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Some Fractional Corrected Dual Euler-Simpson Type Inequalities for Differentiable Strongly Preinvex Functions

Lilia Zenkoufi[1], Abdelkader Moumen[2*], Hicham Saber[2], Badreddine Meftah[1]

[1]Department of Mathematics, Faculty of Mathematics, Computer Science and Material Sciences, University 8 may 1945 Guelma, Guelma 24000, Algeria
[2]Department of Mathematics, College of Science, University of Hail, Hail 55473, Saudi Arabia
E-mail: mo.abdelkader@uoh.edu.sa

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Abstract: In this paper, we introduce a new integral identity involving Riemann-Liouville fractional operators. By considering functions with strongly preinvex first derivatives, we derive several corrected dual Euler-Simpson type inequalities in the fractional setting. The results obtained broaden the scope of traditional inequalities by incorporating fractional and preinvex structures, and offer new insights into fractional analysis. Specific cases and applications to various mean-type inequalities are discussed, underscoring the utility of the proposed framework.

***Keywords*:** corrected dual Euler-Simpson inequality, strong preinvex functions, Hölder inequality, power mean inequality

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1. Introduction

Convexity theory constitutes a critical framework across diverse areas such as optimization, economics, finance, and approximation theory. Recent advances in error estimation for quadrature formulas and the construction of generalized integral inequalities reflect the continued interest in and applicability of convexity-based methods (see [1–12]).

Over recent decades, considerable effort has been devoted to generalizing the classical notion of convexity. Among these developments convex functions introduced by Hanson [13] represent a significant class of generalized convex functions. Their foundational properties and applications have been extensively studied by Noor [14, 15], Pini [16], Yang and Li [17], particularly in the contexts of optimization, variational inequalities, and equilibrium problems.

Recall that function φ is considered preinvex, if

$$\varphi\left(x+v\eta\left(y,x\right)\right)\leq\left(1-v\right)\varphi\left(x\right)+v\varphi(y) \tag{1}$$

is true for every $x,y\in K$ and all $v\in[0,1]$, where K is the invex set (see [18]) and satisfies $x+v\eta\left(y,x\right)\in K$ for all $x,y\in K$.

We note that if we take $\eta\left(y,x\right)=y-x$, (1.1) recapture the classical convexity i.e. $\varphi\left(vx+\left(1-v\right)y\right)\leq v\varphi\left(x\right)+\left(1-v\right)\varphi(y)$.

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Yan and Liu [19] introduced a novel class of generalized convex functions called strong preinvex function

$$\varphi\left(x+v\eta\left(y,x\right)\right)\leq\left(1-v\right)\varphi\left(x\right)+v\varphi(y)-cv\left(1-v\right)\left|\eta\left(y,x\right)\right|^{2}$$

holds for all $x, y \in K$ and all $v \in [0,1]$.

In recent years, scientists have become more interested in fractional calculus. It has been successfully incorporated and used in many technological and scientific fields (see [20–27]). Its main strength is its ability to describe the memory and genetic properties of different materials and processes, which has piqued the curiosity of researchers from a wide range of domains.

In recent years, many researchers have shown increasing interest in the emerging field of fractional calculus, particularly due to its ability to characterize the memory and hereditary properties of various materials and processes. The development of accurate numerical schemes for fractional differential equations has been strongly motivated by this capability. In particular, the Finite Difference Method (FDM) and the β-fractional FDM enable accurate discretization of nonlocal operators. Furthermore, spline-based techniques, such as the Rational Non-Polynomial Spline Method, the Fractional Non-Polynomial Spline Method, and the Logarithmic Non-Polynomial Spline Method, enhance accuracy by combining polynomial and non-polynomial basis functions tailored to the behavior of fractional models. Among these approaches, the Hyperbolic Non-Polynomial Spline Method provides improved stability and flexibility when approximating solutions with boundary layers or sharp gradients.

However, the literature concerning fractional integral inequalities is rich and varied, and several generalizations, extensions, and variants can easily be found (see [28–41]).

It is worth recalling the definition of right and left Riemann-Liouville fractional integrals of order $\alpha > 0$, which are defined as follows:

$$I_{\lambda_1^+}^{\alpha}\varphi(x)=\frac{1}{\Gamma(\alpha)}\int_{\lambda_1}^{x}\left(x-v\right)^{\alpha-1}\varphi(v)dv,\quad x>\lambda_1,$$

$$I_{\lambda_2^-}^{\alpha}\varphi(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{\lambda_2}\left(v-x\right)^{\alpha-1}\varphi(v)dv,\quad \lambda_2>x,$$

respectively, where $\Gamma(\alpha)=\int_{0}^{\infty}e^{-v}v^{\alpha-1}dv$, is the gamma function with $\varphi\in L^1[\lambda_1,\lambda_2]$ and $I_{\lambda_1^+}^{0}\varphi(x)=I_{\lambda_2^-}^{0}\varphi(x)=\varphi(x)$ (see [42]).

Corrected dual Euler-Simpson is the quadrature that follows (see [43]).

$$\int_{\lambda_1}^{\lambda_2}\varphi\left(u\right)du\approx\frac{\lambda_2-\lambda_1}{15}\left(8\varphi\left(\frac{3\lambda_1+\lambda_2}{4}\right)-\varphi\left(\frac{\lambda_1+\lambda_2}{2}\right)+8\varphi\left(\frac{\lambda_1+3\lambda_2}{4}\right)\right).$$

Lakhdari et al. [44] gave some corrected dual Simpson type inequalities for functions whose local fractional derivatives are generalized convex, among the obtained results we have

$$\left|\frac{1}{15^{\alpha}}\left(8^{\alpha}\varphi\left(\frac{3\lambda_1+\lambda_2}{4}\right)-\varphi\left(\frac{\lambda_1+\lambda_2}{2}\right)+8^{\alpha}\varphi\left(\frac{\lambda_1+3\lambda_2}{4}\right)\right)-\frac{\Gamma(1+\alpha)}{\alpha(\lambda_2-\lambda_1)}\left({}_{\lambda_1}\mathscr{I}_{\lambda_2}^{\alpha}\left(\varphi\left(x\right)\right)\right)\right|$$

$$\leq \frac{(\lambda_2-\lambda_1)^{\alpha}}{16^{\alpha}}\left(\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left(\left|\varphi^{(\alpha)}(\lambda_1)\right|+\left|\varphi^{(\alpha)}(\lambda_2)\right|\right)\right.$$

$$+\left(\left(\frac{2}{15}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}+2^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left(\left|\varphi^{(\alpha)}\left(\frac{3\lambda_1+\lambda_2}{4}\right)\right|+\left|\varphi^{(\alpha)}\left(\frac{\lambda_1+3\lambda_2}{4}\right)\right|\right)$$

$$\left.+\left(\left(\frac{34}{15}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-2^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left|\varphi^{(\alpha)}\left(\frac{\lambda_1+\lambda_2}{2}\right)\right|\right).$$

Recently, Munir et al. [45], established some corrected dual Simpson type inequalities for functions whose derivatives are s-convex via Caputo–Fabrizio fractional integral operators, among the obtained results we have

$$\left|\frac{1}{15}\left(8\varphi\left(\frac{3\lambda_1+\lambda_2}{4}\right)-\varphi\left(\frac{\lambda_1+\lambda_2}{2}\right)+8\varphi\left(\frac{\lambda_1+3\lambda_2}{4}\right)\right)\right.$$

$$\left.-\frac{\beta(\alpha)}{(\lambda_2-\lambda_1)^{\alpha}}\left[\left({}^{C\varphi}_{\lambda_1}\mathscr{J}^{\alpha}\varphi\right)(k)+\left({}^{C\varphi}\mathscr{J}^{\alpha}_{\lambda_2}\varphi\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}\varphi(k)\right|$$

$$\leq \frac{\lambda_2-\lambda_1}{240}\times\frac{4^{1-s}-17+15\times 2^{2+s}+15\times 2^{3+2s}-43\times 3^{1+s}+\left(-1+3^{1+s}\right)s}{(s+1)(s+2)}\left(\left|\varphi'(\lambda_1)\right|+\left|\varphi'(\lambda_2)\right|\right).$$

In this study, we propose to study the corrected dual Euler-Simpson type inequalities in the class of strongly preinvex functions, which allows us to establish error bounds for both convex and nonconvex functions. It also allows for better error control by manipulating the constant c. To do this, we first establish a new fractional integral identity. By using this identity we derive some corrected dual Euler-Simpson type inequalities for functions whose first derivatives are strongly preinvex. Particularly cases are discussed. Applications are provided.

2. Main results

In what follows, we assume that $\eta(\lambda_2,\lambda_1)>0$ where $\eta: K\times K\to\mathbb{R}$ is a bifunction and we set $K=[\lambda_1,\lambda_1+\eta(\lambda_2,\lambda_1)]$.

Lemma 1 Let $\varphi: K\to\mathbb{R}$ be a differentiable function on K°, and $\varphi'\in L^1(K)$, then the following equality holds

$$\frac{8\varphi\left(\frac{4\lambda_1+\eta(\lambda_2,\lambda_1)}{4}\right)-\varphi\left(\frac{2\lambda_1+\eta(\lambda_2,\lambda_1)}{2}\right)+8\varphi\left(\frac{4\lambda_1+3\eta(\lambda_2,\lambda_1)}{4}\right)}{15}-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(\eta(\lambda_2,\lambda_1))^{\alpha}}\mathscr{I}^{\alpha}(\lambda_1,\lambda_2,\varphi)$$

$$=\frac{\eta(\lambda_2,\lambda_1)}{16}\left(\int_0^1(1-v^{\alpha})\varphi'\left(\lambda_1+\frac{1-v}{4}\eta(\lambda_2,\lambda_1)\right)dv\right.$$

$$+\int_0^1\left((1-v)^{\alpha}-\frac{17}{15}\right)\varphi'\left(\lambda_1+\frac{2-v}{4}\eta(\lambda_2,\lambda_1)\right)dv$$

$$+\int_0^1 \left(\frac{17}{15}-v^{\alpha}\right)\varphi'\left(\lambda_1+\frac{3-v}{4}\eta\left(\lambda_2,\lambda_1\right)\right)dv$$

$$+\int_0^1 \left(v^{\alpha}-1\right)\varphi'\left(\lambda_1+\frac{3+v}{4}\eta\left(\lambda_2,\lambda_1\right)\right)dv\right),$$

where

$$\mathscr{I}^{\alpha}\left(\lambda_1,\lambda_2,\varphi\right)=I^{\alpha}_{(\lambda_1)^+}\varphi\left(\frac{4\lambda_1+\eta(\lambda_1,\lambda_2)}{4}\right)+I^{\alpha}_{\left(\frac{2\lambda_1+\eta(\lambda_2,\lambda_1)}{2}\right)^-}\varphi\left(\frac{4\lambda_1+\eta(\lambda_1,\lambda_2)}{4}\right)$$

$$+I^{\alpha}_{\left(\frac{2\lambda_1+\eta(\lambda_2,\lambda_1)}{2}\right)^+}\varphi\left(\frac{4\lambda_1+3\eta(\lambda_1,\lambda_2)}{4}\right)+I^{\alpha}_{(\lambda_1+\eta(\lambda_2,\lambda_1))^-}\varphi\left(\frac{4\lambda_1+3\eta(\lambda_1,\lambda_2)}{4}\right).$$

Proof. Let

$$I_1=\int_0^1 \left(1-v^{\alpha}\right)\varphi'\left(\lambda_1+\frac{1-v}{4}\eta\left(\lambda_2,\lambda_1\right)\right)dv,$$

$$I_2=\int_0^1 \left((1-v)^{\alpha}-\frac{17}{15}\right)\varphi'\left(\lambda_1+\frac{2-v}{4}\eta\left(\lambda_2,\lambda_1\right)\right)dv,$$

$$I_3=\int_0^1 \left(\frac{17}{15}-v^{\alpha}\right)\varphi'\left(\lambda_1+\frac{3-v}{4}\eta\left(\lambda_2,\lambda_1\right)\right)dv,$$

and

$$I_4=\int_0^1 \left(v^{\alpha}-1\right)\varphi'\left(\lambda_1+\frac{3+v}{4}\eta\left(\lambda_2,\lambda_1\right)\right)dv.$$

Integrating by parts I_1, we get

$$I_1=-\frac{4}{\eta(\lambda_2,\lambda_1)}\left(1-v^{\alpha}\right)\varphi\left(\lambda_1+\frac{1-v}{4}\eta\left(\lambda_2,\lambda_1\right)\right)dv\Big|_{v=0}^{v=1}$$

$$-\frac{4\alpha}{\eta(\lambda_2,\lambda_1)}\int_0^1 v^{\alpha-1}\varphi\left(\lambda_1+\frac{1-v}{4}\eta\left(\lambda_2,\lambda_1\right)\right)dv$$

$$=\frac{4}{\eta(\lambda_2,\lambda_1)}\varphi\left(\frac{4\lambda_1+\eta(\lambda_2,\lambda_1)}{4}\right)-\frac{4\alpha}{\eta(\lambda_2,\lambda_1)}\int_0^1 v^{\alpha-1}\varphi\left(\lambda_1+\frac{1-v}{4}\eta(\lambda_2,\lambda_1)\right)dv$$

$$=\frac{4}{\eta(\lambda_2,\lambda_1)}\varphi\left(\frac{4\lambda_1+\eta(\lambda_2,\lambda_1)}{4}\right)$$

$$-\frac{4^{\alpha+1}\alpha}{(\eta(\lambda_2,\lambda_1))^{\alpha+1}}\int_{\lambda_1}^{\frac{4\lambda_1+\eta(\lambda_2,\lambda_1)}{4}}\left(\frac{4\lambda_1+\eta(\lambda_2,\lambda_1)}{4}-u\right)^{\alpha-1}\varphi(u)\,du$$

$$=\frac{4}{\eta(\lambda_2,\lambda_1)}\varphi\left(\frac{4\lambda_1+\eta(\lambda_2,\lambda_1)}{4}\right)-\frac{4^{\alpha+1}\Gamma(\alpha+1)}{(\eta(\lambda_2,\lambda_1))^{\alpha+1}}I^{\alpha}_{(\lambda_1)^+}\varphi\left(\frac{4\lambda_1+\eta(\lambda_1,\lambda_2)}{4}\right). \tag{2}$$

Similarly, we get

$$I_2=-\frac{4}{\eta(\lambda_1,\lambda_2)}\left((1-v)^{\alpha}-\frac{17}{15}\right)\varphi\left(\lambda_1+\frac{2-v}{4}\eta\left(\lambda_2,\lambda_1\right)\right)\Bigg|_{v=0}^{v=1}$$

$$-\frac{4\alpha}{\eta\left(\lambda_2,\lambda_1\right)}\int_0^1(1-v)^{\alpha-1}\varphi\left(\lambda_1+\frac{2-v}{4}\eta\left(\lambda_2,\lambda_1\right)\right)dv$$

$$=\frac{68}{15\eta\left(\lambda_2,\lambda_1\right)}\varphi\left(\frac{4\lambda_1+\eta\left(\lambda_2,\lambda_1\right)}{4}\right)-\frac{8}{15\eta\left(\lambda_2,\lambda_1\right)}\varphi\left(\frac{2\lambda_1+\eta\left(\lambda_2,\lambda_1\right)}{2}\right)$$

$$-\frac{4^{\alpha+1}\alpha}{(\eta(\lambda_2,\lambda_1))^{\alpha+1}}\int_{\frac{4\lambda_1+\eta(\lambda_2,\lambda_1)}{4}}^{\frac{2\lambda_1+\eta(\lambda_2,\lambda_1)}{2}}\left(u-\frac{4\lambda_1+\eta(\lambda_1,\lambda_2)}{4}\right)^{\alpha-1}\varphi(u)\,du$$

$$=\frac{68}{15\eta\left(\lambda_2,\lambda_1\right)}\varphi\left(\frac{4\lambda_1+\eta\left(\lambda_2,\lambda_1\right)}{4}\right)-\frac{8}{15\eta\left(\lambda_2,\lambda_1\right)}\varphi\left(\frac{2\lambda_1+\eta\left(\lambda_2,\lambda_1\right)}{2}\right)$$

$$-\frac{4^{\alpha+1}\Gamma(\alpha+1)}{(\eta(\lambda_2,\lambda_1))^{\alpha+1}}I^{\alpha}_{\left(\frac{2\lambda_1+\eta(\lambda_2,\lambda_1)}{2}\right)^-}\varphi\left(\frac{4\lambda_1+\eta(\lambda_1,\lambda_2)}{4}\right), \tag{3}$$

$$I_3=-\frac{4}{\eta\left(\lambda_2,\lambda_1\right)}\left(\frac{17}{15}-v^{\alpha}\right)\varphi\left(\lambda_1+\frac{3-v}{4}\eta\left(\lambda_2,\lambda_1\right)\right)\Bigg|_{v=0}^{v=1}$$

$$
\begin{aligned}
&-\frac{4\alpha}{\eta\left(\lambda_2,\lambda_1\right)}\int\limits_0^1 v^{\alpha-1}\varphi\left(\lambda_1+\frac{3-v}{4}\eta\left(\lambda_2,\lambda_1\right)\right)dv\\
&=-\frac{8}{15\eta(\lambda_1,\lambda_2)}\varphi\left(\frac{2\lambda_1+\eta\left(\lambda_2,\lambda_1\right)}{2}\right)+\frac{68}{15\eta\left(\lambda_2,\lambda_1\right)}\varphi\left(\frac{4\lambda_1+3\eta\left(\lambda_2,\lambda_1\right)}{4}\right)\\
&-\frac{4^{\alpha+1}\alpha}{\left(\eta(\lambda_2,\lambda_1)\right)^{\alpha+1}}\int\limits_{\frac{2\lambda_1+\eta(\lambda_2,\lambda_1)}{2}}^{\frac{4\lambda_1+3\eta(\lambda_2,\lambda_1)}{4}}\left(\frac{4\lambda_1+3\eta\left(\lambda_2,\lambda_1\right)}{4}-u\right)^{\alpha-1}\varphi\left(u\right)du\\
&=-\frac{8}{15\eta(\lambda_1,\lambda_2)}\varphi\left(\frac{2\lambda_1+\eta\left(\lambda_2,\lambda_1\right)}{2}\right)+\frac{68}{15\eta\left(\lambda_2,\lambda_1\right)}\varphi\left(\frac{4\lambda_1+3\eta\left(\lambda_2,\lambda_1\right)}{4}\right)\\
&-\frac{4^{\alpha+1}\Gamma(\alpha+1)}{\left(\eta(\lambda_2,\lambda_1)\right)^{\alpha+1}}I^{\alpha}_{\left(\frac{2\lambda_1+\eta(\lambda_2,\lambda_1)}{2}\right)^+}\varphi\left(\frac{4\lambda_1+3\eta(\lambda_1,\lambda_2)}{4}\right),
\end{aligned}
\tag{4}
$$

and

$$
\begin{aligned}
I_4&=\frac{4}{\eta\left(\lambda_2,\lambda_1\right)}\left(v^{\alpha}-1\right)\varphi\left(\lambda_1+\frac{3+v}{4}\eta\left(\lambda_2,\lambda_1\right)\right)\Bigg|_{v=0}^{v=1}\\
&-\frac{4\alpha}{\eta\left(\lambda_2,\lambda_1\right)}\int\limits_0^1 v^{\alpha-1}\varphi\left(\lambda_1+\frac{3+v}{4}\eta\left(\lambda_2,\lambda_1\right)\right)dv\\
&=\frac{4}{\eta\left(\lambda_2,\lambda_1\right)}\varphi\left(\frac{4\lambda_1+3\eta\left(\lambda_2,\lambda_1\right)}{4}\right)\\
&-\frac{4^{\alpha+1}\alpha}{\left(\eta(\lambda_2,\lambda_1)\right)^{\alpha+1}}\int\limits_{\frac{4\lambda_1+3\eta(\lambda_2,\lambda_1)}{4}}^{\lambda_1+\eta(\lambda_2,\lambda_1)}\left(u-\frac{4\lambda_1+3\eta\left(\lambda_2,\lambda_1\right)}{4}\right)^{\alpha-1}\varphi\left(u\right)du\\
&=\frac{4}{\eta\left(\lambda_2,\lambda_1\right)}\varphi\left(\frac{4\lambda_1+3\eta\left(\lambda_2,\lambda_1\right)}{4}\right)-\frac{4^{\alpha+1}\Gamma(\alpha+1)}{\left(\eta(\lambda_2,\lambda_1)\right)^{\alpha+1}}I^{\alpha}_{(\lambda_1+\eta(\lambda_2,\lambda_1))^-}\varphi\left(\frac{4\lambda_1+3\eta(\lambda_1,\lambda_2)}{4}\right).
\end{aligned}
\tag{5}
$$

Adding (2)–(5), and then multiplying the resulting equality by $\frac{\eta(\lambda_2,\lambda_1)}{16}$, we get the desired result. □

Theorem 1 Let φ be as in Lemma 1. If $|\varphi'|$ is strongly preinvex, then we have

$$\left|\frac{8\varphi\left(\frac{4\lambda_1+\eta(\lambda_2,\lambda_1)}{4}\right)-\varphi\left(\frac{2\lambda_1+\eta(\lambda_2,\lambda_1)}{2}\right)+8\varphi\left(\frac{4\lambda_1+3\eta(\lambda_2,\lambda_1)}{4}\right)}{15}-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(\eta(\lambda_2,\lambda_1))^{\alpha}}\mathscr{I}^{\alpha}(\lambda_1,\lambda_2,\varphi)\right|$$

$$\leq\frac{\eta(\lambda_2,\lambda_1)}{16(\alpha+1)(\alpha+2)}\left(\frac{7\alpha^2+13\alpha}{8}\left(|\varphi'(\lambda_1)|+|\varphi'(\lambda_1+\eta(\lambda_2,\lambda_1))|\right)\right.$$

$$+\frac{5\alpha^2+12\alpha+1}{6}\left(\left|\varphi'\left(\frac{4\lambda_1+\eta(\lambda_2,\lambda_1)}{4}\right)\right|+\left|\varphi'\left(\frac{4\lambda_1+3\eta(\lambda_2,\lambda_1)}{4}\right)\right|\right)$$

$$\left.+\frac{17\alpha^2+31\alpha+4}{20}\left|\varphi'\left(\frac{2\lambda_1+\eta(\lambda_2,\lambda_1)}{2}\right)\right|+\frac{131\alpha^3+696\alpha^2+901\alpha+66}{2880(\alpha+3)}c|\eta(\lambda_2,\lambda_1)|^2\right).$$

Proof. From Lemma 1, modulus and strongly preinvexity of $|\varphi'|$, we have

$$\left|\frac{8\varphi\left(\frac{4\lambda_1+\eta(\lambda_2,\lambda_1)}{4}\right)-\varphi\left(\frac{2\lambda_1+\eta(\lambda_2,\lambda_1)}{2}\right)+8\varphi\left(\frac{4\lambda_1+3\eta(\lambda_2,\lambda_1)}{4}\right)}{15}-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(\eta(\lambda_2,\lambda_1))^{\alpha}}\mathscr{I}^{\alpha}(\lambda_1,\lambda_2,\varphi)\right|$$

$$\leq\frac{\eta(\lambda_2,\lambda_1)}{16}\left(\int_0^1(1-v^{\alpha})\left|\varphi'\left(\lambda_1+\frac{1-v}{4}\eta(\lambda_2,\lambda_1)\right)\right|dv\right.$$

$$+\int_0^1\left(\frac{17}{15}-(1-v)^{\alpha}\right)\left|\varphi'\left(\lambda_1+\frac{2-v}{4}\eta(\lambda_2,\lambda_1)\right)\right|dv$$

$$+\int_0^1\left(\frac{17}{15}-v^{\alpha}\right)\left|\varphi'\left(\lambda_1+\frac{3-v}{4}\eta(\lambda_2,\lambda_1)\right)\right|dv$$

$$\left.+\int_0^1(1-v^{\alpha})\left|\varphi'\left(\lambda_1+\frac{3+v}{4}\eta(\lambda_2,\lambda_1)\right)\right|dv\right)$$

$$\leq\frac{\eta(\lambda_2,\lambda_1)}{16}\left(\int_0^1(1-v^{\alpha})\left(\frac{3+v}{4}|\varphi'(\lambda_1)|+\frac{1-v}{4}\left|\varphi'\left(\frac{4\lambda_1+\eta(\lambda_2,\lambda_1)}{4}\right)\right|\right.\right.$$

$$\left.\left.+c\frac{(1-v)(3+v)}{256}|\eta(\lambda_2,\lambda_1)|^2\right)dv\right.$$

$$+\int_0^1\left(\frac{17}{15}-(1-v)^{\alpha}\right)\left(\frac{2+v}{4}\left|\varphi'\left(\frac{4\lambda_1+\eta(\lambda_2,\lambda_1)}{4}\right)\right|\right.$$

$$\left.+\frac{2-v}{4}\left|\varphi'\left(\frac{2\lambda_1+\eta(\lambda_2,\lambda_1)}{2}\right)\right|+c\frac{4-v^2}{256}|\eta(\lambda_2,\lambda_1)|^2\right)dv$$

$$+\int_0^1\left(\frac{17}{15}-v^{\alpha}\right)\left(\frac{1+v}{4}\left|\varphi'\left(\frac{2\lambda_1+\eta(\lambda_2,\lambda_1)}{2}\right)\right|\right.$$

$$\left.+\frac{3-v}{4}\left|\varphi'\left(\frac{4\lambda_1+3\eta(\lambda_2,\lambda_1)}{4}\right)\right|+c\frac{(1+v)(3-v)}{256}|\eta(\lambda_2,\lambda_1)|^2\right)dv$$

$$+\int_0^1(1-v^{\alpha})\left(\frac{1-v}{4}\left|\varphi'\left(\frac{4\lambda_1+3\eta(\lambda_2,\lambda_1)}{4}\right)\right|\right.$$

$$\left.\left.+\frac{3+v}{4}|\varphi'(\lambda_1+\eta(\lambda_2,\lambda_1))|+c\frac{(1-v)(3+v)}{256}|\eta(\lambda_2,\lambda_1)|^2\right)dv\right)$$

$$=\frac{\eta(\lambda_2,\lambda_1)}{16(\alpha+1)(\alpha+2)}\left(\frac{7\alpha^2+13\alpha}{8}\left(|\varphi'(\lambda_1)|+|\varphi'(\lambda_1+\eta(\lambda_2,\lambda_1))|\right)\right.$$

$$+\frac{5\alpha^2+12\alpha+1}{6}\left(\left|\varphi'\left(\frac{4\lambda_1+\eta(\lambda_2,\lambda_1)}{4}\right)\right|+\left|\varphi'\left(\frac{4\lambda_1+3\eta(\lambda_2,\lambda_1)}{4}\right)\right|\right)$$

$$\left.+\frac{17\alpha^2+31\alpha+4}{20}\left|\varphi'\left(\frac{2\lambda_1+\eta(\lambda_2,\lambda_1)}{2}\right)\right|+\frac{131\alpha^3+696\alpha^2+901\alpha+66}{2880(\alpha+3)}c|\eta(\lambda_2,\lambda_1)|^2\right),$$

where we have used the facts that

$$\int_0^1(1-v^{\alpha})\frac{3+v}{4}dv=\frac{1}{4}\int\left(3+v-3v^{\alpha}-3v^{\alpha+1}\right)dv$$

$$=\frac{7\alpha^2+13\alpha}{8(\alpha+1)(\alpha+2)},\tag{6}$$

$$\int_0^1(1-v^{\alpha})\frac{1-v}{4}dv=\frac{1}{4}\int_0^1\left(1-v-v^{\alpha}+v^{\alpha+1}\right)dv$$

$$= \frac{\alpha^2 + 3\alpha}{8(\alpha+1)(\alpha+2)}, \tag{7}$$

$$\int_0^1 (1 - v^\alpha) \frac{(1-v)(3+v)}{256} dv = \frac{1}{256} \int_0^1 (1 - v^\alpha)(1-v)(3+v)\, dv$$

$$= \frac{1}{256} \int_0^1 \left(3 - 2v - v^2 - 3v^\alpha + 2v^{\alpha+1} + v^{\alpha+2}\right) dv$$

$$= \frac{5\alpha^3 + 30\alpha^2 + 43\alpha}{768(\alpha+1)(\alpha+2)(\alpha+3)}, \tag{8}$$

$$\int_0^1 \left(\frac{17}{15} - (1-v)^\alpha\right) \frac{2+v}{4} dv = \int_0^1 \left(\frac{17}{15} - v^\alpha\right) \frac{3-v}{4} dv$$

$$= \frac{1}{4} \int_0^1 \left(\frac{17}{5} - \frac{17}{15} v - 3v^\alpha + v^{\alpha+1}\right) dv$$

$$= \frac{17\alpha^2 + 39\alpha + 4}{24(\alpha+1)(\alpha+2)}, \tag{9}$$

$$\int_0^1 \left(\frac{17}{15} - (1-v)^\alpha\right) \frac{2-v}{4} dv = \int_0^1 \left(\frac{17}{15} - v^\alpha\right) \frac{1+v}{4} dv$$

$$= \frac{1}{4} \int_0^1 \left(\frac{17}{15} + \frac{17}{15} v - v^\alpha - v^{\alpha+1}\right) dv$$

$$= \frac{17\alpha^2 + 31\alpha + 4}{40(\alpha+1)(\alpha+2)}, \tag{10}$$

and

$$\int_0^1 \left(\frac{17}{15} - (1-v)^\alpha\right) \frac{4-v^2}{256} dv = \int_0^1 \left(\frac{17}{15} - v^\alpha\right) \frac{(1+v)(3-v)}{256} dv$$

$$=\frac{1}{256}\int_{0}^{1}\left(\frac{17}{5}+\frac{34}{15}v-\frac{17}{15}v^{2}-3v^{\alpha}-2v^{\alpha+1}+v^{\alpha+2}\right)dv$$

$$=\frac{187\alpha^{3}+942\alpha^{2}+1157\alpha+132}{11520\left(\alpha+1\right)\left(\alpha+2\right)\left(\alpha+3\right)}. \tag{11}$$

The proof is completed. □

Corollary 1 If we take $\alpha=1$ in Theorem 1, we get

$$\left|\frac{8\varphi\left(\frac{4\lambda_1+\eta(\lambda_2,\lambda_1)}{4}\right)-\varphi\left(\frac{2\lambda_1+\eta(\lambda_2,\lambda_1)}{2}\right)+8\varphi\left(\frac{4\lambda_1+3\eta(\lambda_2,\lambda_1)}{4}\right)}{15}-\frac{1}{\eta\left(\lambda_2,\lambda_1\right)}\int_{\lambda_1}^{\lambda_1+\eta(\lambda_2,\lambda_1)}\varphi\left(u\right)du\right|$$

$$\leq\frac{\eta\left(\lambda_2,\lambda_1\right)}{960}\left(25\left(\left|\varphi'\left(\lambda_1\right)\right|+\left|\varphi'\left(\lambda_1+\eta\left(\lambda_2,\lambda_1\right)\right)\right|\right)+26\left|\varphi'\left(\frac{2\lambda_1+\eta\left(\lambda_2,\lambda_1\right)}{2}\right)\right|\right.$$

$$\left.+30\left(\left|\varphi'\left(\frac{4\lambda_1+\eta\left(\lambda_2,\lambda_1\right)}{4}\right)\right|+\left|\varphi'\left(\frac{4\lambda_1+3\eta\left(\lambda_2,\lambda_1\right)}{4}\right)\right|\right)+\frac{299}{192}c\left|\eta(\lambda_2,\lambda_1)\right|^{2}\right).$$

Corollary 2 If we choose $\eta\left(\lambda_2,\lambda_1\right)=\lambda_2-\lambda_1$ in Theorem 1, we get

$$\left|\frac{1}{15}\left(8\varphi\left(\frac{3\lambda_1+\lambda_2}{4}\right)-\varphi\left(\frac{\lambda_1+\lambda_2}{2}\right)+8\varphi\left(\frac{\lambda_1+3\lambda_2}{4}\right)\right)-\frac{4^{\alpha-1}\Gamma\left(\alpha+1\right)}{\left(\lambda_2-\lambda_1\right)^{\alpha}}\mathscr{J}^{\alpha}\left(\lambda_1,\lambda_2,\varphi\right)\right|$$

$$\leq\frac{\lambda_2-\lambda_1}{16\left(\alpha+1\right)\left(\alpha+2\right)}\left(\frac{7\alpha^{2}+13\alpha}{8}\left(\left|\varphi'\left(\lambda_1\right)\right|+\left|\varphi'\left(\lambda_2\right)\right|\right)\right.$$

$$+\frac{5\alpha^{2}+12\alpha+1}{6}\left(\left|\varphi'\left(\frac{3\lambda_1+\lambda_2}{4}\right)\right|+\left|\varphi'\left(\frac{\lambda_1+3\lambda_2}{4}\right)\right|\right)$$

$$\left.+\frac{17\alpha^{2}+31\alpha+4}{20}\left|\varphi'\left(\frac{\lambda_1+\lambda_2}{2}\right)\right|+\frac{131\alpha^{3}+696\alpha^{2}+901\alpha+66}{2880\left(\alpha+3\right)}c(\lambda_2-\lambda_1)^{2}\right),$$

where

$$\mathscr{J}^{\alpha}\left(\lambda_1,\lambda_2,\varphi\right)=I^{\alpha}_{(\lambda_1)^{+}}\varphi\left(\frac{3\lambda_1+\lambda_2)}{4}\right)+I^{\alpha}_{\left(\frac{\lambda_1+\lambda_2}{2}\right)^{-}}\varphi\left(\frac{3\lambda_1+\lambda_2)}{4}\right)$$

$$+I^{\alpha}_{\left(\frac{\lambda_1+\lambda_2}{2}\right)^{+}}\varphi\left(\frac{\lambda_1+3\lambda_2}{4}\right)+I^{\alpha}_{(\lambda_2)^{-}}\varphi\left(\frac{\lambda_1+3\lambda_2}{4}\right).$$

Corollary 3 In Corollary 2, if we use the strongly convexity of $|\varphi'|$, we get

$$\left|\frac{1}{15}\left(8\varphi\left(\frac{3\lambda_1+\lambda_2}{4}\right)-\varphi\left(\frac{\lambda_1+\lambda_2}{2}\right)+8\varphi\left(\frac{\lambda_1+3\lambda_2}{4}\right)\right)-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(\lambda_2-\lambda_1)^{\alpha}}\mathscr{J}^{\alpha}(\lambda_1,\lambda_2,\varphi)\right|$$

$$\leq\frac{\lambda_2-\lambda_1}{1920(\alpha+1)(\alpha+2)}\left(\left(256\alpha^2+528\alpha+32\right)\left(\left|\varphi'(\lambda_1)\right|+\left|\varphi'(\lambda_2)\right|\right)\right.$$

$$\left.-\frac{1381\alpha^3+7116\alpha^2+9251\alpha+906}{24(\alpha+3)}c(\lambda_2-\lambda_1)^2\right).$$

Corollary 4 In Corollary 1, if we make c tend towards 0, we get

$$\left|\frac{8\varphi\left(\frac{4\lambda_1+\eta(\lambda_2,\lambda_1)}{4}\right)-\varphi\left(\frac{2\lambda_1+\eta(\lambda_2,\lambda_1)}{2}\right)+8\varphi\left(\frac{4\lambda_1+3\eta(\lambda_2,\lambda_1)}{4}\right)}{15}-\frac{1}{\eta(\lambda_2,\lambda_1)}\int_{\lambda_1}^{\lambda_1+\eta(\lambda_2,\lambda_1)}\varphi(u)\,du\right|$$

$$\leq\frac{\eta(\lambda_2,\lambda_1)}{960}\left(25\left(\left|\varphi'(\lambda_1)\right|+\left|\varphi'(\lambda_1+\eta(\lambda_2,\lambda_1))\right|\right)+26\left|\varphi'\left(\frac{2\lambda_1+\eta(\lambda_2,\lambda_1)}{2}\right)\right|\right.$$

$$\left.+30\left(\left|\varphi'\left(\frac{4\lambda_1+\eta(\lambda_2,\lambda_1)}{4}\right)\right|+\left|\varphi'\left(\frac{4\lambda_1+3\eta(\lambda_2,\lambda_1)}{4}\right)\right|\right)\right). \tag{12}$$

Moreover, if we choose $\eta(\lambda_2,\lambda_1)=\lambda_2-\lambda_1$, we get

$$\left|\frac{1}{15}\left(8\varphi\left(\frac{3\lambda_1+\lambda_2}{4}\right)-\varphi\left(\frac{\lambda_1+\lambda_2}{2}\right)+8\varphi\left(\frac{\lambda_1+3\lambda_2}{4}\right)\right)-\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\varphi(u)\,du\right|$$

$$\leq\frac{\lambda_2-\lambda_1}{960}\left(25\left(\left|\varphi'(\lambda_1)\right|+\left|\varphi'(\lambda_2)\right|\right)+26\left|\varphi'\left(\frac{\lambda_1+\lambda_2}{2}\right)\right|\right.$$

$$\left.+30\left(\left|\varphi'\left(\frac{3\lambda_1+\lambda_2}{4}\right)\right|+\left|\varphi'\left(\frac{\lambda_1+3\lambda_2}{4}\right)\right|\right)\right). \tag{13}$$

Corollary 5 Using the convexity of $|\varphi'|$, inequality (4) becomes

$$\left|\frac{1}{15}\left(8\varphi\left(\frac{3\lambda_1+\lambda_2}{4}\right)-\varphi\left(\frac{\lambda_1+\lambda_2}{2}\right)+8\varphi\left(\frac{\lambda_1+3\lambda_2}{4}\right)\right)-\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\varphi(u)\,du\right|$$

$$\leq \frac{17\left(\lambda_2-\lambda_1\right)}{240}\left(\left|\varphi'\left(\lambda_1\right)\right|+\left|\varphi'\left(\lambda_2\right)\right|\right). \tag{14}$$

Remark 1 The result of Corollary 5 can be obtained by Theorem 5 from [16], by letting $\alpha \to 1$ and using the convexity of $|h'|$. Also it represent a refinement of Corollary 2.1 from [25], when $\alpha = 1$.

Theorem 2 Let φ be as in Lemma 1. If $|\varphi'|^q$ is strongly preinvex function where $q>1$ with $\frac{1}{q}+\frac{1}{p}=1$, then we have

$$\left|\frac{8\varphi\left(\frac{4\lambda_1+\eta(\lambda_2,\lambda_1)}{4}\right)-\varphi\left(\frac{2\lambda_1+\eta(\lambda_2,\lambda_1)}{2}\right)+8\varphi\left(\frac{4\lambda_1+3\eta(\lambda_2,\lambda_1)}{4}\right)}{15}-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{\left(\eta\left(\lambda_2,\lambda_1\right)\right)^{\alpha}}\mathscr{I}^{\alpha}\left(\lambda_1,\lambda_2,\varphi\right)\right|$$

$$\leq \frac{\eta\left(\lambda_2,\lambda_1\right)}{16}\left(\left(\frac{B\left(\frac{1}{\alpha},p+1\right)}{\alpha}\right)^{\frac{1}{p}}\right.$$

$$\times\left(\left(\frac{7}{8}\left|\varphi'\left(\lambda_1\right)\right|^q+\frac{1}{8}\left|\varphi'\left(\frac{4\lambda_1+\eta\left(\lambda_2,\lambda_1\right)}{4}\right)\right|^q+\frac{5c}{768}\left|\eta\left(\lambda_2,\lambda_1\right)\right|^2\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{1}{8}\left|\varphi'\left(\frac{4\lambda_1+3\eta\left(\lambda_2,\lambda_1\right)}{4}\right)\right|^q+\frac{7}{8}\left|\varphi'\left(\lambda_1+\eta\left(\lambda_2,\lambda_1\right)\right)\right|^q+\frac{5c}{768}\left|\eta\left(\lambda_2,\lambda_1\right)\right|^2\right)^{\frac{1}{q}}\right)$$

$$+\left(\left(\frac{17}{15}\right)^p\frac{{}_2F_1\left(-p,\frac{1}{\alpha},\frac{1}{\alpha}+1;\frac{15}{17}\right)}{\alpha}\right)^{\frac{1}{p}}$$

$$\times\left(\left(\frac{5}{8}\left|\varphi'\left(\frac{4\lambda_1+\eta\left(\lambda_2,\lambda_1\right)}{4}\right)\right|^q+\frac{3}{8}\left|\varphi'\left(\frac{2\lambda_1+\eta\left(\lambda_2,\lambda_1\right)}{2}\right)\right|^q+\frac{11c}{768}\left|\eta\left(\lambda_2,\lambda_1\right)\right|^2\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\frac{3}{8}\left|\varphi'\left(\frac{2\lambda_1+\eta\left(\lambda_2,\lambda_1\right)}{2}\right)\right|^q+\frac{5}{8}\left|\varphi'\left(\frac{4\lambda_1+3\eta\left(\lambda_2,\lambda_1\right)}{4}\right)\right|^q+\frac{11c}{768}\left|\eta\left(\lambda_2,\lambda_1\right)\right|^2\right)^{\frac{1}{q}}\right)\right),$$

where B and ${}_2F_1$ are beta and hypergeometric function, respectively.

Proof. From Lemma 1, modulus, Hölder's inequality and strongly preinvexity of $|\varphi'|^q$, we have

$$\left|\frac{8\varphi\left(\frac{4\lambda_1+\eta(\lambda_2,\lambda_1)}{4}\right)-\varphi\left(\frac{2\lambda_1+\eta(\lambda_2,\lambda_1)}{2}\right)+8\varphi\left(\frac{4\lambda_1+3\eta(\lambda_2,\lambda_1)}{4}\right)}{15}-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{\left(\eta\left(\lambda_2,\lambda_1\right)\right)^{\alpha}}\mathscr{I}^{\alpha}\left(\lambda_1,\lambda_2,\varphi\right)\right|$$

$$\leq \frac{\eta\left(\lambda_2,\lambda_1\right)}{16}\left(\left(\int_0^1\left(1-v^{\alpha}\right)^p dv\right)^{\frac{1}{p}}\left(\int_0^1\left|\varphi'\left(\lambda_1+\frac{1-v}{4}\eta\left(\lambda_2,\lambda_1\right)\right)\right|^q dv\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^1\left(\frac{17}{15}-(1-v)^{\alpha}\right)^p dv\right)^{\frac{1}{p}}\left(\int_0^1\left|\varphi'\left(\lambda_1+\frac{2-v}{4}\eta\left(\lambda_2,\lambda_1\right)\right)\right|^q dv\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\left(\frac{17}{15}-v^{\alpha}\right)^p dv\right)^{\frac{1}{p}}\left(\int_0^1\left|\varphi'\left(\lambda_1+\frac{3-v}{4}\eta\left(\lambda_2,\lambda_1\right)\right)\right|^q dv\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1(1-v^{\alpha})^p dv\right)^{\frac{1}{p}}\left(\int_0^1\left|\varphi'\left(\lambda_1+\frac{3+v}{4}\eta\left(\lambda_2,\lambda_1\right)\right)\right|^q dv\right)^{\frac{1}{q}}\Bigg)$$

$$\leq\frac{\eta\left(\lambda_2,\lambda_1\right)}{16}\Bigg(\left(\frac{B\left(\frac{1}{\alpha},p+1\right)}{\alpha}\right)^{\frac{1}{p}}\Bigg(\int_0^1\Bigg(\frac{3+v}{4}\left|\varphi'\left(\lambda_1\right)\right|^q$$

$$+\ \frac{1-v}{4}\left|\varphi'\left(\frac{4\lambda_1+\eta\left(\lambda_2,\lambda_1\right)}{4}\right)\right|^q+c\frac{(1-v)(3+v)}{256}\left|\eta(\lambda_2,\lambda_1)\right|^2\Bigg)dv\Bigg)^{\frac{1}{q}}$$

$$+\left(\left(\frac{17}{15}\right)^p\frac{{}_2F_1\left(-p,\frac{1}{\alpha},\frac{1}{\alpha}+1;\frac{15}{17}\right)}{\alpha}\right)^{\frac{1}{p}}\Bigg(\int_0^1\Bigg(\frac{2+v}{4}\left|\varphi'\left(\frac{4\lambda_1+\eta\left(\lambda_2,\lambda_1\right)}{4}\right)\right|^q$$

$$+\ \frac{2-v}{4}\left|\varphi'\left(\frac{2\lambda_1+\eta\left(\lambda_2,\lambda_1\right)}{2}\right)\right|^q+c\frac{4-v^2}{256}\left|\eta(\lambda_2,\lambda_1)\right|^2\Bigg)dv\Bigg)^{\frac{1}{q}}$$

$$+\Bigg(\int_0^1\Bigg(\frac{1+v}{4}\left|\varphi'\left(\frac{2\lambda_1+\eta\left(\lambda_2,\lambda_1\right)}{2}\right)\right|^q+\frac{3-v}{4}\left|\varphi'\left(\frac{4\lambda_1+3\eta\left(\lambda_2,\lambda_1\right)}{4}\right)\right|^q$$

$$+\ c\frac{(3-v)(1+v)}{256}\left|\eta(\lambda_2,\lambda_1)\right|^2\Bigg)dv\Bigg)^{\frac{1}{q}}\Bigg)$$

$$+\left(\frac{B\left(\frac{1}{\alpha},p+1\right)}{\alpha}\right)^{\frac{1}{p}}\Bigg(\int_0^1\Bigg(\frac{1-v}{4}\left|\varphi'\left(\frac{4\lambda_1+3\eta\left(\lambda_2,\lambda_1\right)}{4}\right)\right|^q$$

$$+\ \frac{3+v}{4}\left|\varphi'\left(\lambda_1+\eta\left(\lambda_2,\lambda_1\right)\right)\right|^q+c\frac{(3+v)(1-v)}{256}\left|\eta(\lambda_2,\lambda_1)\right|^2\Bigg)dv\Bigg)^{\frac{1}{q}}\Bigg)$$

$$
\begin{aligned}
&= \frac{\eta(\lambda_2,\lambda_1)}{16}\left(\left(\frac{B\left(\frac{1}{\alpha},p+1\right)}{\alpha}\right)^{\frac{1}{p}}\right.\\
&\times\left(\left(\frac{7}{8}\left|\varphi'(\lambda_1)\right|^q+\frac{1}{8}\left|\varphi'\left(\frac{4\lambda_1+\eta(\lambda_2,\lambda_1)}{4}\right)\right|^q+\frac{5c}{768}\left|\eta(\lambda_2,\lambda_1)\right|^2\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{1}{8}\left|\varphi'\left(\frac{4\lambda_1+3\eta(\lambda_2,\lambda_1)}{4}\right)\right|^q+\frac{7}{8}\left|\varphi'(\lambda_1+\eta(\lambda_2,\lambda_1))\right|^q+\frac{5c}{768}\left|\eta(\lambda_2,\lambda_1)\right|^2\right)^{\frac{1}{q}}\right)\\
&+\left(\left(\frac{17}{15}\right)^p\frac{{}_2F_1\left(-p,\frac{1}{\alpha},\frac{1}{\alpha}+1;\frac{15}{17}\right)}{\alpha}\right)^{\frac{1}{p}}\\
&\times\left(\left(\frac{5}{8}\left|\varphi'\left(\frac{4\lambda_1+\eta(\lambda_2,\lambda_1)}{4}\right)\right|^q+\frac{3}{8}\left|\varphi'\left(\frac{2\lambda_1+\eta(\lambda_2,\lambda_1)}{2}\right)\right|^q+\frac{11c}{768}\left|\eta(\lambda_2,\lambda_1)\right|^2\right)^{\frac{1}{q}}\right.\\
&\left.\left.+\left(\frac{3}{8}\left|\varphi'\left(\frac{2\lambda_1+\eta(\lambda_2,\lambda_1)}{2}\right)\right|^q+\frac{5}{8}\left|\varphi'\left(\frac{4\lambda_1+3\eta(\lambda_2,\lambda_1)}{4}\right)\right|^q+\frac{11c}{768}\left|\eta(\lambda_2,\lambda_1)\right|^2\right)^{\frac{1}{q}}\right)\right),
\end{aligned}
$$

which is the requested result. The proof is completed. □

Corollary 6 In Theorem 2, if we take $\alpha = 1$, we obtain

$$
\begin{aligned}
&\left|\frac{8\varphi\left(\frac{4\lambda_1+\eta(\lambda_2,\lambda_1)}{4}\right)-\varphi\left(\frac{2\lambda_1+\eta(\lambda_2,\lambda_1)}{2}\right)+8\varphi\left(\frac{4\lambda_1+3\eta(\lambda_2,\lambda_1)}{4}\right)}{15}-\frac{1}{\eta(\lambda_2,\lambda_1)}\int\limits_{\lambda_1}^{\lambda_1+\eta(\lambda_2,\lambda_1)}\varphi(u)\,du\right|\\
&\leq \frac{\eta(\lambda_2,\lambda_1)}{16(p+1)^{\frac{1}{p}}}\left(\left(\frac{7}{8}\left|\varphi'(\lambda_1)\right|^q+\frac{1}{8}\left|\varphi'\left(\frac{4\lambda_1+\eta(\lambda_2,\lambda_1)}{4}\right)\right|^q+\frac{5c}{768}\left|\eta(\lambda_2,\lambda_1)\right|^2\right)^{\frac{1}{q}}\right.\\
&+\left(\frac{1}{8}\left|\varphi'\left(\frac{4\lambda_1+3\eta(\lambda_2,\lambda_1)}{4}\right)\right|^q+\frac{7}{8}\left|\varphi'(\lambda_1+\eta(\lambda_2,\lambda_1))\right|^q+\frac{5c}{768}\left|\eta(\lambda_2,\lambda_1)\right|^2\right)^{\frac{1}{q}}\\
&+\left(\frac{17^{p+1}-2^{p+1}}{15^{p+1}}\right)^{\frac{1}{p}}\left(\left(\frac{5}{8}\left|\varphi'\left(\frac{4\lambda_1+\eta(\lambda_2,\lambda_1)}{4}\right)\right|^q\right.\right.\\
&+\frac{3}{8}\left|\varphi'\left(\frac{2\lambda_1+\eta(\lambda_2,\lambda_1)}{2}\right)\right|^q+\frac{11c}{768}\left|\eta(\lambda_2,\lambda_1)\right|^2\right)^{\frac{1}{q}}
\end{aligned}
$$

$$+\left(\frac{3}{8}\left|\varphi'\left(\frac{2\lambda_1+\eta(\lambda_2,\lambda_1)}{2}\right)\right|^q+\frac{5}{8}\left|\varphi'\left(\frac{4\lambda_1+3\eta(\lambda_2,\lambda_1)}{4}\right)\right|^q+\frac{11c}{768}|\eta(\lambda_2,\lambda_1)|^2\right)^{\frac{1}{q}}\Bigg)\Bigg).$$

Corollary 7 In Corollary 6, if we take $\eta(\lambda_2,\lambda_1)=\lambda_2-\lambda_1$, we obtain

$$\left|\frac{1}{15}\left(8\varphi\left(\frac{3\lambda_1+\lambda_2}{4}\right)-\varphi\left(\frac{\lambda_1+\lambda_2}{2}\right)+8\varphi\left(\frac{\lambda_1+3\lambda_2}{4}\right)\right)-\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\varphi(u)\,du\right|$$

$$\leq\frac{\lambda_2-\lambda_1}{16(p+1)^{\frac{1}{p}}}\Bigg(\left(\frac{7}{8}|\varphi'(\lambda_1)|^q+\frac{1}{8}\left|\varphi'\left(\frac{3\lambda_1+\lambda_2}{4}\right)\right|^q+\frac{5c}{768}(\lambda_2-\lambda_1)^2\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{8}\left|\varphi'\left(\frac{\lambda_1+3\lambda_2}{4}\right)\right|^q+\frac{7}{8}|\varphi'(\lambda_2)|^q+\frac{5c}{768}(\lambda_2-\lambda_1)^2\right)^{\frac{1}{q}}$$

$$+\left(\frac{17^{p+1}-2^{p+1}}{15^{p+1}}\right)^{\frac{1}{p}}$$

$$\times\Bigg(\left(\frac{5}{8}\left|\varphi'\left(\frac{3\lambda_1+\lambda_2}{4}\right)\right|^q+\frac{3}{8}\left|\varphi'\left(\frac{\lambda_1+\lambda_2}{2}\right)\right|^q+\frac{11c}{768}(\lambda_2-\lambda_1)^2\right)^{\frac{1}{q}}$$

$$+\left(\frac{3}{8}\left|\varphi'\left(\frac{\lambda_1+\lambda_2}{2}\right)\right|^q+\frac{5}{8}\left|\varphi'\left(\frac{\lambda_1+3\lambda_2}{4}\right)\right|^q+\frac{11c}{768}(\lambda_2-\lambda_1)^2\right)^{\frac{1}{q}}\Bigg)\Bigg).$$

Moreover, if we tend c towards 0, we find

$$\left|\frac{1}{15}\left(8\varphi\left(\frac{3\lambda_1+\lambda_2}{4}\right)-\varphi\left(\frac{\lambda_1+\lambda_2}{2}\right)+8\varphi\left(\frac{\lambda_1+3\lambda_2}{4}\right)\right)-\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\varphi(u)\,du\right|$$

$$\leq\frac{\lambda_2-\lambda_1}{16(p+1)^{\frac{1}{p}}}\left(\left(\frac{7|\varphi'(\lambda_1)|^q+\left|\varphi'\left(\frac{3\lambda_1+\lambda_2}{4}\right)\right|^q}{8}\right)^{\frac{1}{q}}+\left(\frac{\left|\varphi'\left(\frac{\lambda_1+3\lambda_2}{4}\right)\right|^q+7|\varphi'(\lambda_2)|^q}{8}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{17^{p+1}-2^{p+1}}{15^{p+1}}\right)^{\frac{1}{p}}\left(\left(\frac{5\left|\varphi'\left(\frac{3\lambda_1+\lambda_2}{4}\right)\right|^q+3\left|\varphi'\left(\frac{\lambda_1+\lambda_2}{2}\right)\right|^q}{8}\right)^{\frac{1}{q}}\right.$$

$$
+\left(\frac{3\left|\varphi'\left(\frac{\lambda_1+\lambda_2}{2}\right)\right|^q+5\left|\varphi'\left(\frac{\lambda_1+3\lambda_2}{4}\right)\right|^q}{8}\right)^{\frac{1}{q}}\Bigg)\Bigg)\Bigg).
$$

Theorem 3 Let φ be as in Lemma 1. If $|\varphi'|^q$ is strongly preinvex function where $q \geq 1$, then we have

$$
\left|\frac{8\varphi\left(\frac{4\lambda_1+\eta(\lambda_2,\lambda_1)}{4}\right)-\varphi\left(\frac{2\lambda_1+\eta(\lambda_2,\lambda_1)}{2}\right)+8\varphi\left(\frac{4\lambda_1+3\eta(\lambda_2,\lambda_1)}{4}\right)}{15}-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(\eta(\lambda_2,\lambda_1))^{\alpha}}\mathscr{I}^{\alpha}(\lambda_1,\lambda_2,\varphi)\right|
$$

$$
\leq\frac{\eta(\lambda_2,\lambda_1)}{16}\left(\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\left(\frac{7\alpha^2+13\alpha}{8(\alpha+1)(\alpha+2)}|\varphi'(\lambda_1)|^q\right.\right.\right.
$$

$$
\left.+\frac{\alpha^2+3\alpha}{8(\alpha+1)(\alpha+2)}\left|\varphi'\left(\frac{4\lambda_1+\eta(\lambda_2,\lambda_1)}{4}\right)\right|^q+\frac{c\left(5\alpha^3+30\alpha^2+43\alpha\right)|\eta(\lambda_2,\lambda_1)|^2}{768(\alpha+1)(\alpha+2)(\alpha+3)}\right)^{\frac{1}{q}}
$$

$$
+\left(\frac{\alpha^2+3\alpha}{8(\alpha+1)(\alpha+2)}\left|\varphi'\left(\frac{4\lambda_1+3\eta(\lambda_2,\lambda_1)}{4}\right)\right|^q+\frac{7\alpha^2+13\alpha}{8(\alpha+1)(\alpha+2)}|\varphi'(\lambda_1+\eta(\lambda_2,\lambda_1))|^q\right.
$$

$$
\left.\left.+\frac{c\left(5\alpha^3+30\alpha^2+43\alpha\right)|\eta(\lambda_2,\lambda_1)|^2}{768(\alpha+1)(\alpha+2)(\alpha+3)}\right)^{\frac{1}{q}}\right)
$$

$$
+\left(\frac{17\alpha+2}{15(\alpha+1)}\right)^{1-\frac{1}{q}}\left(\left(\frac{17\alpha^2+39\alpha+4}{24(\alpha+1)(\alpha+2)}\left|\varphi'\left(\frac{4\lambda_1+\eta(\lambda_2,\lambda_1)}{4}\right)\right|^q\right.\right.
$$

$$
\left.+\frac{17\alpha^2+31\alpha+4}{40(\alpha+1)(\alpha+2)}\left|\varphi'\left(\frac{2\lambda_1+\eta(\lambda_2,\lambda_1)}{2}\right)\right|^q+\frac{c\left(187\alpha^3+942\alpha^2+1157\alpha+132\right)|\eta(\lambda_2,\lambda_1)|^2}{11520(\alpha+1)(\alpha+2)(\alpha+3)}\right)^{\frac{1}{q}}
$$

$$
+\left(\frac{17\alpha^2+31\alpha+4}{40(\alpha+1)(\alpha+2)}\left|\varphi'\left(\frac{2\lambda_1+\eta(\lambda_2,\lambda_1)}{2}\right)\right|^q+\frac{17\alpha^2+39\alpha+4}{24(\alpha+1)(\alpha+2)}\left|\varphi'\left(\frac{4\lambda_1+3\eta(\lambda_2,\lambda_1)}{4}\right)\right|^q\right.
$$

$$
\left.\left.+\frac{c\left(187\alpha^3+942\alpha^2+1157\alpha+132\right)|\eta(\lambda_2,\lambda_1)|^2}{11520(\alpha+1)(\alpha+2)(\alpha+3)}\right)^{\frac{1}{q}}\right).
$$

Proof. From Lemma 1, modulus, power mean inequality and strongly preinvexity of $|\varphi'|^q$, we have

$$
\left|\frac{8\varphi\left(\frac{4\lambda_1+\eta(\lambda_2,\lambda_1)}{4}\right)-\varphi\left(\frac{2\lambda_1+\eta(\lambda_2,\lambda_1)}{2}\right)+8\varphi\left(\frac{4\lambda_1+3\eta(\lambda_2,\lambda_1)}{4}\right)}{15}-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{(\eta(\lambda_2,\lambda_1))^{\alpha}}\mathscr{I}^{\alpha}(\lambda_1,\lambda_2,\varphi)\right|
$$

$$\leq \frac{\eta\left(\lambda_2,\lambda_1\right)}{16}\Bigg(\left(\int_0^1 \left(1-v^{\alpha}\right)dv\right)^{1-\frac{1}{q}}\left(\int_0^1 \left(1-v^{\alpha}\right)\left|\varphi'\left(\lambda_1+\frac{1-v}{4}\eta\left(\lambda_2,\lambda_1\right)\right)\right|^q dv\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 \left(\frac{17}{15}-(1-v)^{\alpha}\right)dv\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 \left(\frac{17}{15}-(1-v)^{\alpha}\right)\left|\varphi'\left(\lambda_1+\frac{2-v}{4}\eta\left(\lambda_2,\lambda_1\right)\right)\right|^q dv\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 \left(\frac{17}{15}-v^{\alpha}\right)dv\right)^{1-\frac{1}{q}}\left(\int_0^1 \left(\frac{17}{15}-v^{\alpha}\right)\left|\varphi'\left(\lambda_1+\frac{3-v}{4}\eta\left(\lambda_2,\lambda_1\right)\right)\right|^q dv\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 \left(1-v^{\alpha}\right)dv\right)^{1-\frac{1}{q}}\left(\int_0^1 \left(1-v^{\alpha}\right)\left|\varphi'\left(\lambda_1+\frac{3+v}{4}\eta\left(\lambda_2,\lambda_1\right)\right)\right|^q dv\right)^{\frac{1}{q}}\Bigg)$$

$$\leq \frac{\eta\left(\lambda_2,\lambda_1\right)}{16}\Bigg(\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\Bigg(\int_0^1 \left(1-v^{\alpha}\right)\bigg(\frac{3+v}{4}\left|\varphi'\left(\lambda_1\right)\right|^q$$

$$+\frac{1-v}{4}\left|\varphi'\left(\frac{4\lambda_1+\eta\left(\lambda_2,\lambda_1\right)}{4}\right)\right|^q+c\frac{(1-v)(3+v)}{256}\left|\eta(\lambda_2,\lambda_1)\right|^2\bigg)dv\Bigg)^{\frac{1}{q}}$$

$$+\left(\frac{17\alpha+2}{15(\alpha+1)}\right)^{1-\frac{1}{q}}\Bigg(\int_0^1 \left(\frac{17}{15}-(1-v)^{\alpha}\right)\bigg(\frac{2+v}{4}\left|\varphi'\left(\frac{4\lambda_1+\eta\left(\lambda_2,\lambda_1\right)}{4}\right)\right|^q$$

$$+\frac{2-v}{4}\left|\varphi'\left(\frac{2\lambda_1+\eta\left(\lambda_2,\lambda_1\right)}{2}\right)\right|^q+c\frac{4-v^2}{256}\left|\eta(\lambda_2,\lambda_1)\right|^2\bigg)dv\Bigg)^{\frac{1}{q}}$$

$$+\left(\frac{17\alpha+2}{15(\alpha+1)}\right)^{1-\frac{1}{q}}\Bigg(\int_0^1 \left(\frac{17}{15}-v^{\alpha}\right)\bigg(\frac{1+v}{4}\left|\varphi'\left(\frac{2\lambda_1+\eta\left(\lambda_2,\lambda_1\right)}{2}\right)\right|^q$$

$$+\frac{3-v}{4}\left|\varphi'\left(\frac{4\lambda_1+3\eta\left(\lambda_2,\lambda_1\right)}{4}\right)\right|^q+c\frac{(3-v)(1+v)}{256}\left|\eta(\lambda_2,\lambda_1)\right|^2\bigg)dv\Bigg)^{\frac{1}{q}}$$

$$+\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\int_{0}^{1}\left(1-v^{\alpha}\right)\left(\frac{1-v}{4}\left|\varphi'\left(\frac{4\lambda_1+3\eta\left(\lambda_2,\lambda_1\right)}{4}\right)\right|^{q}\right.\right.$$

$$\left.\left.+\ \frac{3+v}{4}\left|\varphi'\left(\lambda_1+\eta\left(\lambda_2,\lambda_1\right)\right)\right|^{q}+c\frac{(3+v)(1-v)}{256}\left|\eta\left(\lambda_2,\lambda_1\right)\right|^{2}\right)dv\right)^{\frac{1}{q}}\right)$$

$$=\frac{\eta\left(\lambda_2,\lambda_1\right)}{16}\left(\left(\frac{\alpha}{\alpha+1}\right)^{1-\frac{1}{q}}\left(\frac{7\alpha^{2}+13\alpha}{8(\alpha+1)(\alpha+2)}\left|\varphi'\left(\lambda_1\right)\right|^{q}\right.\right.$$

$$\left.+\ \frac{\alpha^{2}+3\alpha}{8(\alpha+1)(\alpha+2)}\left|\varphi'\left(\frac{4\lambda_1+\eta\left(\lambda_2,\lambda_1\right)}{4}\right)\right|^{q}+\frac{c\left(5\alpha^{3}+30\alpha^{2}+43\alpha\right)\left|\eta\left(\lambda_2,\lambda_1\right)\right|^{2}}{768(\alpha+1)(\alpha+2)(\alpha+3)}\right)^{\frac{1}{q}}$$

$$+\left(\frac{\alpha^{2}+3\alpha}{8(\alpha+1)(\alpha+2)}\left|\varphi'\left(\frac{4\lambda_1+3\eta\left(\lambda_2,\lambda_1\right)}{4}\right)\right|^{q}\right.$$

$$\left.\left.+\ \frac{\left(7\alpha^{2}+13\alpha\right)}{8(\alpha+1)(\alpha+2)}\left|\varphi'\left(\lambda_1+\eta\left(\lambda_2,\lambda_1\right)\right)\right|^{q}+\frac{c\left(5\alpha^{3}+30\alpha^{2}+43\alpha\right)\left|\eta\left(\lambda_2,\lambda_1\right)\right|^{2}}{768(\alpha+1)(\alpha+2)(\alpha+3)}\right)^{\frac{1}{q}}\right)$$

$$+\left(\frac{17\alpha+2}{15(\alpha+1)}\right)^{1-\frac{1}{q}}\left(\left(\frac{17\alpha^{2}+39\alpha+4}{24(\alpha+1)(\alpha+2)}\left|\varphi'\left(\frac{4\lambda_1+\eta\left(\lambda_2,\lambda_1\right)}{4}\right)\right|^{q}\right.\right.$$

$$\left.+\ \frac{17\alpha^{2}+31\alpha+4}{40(\alpha+1)(\alpha+2)}\left|\varphi'\left(\frac{2\lambda_1+\eta\left(\lambda_2,\lambda_1\right)}{2}\right)\right|^{q}+\frac{c\left(187\alpha^{3}+942\alpha^{2}+1157\alpha+132\right)\left|\eta\left(\lambda_2,\lambda_1\right)\right|^{2}}{11520(\alpha+1)(\alpha+2)(\alpha+3)}\right)^{\frac{1}{q}}$$

$$+\left(\frac{17\alpha^{2}+31\alpha+4}{40(\alpha+1)(\alpha+2)}\left|\varphi'\left(\frac{2\lambda_1+\eta\left(\lambda_2,\lambda_1\right)}{2}\right)\right|^{q}+\frac{17\alpha^{2}+39\alpha+4}{24(\alpha+1)(\alpha+2)}\left|\varphi'\left(\frac{4\lambda_1+3\eta\left(\lambda_2,\lambda_1\right)}{4}\right)\right|^{q}\right.$$

$$\left.\left.+\ \frac{c\left(187\alpha^{3}+942\alpha^{2}+1157\alpha+132\right)\left|\eta\left(\lambda_2,\lambda_1\right)\right|^{2}}{11520(\alpha+1)(\alpha+2)(\alpha+3)}\right)^{\frac{1}{q}}\right),$$

where we have used (6)–(11). The proof is completed. □

Corollary 8 In Theorem 3, if we take $\alpha=1$, we obtain

$$\left|\frac{8\varphi\left(\frac{4\lambda_1+\eta\left(\lambda_2,\lambda_1\right)}{4}\right)-\varphi\left(\frac{2\lambda_1+\eta\left(\lambda_2,\lambda_1\right)}{2}\right)+8\varphi\left(\frac{4\lambda_1+3\eta\left(\lambda_2,\lambda_1\right)}{4}\right)}{15}-\frac{4^{\alpha-1}\Gamma(\alpha+1)}{\left(\eta\left(\lambda_2,\lambda_1\right)\right)^{\alpha}}\mathscr{J}^{\alpha}\left(\lambda_1,\lambda_2,\varphi\right)\right|$$

$$\leq \frac{\eta(\lambda_2,\lambda_1)}{32}\left(\left(\frac{5}{6}\left|\varphi'(\lambda_1)\right|^q+\frac{1}{6}\left|\varphi'\left(\frac{4\lambda_1+\eta(\lambda_2,\lambda_1)}{4}\right)\right|^q+\frac{13c\left|\eta(\lambda_2,\lambda_1)\right|^2}{1536}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{1}{6}\left|\varphi'\left(\frac{4\lambda_1+3\eta(\lambda_2,\lambda_1)}{4}\right)\right|^q+\frac{5}{6}\left|\varphi'(\lambda_1+\eta(\lambda_2,\lambda_1))\right|^q+\frac{13c\left|\eta(\lambda_2,\lambda_1)\right|^2}{1536}\right)^{\frac{1}{q}}$$

$$+\frac{19}{15}\left(\left(\frac{25}{38}\left|\varphi'\left(\frac{4\lambda_1+\eta(\lambda_2,\lambda_1)}{4}\right)\right|^q+\frac{13}{38}\left|\varphi'\left(\frac{2\lambda_1+\eta(\lambda_2,\lambda_1)}{2}\right)\right|^q+\frac{403c\left|\eta(\lambda_2,\lambda_1)\right|^2}{29184}\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\frac{13}{38}\left|\varphi'\left(\frac{2\lambda_1+\eta(\lambda_2,\lambda_1)}{2}\right)\right|^q+\frac{25}{38}\left|\varphi'\left(\frac{4\lambda_1+3\eta(\lambda_2,\lambda_1)}{4}\right)\right|^q+\frac{403c\left|\eta(\lambda_2,\lambda_1)\right|^2}{29184}\right)^{\frac{1}{q}}\right)\right).$$

Moreover, if we choose $\eta(\lambda_2,\lambda_1)=\lambda_2-\lambda_1$ and we tend c towards 0, we find

$$\left|\frac{1}{15}\left(8\varphi\left(\frac{3\lambda_1+\lambda_2}{4}\right)-\varphi\left(\frac{\lambda_1+\lambda_2}{2}\right)+8\varphi\left(\frac{\lambda_1+3\lambda_2}{4}\right)\right)-\frac{1}{\lambda_2-\lambda_1}\int_{\lambda_1}^{\lambda_2}\varphi(u)\,du\right|$$

$$\leq \frac{\lambda_2-\lambda_1}{32}\left(\left(\frac{5\left|\varphi'(\lambda_1)\right|^q+\left|\varphi'\left(\frac{3\lambda_1+\lambda_2}{4}\right)\right|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{\left|\varphi'\left(\frac{\lambda_1+3\lambda_2}{4}\right)\right|^q+5\left|\varphi'(\lambda_2)\right|^q}{6}\right)^{\frac{1}{q}}\right.$$

$$+\frac{19}{15}\left(\left(\frac{25\left|\varphi\left(\frac{3\lambda_1+\lambda_2}{4}\right)\right|^q+'13\left|\varphi'\left(\frac{\lambda_1+\lambda_2}{2}\right)\right|^q}{38}\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\frac{13\left|\varphi'\left(\frac{\lambda_1+\lambda_2}{2}\right)\right|^q+25\left|\varphi'\left(\frac{\lambda_1+3\lambda_2}{4}\right)\right|^q}{38}\right)^{\frac{1}{q}}\right)\right).$$

3. Applications

Let $\mathscr{P}$ be the partition of the points $\lambda_1<\lambda_1+t_1\eta(\lambda_2,\lambda_1)<\lambda_1+t_2\eta(\lambda_2,\lambda_1)<\ldots<t_k=\lambda_1+\eta(\lambda_2,\lambda_1)$ of the interval $[\lambda_1,\lambda_1+\eta(\lambda_2,\lambda_1)]$ with $0=t_0<t_1<t_2<\ldots<t_k=1$, and consider the quadrature formula

$$\int_{\lambda_1}^{\lambda_1+\eta(\lambda_2,\lambda_1)}\varphi(u)\,du=Q(\varphi,\mathscr{P})+\mathscr{E}(\varphi,\mathscr{P}),$$

where

$$Q(\varphi,\mathscr{P})=\sum_{n=0}^{k-1}\frac{8\varphi\left(\lambda_1+\frac{t_{i+1}-t_i}{4}\eta(\lambda_2,\lambda_1)\right)-\varphi\left(\lambda_1+\frac{t_{i+1}-t_i}{2}\eta(\lambda_2,\lambda_1)\right)+8\varphi\left(\lambda_1+\frac{3(t_{i+1}-t_i)}{4}\eta(\lambda_2,\lambda_1)\right)}{15},$$

and the corresponding approximation error is shown by $\mathscr{E}(\varphi,\mathscr{P})$.

Proposition 1 Let $n\in\mathbb{N}$ and let φ be as in Lemma 1. If the function $|\varphi'|^q$ is strongly preinvex, we get

$$\begin{aligned}
|\mathscr{E}(\tau,\mathscr{P})|\le\sum_{n=0}^{k-1}\frac{((t_{i+1}-t_i)\eta(\lambda_2,\lambda_1))^2}{16(p+1)^{\frac{1}{p}}}&\left(\left(\frac{7}{8}|\varphi'(\lambda_1+t_i\eta(\lambda_2,\lambda_1))|^q\right.\right.\\
&+\frac{1}{8}\left|\varphi'\left(\lambda_1+\frac{t_{i+1}-t_i}{4}\eta(\lambda_2,\lambda_1)\right)\right|^q+\left.\frac{5c|(t_{i+1}-t_i)\eta(\lambda_2,\lambda_1)|^2}{768}\right)^{\frac{1}{q}}\\
&+\left(\frac{1}{8}\left|\varphi'\left(\lambda_1+\frac{3(t_{i+1}-t_i)}{4}\eta(\lambda_2,\lambda_1)\right)\right|^q+\frac{7}{8}|\varphi'(\lambda_1+t_{i+1}\eta(\lambda_2,\lambda_1))|^q\right.\\
&+\left.\frac{5c|(t_{i+1}-t_i)\eta(\lambda_2,\lambda_1)|^2}{768}\right)^{\frac{1}{q}}\\
&+\left(\frac{17^{p+1}-2^{p+1}}{15^{p+1}}\right)^{\frac{1}{p}}\left(\left(\frac{5}{8}\left|\varphi'\left(\left(\lambda_1+\frac{t_{i+1}-t_i}{4}\eta(\lambda_2,\lambda_1)\right)\right)\right|^q\right.\right.\\
&+\frac{3}{8}\left|\varphi'\left(\lambda_1+\frac{t_{i+1}-t_i}{2}\eta(\lambda_2,\lambda_1)\right)\right|^q+\left.\frac{11c|(t_{i+1}-t_i)\eta(\lambda_2,\lambda_1)|^2}{768}\right)^{\frac{1}{q}}\\
&+\left(\frac{3}{8}\left|\varphi'\left(\lambda_1+\frac{t_{i+1}-t_i}{2}\eta(\lambda_2,\lambda_1)\right)\right|^q\right.\\
&+\left.\left.\left.\frac{5}{8}\left|\varphi'\left(\lambda_1+\frac{3(t_{i+1}-t_i)}{4}\eta(\lambda_2,\lambda_1)\right)\right|^q+\frac{11c|(t_{i+1}-t_i)\eta(\lambda_2,\lambda_1)|^2}{768}\right)^{\frac{1}{q}}\right)\right).
\end{aligned}$$

Proof. Applying Corollary 6 on the subintervals $[\lambda_1+t_n\eta(\lambda_2,\lambda_1),\lambda_1+t_{n+1}\eta(\lambda_2,\lambda_1)]$ $(n=0$ to $(k-1))$ of the partition $\mathscr{P}$, we get

$$\left|\frac{8\varphi\left(\lambda_1+\frac{t_{i+1}-t_i}{4}\eta(\lambda_2,\lambda_1)\right)-\varphi\left(\lambda_1+\frac{t_{i+1}-t_i}{2}\eta(\lambda_2,\lambda_1)\right)+8\varphi\left(\lambda_1+\frac{3(t_{i+1}-t_i)}{4}\eta(\lambda_2,\lambda_1)\right)}{15}\right.$$

$$-\frac{1}{(t_{i+1}-t_i)\eta(\lambda_2,\lambda_1)}\int_{\lambda_1+t_i\eta(\lambda_2,\lambda_1)}^{\lambda_1+t_{i+1}\eta(\lambda_2,\lambda_1)}\varphi(u)du\Bigg|$$

$$\leq \frac{(t_{i+1}-t_i)\eta(\lambda_2,\lambda_1)}{16(p+1)^{\frac{1}{p}}}\left(\left(\frac{7}{8}|\varphi'(\lambda_1+t_i\eta(\lambda_2,\lambda_1))|^q\right.\right.$$

$$\left.+\frac{1}{8}\left|\varphi'\left(\lambda_1+\frac{t_{i+1}-t_i}{4}\eta(\lambda_2,\lambda_1)\right)\right|^q+\frac{5c|(t_{i+1}-t_i)\eta(\lambda_2,\lambda_1)|^2}{768}\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{8}\left|\varphi'\left(\lambda_1+\frac{3(t_{i+1}-t_i)}{4}\eta(\lambda_2,\lambda_1)\right)\right|^q+\frac{7}{8}|\varphi'(\lambda_1+t_{i+1}\eta(\lambda_2,\lambda_1))|^q\right.$$

$$\left.+\frac{5c|(t_{i+1}-t_i)\eta(\lambda_2,\lambda_1)|^2}{768}\right)^{\frac{1}{q}}$$

$$+\left(\frac{17^{p+1}-2^{p+1}}{15^{p+1}}\right)^{\frac{1}{p}}\left(\left(\frac{5}{8}\left|\varphi'\left(\left(\lambda_1+\frac{t_{i+1}-t_i}{4}\eta(\lambda_2,\lambda_1)\right)\right)\right|^q\right.\right.$$

$$\left.+\frac{3}{8}\left|\varphi'\left(\lambda_1+\frac{t_{i+1}-t_i}{2}\eta(\lambda_2,\lambda_1)\right)\right|^q+\frac{11c|(t_{i+1}-t_i)\eta(\lambda_2,\lambda_1)|^2}{768}\right)^{\frac{1}{q}}$$

$$+\left(\frac{3}{8}\left|\varphi'\left(\lambda_1+\frac{t_{i+1}-t_i}{2}\eta(\lambda_2,\lambda_1)\right)\right|^q\right.$$

$$\left.\left.+\frac{5}{8}\left|\varphi'\left(\lambda_1+\frac{3(t_{i+1}-t_i)}{4}\eta(\lambda_2,\lambda_1)\right)\right|^q+\frac{11c|(t_{i+1}-t_i)\eta(\lambda_2,\lambda_1)|^2}{768}\right)^{\frac{1}{q}}\right)\right).$$

Multiplying the previous inequality by $(t_{n+1}-t_n)\eta(\lambda_2,\lambda_1)$, summing the result across $n=0$ to $(k-1)$, then applying the triangle inequality yields the required inequality. □

Now, we propose some applications to inequalities involving means.

Given arbitrary real numbers λ_1,λ_2 we have: The arithmetic mean: $A(\lambda_1,\lambda_2)=\frac{\lambda_1+\lambda_2}{2}$. The weighted arithmetic mean: $A(m,n,\lambda_1,\lambda_2)=\frac{m\lambda_1+n\lambda_2}{m+n}$. The harmonic mean: $H(\lambda_1,\lambda_2)=\frac{2\lambda_1\lambda_2}{\lambda_1+\lambda_2}$. The geometric mean: $G(\lambda_1,\lambda_2)=\sqrt{\lambda_1\lambda_2}$, $\lambda_1,\lambda_2>0$. The logarithmic mean: $L(\lambda_1,\lambda_2)=\frac{\lambda_2-\lambda_1}{\ln\lambda_2-\ln\lambda_1}$, $\lambda_1,\lambda_2>0$ with $\lambda_1\neq\lambda_2$. The p-logarithmic mean: $L_p(\lambda_1,\lambda_2)=\left(\frac{\lambda_2^{p+1}-\lambda_1^{p+1}}{(p+1)(\lambda_2-\lambda_1)}\right)^{\frac{1}{p}}$, $\lambda_1,\lambda_2>0,\lambda_1\neq\lambda_2$ and $p\in\mathbb{R}\setminus\{-1,0\}$.

Proposition 2 For $1<\lambda_1<\lambda_2$ with $\lambda_1,\lambda_2\in\mathbb{R}$, we have

$$\left|8A^3(3,1,\lambda_1,\lambda_1+G(\lambda_2,\lambda_1))-A^3(\lambda_1,\lambda_1+G(\lambda_2,\lambda_1))\right.$$

$$+\,8A^{3}\left(1,3,\lambda_1,\lambda_1+G\left(\lambda_2,\lambda_1\right)\right)-15L_3^3\left(\lambda_1,\lambda_1+G\left(\lambda_2,\lambda_1\right)\right)\Big|$$

$$\leq\frac{3\lambda_1\sqrt{\lambda_1\lambda_2}}{256}\left(544\lambda_1+544\sqrt{\lambda_1\lambda_2}+201\lambda_2\right).$$

Proof. When applied to the function $\varphi(u)=\frac{1}{3}u^3$ with $\eta(\lambda_2,\lambda_1)=G(\lambda_1,\lambda_2)$, the claim is derived from inequality (12). □

Proposition 3 For $1<\lambda_1<\lambda_2$ with $\lambda_1,\lambda_2\in\mathbb{R}$, we have

$$\left|\frac{16H\left(3\lambda_1,\lambda_2\right)-3H\left(\lambda_1,\lambda_2\right)+16H\left(\lambda_1,3\lambda_2\right)}{3}-15G^2\left(\lambda_1,\lambda_2\right)L^{-1}\left(\lambda_1,\lambda_2\right)\right|$$

$$\leq\frac{17\left(\lambda_2-\lambda_1\right)}{8}G^2\left(\lambda_1,\lambda_2\right)H^{-1}\left(\lambda_1^2,\lambda_2^2\right).$$

Proof. When inequality (14), applied to the function $\varphi(u)=\frac{1}{u}$ on $\left[\frac{1}{\lambda_2},\frac{1}{\lambda_1}\right]$, yields the claim. □

4. Conclusion

This paper introduces a new integral identity based on Riemann-Liouville fractional operators. By applying this identity to functions with strongly preinvex first derivatives, we establish a series of fractional corrected dual Euler-Simpson inequalities. Specific cases are discussed to illustrate the general framework, and applications to inequalities involving different types of means are provided. The findings are expected to stimulate continued exploration in fractional analysis and inequality theory.

Author contributions

All authors contributed to the content and writing of the main manuscript. All authors reviewed the manuscript.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this research.

Confilict of interest

The authors declare no competing financial interest.

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