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Research Article

The Exponential Integral Ei(x) and Numerical Calculation of Its Inverse

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Abstract: The exponential integral $y = \mathrm{Ei}(x) = \int (\frac{\exp(x)}{x})dx$ represents a transcendental function that can be applied as a growth function with interesting properties. Representing an infinite series, its numerical evaluation can be carried out with assymptotic expansion or summing up a finite number of elements with a sufficiently small remainder. Its inverse results in two subfunctions that are defined here as $x = \mathrm{Ei}^{(-1)}_{x<0}(y)$ for $x < 0$ (and always $y < 0$), and $x = \mathrm{Ei}^{(-1)}_{x>0}(y)$ for $x > 0$, where x is a real-valued variable. There is no direct formula to calculate numerical values for the two inverse subfunctions. In this article, we first show that for x being close to 0, the inverse function with sufficiently small error is represented by Lambert's W-function. Then we compare two methods. In the first method, two versions of a generic root-finding algorithm search for a number X that solves $Y = \mathrm{Ei}(X)$ for given number Y, distinguishing $x < 0$ and $x > 0$ in the case of $Y < 0$. The second method is a Taylor expansion with a 10th-degree polynomial. For the Y-value of interest, a corresponding reference interval is taken by the algorithm from a list that is here developed, and the associated X_0 together with Y enters the polynomial to calculate X directly. The chosen remainder or maximum error ε_Y, that results from cutting the infinite Taylor series, is taken into account in the interval limits around X_0. The relationship between ε_Y and corresponding ε_X (the resulting maximum error for X) is analyzed. Iterative formulas for the determination of optimal intervals as a function of X_0 are derived for given $|\varepsilon_Y|$ or relative error $\left|\frac{\varepsilon_Y}{Y}\right|$. The methods were tested with *Mathematica* notebooks that are provided as supporting information. The more flexible root-finding algorithm will generally be the best choice. However, the Taylor expansion can be implemented without any special functions, as long as the corresponding polynomial can be solved accurately.

***Keywords*:** accuracy, inverse function, Lambert's W-function, remainder of infinite series, root-finding algorithm, Taylor expansion, transcendental function

MSC: 33-02, 33-04, 33F05, 65R32

1. Introduction

The exponential-integral function $\mathrm{Ei}(x)$ is defined as the transcendental function

$$\mathrm{Ei}(x) = \int \frac{\exp(x)}{x} dx, \text{ with } x \neq 0. \tag{1}$$

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The division by x in the integrand of (1) causes the integral to be improper, with a singular point at $x = 0$. The classical form of presentation of $\mathrm{Ei}(x)$ as a definite-integral function (page 367 in [1], page 228 in [2]) is

$$\mathrm{Ei}(x) = \int_{\infty}^{-x} \frac{\exp(-t)}{t} dt = -\int_{-x}^{\infty} \frac{\exp(-t)}{t} dt = \int_{-\infty}^{x} \frac{\exp(t)}{t} dt,$$

where t is used as a dummy variable to pass through the singular value at $t = 0$. For $x > 0$, the Cauchy principal value has to be used conceptually, even though it turns out to have no direct implications for numerical calculations:

$$\mathrm{Ei}(x) = \begin{cases} \displaystyle\int_{-\infty}^{x} \frac{\exp(t)}{t}\, dt & \text{for } x < 0, \\ \\ \displaystyle\lim_{\varepsilon \to 0} \left\{ \int_{-\infty}^{-\varepsilon} \frac{\exp(t)}{t}\, dt + \int_{\varepsilon}^{x} \frac{\exp(t)}{t}\, dt \right\} & \text{for } x > 0. \end{cases}$$

The function was introduced with its characteristic symbol $\mathrm{Ei}(x)$ in 1846 by the German mathematician Oskar Schlömilch (1823-1901) (page 324 in [3]). Its origin from Schlömilch is reported also 24 years later in 1870 by Glaisher (page 567 in [1]). One can speculate that "Ei" stands for "exponential integral". Maybe surprisingly, Schlömilch developed the function from its relationship with the "logarithmic integral"

$$\mathrm{li}\,(\exp(x)) = \mathrm{Ei}(x), \text{ where } \mathrm{li}(x) = \int_{0}^{x} \frac{1}{\ln(t)} dt,$$

rather than from the exponential function, as in (1). Maybe a natural way to think of $\mathrm{Ei}(x)$ is as the solution to the differential equation

$$\frac{\partial y}{\partial x} = \frac{\exp\,(c_1 \cdot x)}{x} \;\Rightarrow\; y(x) = c_2 + \mathrm{Ei}\,(c_1 \cdot x)\,,$$

where c_1 and c_2 are constants. Taking the integral of $\frac{\exp(c_1 \cdot x)}{x}$ with respect to x leads to the series representation of $\mathrm{Ei}\,(c_1 \cdot x)$ (see (4) and (5)).

Exponential(-type) integrals are treated as a group of several interrelated functions, including the logarithmic integral $\mathrm{li}(x)$ (chapter 5 in [2], chapter 6 in [4, 5]). A relatively detailed exposition specifically about $\mathrm{Ei}(x)$ is found in chapter 37 in [6]. The function $\frac{\exp(x)}{x}$, with the to-be-integrated area is shown in Figure 1a for $-2 \le x \le 3$. Figure 1b shows the function $y = \mathrm{Ei}(x)$, and Figure 1c (in the corresponding interval) the inverse subfunctions ("branches")

$$x = \begin{cases} \mathrm{Ei}_{x<0}^{(-1)}(y) & \text{for } -\infty \le x < 0 \quad \text{and } -\infty \le y < 0\,, \\ \\ \mathrm{Ei}_{x>0}^{(-1)}(y) & \text{for } 0 < x \le \infty \quad \text{and } -\infty \le y \le \infty\,. \end{cases} \tag{2}$$

The first subfunction has the domain $x < 0$, and the second one the domain $x > 0$, as indicated by the subindex. The superscript (-1) indicates the inverse function of $\mathrm{Ei}(x)$ in (1). The distinction of the two inverse subfunctions is relevant for $y < 0$, and corresponds to the two cases

$$\mathrm{Ei}_{x<0}(x) = \gamma + \ln(-x) + \sum_{i=1}^{\infty} \frac{x^i}{i! \cdot i} \quad \text{for } x < 0,$$

$$\mathrm{Ei}_{x>0}(x) = \gamma + \ln(x) + \sum_{i=1}^{\infty} \frac{x^i}{i! \cdot i} \quad \text{for } x > 0.$$

The two subfunctions for $y < 0$ do not only have opposite signs, but are numerically different. For example, $\mathrm{Ei}_{x<0}^{(-1)}(-0.8) \approx -0.3471$, whereas $\mathrm{Ei}_{x>0}^{(-1)}(-0.8) \approx 0.2036$. The numbers given here with decimals are not exact, but are rounded to the last shown decimal (hence $\approx$).

The exponential-integral function $\mathrm{Ei}(x)$ is needed or applied in diverse fields, such as obtaining the general integral of motion of a meteoroid body in astrophysics [7], solving the Schrödinger equation of helium in chemistry [8], computing the nonrelativistic energies of the hydrogen molecule in quantum physics [9], or calculating the well function in hydrology [10]. Our motivation here is to work with a generalization of the exponential function for the relationship between a quantity $q(t)$ at time t that is the solution to the differential equation $\frac{\partial q}{\partial t} \cdot \frac{1}{q} = \exp(a + b \cdot q)$. Our generalized exponential function [11] can be expressed in the form

$$t(q) = \begin{cases} t_C + \dfrac{\mathrm{Ei}(-b \cdot q) - \mathrm{Ei}(-b \cdot q_C)}{\exp(a)} & \text{for } b \neq 0,\ q_C > 0, \\ \\ t_C + \dfrac{\ln(q) - \ln(q_C)}{\exp(a)} & \text{for } b = 0. \end{cases}$$

Solved for $q(t)$, as preferred for applications, it involves the inverse subfunctions from (2) with $x = -b \cdot q$ and $q > 0$:

$$q(t) = \begin{cases} -\dfrac{1}{b} \cdot \mathrm{Ei}_{x>0}^{(-1)}\left((t - t_C) \cdot \exp(a) + \mathrm{Ei}(-b \cdot q_C)\right) & \text{for } b < 0,\ q_C > 0, \\ \\ q_C \cdot \exp\left((t - t_C) \cdot \exp(a)\right) & \text{for } b = 0, \\ \\ -\dfrac{1}{b} \cdot \mathrm{Ei}_{x<0}^{(-1)}\left((t - t_C) \cdot \exp(a) + \mathrm{Ei}(-b \cdot q_C)\right) & \text{for } b > 0,\ q_C > 0,\ t < t_C - \dfrac{\mathrm{Ei}(-b \cdot q_C)}{\exp(a)}, \end{cases} \tag{3}$$

where q_C is a calibrating quantity at time t_C, and a and b are parameters. The three exponential-type growth forms correspond, respectively, to sigmoid ($b < 0$), exponential ($b = 0$), and explosive growth ($b > 0$). Sigmoid growth has an inflection point ("IP") at $q_{IP} = \frac{-1}{b}$, *id est*, the inflection point quantity depends on a single parameter. The term "explosive growth" refers to a relative growth rate that increases with time, so that time for increasing quantity becomes restricted to stay below $t_C - \frac{\mathrm{Ei}(-b \cdot q_C)}{\exp(a)}$. The restriction comes from $y = (t - t_C) \cdot \exp(a) + \mathrm{Ei}(-b \cdot q_C)$ having to be negative in $-\frac{1}{b} \cdot \mathrm{Ei}_{x<0}^{(-1)}(y)$, as one can see in Figure 1c. Equation (3) is of interest, for example, for statistical modeling of tree growth [12, 13].

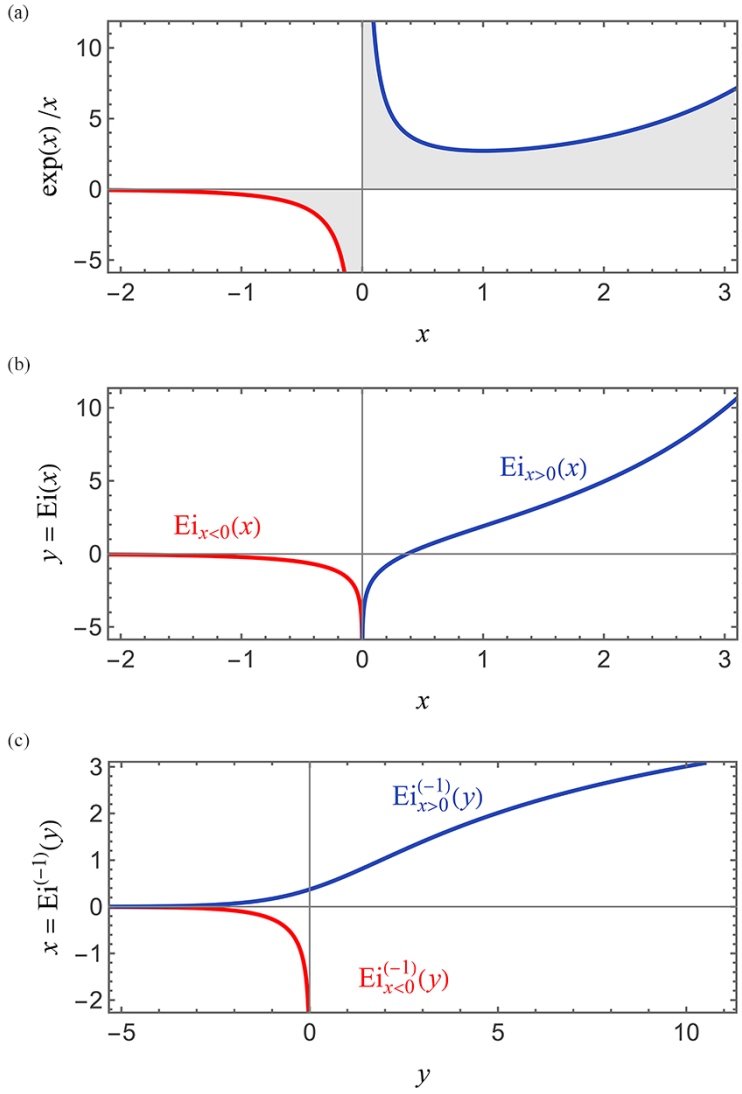


Figure 1. (a) The function $\frac{\exp(x)}{x}$ with the area to be integrated, shown for $-2 \le x \le 3$, with the singular point at $x = 0$. For $x > 0$, there is a minimum at $X = 1$, which corresponds to the inflection point of $\mathrm{Ei}(x)$; (b) The corresponding integral function $\mathrm{Ei}(x)$; (c) For the corresponding inverse function $\mathrm{Ei}^{(-1)}(y)$ and $y < 0$, two separate subfunctions have to be distinguished. The codomain of $\mathrm{Ei}(x)$ for $x < 0$ is $y < 0$, so that the argument y in $\mathrm{Ei}^{(-1)}_{x<0}(y)$ has to be negative

Apparently the only article that treats the estimation of the inverse $\mathrm{Ei}(x)$ is from Pecina in 1986 [14], who derived approximation formulas with series expansion in Chebyshev polynomials in five intervals from -40 to 10^{75}. For $-\infty < x \le -40$, the formula $y \approx \exp(x-\gamma)$ is recommended, which could be used as $x \approx \ln(-y) - \gamma$. For example, $\mathrm{Ei}(-40) \approx -1.04 \times 10^{-19}$, but $\ln\left(-\left(-1.04 \times 10^{-19}\right)\right) - \gamma \approx -43.7$, a rather large difference of $-40 - (-43.7) = 3.7$. Also, the two subfunctions for $x < 0$ are not distinguished in [14]. The involved coefficients are not numerically provided, so that the method could not be implemented and compared further here.

The function $\mathrm{Ei}(x)$ can be regarded as a special case of a Volterra-Fredholm integral equation, by separating the original integrand in (1) into the two functions $\exp(x)$ and $\frac{1}{x}$. Volterra-Fredholm integral equations are often solved by transformation into a system of linear equations and using standard linear algebra techniques (see [15, 16] and references therein for more details). We are, however, not aware of a way to use a Volterra-Fredholm integral equation for finding the numerical solution of the inverse of $\mathrm{Ei}(x)$.

Following, we will review and explore the possibilities for numerical calculation of $Y = \mathrm{Ei}(X)$ and especially its inverse $X = \mathrm{Ei}_{x<0}^{(-1)}(Y)$ or $X = \mathrm{Ei}_{x>0}^{(-1)}(Y)$. Capital-letters X and Y will be used for specific numerical values, whereas lower-case x and y refer to variables. We will go through sections about the series representation and functional relationships of $\mathrm{Ei}(x)$, numerical calculation of $\mathrm{Ei}(X)$, Lambert's W-function for the inverse close to $x = 0$, root-finding algorithms for calculating a numerical value of the inverse function, Taylor expansion of $\mathrm{Ei}(x)$, the propagation effect of the Y-error to become an X-error for the inverse function, the determination of a list of input values for Taylor expansion, and final conclusions.

2. Series representation and functional relationships of Ei(x)

The exponential integral from (1) can be converted into an infinite series, which is derived from the series representation of the exponential function (page 229 in [2], page 377 in [6], page 123 in [17]):

$$\exp(x) = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... = \sum_{i=0}^{\infty} \frac{x^i}{i!}. \tag{4}$$

Dividing both sides of the equation by x, integrating each term of the right-hand side, adding Euler(-Mascheroni)'s constant γ, and using i as an integer counting-variable up to number K, results in

$$\begin{aligned} \mathrm{Ei}(x) &= \gamma + \ln|x| + x + \frac{x^2}{2! \cdot 2} + \frac{x^3}{3! \cdot 3} + \ldots \\ &= \gamma + \ln|x| + \sum_{i=1}^{\infty} \frac{x^i}{i! \cdot i} \\ &= \gamma + \ln|x| + \sum_{i=1}^{K} \frac{x^i}{i! \cdot i} + R_{Ei}(x, K) \end{aligned} \tag{5}$$

with $x \neq 0$, and where $R_{Ei}(x, K)$ is analyzed in the next section (see (10)). If one admits calculation with complex numbers (z), then $\ln|x|$ can be replaced with $\frac{\left(\ln(z) - \ln\left(\frac{1}{z}\right)\right)}{2}$ [5]. Euler's constant (page 232 in [17], [18, 29], page 243 in [20]) is

$$\gamma = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - \ln(n)\right) \approx 0.577216.$$

There seems to be no explicit explanation in the literature for why γ is added, but it goes back to the function's origin, where it is already included (page 324 in [3]). Glaisher mentioned 153 years ago its "connection with the Gamma Function, and its occurrence in the expansions of the … exponential-integral" (page 25 in [21]). The addition of γ cancels out in the definite integral $\int_{X_1}^{X_2} \left(\frac{\exp(x)}{x}\right) dx$. An intuitive "advantage" is that for x going to minus infinity, $\mathrm{Ei}(x)$ becomes zero, rather than $-\gamma$, so that adding γ can be considered a more natural calibration of the function's horizontal position. Furthermore, the relationship of $\mathrm{Ei}(x)$ with the incomplete gamma function becomes directly

$$\begin{aligned} \mathrm{Ei}_{x<0}(x) &= -\Gamma(0,-x), \\ \mathrm{Ei}_{x>0}(x) &= -\ln(-1) - \Gamma(0,-x), \end{aligned} \tag{6}$$

with complex calculations resulting in real-valued numbers for $x > 0$ (page 155 in [11]).

Often one has $c \cdot x$, rather than x, where c is a constant. In that case, (1) converts to

$$\mathrm{Ei}(c \cdot x) = \int \frac{\exp(c \cdot x)}{x} dx,$$

without c appearing in the denominator of the integrand. To see the reason for this, one integrates the series representation of $\frac{\exp(c \cdot x)}{x} = \sum_{i=0}^{\infty} \frac{(c \cdot x)^i}{(i! \cdot x)}$ with respect to x.

Finally, the relationship of $\mathrm{Ei}(x)$ with the other exponential integral

$$E_n(x) = \int_1^{\infty} \frac{\exp(-x \cdot t)}{t^n} dt$$

should be mentioned. With $n = 1$, it is shown on pages 143–144 in [22] that

$$\begin{aligned} \mathrm{Ei}(-x) &= -E_1(x) \quad \text{for} \quad x > 0, \\ \mathrm{Ei}(x) &= -\frac{E_1(x \cdot \exp(\pi \cdot I)) + E_1(x \cdot \exp(-\pi \cdot I))}{2} \quad \text{for} \quad x < 0, \end{aligned} \tag{7}$$

where I refers to the imaginary part of a complex number. Whereas $E_n(x)$ is sometimes called "generalized exponential integral" (page 87 in [23]), (7) shows that it is related, but not truly a generalization of $\mathrm{Ei}(x)$, like $E_n(x)$ is a generalization of $E_1(x)$.

3. Numerical calculation of Ei(X)

The algorithm recommended on page 169 in [22], and more recently on page 269 in [24], combine the series presentation from (5) of $\mathrm{Ei}(x)$ for small $|X|$ and $X \neq 0$ with the asymptotic expansion

$$\mathrm{Ei}(X) \sim \frac{\exp(X)}{X} \cdot \left(1 + \frac{1!}{X} + \frac{2!}{X^2} + \ldots + \frac{k_{AE}!}{X^k}\right) + R_{AE}(k_{AE}, X) \tag{8}$$

for large X, where $R_{AE}(k_{AE}, X)$ is the remainder term after truncation of the series ("AE" standing for asymptotic expansion). Note that the formula includes the addition of γ in the result, without γ appearing in (8). An approximate lower limit for $|X|$ with the asymptotic expansion from (8) is found with

$$|X| > |\ln|RRE|| \Rightarrow |RRE| < \exp(|X|),$$

where *RRE* is the "Required Relative Error" (page 296 in [24]). For example, if a relative error of less than 0.001 is desired, asymptotic expansion can be used for approximately $|X| > \left|\ln\left(10^{-6}\right)\right| \approx 14$. For a relative error of less than 10^{-16}, one gets $|X| > \left|\ln\left(10^{-16}\right)\right| \approx 37$.

The derivation of (8) is explained on pages 32–33 in [25], with complex numbers. A frequently applied rule for divergent series, such as (8), is to stop adding terms at the smallest term of the series (page 524 in [26], [27]). The smallest term in (8) is at $k \approx |x|$ ([27], page 33 in [25]). In practice, one uses

$$k_{AE} = \text{Int}|X|,$$

id est, the integer part of the absolute value of X (page 658 in [27]). This rule is motivated by using Stirling's approximation for factorials on page 38 in [28].

To show here that indeed $k_{AE} \approx |x|$ for (4), one can analyze the first derivative of $\frac{i!}{x^i}$ with respect to i, which is $x^{-i} \cdot \left(-i! \cdot \ln(x) + i! \cdot \left(-\gamma + \sum_{j=1}^{i-1} \frac{1}{j}\right)\right)$ for integer i, where the term within the inside parentheses represents the Digamma (Psi) function $\Psi(i)$ for integer values (page 258 in [29]). Setting the derivative equal to zero, to find a minimum, and simplifying yields $\ln(x) = -\gamma + H(i) - \frac{1}{i+1}$, where $H(i)$ is the harmonic number of i. Since $H(i) = \ln(i) + \gamma + \frac{1}{2 \cdot i} - \varepsilon(i)$ according to page 866 in [30], with $0 \leq \varepsilon(i) \leq \frac{1}{8 \cdot i^2}$ the result is $\ln(x) = \ln(i) + \frac{1}{2 \cdot i} - \frac{1}{i+1} - \varepsilon(i)$. With $\frac{1}{2 \cdot i} - \frac{1}{i+1} - \varepsilon(i)$ being ever smaller for larger i, this simplifies to $k_{AE} \approx x$.

A bound formula for the corresponding absolute value of the remainder in (8) only for negative X is provided on page 33 in [25]:

$$|R_{AE}(k_{AE}, X)| \leq \frac{\exp(X)}{|X|} \cdot \frac{(k_{AE}+1)!}{|X|^{k_{AE}+1}} = \frac{\exp(X) \cdot (k_{AE}+1)!}{|X|^{k_{AE}+2}} \text{ for } X < 0.$$

We could not confirm numerically the second formula for the remainder provided in [25, equation (2.10)]. Consequently, for (8) and $X > 0$ one has a stopping rule, but no formula to calculate the remainder $|R_{AE}(k_{AE}, X)|$.

On the other hand, when using the normal series representation from (5) for smaller x, a standard practice is to stop adding terms after the last included terms becomes smaller in magnitude than some small ε times the magnitude of the already accumulated sum, but that procedure is "not mathematically rigorous in all cases" (page 210 in [24]. Press et al. state that for monotonic asymptotic series "there is nothing better than direct summation without acceleration" (page 216 in [24]). A method with excact accuracy control is to use the series representation from (5)

$$\text{Ei}(X) \approx \gamma + \ln|X| + \sum_{i=1}^{K} \frac{X^i}{i! \cdot i}, \tag{9}$$

with the rigorous bound formula for the remainder

$$R_{Ei}(X, K) < \frac{X^{K+1}}{K! \cdot (K+1)^2 \cdot \left(1 - \frac{X}{K+2}\right)} \text{ for } K > |X| - 2, \tag{10}$$

which applies to both positive and negative x. We use capital K to distinguish this k for the number of terms from k_{AE} for the asymptotic expansion, and later k for the Taylor expansion. The remainder $R_{Ei}(X, K)$ can be positive or negative. Inequality (10) was contributed by Rafael del Río on page 224 in [10], and its derivation is brief. Using lower-case x as a continous real-valued variable, the remainder from (5) can be expressed as

$$R_{Ei}(x,K) = \frac{x^{K+1}}{(K+1)!\cdot(K+1)} + \frac{x^{K+2}}{(K+2)!\cdot(K+2)} + \ldots + \frac{x^{K+1+i}}{(K+1+i)!\cdot(K+1+i)} + \ldots$$

Next, this equation is transformed into the following inequality:

$$R_{Ei}(x,K) < \frac{x^{K+1}}{(K+1)!\cdot(K+1)} + \frac{x^{K+2}}{(K+1)!\cdot(K+2)} + \ldots + \frac{x^{K+1+i}}{(K+1)!\cdot(K+1+i)} + \ldots$$

$$\Rightarrow R_{Ei}(x,K) < \frac{x^{K+1}}{(K+1)!\cdot(K+1)} \cdot \left(1 + \frac{x}{K+2} + \frac{x^2}{(K+3)\cdot(K+2)} + \ldots + \frac{x^i}{\frac{(K+1+i)!}{(K+1)!}} + \ldots\right).$$

In this arrangement, one can apply the following formula of a geometric series to the term in parenthesis:

$$\sum_{i=1}^{\infty}\left(\frac{x}{K+2}\right)^i = \frac{1}{1-\frac{x}{K+2}} \qquad \text{for } \frac{|x|}{K+2} < 1.$$

The latter condition is equivalent to $K > |x| - 2$ polynomial terms, and one gets

$$R_{Ei}(x,K) < \frac{\frac{x^{K+1}}{(K+1)!\cdot(K+1)}}{1-\frac{x}{K+2}}.$$

Rearranging results in (10).

One can also search the number of terms for fulfilling the absolute bound of the error relative to the calculated $\widehat{Y}$. Dividing the right-hand side of (10) by the series representation from (9) without the remainder $R_{Ei}(x,K)$ results in

$$\left|\frac{R_{Ei}(X,K)}{\widehat{Y}}\right| < \left|X^{-K-1}\cdot\left(\gamma + \ln|X| + \sum_{i=1}^{K}\frac{X^i}{i!\cdot i}\right)\cdot K!\cdot(K+1)^2\cdot\left(1-\frac{X}{K+2}\right)\right|. \tag{11}$$

For numerical evaluation, in each iterative step i and K are increased by 1, adding another term $\frac{X^i}{(i!\cdot i)}$ in (9), until the desired value of $\left|\frac{R_{Ei}(X,K)}{\widehat{Y}}\right|$ is reached.

The three methods to calculate $\mathrm{Ei}(X)$ with (8), with (9) and (10), or with (9) and (11) were implemented in the *Mathematica* notebooks "Ei(X) with asymptotic expansion" and "Ei(X) with series summation", respectively, which are available at https://doi.org/10.17632/9f8dmwyspb.5 (see Appendix). The notebook "Ei(X) with series summation" includes both forms of calculating $\mathrm{Ei}(X)$ with (9).

How does applying the series of (9) compare with the asymptotic expansion in (8) for large x? As an arbitrary example, consider $\mathrm{Ei}(40.3) \approx 8.09\cdot 10^{15}$. With (8), $k_{AE} = Int|40.3| = 40$ terms, the Y-error results to be 0.43, and the relative Y-error is $5.0\cdot 10^{-17}$. With (9) and (10), one needs 102 terms to get an error of 0.36, and the relative Y-error is $4.4\cdot 10^{-17}$. Consequently, one needs 2.5 times the number of terms with the series of $\mathrm{Ei}(x)$ from (9), than with the asymptotic expansion from (8), an increase of summands that is not problematic nowadays (the result is similar for negative -40.3). An advantage of using (9) with the remainder from (10), however, is that the error can be chosen freely, and the

number of terms adapted correspondingly. For example, if one sets the error bound $\leq 10^{-16}$ for calculating $\mathrm{Ei}(40.3)$, then the result is $8.0904438608288838176633857975568 \cdot 10^{15}$ with $K = 135$ in (9), a relatively modest increase from 102 terms.

With a single polynomial term $K = 1$ in the series of $\mathrm{Ei}(x)$, (5) becomes

$$\mathrm{Ei}(x) = \gamma + \ln|x| + x + R_{Ei}(x,1), \tag{12}$$

and the remainder in (10) simplifies to

$$R_{Ei}(x,1) < \frac{3 \cdot x^2}{12 - 4 \cdot x} \text{ for } |x| < 3. \tag{13}$$

Since (12) can be arranged as $\mathrm{Ei}(x) - R_{Ei}(x,1) = \gamma + \ln|x| + x$, $\mathrm{Ei}(x) - R_{Ei}(x,1)$ can be interpreted as $y - \varepsilon_Y$, where ε_Y ("epsilon") is the error of y from cutting the series. With $R_{Ei}(x,1) = \varepsilon_Y$ in (13), and solving for x, one gets the following inequality:

$$-\frac{2}{3} \cdot \left(\varepsilon_Y + \sqrt{\varepsilon_Y \cdot (9 + \varepsilon_Y)}\right) < x < \frac{2}{3} \cdot \left(\varepsilon_Y + \sqrt{\varepsilon_Y \cdot (9 + \varepsilon_Y)}\right). \tag{14}$$

For example, with $\varepsilon_Y = 10^{-7}$, (14) becomes $-6.325222 \cdot 10^{-4} < x < 6.325222 \cdot 10^{-4}$. Consequently, for numerical evaluation sufficiently close to $x = 0$, equation (12) can be used. For example, $\mathrm{Ei}\left(-10^{-4}\right) \approx \gamma + \ln\left|-10^{-4}\right| - 10^{-4} \approx -8.633225$, and $\mathrm{Ei}\left(10^{-4}\right) \approx \gamma + \ln\left|10^{-4}\right| + 10^{-4} \approx -8.633025$.

Finally, the relationships with the incomplete gamma function $\Gamma(a,x)$ in (6), or also with the exponential integral $E_n(x)$ in (7) could be used to determine $\mathrm{Ei}(x)$. However, $\Gamma(a,x)$ and $E_n(x)$ represent more general infinite series, so that there is no simplification for algorithms to calculate $\Gamma(a,x)$ and $E_n(x)$ (see pages 259–263 and 266–268 in [24]).

4. Lambert's W-function for $x = \mathrm{Ei}^{(-1)}(y)$ close to $x = 0$

We are now turning to the inverse from (2). There is one situation, where the exposition in the last section leads us to a formula for the inverse. Converting (12) to $y = \gamma + \ln|x| + x$, and solving for x yields

$$x = \begin{cases} W\left(-\exp(-\gamma + y)\right) & \text{for } x < 0, \\ \\ W\left(\exp(-\gamma + y)\right) & \text{for } x > 0, \end{cases} \tag{15}$$

where $W(z)$ refers to Lambert's W-function, which is defined as the solution to $W(z) \cdot \exp(W(z)) = z$ [31–33]. The numerical evaluation is treated in [31]. A real solution exists always for $z > 0$. For $z \leq 0$, the solution of the so-called principal branch has to be taken, and the condition $z \geq -\exp(-1)$ has to be met to avoid complex numbers as results (page 1111 in [33]). In the case of $\mathrm{Ei}(x)$ and $x < 0$, with $z = -\exp(-\gamma + y)$, the condition converts to

$$-\exp(-\gamma + y) \geq -\exp(-1) \;\Rightarrow\; y \leq \gamma - 1 \approx -0.423,$$

which is generally not restrictive for x close to zero (see Figure 1b,c). The desired accuracy for x determines the limits for x in (14), up to which one can use (15).

Since the input in the inverse is y, we convert (14) to

$$y < \mathrm{Ei}\left(-\frac{2}{3}\cdot\left(\varepsilon_Y + \sqrt{\varepsilon_Y\cdot(9+\varepsilon_Y)}\right)\right) \quad \text{for } x<0,$$

$$y < \mathrm{Ei}\left(\frac{2}{3}\cdot\left(\varepsilon_Y + \sqrt{\varepsilon_Y\cdot(9+\varepsilon_Y)}\right)\right) \quad \text{for } x>0. \tag{16}$$

to calculate the limits, for which (15) can be used with given ε_Y. For example, with $\varepsilon_Y = 10^{-7}$, (16) becomes $y_{x<0} < -6.789212$ and $y_{x>0} < -6.787947$. In *Mathematica*, Lambert's W-function is implemented as the function ProductLog.

5. Root-finding algorithms for calculating a numerical value of the inverse function $x = \mathrm{Ei}^{(-1)}(y)$

Except for x being close to zero, the exposition of the numerical calculation of $\mathrm{Ei}(X)$ in Section 3 does not lead to a method for calculating inverse values $X = \mathrm{Ei}^{(-1)}(Y)$. One method that works generically, is a root-finding algorithm, where X is being searched, such that $\mathrm{Ei}(X) - Y = 0$ (see chapter 9 in [24]). The two branches for $x<0$ and $x>0$ have to be treated separately (Figure 1b). For example, *Mathematica*'s function FindRoot can be used, specifying that the search should be either between 0 and a large positive value, or between 0 and a large negative value (see also the *Mathematica* notebook "Inverse Ei(Y) with FindRoot" at https://doi.org/10.17632/9f8dmwyspb.5).

Here, we programmed alternatively our own efficient trial-and-error algorithm, where X_{Try} with some starting value is iteratively adjusted in a way that $\mathrm{Ei}(X_{Try})$ approaches the desired Y with ever smaller error. To calculate repeatedly $\mathrm{Ei}(x)$, *Mathematica*'s function ExpIntegralEi was used. Here is an outline of the algorithm (see also the *Mathematica* notebook "Inverse Ei(Y) with converging search" at https://doi.org/10.17632/9f8dmwyspb.5):

1) The user provides Y, and decides if the inverse is to be determined for $x<0$ (left branch in Figure 1b) or $x>0$ (right branch in Figure 1b). For $x<0$, Y has to be negative.

2) Following the recommendation by Ricker & von Rosen (page 155 in [11]), the starting value for $x>0$ and $Y \le 1$ is $X_{Start} = 10^Y$, whereas for $Y>1$ it is $X_{Start} = \ln(Y)$. The starting value for $x<0$ and $Y \le -0.5$ is $X_{Start} = -\exp(Y)$, whereas for $-0.5 < Y < 0$ it is $X_{Start} = \ln(-Y)$.

3) If $\mathrm{Ei}(X_{Start}) < Y$, then increase repeatedly X_{Start} by 1, until $\mathrm{Ei}(X_{Start} + \mathrm{sum}) > Y$. If $\mathrm{Ei}(X_{Start}) > Y$, then decrease repeatedly X_{Start} by 1, until $\mathrm{Ei}(X_{Start} - \mathrm{sum}) < Y$. In either case register the new $X_{New} = X_{Start} + \mathrm{sum}$.

4) Now go in the opposite direction, but decrease or increase in steps of only 0.1, until $\mathrm{Ei}(X_{New} - \mathrm{sum}) < Y$ or $\mathrm{Ei}(X_{New} + \mathrm{sum}) > Y$. Register again $X_{Next} = X_{New} + \mathrm{sum}$.

5) Repeat the last step, but changing again the direction and using 0.01. In each additional iteration, the step direction is switched, and the step size divided by 10. The number of iterations depends on the desired accuracy for X (including 16 or more, if the software can work with over 15 decimals). The last X_{Next} is the solution.

Note that the two branches for $x<0$ and $x>0$ are also distinguished in (6) as complex versus real-valued numbers. For example, for $Y = -0.8$ one gets:

$$-\ln(-1) - \Gamma(0, -(0.2036)) - (-0.8) \approx 0.00007, \text{ and } -\Gamma(0, -X) - (-0.3471) - (-0.8) \approx -0.00009.$$

However, if one exchanges the resulting X-values 0.2036 and −0.3471 in the formulas, they are ”eliminated” as complex numbers:

$$-\ln(-1)-\Gamma(0,-(-0.3471))-(-0.8)\approx 0.00009-\pi\cdot I,\ \text{and}\ -\Gamma(0,-X)-(0.2036)-(-0.8)\approx 0.00007+\pi\cdot I,$$

where I is the imaginary unit of a complex number, and $\pi\approx 3.14159$. This variation of the FindRoot algorithm is included in the *Mathematica* notebook “Inverse Ei(Y) with FindRoot” (https://doi.org/10.17632/9f8dmwyspb.5).

Table 1 shows the results from testing the described converging-search algorithms and the mentioned FindRoot algorithms (the version with the gamma function performs identically to the version with the exponential integral). We established four Y-ranges for numerical comparisons, with the specifications given in the first three columns. The four ranges for Y in the second column were each divided on a logarithmic scale into equidistant intervals to get 1000 Y-values. For each value, our corresponding function xNegConvergingSearchInvEi, xPosConvergingSearchInvEi, xNegFindRootInvEi, or xPosFindRootInvEi was used to calculate the corresponding X. In each case, the absolute error was calculated as

$$|\varepsilon_Y|=\left|Y-Ei\left(\widehat{X}\right)\right|, \tag{17}$$

where $\widehat{X}$ is the calculated value for X, *id est*, $\widehat{X}=X-\varepsilon_X$, with ε_X being the corresponding error for X. For example, for $Y=-10^{-7}$ and $x<0$, the result was $\widehat{X}=-13.45156569744$ with 13 significant digits, and

$$\begin{aligned}\varepsilon_Y&=\left|-10^{-7}-Ei(-13.45156569744)\right|\\&=\left|-1\cdot 10^{-7}-\left(1+10^{-14}\right)\cdot 10^{-7}\right|\\&=1\cdot 10^{-21}.\end{aligned}$$

Table 1. Testing of the numerical calculation of the inverse of $\mathrm{Ei}(x)$ with a root-finding algorithms

Sub-function	Range of $y=\mathrm{Ei}(x)$	Range of x	Converging search		Mathematica’s FindRoot with ExpIntegralEi	
			Range and mean of $\lvert\varepsilon_Y\rvert$ for 1000 calculated X-values	Mean time in seconds per value	Range and mean of $\lvert\varepsilon_Y\rvert$ for 1000 calculated X-values	Mean time in seconds per value
$x<0$	$-10^{-7}\geq y\geq -1$	$-13.45156569744346\leq x\leq -0.2647370104515431$	$1\cdot 10^{-21}$ to $3\cdot 10^{-13}$, $4\cdot 10^{-14}$	0.0024	≈ 0 to $3\cdot 10^{-15}$, $1\cdot 10^{-17}$	0.0009
	$-1\geq y\geq -15$	$-0.2647370104515431\leq x\leq -1.717517883148904\cdot 10^{-7}$	≈ 0 to $3\cdot 10^{-10}$, $6\cdot 10^{-12}$	0.0011	≈ 0 to $1\cdot 10^{-14}$, $7\cdot 10^{-15}$	0.0005
$x>0$	$-15\leq y\leq 10$	$1.717517293175571\cdot 10^{-7}\leq x\leq 3.009850414759061$	≈ 0 to $3\cdot 10^{-11}$, $2\cdot 10^{-13}$	0.0018	≈ 0 to $9\cdot 10^{-9}$, $2\cdot 10^{-11}$	0.0006
	$10\leq y\leq 10^7$	$3.009850414759061\leq x\leq 19.00521318217205$	$1\cdot 10^{-13}$ to $1\cdot 10^{-7}$, $9\cdot 10^{-9}$	0.0034	≈ 0 to $3\cdot 10^{-8}$, $2\cdot 10^{-10}$	0.0011

Numbers for the range of y (2nd column) are exact; the corresponding numbers for the range of x (3rd column) are rounded to 16 significant digits. Errors in columns 5 to 7 are rounded to one significant digit, and time values to four decimals. The time was measured with an Intel Core i7-12700H processor

In case of the result $\varepsilon_Y \approx 0$, there was no difference between $\widehat{X}$ and X with 16 significant digits. Table 1 presents the smallest and largest absolute error, as well as the mean value for the 1000 values of ε_Y. Furthermore, the time of the process to calculate the 1000 values and its errors was measured with the *Mathematica* function Timing (on a computer with an Intel Core i7-12700H processor). The mean time for the 1000 iterations is reported. The calculations were carried out with 16 significant digits for all numbers (machine precision was $\frac{53}{\log_2(10)} \approx 15.95$, corresponding to 53 binary digits of precision).

The results in Table 1 show that the two algorithms are fairly similar in efficiency and outcome. The absolute error of the converging search is higher on average by a factor over 1000, and the time is 2.2 times as high. One could increase the precision for each method in *Mathematica*, but the calculation time increases as well. The comparison here is not completely fair, because *Mathematica*'s FindRoot function is programmed at a lower level. Doing the same with the converging-search algorithm (such as programming it in C++) would speed it up a lot.

6. Taylor expansion of Ei(x)

With Taylor expansion, one can derive a polynomial that solves numerically the inverses $X = \mathrm{Ei}_{x<0}^{(-1)}(Y)$ and $X = \mathrm{Ei}_{x>0}^{(-1)}(Y)$ for desired error $|\varepsilon_Y|$ or relative error $\left|\frac{\varepsilon_Y}{Y}\right|$, within a predefined overall range for Y. As a guiding example, let us take $Y = -0.31$, with $|\varepsilon_Y| = 10^{-7}$. Whereas the fixed ε_Y and restricted overall range (here $-15 \le y < 0$ for $x < 0$, and $-15 \le y < 10^7$ for $x > 0$) are limitations of this method, the advantage is that no special function is required, when the basic parameters are provided in a list. Only software for solving polynomials is required.

According to the Abel–Ruffini theorem [34], unfortunately there are no explicit formulas for the roots of polynomial equations of degree five or higher, in order to solve them directly. Nevertheless, accurate root finding for polynomials of different degrees is nowadays well-developed, despite the existence of ill-conditioned polynomials, where the result can present large numerical errors (chapter 9.5 in [24], [35], Appendix A1 in [36]). The polynomial to be solved here for X is

$$b_0 + b_1 \cdot X^1 + b_2 \cdot X^2 + \ldots + b_k \cdot X^k + R_T(X, X_0) = Y, \tag{18}$$

where k is the degree of the polynomial, and $R_T(X, X_0)$ stands for the remainder of the Taylor expansion from (21) that is not taken into account. Rather than calling subsequently the absolute error $R_T(X, X_0)$, we will call it ε_Y, so that (18) can be expressed as

$$(b_0 - Y) + b_1 \cdot X^1 + b_2 \cdot X^2 + \ldots + b_k \cdot X^k = -\varepsilon_Y \approx 0. \tag{19}$$

For our example, let us take $k = 10$, and the polynomial to be solved becomes

$$(b_0 + 0.31) + b_1 \cdot X + b_2 \cdot X^2 + \ldots + b_k \cdot X^{10} = 0,$$

where the formulas for b_0 to b_{10} are derived next, and are presented in (32).

Following page 86 in [17], page 92 in [20], and page 228 in [24], the Taylor-series expansion with its multiple derivatives is

$$\begin{aligned}\mathrm{Ei}\,(x,X_0,k) = &\mathrm{Ei}\,(X_0) + \frac{(x-X_0)^1}{1!}\cdot\frac{\partial\,\mathrm{Ei}\,(X_0)}{\partial X_0}\\ &+\frac{(x-X_0)^2}{2!}\cdot\frac{\partial^2\,\mathrm{Ei}\,(X_0)}{\partial X_0^2}+\ldots\\ &+\frac{(x-X_0)^k}{k!}\cdot\frac{\partial^k\,\mathrm{Ei}\,(X_0)}{\partial X_0^k}+R_T\,(x,X_0,k)\,,\end{aligned}\tag{20}$$

where X_0 is a fixed value, x the variable for X near X_0, k the number of terms that are used, and $R_T\,(x,X_0,k)$ the remainder from truncating the series. For any X_0 and k, (20) without the remainder represents a polynomial of x with degree k, which can be formed properly by expanding all terms $x-X_0$ in parentheses, and subsequently summing up terms of the same power. Note that the absolute error ε_Y does not appear in (22), but enters in (23) and subsequently in Section 8 to determine the X_{0i} values with their intervals.

For the Taylor series of (20), there are alternative expressions for the remainder $R_T\,(x,X_0,k)$ from truncation (page 87 in [17], page 94 in [20]). Here, Cauchy's form will be used, adapted from page 272 in [37]:

$$R_T\,(x,X_0,k) = \frac{\partial^{k+1}\,\mathrm{Ei}\,(X_0+\theta\cdot(x-X_0))}{\partial X_0^{k+1}}\cdot\frac{(x-X_0)^{k+1}}{k!}\cdot(1-\theta)^k,\tag{21}$$

with $0\le\theta\le 1$. The largest possible remainder with $\theta=0$ will be used here, so that (21) becomes

$$R_T\,(x,X_0,k) = \frac{\partial^{k+1}\mathrm{Ei}\,(X_0)}{\partial X_0^{k+1}}\cdot\frac{(x-X_0)^{k+1}}{k!}.\tag{22}$$

Define L as the limit for x, where the absolute value of $|x-X_0|$ is largest for desired $R_T\,(x,X_0,k)$. Calling the largest possible remainder "error ε_Y", which represents the minimum accuracy for Y, and i referring to the index number of the appropriate value from a list, (22) becomes

$$\varepsilon_Y = \frac{\partial^{k+1}\mathrm{Ei}\,(X_{0i})}{\partial X_{0i}^{k+1}}\cdot\frac{(L_i-X_{0i})^{k+1}}{k!},\tag{23}$$

which is always positive for $L_i > X_{0i}$ and $\frac{\partial^{k+1}\mathrm{Ei}(X_{0i})}{\partial X_{0i}^{k+1}} > 0$. We will use $k=10$. For even k, the term $\frac{\partial^{k+1}\mathrm{Ei}(X_{0i})}{\partial X_{0i}^{k+1}}$ is only negative for $X_{0i}<0$, so that ε_Y is negative for $X_{0i}<0$ and $(L_i-X_{0i})^{k+1}>0$, or $X_{0i}>0$ and $(L_i-X_{0i})^{k+1}<0$. The situation is different for uneven k, because $\frac{\partial^{k+1}\mathrm{Ei}(X_{0i})}{\partial X_{0i}^{k+1}} < 0$ is true also for an initial range of $X_{0i}>0$, whose length depends on k. Furthermore, since for even k it is true that $(L_i-X_{0i})^{k+1} = -(X_{0i}-L_i)^{k+1}$, and for uneven k it is true that $(L_i-X_{0i})^{k+1} = (X_{0i}-L_i)^{k+1}$, there is a symmetric interval around X_{0i} to an upper and to a lower L_i that results for any X_{0i} in the same absolute value of ε_Y.

In summary, the algorithm that uses the Taylor expansion, has three steps:

1) To find $X=\mathrm{Ei}^{(-1)}(Y)$ for given Y-input, the corresponding X_0 is determined according to a maximum Y-interval for a desired accuracy, and into which Y falls; the Y-intervals with their corresponding X_0 are found on a list that is developed in Section 8.

2) The resulting numeric X_0 is used to calculate the polynomial coefficients with (20), and for $k = 10$ with (32).

3) The polynomial $(b_0 - Y) + b_1 \cdot X^1 + ... + b_{10} \cdot X^k = 0$ is solved, and the real root X that is closest to X_0 is the solution.

The root of interest here is the one that is closest to X_0 for the following reason: According to the fundamental theorem of algebra, the polynomial of degree k from (20) has k roots (or zeros) (chapter 1.2 in [36]). Of these, 0 to k roots can result to be real roots. From Figure 1c it is clear that there is always one (and only one) solution for X, which has to be in the interval $L - X_0$. The real-valued root that is closest to X_0 must be the solution, because if the second-closest real-valued root were also in the interval, it would be a second solution, contradicting that there is only one solution. Being outside the interval, it must be farther away from X_0.

7. Propagation effect of the error ε_Y (for Y) to become error ε_X (for X) for the inverse function of Ei(x)

Define ε_Y as the value of the remainder from cutting an infinite series that is used to calculate x as a function of y, where the remainder and hence ε_Y can be positive or negative. The effect of the absolute error ε_Y in (19) is $Y - \varepsilon_Y$. Since the objective is to determine X, the effect of ε_Y on the absolute error ε_X (that affects X) is of interest. In particular, one may want to calculate one given the other. An absolute value of ε_X that is smaller than the corresponding absolute value of ε_Y is preferable, because controlling the latter automatically controls the former.

With (2), the intended X is $\mathrm{Ei}^{(-1)}(Y)$, but the calculated X is $\mathrm{Ei}^{(-1)}(Y - \varepsilon_Y)$, so that expressed as a continuous function

$$\varepsilon_X = \begin{cases} \mathrm{Ei}_{x<0}^{(-1)}(y) - \mathrm{Ei}_{x<0}^{(-1)}(y - \varepsilon_Y) \text{ for } x < 0,\ y < 0, \text{ and } y < \varepsilon_Y, \\ \\ \mathrm{Ei}_{x>0}^{(-1)}(y) - \mathrm{Ei}_{x>0}^{(-1)}(y - \varepsilon_Y) \text{ for } x > 0, \end{cases} \tag{24}$$

where ε_Y and ε_X can be either positive or negative. From (24), three additional formulas can be derived (substituting with $y = \mathrm{Ei}(x)$), so that both ε_X and ε_Y can be calculated as a function of either x or y:

$$\varepsilon_X = \begin{cases} x - \mathrm{Ei}_{x<0}^{(-1)}(\mathrm{Ei}(x) - \varepsilon_Y) \text{ for } x < 0 \text{ and } \mathrm{Ei}(x) < \varepsilon_Y, \\ \\ x - \mathrm{Ei}_{x>0}^{(-1)}(\mathrm{Ei}(x) - \varepsilon_Y) \text{ for } x > 0, \end{cases} \tag{25}$$

$$\varepsilon_Y = \begin{cases} y - \mathrm{Ei}\left(\mathrm{Ei}_{x<0}^{(-1)}(y) - \varepsilon_X\right) \text{ for } x < 0 \text{ and } y < 0, \\ \\ y - \mathrm{Ei}\left(\mathrm{Ei}_{x>0}^{(-1)}(y) - \varepsilon_X\right) \text{ for } x > 0, \end{cases}$$

$$\varepsilon_Y = \mathrm{Ei}(x) - \mathrm{Ei}(x - \varepsilon_X) \text{ for } x \neq 0 \text{ and } x \neq \varepsilon_X. \tag{26}$$

In order to analyze x, where $|\varepsilon_X| < |\varepsilon_Y|$, one inserts (25) into the inequality, and rearranges to

$$
\begin{aligned}
|\varepsilon_Y| - \left| x - \mathrm{Ei}_{x<0}^{(-1)}\left(\mathrm{Ei}(x) - \varepsilon_Y\right)\right| > 0 \text{ for } x < 0, \\
|\varepsilon_Y| - \left| x - \mathrm{Ei}_{x>0}^{(-1)}\left(\mathrm{Ei}(x) - \varepsilon_Y\right)\right| > 0 \text{ for } x > 0.
\end{aligned}
\tag{27}
$$

The left-hand side of (27) is graphed for $\varepsilon_Y = 10^{-7}$ for negative and positive x in Figure 2a. One sees that for $x > 0$, $|\varepsilon_X|$ is always smaller than $|\varepsilon_Y|$, but for $x < 0$ there is a switching point X_S, where to the left $|\varepsilon_X| > |\varepsilon_Y|$ and to the right $|\varepsilon_X| < |\varepsilon_Y|$. To find the switching point, where the red line goes through zero, it is easiest to table values for the left-hand side of (27) with a desired error, and detect, where the left-hand side is closest to zero. For example, the switching point for $|\varepsilon_Y| = |\varepsilon_X| = 10^{-7}$ is $X_S \approx -0.567$, and for $|\varepsilon_Y| = |\varepsilon_X| = 0.01$, it is $X_S \approx -0.572$.

Next, one can also work with relative errors

$$
RE_Y = \frac{\varepsilon_Y}{Y}, \tag{28}
$$

$$
RE_X = \frac{\varepsilon_X}{X}, \tag{29}
$$

which can be positive or negative. Dividing (26) by $y = \mathrm{Ei}(x)$, expressed as function with lower-case letters, on gets

$$
\frac{\varepsilon_Y}{\mathrm{Ei}(x)} = \frac{\mathrm{Ei}(x) - \mathrm{Ei}(x - \varepsilon_X)}{\mathrm{Ei}(x)}
$$

$$
\Rightarrow RE_Y = 1 - \frac{\mathrm{Ei}(x - \varepsilon_X)}{\mathrm{Ei}(x)}
$$

$$
\Rightarrow (1 - RE_Y) \cdot \mathrm{Ei}(x) = \mathrm{Ei}(x - \varepsilon_X)
$$

$$
\Rightarrow RE_X = \begin{cases} 1 - \dfrac{\mathrm{Ei}_{x<0}^{(-1)}\left((1 - RE_Y) \cdot \mathrm{Ei}(x)\right)}{x} & \text{for } x < 0, \\ 1 - \dfrac{\mathrm{Ei}_{x>0}^{(-1)}\left((1 - RE_Y) \cdot \mathrm{Ei}(x)\right)}{x} & \text{for } x > 0. \end{cases} \tag{30}
$$

To avoid the inverse and calculate RE_Y, one can rearrange (30) as

$$
\Rightarrow (1 - RE_Y) \cdot \mathrm{Ei}(x) = \mathrm{Ei}\left(x \cdot (1 - RE_X)\right)
$$

$$
\Rightarrow RE_Y = 1 - \frac{\mathrm{Ei}\left(x \cdot (1 - RE_X)\right)}{\mathrm{Ei}(x)}.
$$

With (30), the inequality $|RE_X| < |RE_Y|$ becomes

$$
\begin{aligned}
|RE_Y| - \left| 1 - \frac{\mathrm{Ei}^{(-1)}_{x<0}\left((1-RE_Y)\cdot \mathrm{Ei}(x)\right)}{x} \right| &> 0 \text{ for } x < 0, \\
|RE_Y| - \left| 1 - \frac{\mathrm{Ei}^{(-1)}_{x>0}\left((1-RE_Y)\cdot \mathrm{Ei}(x)\right)}{x} \right| &> 0 \text{ for } x > 0,
\end{aligned}
\tag{31}
$$

which can be analyzed graphically for given RE_Y. The left-hand side of (31) is graphed for $RE_Y = 10^{-7}$ for negative and positive x in Figure 2b. One sees that, in contrast to $|\varepsilon_X|$ and $|\varepsilon_Y|$ in Figure 2a, $|RE_X|$ is larger than $|RG_Y|$ only close to zero, for both $x < 0$ and $x > 0$. To find the two switching points, where the lines go through zero, it is again easiest to table values for the left-hand side of (31) with a desired relative error, and detect one negative and one positive switching point, where the left-hand side is closest to zero. For example, the switching points for $|RE_Y| = |RE_X| = 10^{-7}$ are $X_S \approx -0.435$ and $X_S \approx 0.150$. For $|RE_Y| = |RE_X| = 0.01$, the values change slightly to -0.440 and 0.151.

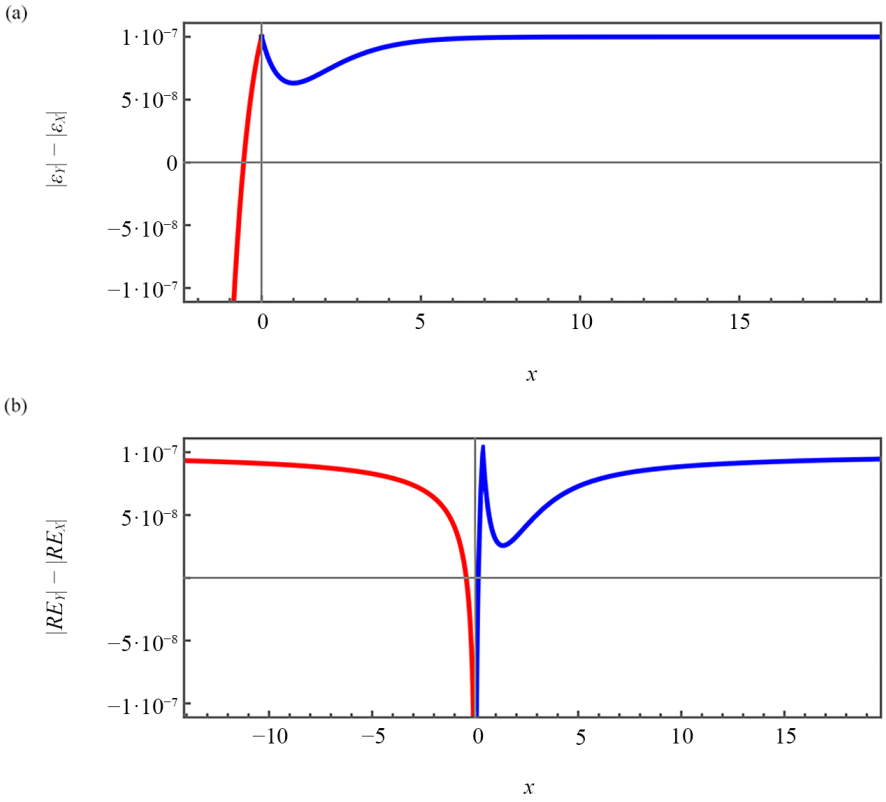


Figure 2. (a) The function $|\varepsilon_Y| - |\varepsilon_X(x)|$ (left-hand side of (27)) is positive, where the absolute error $|\varepsilon_Y|$ is larger than $|\varepsilon_X|$. For $\varepsilon_Y = 10^{-7}$, this is the case for $x > -0.567$. For $x \to 0$ and $x \to \infty$, the function goes towards $\varepsilon_Y = 10^{-7}$. For $x \to -\infty$. the function goes towards $|\varepsilon_Y| - \left|\infty + \mathrm{Ei}^{(-1)}_{x<0}(-\varepsilon_Y)\right| \approx -\infty$; (b) The function $|RE_Y| - |RE_X(x)|$ (left-hand side of (31)) is positive, where the absolute value of the relative error $|RE_Y|$ is larger than $|RE_X|$. For $RE_Y = 10^{-7}$, this is the case for $x < -0.435$ and $x > 0.150$. For $x \to -\infty$ and $x \to \infty$, $|RE_X(x)| \to 0$, and the function goes towards $RE_Y = 10^{-7}$. For $x \to 0$, the function goes towards $|RE_Y| - \left(1 - \frac{1}{x}\right)$

8. Determination of a list of X_{0i} and optimal L_i with fixed ε_Y for Taylor expansion with $k = 10$

With an increasing number of terms k in (20), the remainder of (21) becomes smaller, which means that the absolute error ε_Y becomes smaller. Furthermore, for desired maximum ε_Y, the interval $L_i - X_{0i}$ becomes wider, so that a list with X_{0i} becomes shorter. On the other hand, the terms and polynomial equation become larger. Considering those two consequences, we chose $k = 10$, and polynomial coefficients from (20) for (19) are the following:

$$b_0 = \mathrm{Ei}(X_0) + \frac{\exp(X_0)}{3628800} \cdot \begin{pmatrix} -10628640 + 6999840 \cdot X_0 - 2592720 \cdot X_0^2 \\ +662640 \cdot X_0^3 - 127860 \cdot X_0^4 + 19524 \cdot X_0^5 \\ -2414 \cdot X_0^6 + 242 \cdot X_0^7 - 19 \cdot X_0^8 + X_0^9 \end{pmatrix},$$

$$b_1 = \frac{-\exp(X_0)}{362880 \cdot X_0} \cdot \begin{pmatrix} -3628800 + 3265920 \cdot X_0 - 1451520 \cdot X_0^2 \\ +423360 \cdot X_0^3 - 90720 \cdot X_0^4 + 15120 \cdot X_0^5 \\ -2016 \cdot X_0^6 + 216 \cdot X_0^7 - 18 \cdot X_0^8 + X_0^9 \end{pmatrix},$$

$$b_2 = \frac{\exp(X_0)}{80640 \cdot X_0^2} \cdot \begin{pmatrix} -1814400 + 1814400 \cdot X_0 - 887040 \cdot X_0^2 \\ +282240 \cdot X_0^3 - 65520 \cdot X_0^4 + 11760 \cdot X_0^5 \\ -1680 \cdot X_0^6 + 192 \cdot X_0^7 - 17 \cdot X_0^8 + X_0^9 \end{pmatrix},$$

$$b_3 = \frac{-\exp(X_0)}{30240 \cdot X_0^3} \cdot \begin{pmatrix} -1209600 + 1209600 \cdot X_0 - 604800 \cdot X_0^2 \\ +199920 \cdot X_0^3 - 48720 \cdot X_0^4 + 9240 \cdot X_0^5 \\ -1400 \cdot X_0^6 + 170 \cdot X_0^7 - 16 \cdot X_0^8 + X_0^9 \end{pmatrix},$$

$$b_4 = \frac{\exp(X_0)}{17280 \cdot X_0^4} \cdot \begin{pmatrix} -907200 + 907200 \cdot X_0 - 453600 \cdot X_0^2 \\ +151200 \cdot X_0^3 - 37620 \cdot X_0^4 + 7380 \cdot X_0^5 \\ -1170 \cdot X_0^6 + 150 \cdot X_0^7 - 15 \cdot X_0^8 + X_0^9 \end{pmatrix},$$

$$b_5 = \frac{-\exp(X_0)}{14400 \cdot X_0^5} \cdot \begin{pmatrix} -725760 + 725760 \cdot X_0 - 362880 \cdot X_0^2 \\ +120960 \cdot X_0^3 - 30240 \cdot X_0^4 + 6024 \cdot X_0^5 \\ -984 \cdot X_0^6 + 132 \cdot X_0^7 - 14 \cdot X_0^8 + X_0^9 \end{pmatrix},$$

$$b_6 = \frac{\exp(X_0)}{17280 \cdot X_0^6} \cdot \begin{pmatrix} -604800 + 604800 \cdot X_0 - 302400 \cdot X_0^2 \\ +100800 \cdot X_0^3 - 25200 \cdot X_0^4 + 5040 \cdot X_0^5 \\ -836 \cdot X_0^6 + 116 \cdot X_0^7 - 13 \cdot X_0^8 + X_0^9 \end{pmatrix},$$

$$b_7 = \frac{-\exp(X_0)}{30240 \cdot X_0^7} \cdot \begin{pmatrix} -518400 + 518400 \cdot X_0 - 259200 \cdot X_0^2 \\ +86400 \cdot X_0^3 - 21600 \cdot X_0^4 + 4320 \cdot X_0^5 \\ -720 \cdot X_0^6 + 102 \cdot X_0^7 - 12 \cdot X_0^8 + X_0^9 \end{pmatrix},$$

$$b_8 = \frac{\exp(X_0)}{80640 \cdot X_0^8} \cdot \begin{pmatrix} -453600 + 453600 \cdot X_0 - 226800 \cdot X_0^2 \\ +75600 \cdot X_0^3 - 18900 \cdot X_0^4 + 3780 \cdot X_0^5 \\ -630 \cdot X_0^6 + 90 \cdot X_0^7 - 11 \cdot X_0^8 + X_0^9 \end{pmatrix},$$

$$b_9 = \frac{-\exp(X_0)}{362880 \cdot X_0^9} \cdot \begin{pmatrix} -403200 + 403200 \cdot X_0 - 201600 \cdot X_0^2 \\ +67200 \cdot X_0^3 - 16800 \cdot X_0^4 + 3360 \cdot X_0^5 \\ -560 \cdot X_0^6 + 80 \cdot X_0^7 - 10 \cdot X_0^8 + X_0^9 \end{pmatrix},$$

$$b_{10} = \frac{\exp(X_0)}{3628800 \cdot X_0^{10}} \cdot \begin{pmatrix} -362880 + 362880 \cdot X_0 - 181440 \cdot X_0^2 \\ +60480 \cdot X_0^3 - 15120 \cdot X_0^4 + 3024 \cdot X_0^5 \\ -504 \cdot X_0^6 + 72 \cdot X_0^7 - 9 \cdot X_0^8 + X_0^9 \end{pmatrix}. \tag{32}$$

We will show next how X_0 is found, given Y and desired accuracy (error ε_Y). In our guiding example, with $Y = -0.31$ and $|\varepsilon_Y| = 10^{-7}$, we left it open, if $x < 0$ or $x > 0$. It turns out with (38) that $X_0 = -0.834406$ (rounded to six significant digits) for calculating the inverse $X = \mathrm{Ei}^{(-1)}_{x<0}(-0.31)$ and $X_0 = 0.372387$ for $X = \mathrm{Ei}^{(-1)}_{x>0}(-0.31)$, with $|\varepsilon_Y| = 10^{-7}$. With (32), our polynomial from (19) in the former case $(x < 0)$ becomes now

$$\begin{aligned} &-2.22279 - 10.9846 \cdot X - 32.0667 \cdot X^2 - 68.7981 \cdot X^3 - 108.295 \cdot X^4 - 124.605 \cdot X^5 \\ &- 103.705 \cdot X^6 - 60.8752 \cdot X^7 - 23.9388 \cdot X^8 - 5.66708 \cdot X^9 - 0.611258 \cdot X^{10} = 0, \end{aligned}$$

which (using *Mathematica*'s NSolve function) has two real-valued solutions at $X = -1.81$ and -0.801. The second solution is much closer to $X_0 = -0.834406$ and thus is the desired one. In the other case $(x > 0)$, (19) becomes

$$\begin{aligned} &-3.02958 + 27.8538 \cdot X - 162.004 \cdot X^2 + 774.656 \cdot X^3 - 2730.12 \cdot X^4 + 7038.18 \cdot X^5 \\ &- 13125.1 \cdot X^6 + 17263.3 \cdot X^7 - 15211.4 \cdot X^8 + 8068.84 \cdot X^9 - 1950.11 \cdot X^{10} = 0, \end{aligned}$$

which has the two real-valued solutions at $X = 0.298$ and 0.903, of which the first solution is closer to $X_0 = 0.372387$ and thus is the desired one.

Returning to the derivation of the formulas, define the i^{th} limit L_i as one of two limits that is used to determine X_{0i}, either leftwards or rightwards on the x-axis, and L_{i+1} as the complementary limit of the interval for given X_{0i}. The symmetric semi-intervals $|L_i - X_{0i}|$ around X_{0i} should be as large as possible, to minimize the number of necessary intervals and fixed points X_{0i} over the established overall range for x. With larger intervals, however, the error ε_Y at the limits increases. Furthermore, the intervals that fulfill the minimum desired accuracy, vary with varying X_{0i}, *id est*, there is an optimal interval for each X_{0i}, where the error at the limits L_i and L_{i+1} is equal to the desired maximum error ε_Y, and smaller between L_i and X_{0i}, as well as between X_{0i} and L_{i+1}.

The formulas to calculate the X_{0i} and L_i are based on (23). For going with even k and $x < 0$ rightwards from a lower L_i, the two formulas for finding iteratively the next X_{0i} and L_i are

$$\varepsilon_Y - \frac{\partial^{11}\mathrm{Ei}(X_{0i})}{\partial X_{0i}^{11}} \cdot \frac{(L_i - X_{0i})^{11}}{10!} = 0 \text{ for finding } X_{0i}(\varepsilon_Y, L_i), \tag{33}$$

$$\varepsilon_Y - \frac{\partial^{11}\mathrm{Ei}\,(X_{0i})}{\partial X_{0i}^{11}} \cdot \frac{(X_{0i}-L_i)^{11}}{10!} = 0 \text{ for finding } L_i(\varepsilon_Y, X_0), \tag{34}$$

with

$$\frac{\partial^{11}\mathrm{Ei}\,(X_{0i})}{\partial X_{0i}^{11}} = \frac{\exp(X_{0i})}{X_{0i}^{11}} \cdot \left(\begin{array}{l} 3628800 - 3628800 \cdot X_{0i} + 1814400 \cdot X_{0i}^2 \\ -604800 \cdot X_{0i}^3 + 151200 \cdot X_{0i}^4 - 30240 \cdot X_{0i}^5 \\ +5040 \cdot X_{0i}^6 - 720 \cdot X_{0i}^7 + 90 \cdot X_{0i}^8 - 10 \cdot X_{0i}^9 + X_{0i}^{10} \end{array} \right). \tag{35}$$

The same formulas (33) and (34) are used for $x > 0$ to go leftwards from an upper limit.

For increasingly negative x, the resulting absolute value $|y|$ becomes ever smaller (see Figure 1c). Eventually, $|\varepsilon_Y|$ becomes larger than $|y|$, which for $\varepsilon_Y = 10^{-9}$ is to the left of $x \approx -26.62$, so that it makes sense to use relative errors from (28) and (29). For $x < 0$ going leftwards from an upper L_i, the two formulas for finding iteratively the next X_{0i} and L_i, are

$$RE_Y - \frac{\partial^{11}\mathrm{Ei}\,(X_{0i})}{\partial X_{0i}^{11}} \cdot \frac{(L_i - X_{0i})^{11}}{\mathrm{Ei}\,(L_i) \cdot 10!} = 0 \text{ for finding } X_{0i}(RE_Y, L_i), \tag{36}$$

$$RE_Y - \frac{\partial^{11}\mathrm{Ei}\,(X_{0i})}{\partial X_{0i}^{11}} \cdot \frac{(X_{0i} - L_i)^{11}}{\mathrm{Ei}\,(L_i) \cdot 10!} = 0 \text{ for finding } L_i(RE_Y, X_{0i}), \tag{37}$$

with $L_i \neq 0$, and $\frac{\partial^{11}\mathrm{Ei}(X_{0i})}{\partial X_{0i}^{11}}$ from (35). For $x > 0$ and $y > 0$, or going into the opposite direction, $L_i - X_{0i}$ becomes $X_{0i} - L_i$, and $X_{0i} - L_i$ becomes $L_i - X_{0i}$. The change from negative to positive y occurs at $X = \mathrm{Ei}_{x>0}^{(-1)}(0) \approx 0.3725$.

For (36) with $x < 0$, and for (37) with $x > 0$, there are slight complications. For (36) to find X_{0i} with $x < 0$, the left-hand side of (36) goes through zero close to L_i, then reaches a minimum and goes towards zero to the left (Figure 3a). For that reason, one has to indicate in the root-finding algorithm to search between L_i (dashed line on the right in Figure 3a) and the potential X_{0i} that corresponds to the minimum (dashed line on the left in Figure 3b). Taking the first derivative with respect to X_{0i}, and setting the result equal to zero results in

$$\frac{\exp(X_{0i}) \cdot (L_i - X_{0i})^{10} \cdot \left(X_{0i}^{12} - L_i \cdot \left(\begin{array}{l} -39916800 + 39916800 \cdot X_{0i} \\ -19958400 \cdot X_{0i}^2 + 6652800 \cdot X_{0i}^3 \\ -1663200 \cdot X_{0i}^4 + 332640 \cdot X_{0i}^5 \\ -55440 \cdot X_{0i}^6 + 7920 \cdot X_{0i}^7 - 990 \cdot X_{0i}^8 \\ +110 \cdot X_{0i}^9 - 11 \cdot X_{0i}^{10} + X_{0i}^{11} \end{array} \right) \right)}{3628800 \cdot X_{0i}^{12} \cdot \mathrm{Ei}\,(L_i)} = 0.$$

The solution is found, where the term in the outer parentheses is zero. Expanding the term in parentheses results in

$$\begin{array}{l} 39916800 \cdot L_i - 39916800 \cdot L_i \cdot X_{0i} + 19958400 \cdot L_i \cdot X_{0i}^2 - 6652800 \cdot L_i \cdot X_{0i}^3 \\ +1663200 \cdot L_i \cdot X_{0i}^4 - 332640 \cdot L_i \cdot X_{0i}^5 + 55440 \cdot L_i \cdot X_{0i}^6 - 7920 \cdot L_i \cdot X_{0i}^7 \\ +990 \cdot L_i \cdot X_{0i}^8 - 110 \cdot L_i \cdot X_{0i}^9 + 11 \cdot L_i \cdot X_{0i}^{10} - L_i \cdot X_{0i}^{11} + X_{0i}^{12} = 0, \end{array} \tag{38}$$

which can be solved for given L_i as a 12^{th}-degree polynomial for real-valued, negative X_{0i}. The real-valued solution that is closest to L_i represents the lower search limit for X_{0i}.

For (37) to find L_i with $x > 0$, the situation is similar. The left-hand side of (37) becomes negative at $L_i = X_{0i}$, then reaches a minimum and goes towards zero to the right (Figure 3b). For that reason, one has to indicate in the root-finding algorithm to search between X_{0i} and the L_i that corresponds to the minimum. Taking the first derivative with respect to L_i, and setting the result equal to zero results in

$$\frac{\exp(X_{0i}) \cdot (L_i - X_{0i})^{10} \cdot \begin{pmatrix} 3628800 - 3628800 \cdot X_{0i} \\ +1814400 \cdot X_{0i}^2 - 604800 \cdot X_{0i}^3 \\ +151200 \cdot X_{0i}^4 - 30240 \cdot X_{0i}^5 \\ +5040 \cdot X_{0i}^6 - 720 \cdot X_{0i}^7 \\ +90 \cdot X_{0i}^8 - 10 \cdot X_{0i}^9 + \cdot X_{0i}^{10} \end{pmatrix} \cdot \begin{pmatrix} \exp(L_i) \cdot (X_{0i} - L_i) \\ +11 \cdot Li \cdot \mathrm{Ei}(L_i) \end{pmatrix}}{-3628800 \cdot L_i \cdot X_{0i}^{11} \cdot \mathrm{Ei}(L_i)^2} = 0. \tag{39}$$

The solution is found, where the last term $\exp(L_i) \cdot (X_{0i} - L_i) + 11 \cdot L_i \cdot \mathrm{Ei}(L_i)$ is zero. For using a root-finding algorithm to find L_i, it is numerically much better to rearrange the equation into the following form:

$$\frac{11 \cdot L_i \cdot \mathrm{Ei}(L_i)}{\exp(L_i) \cdot (L_i - X_{0i})} - 1 = 0, \text{ with } L_i > X_{0i}.$$

The solution represents the lower search limit for L_i.

Once one has a list of all values of L_i, X_{0i}, and L_{i+1} for a given range, the limits L_i should be tested by calculating

$$\left| \frac{\partial^{11} \mathrm{Ei}(X_{0i})}{\partial X_{0i}^{11}} \cdot \frac{(L_i - X_{0i})^{11}}{10!} \right| \text{ from (23), or } \left| \frac{\partial^{11} \mathrm{Ei}(X_{0i})}{\partial X_{0i}^{11}} \cdot \frac{(L_i - X_{0i})^{11}}{\mathrm{Ei}(L_i) \cdot 10!} \right| \text{ for relative errors,}$$

for each L_i and L_{i+1} with the corresponding X_{0i}. The result should always be the chosen $|\varepsilon_Y|$ or $|RE_Y|$.

A computer algorithm to calculate the inverse exponential integral needs at least $\mathrm{Ei}(L_i)$ and $\mathrm{Ei}(L_{i+1})$ with the corresponding X_{0i} as input, in order to find the necessary X_{0i} in the range of $\mathrm{Ei}(L_i) \le Y \le \mathrm{Ei}(L_{i+1})$. Our version that is available at https://doi.org/10.17632/9f8dmwyspb.5 (see Appendix), includes also the values for $\mathrm{Ei}(X_{0i})$ and $\exp(X_{0i})$, such that only algebraic operations are needed to calculate the polynomial coefficients from (32). For our guiding example, one can look up at the beginning of the file "Inverse Ei(Y) with Taylor expansion" in the lists "xNegLimitsRelErr7" and "xPosLimitsAbsErr7", into which intervals $Y = -0.31$ falls. In the former case, the interval is from -0.406 to -0.219, and in the latter from 0.314 to -0.365, for $|\varepsilon_Y| = 10^{-7}$. The interval then determines X_0 for (32) in the list.

Table 2 shows the continuation of Table 1 for the calculations with the Taylor expansion. The third column shows the maximum-error specification, and the fourth column the number of intervals that resulted with (33) to (37) for the given input ranges (in total 90 intervals). In addition to the range and mean of $|\varepsilon_Y|$ from (17) for 1000 calculated X-values, the corresponding values for $|RE_Y|$ and $|\varepsilon_X|$ are given in columns 6 and 7, calculated with (28) and (26) (the latter was used with *Mathematica*'s FindRoot function). Table 2 shows that the desired maximum-error specifications are fulfilled. The time values in the last column are in between the values with the two algorithms in Table 1. As with the converging search, the Taylor method could well be the fastest method, if implemented at a lower programming level (such as with C++).

Figure 4 shows the interval lengths that were determined with (33) to (37), as a function of X_{0i}. One sees that the absolute interval length is close to zero for X_{0i} being close to zero. To the left and to the right with $\varepsilon_Y = 10^{-7}$, the length becomes increasingly larger, but with relative error $RE_Y = 10^{-7}$ only to the left. For the relative error to the right (orange squares), if one had started at $\mathrm{Ei}_{x>0}^{(-1)}(0) \approx 0.3725$, where y becomes positive, the points would increase to the

shown maximum, before they decrease slightly as shown, and stabilize at ≈ 1.757 (calculated after 10,000 iterations for $X_{0i} \approx 3511.7$).

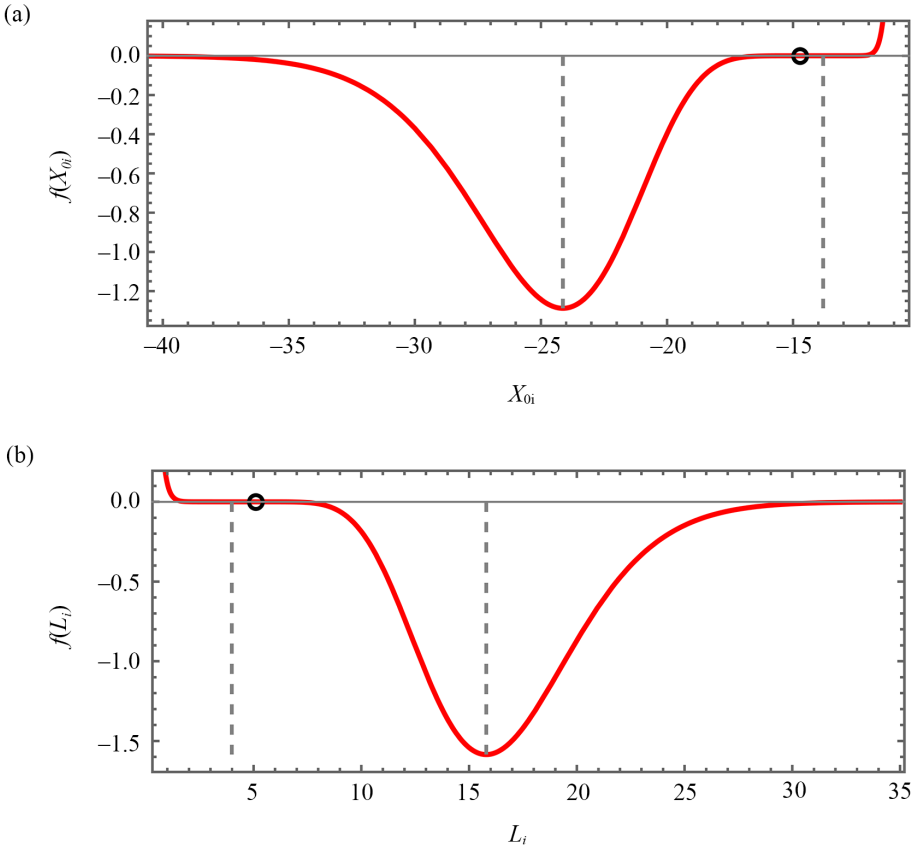


Figure 3. (a) The left-hand side of (36) for $x < 0$ is shown as a function of X_{0i}. The X_{0i} corresponding to the minimum can be calculated with (38), and represents the lower limit for the search of X_{0i} for given L_i. A root-finding algorithm has to search between the two dashed lines, the right one at L_i and the left one at the minimum. The correct solution is indicated with a black circle; (b) The same situation occurs with the left-hand side of (37) for $x > 0$ to the right, when one searches the next L_i with given X_{0i}

Table 2. Testing of the numerical calculation of the inverse of $\mathrm{Ei}(x)$ with Taylor expansion ($k = 10$)

Sub-function	Range of $y = \mathrm{Ei}(x)$	Maximum-error specification	Number of intervals	Range and mean of $\lvert\varepsilon_Y\rvert$ for 1000 calculated X-values	Range and mean of RE_Y for 1000 calculated X-values	Range and mean of $\lvert\varepsilon_X\rvert$ for 1000 calculated X-values	Mean time in seconds per X-value
$x < 0$	$-10^{-7} \geq y \geq -1$	$\lvert RE_Y\rvert \leq 10^{-7}$	14	≈ 0 to $1 \cdot 10^{-8}$, $6 \cdot 10^{-11}$	≈ 0 to $1 \cdot 10^{-8}$, $1 \cdot 10^{-9}$	≈ 0 to $9 \cdot 10^{-9}$, $1 \cdot 10^{-9}$	0.0018
	$-1 \geq y \geq -15$	$\lvert\varepsilon_Y\rvert \leq 10^{-7}$	31	≈ 0 to $1 \cdot 10^{-8}$, $8 \cdot 10^{-10}$	≈ 0 to $1 \cdot 10^{-8}$, $3 \cdot 10^{-10}$	≈ 0 to $2 \cdot 10^{-9}$, $5 \cdot 10^{-11}$	0.0017
$x > 0$	$-15 \leq y \leq 10$	$\lvert\varepsilon_Y\rvert \leq 10^{-7}$	36	≈ 0 to $1 \cdot 10^{-8}$, $6 \cdot 10^{-10}$	≈ 0 to $3 \cdot 10^{-8}$, $5 \cdot 10^{-10}$	≈ 0 to $4 \cdot 10^{-9}$, $1 \cdot 10^{-10}$	0.0018
	$10 \leq y \leq 10^7$	$\lvert RE_Y\rvert \leq 10^{-7}$	9	$3 \cdot 10^{-13}$ to 0.9, 0.007	$2 \cdot 10^{-14}$ to $9 \cdot 10^{-8}$, $4 \cdot 10^{-9}$	$3 \cdot 10^{-14}$ to $1 \cdot 10^{-7}$, $4 \cdot 10^{-9}$	0.0018

Numbers for the range of y (2nd column) are exact; the corresponding numbers for the range of x are given in Table 1. Errors in columns 5 to 7 are rounded to one significant digit, and time values to four decimals. The time was measured with an Intel Core i7-12700H processor

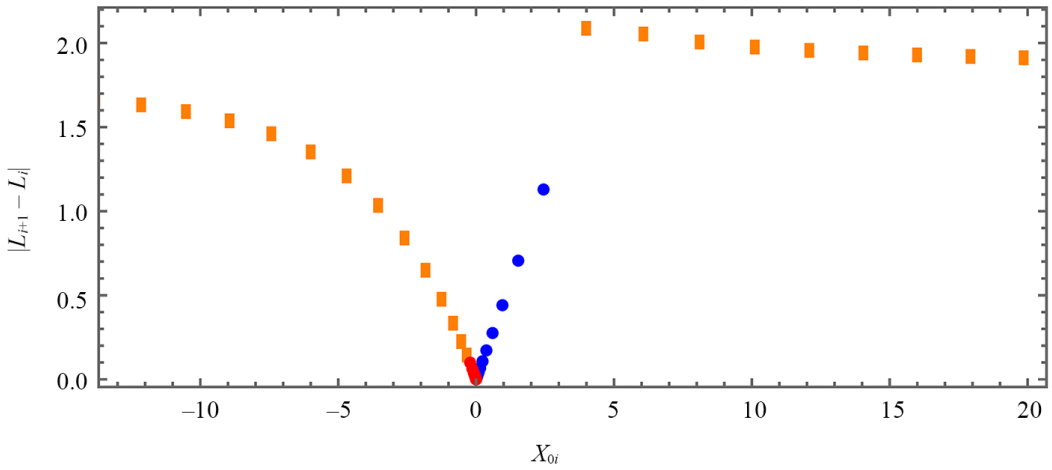


Figure 4. The absolute interval lengths from L_i to L_{i+1} (symmetrically around X_{0i}) are shown as a function of X_{0i}, in the range from $X_{0i} \approx -12.15$ to 19.77. The corresponding range for y is -10^{-7} to -15 for $x < 0$, and $y = -15$ to 10^7 for $x > 0$. Orange squares are for intervals with relative error $|RE_Y| = 10^{-7}$, and circles for intervals with $|\varepsilon_Y| = 10^{-7}$. Red circles are for $x < 0$, and blue circles for $x > 0$

9. Conclusions

The exponential integral $y = \mathrm{Ei}(x) = \int \left(\frac{\exp(x)}{x}\right) dx$ is a transcendental function, representing the infinite series from (5). Its numerical evaluation can be carried out with the asymptotic expansion from (8), or summing up a finite number of elements with a sufficiently small remainder, calculated with (5) and (10). Its inverse results in two subfunctions that are defined here as $x = \mathrm{Ei}^{(-1)}_{x<0}(y)$ for $x < 0$ (and always $y < 0$), and $x = \mathrm{Ei}^{(-1)}_{x>0}(y)$ for $x > 0$. Unfortunately, there is no direct formula to calculate numerical values for the two inverse subfunctions. It is unknown, if a series representation for the inverse of the exponential integral could exist, and how one would find it. Series representations for the inverse of transcendental functions exist, for example, for the logarithmic and the exponential functions, which are inverses of each other.

We showed here that for x being close to 0, the inverse with sufficiently small error can be calculated with Lambert's W-function from (15). For the more usual other cases, we presented two generic methods: a root-finding algorithm in two variants and a Taylor expansion that leads to a (here 10^{th} degree) polynomial. The more flexible root-finding algorithm will generally be the best choice. However, the Taylor expansion can be implemented without any special functions, as long as the polynomial can be solved accurately. Providing $\mathrm{Ei}(X_{0i})$ and $\exp(X_{0i})$ also in the interval list, only algebraic operations are involved in that case. The corresponding algorithms were programmed in *Mathematica* notebooks, and are available at https://doi.org/10.17632/9f8dmwyspb.5 (see Appendix).

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Conflict of interest

The authors declare no competing financial interest.

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Appendix

The following six *Mathematica* notebooks with the iimplemented methods that were presented here, are available at https://doi.org/10.17632/9f8dm
wyspb.5, either with extension "nb" to run them in the *Mathematica* software, or as PDF files for inspection only. In *Mathematica*, the numerical value of $\mathrm{Ei}(X)$ is best calculated with the build-in function ExpIntegralEi; the first two notebooks are for demonstration of the possible methods. For calculating the inverse, the most efficient function is in the fourth notebook "Inverse Ei(Y) with FindRoot":

1) Ei(X) with asymptotic expansion (Ricker Oct 2025).nb (or pdf);
2) Ei(X) with series summation (Ricker Oct 2025).nb (or pdf);
3) Inverse Ei(Y) with converging search (Ricker Oct 2025).nb (or pdf);
4) Inverse Ei(Y) with FindRoot (Ricker Oct 2025).nb (or pdf);
5) Inverse Ei(Y) with Taylor expansion (Ricker Oct 2025).nb (or pdf);
6) Inverse Ei(Y) with Taylor expansion-LIMIT GENERATION (Ricker Oct 2025).nb (or pdf).