr{r}^*}{k}-1}\\
&\quad\frac{(-1)^{\mathscr{m}+\mathscr{n}}\ \ {}^{\mathscr{m}+\mathscr{n}}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}f(\varsigma)\ \vartheta'(\varsigma)d\varsigma}{(\ \vartheta(b)-\vartheta(\varsigma)\ )^{1-\mathbb{w}}}\\
&={}^{\mathscr{m}k+\mathscr{n}k-\mathscr{r}-\mathscr{r}^*}\mathscr{I}_{b,k}^{\mathbb{w},\vartheta}\ \left((-1)^{\mathscr{m}+\mathscr{n}}\ {}^{\mathscr{m}+\mathscr{n}}\mathbb{T}_{b,k}^{\mathbb{w},\vartheta}f(t)\right),
\end{aligned}$$

which completes the proof of the second part of Theorem. □

5. Discussion

Figures 1 and 2 present the left-handed (HD) visualizations of equation (30) corresponding to different values of k, and different fractional orders. Figures 3 and 4 present the right-handed (HD) visualizations of equation (31) for different values of k, and different fractional orders. Figures 5 and 6 present the left-handed (HD) visualizations of equation (32) corresponding to different values of k, and different fractional orders. Figures 7 and 8 present the left-handed (HD) visualizations of equation (33) for different values of k, and different fractional orders. In both the left-handed and right-handed cases, the behavior is the same. When the value of k is large, the result becomes very small, almost zero. When the value of k is small, the graph changes smoothly and steadily. Also, when the fractional order is small, the output stays near zero. But when we increase the fractional order, the output starts to go down.

6. Conclusion

In this work, distinct fractional derivatives and integrals that were generated by an iterative procedure using conformable integrals with respect to another function are presented and analyzed. We were able to generate both left and right fractional (k,ϑ)-conformable derivatives and infer both left and right fractional (k,ϑ)-conformable integrals based on the Riemann-Liouville and Caputo definitions. Rigorous mathematical analysis indicates that the features shown by these fractional operators are consistent with those of their classical counterparts. In an effort to clarify the connection between these unique fractional differential operators, we also suggested fractional derivatives for functions inside certain spaces.

We believe that these new fractional operators could provide new perspectives on optimum control problems, fractional variational problems, and complex system modeling. One major advantage of these operators is their dependence on two factors, including the (k,ϑ)-conformable operator parameter that improves the detection of memory effects. Overall, this work lays the groundwork for future research into these generalized fractional operators and how they could be used to comprehend complicated system dynamics.

Author contributions

The authors contributed equally and significantly in writing this paper.

Acknowledgment

The authors A. Aloqaily and N. Mlaiki would like to thank the Prince Sultan University for paying the publication fees for this work through TAS LAB.

Conflict of interest

The authors declare no competing financial interest.

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