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Research Article

Associated Statistics' Extensions in the p, q-Rung Orthopair Fuzzy Environment: Application in Interactive Multi-Attribute Decision Making

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Abstract: In order to systematize statistical analysis in interactive Multi-Attribute Decision Making (MADM) models, the concept of "Monotone Expectation" is considered for the extension of classical statistics in the p, q-Rung Orthopair Fuzzy (p, q-ROF) environment, concretely, the probability mean (expectation), variance, covariance, and correlation coefficient. In the classical view, the operations of subtraction and division are not defined in p, q-ROF values. The operations of pseudo-subtraction and pseudo-division are introduced to define the covariance and correlation coefficient. Based on the Choquet finite integral, extensions of monotone statistics such as F-associated expectation, variance, covariance and correlation coefficient for the p, q-ROF environment are presented. The fuzzy measure Associated Probabilities Class (APC) of a fuzzy measure and the classes of associated statistics: probability averaging class, variances class, covariances class and correlation coefficients class are considered. The relationship between monotone and associated statistics is studied. It is established that monotone statistics can be represented by unique corresponding associated statistics if there is a partial ordering between the arguments of the statistics, as between the p, q-ROF values. That is, the remaining $(n! - 1)$ associated statistics of the corresponding associated class, where n is the number of interacting attributes, become useless in the definitions of monotone statistics. In the interactive MADM environment, in decision-making modeling, the aggregations of monotone statistics cannot fully reflect all possible degrees of interaction of attributes. In more detail: the aggregations with monotone statistical parameters under p, q-ROF environment used in interactive MADM models take into account interactions of the focal elements of only one consonant structure from the $n!$ consonant structures of attributes. The extensions of monotone statistics, called F-associated statistics, which take into account the degrees of interaction of elements of all $n!$ consonant structures of attributes in aggregations are defined. The issue of the correctness of the extensions is discussed. The values of monotone and F-associated statistics coincide if there is a partial ordering between the arguments of the statistics and the Choquet second-order extreme capacities are taken in the role of a fuzzy measure. A calculation scheme for monotone and F-associated statistics is constructed for a simple interactive MADM model under p, q-ROF environment. A simple numerical example is presented to illustrate the obtained results.

***Keywords*:** fuzzy statistics, fuzzy measure, Choquet integral, associated probability, monotone statistics, associated statistics, F-associated statistics, Multi-Attribute Decision Making (MADM)

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Abbreviation

MADM	Multi-Attribute Decision Making
MCDM	Multi-Criteria Decision Making
FSs	Fuzzy Sets
q-ROF	q-Rung Orthopair Fuzzy
q-ROFSs	q-Rung Orthopair Fuzzy Sets
q-ROFNs	q-Rung Orthopair Fuzzy Numbers
p, q-ROF	p, q-Rung Orthopair Fuzzy
p, q-ROFSs	p, q-Rung Orthopair Fuzzy Sets
p, q-ROFNs	p, q-Rung Orthopair Fuzzy Numbers
CC	Correlation Coefficient
APC	Associated Probabilities Class
AEC	Associated Expectations Class
ACC	Associated Covariances Class
ACCC	Associated Correlation Coefficients Class
IFSs	Intuitionistic Fuzzy Sets
PFSs	Pythagorean Fuzzy Sets
OWA	Ordered Weighted Averaging
ME	Monotone Expectation
MVar	Monotone Variation
MCov	Monotone Covariation
MCor	Monotone Correlation
E	Expectation
AsPA	Associated Probability Average
AsCov	Associated Covariance
AsCor	Associated Correlation
F-As-p, q-ROFPA	F-Associated p, q-Rung Orthopair Fuzzy Probability Averaging
F-As-p, q-ROFVar	F-Associated p, q-Rung Orthopair Fuzzy Variance
F-As-p, q-ROFCov	F-Associated p, q-Rung Orthopair Fuzzy Covariance
F-As-p, q-ROFCor	F-Associated p, q-Rung Orthopair Fuzzy Correlation
SII	Statistical Independence Index

1. Introduction

1.1 *Problem discussion and research motivation*

Today, the role of MADM models in the analysis problems of complex systems or processes is well known. It is also known that often in MADM modeling of such problems, fuzziness is introduced, when the expert's knowledge and assessments of possible alternatives in relation to the attributes of the decision are the only source of information. In such cases, reducing the degree of uncertainty in expert assessments is an important problem in modeling reliable and credible decision-making. Fuzzy mathematics and fuzzy logic tools play a crucial role in overcoming this problem. Thus, the development of fuzzy modeling approaches is an important problem of MADM [1–3]. Our approach in this study concerns the creation of statistical parameter constructions for the aggregation of fuzzy-probability models in order to reduce the uncertainty of decision-making in various types of fuzzy semantic environments.

Without loss of generality, we consider a relatively simple fuzzy MADM model, which we will conventionally denote by $\langle D, S\rangle$, where $D = \{d_1, d_2, \ldots, d_m\}$ are possible alternatives, and $S = \{s_1, s_2, \ldots, s_n\}$ are the set of attributes that affect the decision-making. Monotone elements $\{\eta_{ij}\}$ in the decision matrix $D \times S$ represent the expert assessment of the alternative d_i, its rating and other with respect to attribute s_j. Often in MADM problems it is important to rank alternatives

$\{d_i\}$, $i = 1, ..., m$, by comparing their evaluation vectors $\eta_i = \{\eta_{i1}, \eta_{i2}, ..., \eta_{in}\}$, $i = 1, 2, ..., m$. For this purpose, we use monotone aggregation tools (schemes and operators), which we will conventionally denote by F, and which map the vectors η_i, $i = 1, 2, ..., m$ to scalar values, where it is possible to use the ranking relation. Simply put, we will say that a decision d_i is no less dominant (worse) than an alternative d_j if $F(\eta_{i1}, \eta_{i2}, ..., \eta_{in}) \geq F(\eta_{j1}, \eta_{j2}, ..., \eta_{jn})$. Therefore, the possibility of obtaining a Pareto optimal solution is created [4].

Often, MADM models allow for the interaction between attributes [5–8]. It is necessary to take this phenomenon into account in F-aggregation tools. In such cases, it is assumed that the role of the uncertainty index on the set of attributes S is played not by additive, but by monotone, fuzzy measures [9–11]. In general, the use of additive and linear aggregation operators in such MADM is not recommended. As usual, fuzzy integral operators are used, in which fuzzy measures play an important role. In practical problems, Choquet [10] or Sugeno [9] type aggregation operators are often used. In this study, we consider monotone statistics based on the Choquet finite integral aggregation operators, such as expectation, variance, covariance and correlation coefficient [12, 13]. The p, q-ROF environment was chosen because it is the most general for orthopair fuzzy sets. It contains fuzzy sets such as Atanassov intuitionistic fuzzy sets [14], Yager Pythagorean [15], and q-rung orthopair fuzzy sets [16].

Similar aggregation statistics, but for arguments of Zadeh's Fuzzy Sets (FSs) agreement levels from [0, 1], are discussed in [17, 18]. In [19, 20], an extension of the so-called "monotone expectation" concept in the case of intuitionistic fuzzy information is presented. Aggregation operators, which are formed using fuzzy measures in MADM models [21–34], are known for their use to reduce the risks of uncertainty in aggregated values, especially when there is interaction between attributes [5–7]. The probabilistic interpretation of the Choquet integral aggregation is related to population expectations [35–37]. Additive probabilistic aggregations are often effective in MADM problems, except for interactive MADM models. Here, the theory of fuzzy measure, as an index describing the uncertainty of data, is not a systematized theory like the theories of probability and statistics, and, therefore, the theory of fuzzy statistics based on fuzzy measure will not be well-structured for interactive MADM models. This applies to the so-called "monotone statistics" based on the fuzzy measure and the Choquet finite integral, in the definition of which the additive probability measure is replaced by a non-additive, but monotone measure (fuzzy measure).

In this study, our objective is to create probabilistic interpretations of monotone statistics with new F-associated statistical aggregation operators, which will facilitate the systematic representation of fuzzy measure-based statistics in the p, q-ROF environment. As we have noted, this problem has been discussed in [17] for arguments of Zadeh's compatibility levels from [0, 1]. A generalization of the concept of "monotone expectation" developed here is presented for the p, q-ROF environment.

1.2 *A brief overview of fuzzy statistics based on the Choquet integral*

As already mentioned, systematic statistical analysis is not designed for q-rung orthopair fuzzy data, of course, due to the complex nature of the data source. Because these data are characterized by interaction/interdependence and at the same time, additive and linear analogs of classical statistics cannot be used for their aggregation. The development of q-rung orthopair fuzzy statistics is mainly in the direction of correlation analysis, where the degree of proximity and their similarity between two or more q-Rung Orthopair Fuzzy Sets (q-ROFSs) or samples is measured. In this regard, we will highlight the following studies. Correlation Coefficients (CCs) have been widely used to measure relationships between FSs and have found applications in data analysis, pattern recognition, machine learning, and Multi-Criteria Decision-Making (MCDM) in the literature. Numerous researchers have employed CCs to facilitate MCDM in fuzzy environments and to analyze the interrelations among FSs extensions. The new CC for q-ROFSs is demonstrated in [38], where bits versatility by showcasing real examples for two special cases, intuitionistic and Pythagorean Fuzzy Sets (PFSs) is presented, and the results with existing methods to highlight its superiority is compared. Examples of the CC formula for Intuitionistic Fuzzy Sets (IFSs) are included in [39], the CC for PFSs is presented in [40], the CC for q-rung orthopair fuzzy sets is described in [41], the two novel λ-CCs for q-ROFSs [42] and a partial CC technique for IFSs applied to pattern recognition are presented in [42, 43].

The difficulty of systematizing fuzzy data and fuzzy set statistics is the integration of expert data and their uncertainty index–the fuzzy measure–into the corresponding population statistics. This is related to the constructions of non-additive, but monotone aggregation statistics. It is known that the Choquet finite integral is distinguished among these integration tools. In this study, based on the Choquet integral, we tried to understand the extensions of classical statistics (expectation, variance, covariance and correlation) for q-ROFSs data, which forms a certain orderly statistical analysis for researchers and practitioners working in the q-ROFSs data environment.

The Choquet integral, or monotone expectation, as it is called in MADM schemes [35–37], has many interesting statistical properties. One of them, based on our research presented here, is its ability to take into account attribute interactions in the decision process [5–7]. This phenomenon has been described in many works for various fuzzy environments [27–34]. However, it should also be noted that the Choquet integral in aggregations takes into account only a certain number of attribute interactions, and not the full set of attribute interactions [27–34] for various types of fuzzy arguments.

It should be noted that the mathematical statistical foundations of fuzzy uncertainty modeling have been developed for a long time. There are mainly two directions: 1) the theory of imprecise probabilities [44, 45], where extreme (upper and lower) expectations are built, and which have a deep semantic interpretation in their applications; 2) the Dempster-Shafer belief structure [46, 47], which is later interpreted as a special case of imprecise probabilities. Later, fuzzy probabilistic and fuzzy statistical approaches [48, 49] took an important place in the problems of statistical modeling of complex phenomena. These approaches developed mainly in two directions. The first includes methods that develop classical data analysis methods within the framework of modern fuzzy set theories. For example: fuzzy clustering [50], multiple fuzzy linear regression [51], fuzzy hypotheses under non-fuzzy population [52], fuzzy logic-based time series prediction problems [53], fuzzy utility based statistical theory of decision-making [54], control's problems in fuzzy metric spaces [55, 56] and others. The second direction includes such approaches as: maximal confidence approach under fuzzy information [57], methods of classification and identification for the Dempster-Shafer Belief Structure (DSBS) [58], the approach of statistical hypotheses for fuzzy data [59], discrimination analysis [33, 34, 60], fuzzy data cluster approaches [61] and others.

As mentioned, in this study, our interest lies in constructing classical statistical extensions for interactive MADM under p, q-ROF information, when the uncertainty index is described by a fuzzy measure. We share the concept of the so-called "Monotone Expectation" of such an approach, which was developed in a number of studies [17, 18, 27–34, 62, 63].

We especially highlight the concept of "Monotone Expectation" developed in [17] for L. Zadeh's compatibility levels from [0, 1]. Here, the authors developed monotone aggregation operators, in which the associated probabilities of the fuzzy measure are used instead of the values of the fuzzy measure [35, 36, 64]. The studies [17, 18, 27–34, 62, 63] are good examples of the modern direction of imprecise probabilities. In [27] the concept of "Monotone Expectation" is developed for different fuzzy environments. In [28] the concept is presented for immediate associated probabilities and triangular fuzzy arguments. The same concept is discussed for intuitionistic fuzzy arguments in [29–31]. The concept is developed in the extensions of the Yager Ordered Weighted Averaging (OWA) operator for intuitionistic fuzzy arguments in [32, 33], where the concept of "Monotone Expectation" is developed for the q-ROF environment. In [17] the concept is developed not for any known operator and not for any fuzzy environment, but for the environment of arguments of real numbers from [0, 1], but for the expectation, variation and moments of the k-order and covariance of classical statistical parameters. For these parameters, the corresponding monotone parameters are defined based on the Choquet integral. In the same way are defined the so-called F-associated statistical parameters, in which the class of associated probabilities of the fuzzy measure is used in the aggregations instead of the fuzzy measure. It should be noted that, unlike the case of monotone statistical parameters, in the definitions of which only one associated probability participates, all possible $n!$ number of associated probabilities participate in F-associated statistical parameters, which ensures that all attribute interactions are taken into account in the aggregations. The relationship between monotone and F-associated statistics is studied. The issue of extension's correctness is also discussed, and a practical example of interactive MADM for ranking alternatives by aggregating monotone and F-associated statistics is given.

In this study, we will limit ourselves to a finite universe, since in real MADM problems the set of attributes is finite. The study presented here develops the concept of "Monotone Expectation" for the p, q-ROF environment, but for such classical statistics as expectation, variance, covariance and correlation coefficient. The obtained results can be considered as a matter of systematization of p, q-ROF statistical theory. The proof is given for the correctness of the extension of

F-associated statistics as monotone statistics. It is proved that for a rather wide class of fuzzy measures, for second-order extreme capacities, min-associated and max-associated statistics coincide with the corresponding monotone statistics.

The first part of the second section presents the connection of the fuzzy measure and its associated probability class with the Choquet finite integral. The main principles of the concept of "Monotone Expectation" are given. The second part of the second section presents fundamental issues of the theory of orthopair fuzzy sets. The third section presents the definitions and properties of pseudo-subtraction and pseudo-division operations for p, q-Rung Orthopair Fuzzy Numbers (p, q-ROFNs). The fourth section presents the definitions and properties of monotone and corresponding associated statistics based on the Choquet integral for the p, q-ROF environment. The relationship between these statistics is presented. The fifth section presents the concepts of F-associated statistics and their relationship to monotone statistics. Proofs are presented regarding correct extensions. Binary relations for ranking alternatives with F-associated and monotone statistics are constructed. Section 6 presents a practical example of MADM using the new aggregation statistics. A comparative analysis is performed. Section 7 discusses conclusions and future perspectives.

2. Preliminary and notations

2.1 *Fuzzy measure and "Monotone Expectation" principle*

As we have noted, the fuzzy measure is an index of the uncertainty of expert data in decision-making analysis. Therefore, its identification and generation are important problems. We will consider only certain aspects of the fuzzy measure. Based on our problems, we present a definition of the fuzzy measure for the finite case [3–5, 11].

Definition 1 Let $S = \{s_1, s_2, ..., s_n\}$ be some finite set and g: $2^S \to [0, 1]$ be some function. We say that g is a fuzzy measure if it satisfies the following conditions:

$$g(\varnothing) = 0,\ \ g(S) = 1,\ \ \forall M, N \subseteq S\ \text{ if }\ M \subseteq N\ \text{ then }\ g(M) \le g(N). \tag{1}$$

A fuzzy measure is considered as a probability measure extension where additivity is replaced by a weaker property, monotonicity.

Let us denote by S_n the group of all possible permutations of the indices $\{1, 2, ..., n\}$. Consider the class of probabilities associated with a fuzzy measure g on the group S_n.

Definition 2 [35] Probability function Asp_σ: $2^S \to [0, 1]$ by definition

$$Asp_\sigma(s_{\sigma(1)}) = g(\{s_{\sigma(1)}\},$$

$$Asp_\sigma(s_{\sigma(j)}) = g(\{s_{\sigma(1)}, ..., s_{\sigma(j)}\} - g(\{s_{\sigma(1)}, ..., s_{\sigma(j-1)}\},$$

$$Asp_\sigma(s_{\sigma(n)}) = 1 - g(\{s_{\sigma(1)}, ..., s_{\sigma(n-1)}\},\ Asp_\sigma(s_{\sigma(0)}) \equiv 0, \tag{2}$$

for each permutation $\sigma \in S_n$ is called the associated probability, while class

$$\langle Asp_\sigma = \{Asp_\sigma(s_{\sigma(1)}), ..., Asp_\sigma(s_{\sigma(n)})\}\rangle_{\sigma \in S_n}$$

is called Associated Probability Class (APC).

Definition 3 [35] Let $S = \{s_1, s_2, ..., s_n\}$ be some finite set, and let g: $2^S \to [0, 1]$ be some fuzzy measure with its APC $\{Asp_\sigma\}_{\sigma \in S_n}$. Let η: $S \to (0, +\infty)$ be some function, $\eta_i \equiv \eta(s_i)$, $i = 1, 2, ..., n$, and let $\tau = \{\tau(1), \tau(2), ..., \tau(n)\} \in S_n$ be a permutation such that $\eta_{\tau(1)} \geq \eta_{\tau(2)} \geq ...\eta_{\tau(n)}$. The Choquet integral of the function η with respect to the fuzzy measure g is called

$$\text{(CH)} \int_S \eta \odot g(\cdot) \triangleq \int_0^\infty g(\{s \in S/\eta(s) \geq \beta\})d\beta = \sum_{j=1}^{n} Asp_\tau\left(s_{\tau(j)}\right) \cdot \eta_{\tau(j)}. \tag{3}$$

Clearly (CH) $\int_S \eta \odot g(\cdot) = E_{Asp_\tau}(\eta)$, where $E_{Asp_\tau}(\eta)$ is the expectation of the value η with respect to the associated probability Asp_τ.

Definition 4 [35] Let $S = \{s_1, s_2, ..., s_n\}$ be some finite set, and let g: $2^S \to [0, 1]$ be some fuzzy measure with its APC $\{Asp_\sigma\}_{\sigma \in S_n}$. Let η: $S \to (0, +\infty)$ be some function. The Monotone Expectation (ME) of η with respect to the fuzzy measure g is called the value of the Choquet integral of η

$$\text{ME}_g(\eta) \triangleq \text{(CH)} \int_S \eta \odot g(\cdot). \tag{4}$$

It becomes clear that the monotone expectation coincides with the probabilistic expectation with respect to some Asp_τ, $\tau \in S_n$, associated probability

$$\text{ME}_g(\eta) = E_{Asp_\tau}(\eta) = \sum_{j=1}^{n} Asp_\tau(s_{\tau(j)}) \cdot \eta_{\tau(j)}, \tag{5}$$

where $\eta(s_{\tau(1)}) \geq \eta(s_{\tau(2)}) \geq ... \geq \eta(s_{\tau(n)})$.

Unlike a single probability distribution describing the uncertainty of a random experiment, the uncertainty index of experimental data–the fuzzy measure is associated with a class of probability distributions, and hence a class of associated expectations.

Definition 5 [17] For each permutation $\sigma \in S_n$, the expectation–$E_{Asp_\sigma}(\eta)$ is called the associated expectation, and the class $\left\{E_{Asp_\sigma}(\eta)\right\}_{\sigma \in S_n}$ is called the Associated Expectations Class (AEC).

The general relationship between the AEC class and ME is as follows.

Proposition 1 [35] Let us say there is some finite set $S = \{s_1, s_2, ..., s_n\}$, some fuzzy measure g: $2^S \to [0, 1]$ with its APC and AEC classes. Then for any function η: $S \to (0, +\infty)$ it is valid

$$\min_{\sigma \in S_n} E_{Asp_\sigma}(\eta) \leq \text{ME}_g(\eta) \leq \max_{\sigma \in S_n} E_{Asp_\sigma}(\eta). \tag{6}$$

There is a certain class of fuzzy measures for which the equalities in (6) are achieved.

Definition 6 [35] Let us say that there is some finite set $S = \{s_1, s_2, ..., s_n\}$ and there are two fuzzy measures g_*, g^*: $2^S \to [0, 1]$ such that for any set $\forall M \subseteq S$

$$g_*(M) = 1 - g^*(\bar{M}). \tag{7}$$

Then g_* and g^* fuzzy measures are called dual measures, $\bar{M} = S \backslash M$.

It is not difficult to prove that APCs of dual fuzzy measures g_* and g^*coincide. Denote this APC with $\{Asp_\sigma\}_{\sigma \in S_n}$.

We now consider one important class of fuzzy measures, which plays an essential role in our future presentations.

Definition 7 [35] The dual g_* and g^* fuzzy measures on 2^S are called the lower and upper second-order Choquet capacities, respectively, if for the pair of sets $\forall M, N \subseteq S$ it is valid the following

$$g_*(M \cup N) + g_*(M \cap N) \geq g_*(M) + g_*(N),$$

$$g^*(M \cup N) + g^*(M \cap N) \leq g^*(M) + g^*(N). \tag{8}$$

Proposition 2 [35] Let g_* and g^* be some dual Choquet second-order lower and upper capacities on 2^S, then
(1) for any function $\eta: S \to (0, +\infty)$

$$\mathrm{ME}_{g_*}(\eta) = \min_{\sigma \in s_n} E_{Asp_\sigma}(\eta),$$

$$\mathrm{ME}_{g^*}(\eta) = \max_{\sigma \in s_n} E_{Asp_\sigma}(\eta); \tag{9}$$

(2) for any set $N \subseteq S$

$$g_*(N) = \min_{\sigma \in s_n} Asp_\sigma(N),$$

$$g^*(N) = \max_{\sigma \in s_n} Asp_\sigma(N). \tag{10}$$

The concept of "Monotone Expectation" presented here was first introduced by Campos and Bolanos [35] and has been developed in a number of papers [8, 17, 18, 27–34, 64] for various fuzzy environments, but only for population expectation. For population variance, the concept is introduced in [17, 18], but for arguments of Zadeh consistency from [0, 1]. In [17], the concept was also developed for the k-order moment and covariance, but again for arguments from [0, 1].

2.2 *Orthopair fuzzy sets and p, q-orthopair fuzzy sets*

L.A. Zadeh's FSs and fuzzy logic-based decision-making modeling approaches are well known. This applies to cases when the objective data included in MADM models are small or do not exist at all. In such cases, as always, expert knowledge and its evaluations are the only source of MADM input data formation. Fuzzy decision-making models are today a successful tool in the analysis and synthesis of complex and difficult phenomena. Expert assessments, his intellectual activity determine the formation of reliable input structures. The formation of expert data in modern semantic forms is a rapidly developing direction today, which implies the creation of new extensions of Zadeh's fuzzy sets. We will consider one of the extensions, which is called the concept of orthopair fuzzy sets [16]. Atanassov extended Zadeh FSs to intuitionistic fuzzy sets [14]. Each element of this set (u, v)–the intuitionistic fuzzy number–denotes the level of compatibility u and the level of incompatibility v of the element in the set. Obviously, the tool of IFSs is more useful for better representation of uncertainty in expert evaluations. These levels have the requirement $u + v \leq 1$. But in many cases this condition is violated and the problem goes beyond the IFS environment. Yager [15] introduced Pythagorean orthopair fuzzy sets, the pair (u, v) of which satisfies the condition $u^2 + v^2 \leq 1$. And if this condition is also violated, then we use again Yager's q-ROFN [16]. (u, v) is q-ROFN, if $u^q + v^q \leq 1$, $q \geq 1$. And finally, the extension of q-ROFN numbers are p, q-ROFN numbers [12, 13]. We say that the pair (u, v) is p, q-ROFNs, if $u^p + v^q \leq 1$, $p, q \geq 1$. It is clear that it includes IFNs, PFNs, q-ROFNs numbers.

Let us briefly review the definitions of p, q-ROFNs.

Definition 8 [12] Let Z be any set. A set M is called a p, q-rung orthopair fuzzy subset of Z if

$$M=\{\langle u_M(z),\, v_M(z)\rangle,\, z\in Z\},\tag{11}$$

where $u_M(z)$ denotes the degree of belonging of the element z to the set M, and $v_M(z)$ of not belonging to M, where

$$u_M(z)\in[0.1],\, v_M(z)\in[0.1],\, [u_M(z)]^p+[v_M(z)]^q\leq 1,\, p,\, q\geq 1.\tag{12}$$

Here for each z, $\eta=\langle z,\, u_\eta(z),\, v_\eta(z)\rangle$ is called p, q-ROFN and is denoted by $\eta=\langle u_\eta,\, v_\eta\rangle$.

Here we give the definitions of arithmetic operations introduced on p, q-ROFNs [12, 13].

Definition 9 [12] Let $\xi=\langle u_\xi,\, v_\xi\rangle$ and $\eta=\langle u_\eta,\, v_\eta\rangle$ be some two p, q-ROFNs. Then

$$\begin{aligned}
&\eta^c=\langle v_\eta,\, u_\eta\rangle;\\
&\eta\oplus\xi=\left\langle\left(u_\eta^p+u_\xi^p-u_\eta^p\cdot u_\xi^p\right)^{1/p},\, v_\eta\cdot v_\xi\right\rangle;\\
&\eta\otimes\xi=\left\langle u_\eta\cdot u_\xi,\, \left(v_\eta^q+v_\xi^q-v_\eta^q\cdot v_\xi^q\right)^{1/q}\right\rangle;\\
&\eta\wedge\xi=\langle\min\{u_\eta,\, u_\xi\},\, \max\{v_\eta,\, v_\xi\}\rangle;\\
&\eta\vee\xi=\langle\max\{u_\eta,\, u_\xi\},\, \min\{v_\eta,\, v_\xi\}\rangle;\\
&\lambda\cdot\eta=\left\langle\left(1-\left(1-u_\eta^p\right)^\lambda\right)^{1/p},\, v_\eta^\lambda\right\rangle,\quad \lambda>0;\\
&\eta^\lambda=\left\langle u_\eta^\lambda,\, \left(1-\left(1-v_\eta^q\right)^\lambda\right)^{1/q}\right\rangle,\quad \lambda>0.
\end{aligned}\tag{13}$$

Definition 10 [12] Let $\eta=\langle u_\eta,\, v_\eta\rangle$ be some p, q-ROFN.

(1) A score function of η is $Score(\eta)=u_\eta^q-v_\eta^q$;

(2) An accuracy function of η is $Accuracy(\eta)=u_\eta^p+v_\eta^q$.

Definition 11 [12] Let $\eta=\langle u_\eta,\, v_\eta\rangle$ and $\xi=(u_\xi,\, v_\xi)$ be some two p, q-ROFNs, then

(1) The partial ordering relation $\underset{pt}{\leq}$ on p, q-ROFNs is defined as

$$\text{If } Score(\eta)\geq Score(\xi) \text{ and } Accuracy(\eta)\leq Accuracy(\xi) \text{ then } \xi\underset{pt}{\leq}\eta;$$

(2) The complete ordering relation $\underset{t}{\leq}$ on p, q-ROFNs is defined as

a) if $Score(\eta) > Score(\xi)$ then $\xi \underset{t}{<} \eta$;

b) if $Score(\eta) = Score(\xi)$ then

if $Accuracy(\eta) > Accuracy(\xi)$ then $\xi \underset{t}{<} \eta$;

$$\text{if } Accuracy(\eta) = Accuracy(\xi) \text{ then } \xi \underset{t}{=} \eta. \tag{14}$$

$\theta = (0, 1)$ is a zero element, and $e = (1, 0)$ is a unit element. Note that the addition $\oplus$ and multiplication $\otimes$ operations are implemented by the bivariate functions $k_1(x, y) = (x^s + y^s - x^s y^s)^{1/s}$, $s \in \{p, q\}$, and $k_2(x, y) = xy$ defined on $[0, 1]^2$. As is known, functions k_1 and k_2 are Archimedean t-conorm and t-norm on $[0, 1]$, respectively, and are defined by the additive dual generators $t_1(x) = -\ln(1 - x^s)$ and $t_2(x) = -\ln(x^s)$, $s \in \{p, q\}$ [16]. We get important inclusions: $\eta \otimes \xi \underset{pt}{\leq} \eta \wedge \xi \underset{pt}{\leq} \eta$, $\xi \underset{pt}{\leq} \eta \vee \xi \underset{pt}{\leq} \eta \oplus \xi$, $\forall \eta, \xi \in p, q$-ROFNs. It is also easy to check that the functions k_1 and k_2 are dual to the function $N_s(x) = (1 - x^s)^{1/s}$, $x \in [0, 1]$, $s \in \{p, q\}$:

$$N_s(k_1(N_s(x), k_1(N_s(y)) = xy = k_2(x, y), x \in [0, 1], s \in \{p, q\}.$$

3. Pseudo-operations of subtraction and division in *p, q*-ROFNs

To develop correlation analysis on p, q-ROFNs data, it is necessary to have the arithmetic operations of subtraction and division. As can be seen, the operations of subtraction and division are not generally defined on p, q-ROFNs due to the nature of these numbers. Since there are no opposites and inverses in the classical sense. In this section, we will introduce pseudo-definitions of these operations and use them in the calculations of the distributions of the statistics of variation, covariance, and correlation coefficient for the p, q-ROFNs environment.

Let $a = \langle u_a, v_a \rangle$ and $b = \langle u_b, v_b \rangle$ be p, q-ROFNs. Then

$$u_a^p + v_a^q \leq 1,\ u_b^p + v_b^q \leq 1,\ 0 \leq u_a, v_a, u_b, v_b \leq 1,\ p, q \geq 1.$$

As we noted in previous paragraph, generation of the addition $\oplus$ and multiplication $\otimes$ operations in p, q-ROFNs is based on Archimedean t-conorm and t-norm–the bivariate functions $k_1(x, y) = (x^s + y^s - x^s y^s)^{1/s}$, $s \in \{p, q\}$, and $k_2(x, y) = xy$ defined on $[0, 1]^2$. Their dual generators are functions $t_1(x) = -\ln(1 - x^s)$ and $t_2(x) = -\ln(x^s)$, $s \in \{p, q\}$. This was the motivation for introducing subtraction and division operations into p, q-ROFNs.

Formula (13) presents the basic arithmetic operations on p, q-ROFNs numbers. Let us introduce the operations of subtraction $\ominus$ and division $\oslash$ on p, q-ROFNs numbers in such a way that the basic properties of these operations are fulfilled, such as

$$b \ominus b = \theta,\ b \ominus \theta = b,\ e \ominus b = e,\ (b \oplus a) \ominus b = a,$$

$$b \oslash e = b,\ b \oslash b = e,\ \theta \oslash b = \theta,\ (b \otimes a) \oslash b = a. \tag{15}$$

It is not difficult to show the following proposition.

Proposition 3 The following operations satisfy the properties (15), preserving the results in p, q-ROFNs and with additional conditions $b \ominus a = c \Leftrightarrow b = a \oplus c$, $b \oslash a = c \Leftrightarrow b = a \otimes c$, $b \neq \theta$, $0/0 = 1$:

$$b \ominus a = \left\langle \left(\frac{u_b^p - u_a^p}{1 - u_a^p} \right)^{1/p}, \frac{v_b}{v_a} \right\rangle \quad \text{if } 0 \leq \frac{v_b}{v_a} \leq \left(\frac{1 - u_b^p}{1 - u_a^p} \right)^{1/q} \leq 1,$$

$$b \oslash a = \left\langle \frac{u_b}{u_a}, \left(\frac{v_b^q - v_a^q}{1 - v_a^q} \right)^{1/q} \right\rangle \quad \text{if } 0 \leq \frac{u_b}{u_a} \leq \left(\frac{1 - v_b^q}{1 - v_a^q} \right)^{1/p} \leq 1. \tag{16}$$

A similar proposition of this proposition is given in [65], but for q-ROFNs numbers. When $p = q$, then the definitions of the subtraction and division operations given here and in [65] coincide. Therefore, we will not give the proof here.

Obviously, the operations defined by formulas (16) are not valid for all p, q-ROFNs numbers. Therefore, let us make extensions of the subtraction and division operations for any p, q-ROFNs numbers. Let us extend the subtraction and division operations to the real numbers by the following principle:

$$b - a = \bigwedge_{c \in \mathbb{R}^1} \{a + c \geq b\} \quad \text{and} \quad b/a = \bigvee_{c \in \mathbb{R}^1} \{a \times c \leq b\}. \tag{17}$$

Then this principle extends to p, q-ROFNs numbers as follows: if a, $b \in p$, q-ROFNs, then

$$b \widetilde{\ominus} a = \bigwedge_{c \in p,\, q\text{-ROFNs}} \{a \oplus c \underset{t}{\geq} b\} \quad \text{and} \quad b \widetilde{\oslash} a = \bigvee_{c \in p,\, q\text{-ROFNs}} \{a \otimes c \underset{t}{\leq} b\}, \tag{18}$$

where

$$e \widetilde{\ominus} e = \theta, \quad e \widetilde{\ominus} b = e, \forall b \neq e \quad \text{and} \quad \theta \widetilde{\oslash} \theta = e, \quad \theta \widetilde{\oslash} b = \theta, \forall b \neq \theta. \tag{19}$$

Proposition 4 Properties (18) and (19) are possessed by the following extended operations of subtraction $\widetilde{\ominus}$ and division $\widetilde{\oslash}$, with the additional condition $0/0 = 1$

$$b \widetilde{\ominus} a = \left\langle \left(0 \vee \left(\frac{u_b^p - u_a^p}{1 - u_a^p} \right) \right)^{1/p}, 1 \wedge \left(\frac{v_b}{v_a} \right) \wedge \left(\frac{1 - u_b^p}{1 - u_a^p} \right)^{1/q} \right\rangle,$$

$$b \widetilde{\oslash} a = \left\langle 1 \wedge \left(\frac{u_b}{u_a} \right) \wedge \left(\frac{1 - v_b^q}{1 - v_a^q} \right)^{1/p}, \left(0 \vee \left(\frac{v_b^q - v_a^q}{1 - v_a^q} \right) \right)^{1/q} \right\rangle. \tag{20}$$

We will not give a proof of this proposition, due to its simplicity. Equalities (20) are correctly related to operations (16) by the following proposition (for simplicity, the proof is also not given).

Proposition 5

If $0 \leq \frac{v_b}{v_a} \leq \left(\frac{1-u_b^p}{1-u_a^p}\right)^{1/q} \leq 1$ then $a\widetilde{\ominus}b = a \ominus b$;

If $0 \leq \frac{u_b}{u_a} \leq \left(\frac{1-v_b^q}{1-v_a^q}\right)^{1/p} \leq 1$ then $a\widetilde{\oslash}b = a \oslash b$.

Note that when $p = q$, the extensions (20) coincide with the analogous extensions for q-ROFNs numbers given in [65].

It is not difficult to show the following algebraic properties of the extended pseudo-operations:

Proposition 6 Let $a = \langle u_a, v_a\rangle$, $b = \langle u_b, v_b\rangle$ and $c = \langle u_c, v_c\rangle$ are p, q-ROFNs numbers, β, β_1, $\beta_2 > 0$ and $\beta_1 > \beta_2$. Then some following equalities and inequalities are valid:

1. $\beta(b\widetilde{\ominus}a) = \beta b\widetilde{\ominus}\beta a$;

2. $(\beta_1 - \beta_2)b = \beta_1 b\widetilde{\ominus}\beta_2 b$;

3. $b^{\beta_1}\widetilde{\oslash}b^{\beta_2} = b^{\beta_1-\beta_2}$;

4. $b^{\beta}\widetilde{\oslash}a^{\beta} = (b\widetilde{\oslash}a)^{\beta}$;

5. $a \oplus (b\widetilde{\ominus}a) \underset{pt}{\geq} b$;

6. $a \otimes (b\widetilde{\oslash}a) \underset{pt}{\leq} b$;

7. $(b \oplus a)\widetilde{\ominus}a = b$;

8. $(b \otimes a)\widetilde{\oslash}a = b$;

9. $b \oplus a \underset{pt}{\geq} c \Leftrightarrow c\widetilde{\ominus}b \underset{pt}{\leq} a$;

10. $b \otimes a \underset{pt}{\leq} c \Leftrightarrow c\widetilde{\oslash}b \underset{pt}{\geq} a$;

11. $c\widetilde{\ominus}(b \oplus a) = c\widetilde{\ominus}b\widetilde{\ominus}a$;

12. $c\widetilde{\oslash}(b \otimes a) = c\widetilde{\oslash}b\widetilde{\oslash}a$;

13. $c\widetilde{\ominus}(c\widetilde{\ominus}a) \underset{pt}{\leq} a$;

14. $c\widetilde{\oslash}(c\widetilde{\oslash}a) \underset{pt}{\geq} a$;

$$15.\ c \oplus b \widetilde{\ominus} a \underset{pt}{\leq} c \widetilde{\ominus} a \oplus b;$$

$$16.\ c \otimes b \widetilde{\oslash} a \underset{pt}{\geq} c \widetilde{\oslash} a \otimes b. \tag{21}$$

In order to systematize statistical analysis in interactive Multi-Attribute Decision Making (MADM) models, the concept of "Monotone Expectation" is considered for the extension of classical statistics in the p, q-Rung Orthopair Fuzzy (p, q-ROF) environment. In the classical view, the arithmetic operations of subtraction and division are not defined in p, q-ROF values. The operations of pseudo-subtraction and pseudo-division are introduced to define the covariance and correlation coefficient. The analytical properties of the new operations were also presented.

4. Monotone statistics based on the Choquet integral and corresponding associated statistics in the *p, q*-ROFNs environment

As we have already noted, in statistics of expert data, including statistics of p, q-ROFNs data, there are no transformations of additive and linear aggregations [66, 67]. In classical statistics, the expected value of the population, i.e., the probability mean, is defined by the Riemann integral aggregation for a finite sample, which is a linear sum along the probability distribution of the data. We, according to the "monotone expectation concept", consider a monotone mean as an extension of the probability mean for a sample of expert data, which is defined by the Choquet integral and actually coincides with the probability mean with respect to one of the associated probability distributions of the fuzzy measure. Specifically, this can be formulated as follows.

Let g: $2^S \to [0, 1]$, $S = \{s_1, s_2, ..., s_n\}$ be a finite fuzzy measure with its associated probabilistic class $\{Asp_\sigma\}_{\sigma \in S_n}$; η: $S \to \{p, q\text{-ROFNs}\}$be some variable with values $\eta = \{\eta_1, \eta_2, ..., \eta_n\}$, $\eta_i = \eta(s_i)$, $i = 1, 2, ..., n$ and $\eta_i \in p, q$-ROFNs. Let us expand the Choquet finite integral for variable η with respect to fuzzy measure g.

Definition 12 The p, q-rung orthopair fuzzy Choquet integral (The monotone expectation) of η with respect to the fuzzy measure g is called the expression

$$\begin{aligned} \mathrm{ME}_g(\eta) &\triangleq \int_S g(\{s \in S \mid \eta = \eta(s) \underset{t}{\geq} \alpha\}) d\alpha \\ &= \bigoplus_{k=1}^{n} \eta_{\tau(k)} \cdot \left[g\left(\left\{s_{\tau(1)}, \ldots, s_{\tau(k)}\right\}\right) - g\left(\left\{s_{\tau(1)}, \ldots, s_{\tau(k-1)}\right\}\right) \right] \\ &= \bigoplus_{k=1}^{n} \eta_{\tau(k)} \cdot \mathrm{Asp}\,\tau\left(s_{\tau(k)}\right) \end{aligned} \tag{22}$$

where $\tau \in S_n$ is a permutation such that $\eta_{\tau(1)} \underset{t}{\geq} \eta_{\tau(2)} \underset{t}{\geq} \cdots \underset{t}{\geq} \eta_{\tau(n)}$.

Definition 13 Let η, ξ: $S \to \{p, q\text{-ROFNs}\}$ be some variables, $\eta = \langle \eta_1, ..., \eta_n \rangle$, $\eta_i = \eta(s_i)$ and $\xi = \langle \xi_1, ..., \xi_n \rangle$, $\xi_i = \xi(s_i)$.

(a) We say that the variables η and ξ are comonotone with respect to partial ordering $\underset{pt}{\leq}$ if there exists a permutation $\tau \in S_n$ such that $\eta_{\tau(1)} \underset{pt}{\geq} \eta_{\tau(2)} \underset{pt}{\geq} \cdots \underset{pt}{\geq} \eta_{\tau(n)}$ and $\xi_{\tau(1)} \underset{pt}{\geq} \xi_{\tau(2)} \underset{pt}{\geq} \cdots \underset{pt}{\geq} \xi_{\tau(n)}$;

(b) We say that the variables η and ξ are comonotone with respect to complete ordering $\underset{t}{\leq}$ if there exists a permutation $\tau \in S_n$ such that $\eta_{\tau(1)} \underset{t}{\geq} \eta_{\tau(2)} \underset{t}{\geq} \cdots \underset{t}{\geq} \eta_{\tau(n)}$ and $\xi_{\tau(1)} \underset{t}{\geq} \xi_{\tau(2)} \underset{t}{\geq} \cdots \underset{t}{\geq} \xi_{\tau(n)}$.

It is clear that if η and ξ are comonotone with respect to partial ordering $\underset{pt}{\leq}$, then they are also comonotone with respect to complete ordering $\underset{t}{\leq}$.

If there is a class of associated probabilities $\{Asp_\sigma\}_{\sigma\in S_n}$ of a fuzzy measure g, then we introduce the following notions of associated classes. A class of associated probabilities induces the classes of associated probability averages (expectations), the associated covariances, associated variations and associated correlation coefficients.

Definition 14 Let $\eta = \langle \eta_1, ..., \eta_n \rangle$, $\eta_i = \eta(s_i)$ and $\xi = \langle \xi_1, ..., \xi_n \rangle$, $\xi_i = \xi(s_i)$ be variables on S.

(a) The class of associated probability averages (expectations) $\{\text{AsPA}_\sigma\}_{\sigma\in S_n}$ of variable η (APAC) is called the class of expectations of η, where $\forall\sigma \in S_n$

$$\text{AsPA}_\sigma(\eta) = \bigoplus_{k=1}^{n} \eta_{\sigma(k)} Asp_\sigma\left(s_{\sigma(k)}\right); \tag{23}$$

(b) $\forall\sigma \in S_n$ permutation the associated covariance of the variables ξ and η is called the probabilistic covariance of the values $\xi_\sigma = \left\langle \xi(s_{\sigma(1)}), ..., \xi(s_{\sigma(n)}) \right\rangle$ and $\eta_\sigma = \left\langle \eta(s_{\sigma(1)}), ..., \eta(s_{\sigma(n)}) \right\rangle$ with respect to the associated probability Asp_σ

$$\text{AsCov}_\sigma(\eta, \xi) = \text{AsPA}_\sigma(\eta \otimes \xi) \widetilde{\ominus} (\text{AsPA}_\sigma(\eta) \otimes \text{AsPA}_\sigma(\xi)), \tag{24}$$

where $\eta \otimes \xi \equiv \left\langle \eta_{\sigma(1)} \otimes \xi_{\sigma(1)}, ..., \eta_{\sigma(n)} \otimes \xi_{\sigma(n)} \right\rangle$, and the class $\{\text{AsCov}_\sigma\}_{\sigma\in S_n}$ is called the Associated Covariances Class (ACC);

(c) $\forall\sigma \in S_n$ permutation, the associated variation of a variable η is called the probabilistic variation of the value $\eta_\sigma = \left\langle \eta(s_{\sigma(1)}), ..., \eta(s_{\sigma(n)}) \right\rangle$ with respect to the associated probability Asp_σ:

$$\text{AsVar}_\sigma(\eta) = \text{AsPA}_\sigma\left(\eta^2\right) \widetilde{\ominus} (\text{AsPA}_\sigma(\eta))^2, \tag{25}$$

and the class $\{\text{AsVar}_\sigma\}_{\sigma\in S_n}$ is called the Associated Variations Class (AVC);

(d) $\forall\sigma \in S_n$ permutation the associated correlation coefficient of the variables ξ and η is called the probability correlation coefficient of the values $\xi_\sigma = \left\langle \xi(s_{\sigma(1)}), ..., \xi(s_{\sigma(n)}) \right\rangle$ and $\eta_\sigma = \left\langle \eta(s_{\sigma(1)}), ..., \eta(s_{\sigma(n)}) \right\rangle$ with respect to the associated probability Asp_σ

$$\text{AsCor}_\sigma(\eta, \xi) = \text{AsCov}_\sigma(\eta, \xi) \widetilde{\oslash} \left[(\text{AsVar}_\sigma(\eta))^{1/2} \otimes (\text{AsVar}_\sigma(\xi))^{1/2}\right], \tag{26}$$

and the class $\{\text{AsCor}_\sigma\}_{\sigma\in S_n}$ is called the Associated Correlation Coefficients Class (ACCC).

We have the following proposition for the extension of the Choquet integral for variables $\eta, \xi \colon S \to \{p, q\text{-ROFNs}\}$.

Proposition 7 The monotone expectation $\text{ME}_g(\eta)$ of a variable η with p, q-ROF values coincides with one of the associated probability average of η:

$$\text{ME}_g(\eta) = \text{AsPA}_\tau(\eta). \tag{27}$$

Proof. This proof becomes clear by comparing formulas (23) and (24). □

Using the principle of the "monotonic expectation concept" for expert data, we derive monotone extensions of basic classical statistics with respect to the fuzzy measure. Now let us introduce the extensions based on the Choquet integral for monotone statistics in the p, q-ROF environment.

Definition 15 Let g: $2^S \to [0, 1]$, $S = \{s_1, s_2, ..., s_n\}$ be a finite fuzzy measure with associated probability class $\{Asp_\sigma\}_{\sigma \in S_n}$; let be given some variables η, ξ: $S \to \{p$, q-ROFNs$\}$, then

(a) the monotone variation of η with respect to fuzzy measure g is called

$$\mathrm{MVar}_g(\eta) = \mathrm{ME}_g\left(\eta^2\right) \widetilde{\ominus} \left(\mathrm{ME}_g(\eta)\right)^2; \tag{28}$$

(b) the monotone covariance of variables η and ξ is called

$$\mathrm{MCov}_g(\eta, \xi) = \mathrm{ME}_g(\eta \otimes \xi) \widetilde{\ominus} (\mathrm{ME}_g(\eta) \otimes \mathrm{ME}_g(\xi)); \tag{29}$$

(c) the monotone correlation coefficient of variables η and ξ is called

$$\mathrm{MCor}_g(\eta, \xi) = \mathrm{MCov}_g(\eta, \xi) \widetilde{\oslash} \left[(\mathrm{MVar}_g(\eta))^{1/2} \otimes (\mathrm{MVar}_g(\xi))^{1/2}\right] \tag{30}$$

Now let us establish the connection between monotone statistics and associated statistics in the form of proposition.

Proposition 8 If η and ξ are variables with values from p, q-ROFN on S, η, ξ: $S \to \{p$, q-ROFNs$\}$ and they are comonotone on some permutation $\tau \in S_n$ with respect to partial ordering $\underset{pt}{\leq}$; and g is some fuzzy measure on 2^S, then

$$\text{a) } \mathrm{ME}_g(\eta) = \mathrm{AsPA}_\tau(\eta),$$

$$\text{b) } \mathrm{MVar}(\eta) = \mathrm{AsVar}_\tau(\eta),$$

$$\text{c) } \mathrm{MCov}(\eta, \xi) = \mathrm{AsCov}_\tau(\eta, \xi),$$

$$\text{d) } \mathrm{MCor}(\eta, \xi) = \mathrm{AsCor}_\tau(\eta, \xi). \tag{31}$$

Proof. The proof of a) is analogous to the proof of the proposition for environment q-ROF given in [17]. Therefore, we will not consider the generalization to numbers p, q-ROFNs.

Consider b). Since η in the variation $\mathrm{MVar}(\eta) = \mathrm{ME}_g(\eta^2) \widetilde{\ominus} [\mathrm{ME}_g(\eta)]^2$ takes only positive values, therefore η and η^2 are comonotone variables. That is $\exists \tau \in S_n$, $\eta_{\tau(1)} \underset{pt}{\geq} \eta_{\tau(2)} \underset{pt}{\geq} ... \underset{pt}{\geq} \eta_{\tau(n)} \Rightarrow \eta^2_{\tau(1)} \underset{pt}{\geq} \eta^2_{\tau(2)} \underset{pt}{\geq} ... \underset{pt}{\geq} \eta^2_{\tau(n)}$. Thus, $\mathrm{ME}_g(\eta^2) = \mathrm{AsPA}_\tau(\eta^2)$ and $\mathrm{ME}_g(\eta) = \mathrm{AsPA}_\tau(\eta)$. Then we have

$$\mathrm{MVar}_\tau(\eta) = \mathrm{AsPA}_\tau(\eta^2) \widetilde{\ominus} [\mathrm{AsPA}_\tau(\eta)]^2 = \mathrm{AsVar}_\tau(\eta). \tag{32}$$

I.e. the monotone variation coincides with one of the associated variations of η.

Consider c). Since η and ξ are comonotone and take positive values, then η, ξ and $\eta \otimes \xi$ will also be comonotone. Therefore

$$\mathrm{ME}_g(\eta \otimes \xi) = \mathrm{AsPA}_\tau(\eta \otimes \xi),\ \ \mathrm{ME}_g(\eta) = \mathrm{AsPA}_\tau(\eta),\ \ \mathrm{ME}_g(\xi) = \mathrm{AsPA}_\tau(\xi)$$

and

$$\mathrm{MCov}(\eta,\, \xi) = \mathrm{AsPA}_\tau(\eta \otimes \xi) \widetilde{\ominus} [\mathrm{AsPA}_\tau(\eta) \otimes \mathrm{AsPA}_\tau(\xi)] = \mathrm{AsCov}_\tau(\eta,\, \xi). \tag{33}$$

That is, the monotone covariance between comonotone variables coincides with one of the associated covariances on p, q-ROFNs.

d) This can be proven similarly. □

Now let us consider issues such as the properties of monotone and associated statistics in the p, q-ROFNs environment.

Proposition 9 Let $\eta\colon S \to \{p,\, q\text{-ROFNs}\}$ be a variable; $\eta_i \equiv \eta(s_i) \in \{p,\, q\text{-ROFNs}\}$, $i = 1,\, ...,\, n$; $g\colon 2^S \to [0,\, 1]$ be some fuzzy measure with its APC class $\{Asp_\sigma\}_{\sigma \in S_n}$, then $\mathrm{ME}_g(\eta)$ is also p, q-ROFN number with the following expression

$$\mathrm{ME}_g(\eta) = \mathrm{AsPA}_\tau(\eta) = \bigoplus_{k=1}^{n} \eta_{\sigma(k)} \cdot Asp_\sigma\left(s_{\sigma(k)}\right) = \left\langle \left[1 - \prod_{k=1}^{n} \left(1 - u^p_{\eta_{\tau(k)}}\right)^{Asp_\tau\left(s_{\tau(k)}\right)}\right]^{1/p},\, \prod_{k=1}^{n} v_{\eta_{\tau(k)}}^{Asp_\tau\left(s_{\tau(k)}\right)} \right\rangle, \tag{34}$$

where $\tau \in S_n$ is a permutation such that $\eta_{\tau(1)} \underset{pt}{\geq} \eta_{\tau(2)} \underset{pt}{\geq} \cdots \underset{pt}{\geq} \eta_{\tau(n)}$.

The proof is equivalent to the analogous proof presented in [32], but for the q-ROFNs environment, and we will not repeat it here.

Now let us consider analogous propositions for other monotone statistics. The proofs are by induction and for simplicity we will not present them here.

Proposition 10 Let $\eta\colon S \to \{p,\, q\text{-ROFNs}\}$ be a variable, $\eta_i \equiv \eta(s_i) \in \{p,\, q\text{-ROFNs}\}$, $i = 1,\, ...,\, n$, and $g\colon 2^S \to [0,\, 1]$ be some fuzzy measure with its APC $\{Asp_\sigma\}_{\sigma \in S_n}$. Then monotone variance $\mathrm{MVar}_g(\eta)$ is p, q-ROFN number with the following expression:

$$\begin{aligned}
&\mathrm{MVar}(\eta) \\
&= \mathrm{AsVar}_\tau(\eta) \\
&= \mathrm{AsPA}_\tau\left(\eta^2\right) \widetilde{\ominus} [\mathrm{AsPA}_\tau(\eta)]^2 \\
&= \left\langle \left[0 \vee \frac{\left(1 - A_\tau^{1-u_\eta^{2p}}\right) - \left(1 - A_\tau^{1-u_\eta^{p}}\right)}{1 - \left(1 - A_\tau^{1-u_\eta^{p}}\right)^2}\right]^{1/p},\, 1 \wedge \left(\frac{A_\tau^{v_\eta^{2q}}}{1 - \left(1 - A_\tau^{v_\eta^{q}}\right)^2}\right)^{1/q} \wedge \left(\frac{A_\tau^{1-u_\eta^{2p}}}{1 - \left(1 - A_\tau^{1-u_\eta^{p}}\right)^2}\right)^{1/p} \right\rangle,
\end{aligned} \tag{35}$$

where

$$A_\tau^{1-u_\eta^{2p}} \equiv \prod_{k=1}^{n}\left(1-u_{\eta_{\tau(k)}}^{2p}\right)^{Asp_\tau\left(s_{\tau(k)}\right)}$$

$$A_\tau^{1-u_\eta^{p}} \equiv \prod_{k=1}^{n}\left(1-u_{\eta_{\tau(k)}}^{p}\right)^{Asp_\tau\left(s_{\tau(k)}\right)}$$

$$A_\tau^{v_\eta^{2q}} \equiv \prod_{k=1}^{n}\left(1-\left(1-v_{\eta_{\tau(k)}}^{q}\right)^{2}\right)^{Asp_\tau\left(s_{\tau(k)}\right)}$$

$$A_\tau^{v_\eta^{q}} \equiv \prod_{k=1}^{n}\left(v_{\eta_{\tau(k)}}^{q}\right)^{Asp_\tau\left(s_{\tau(k)}\right)} \tag{36}$$

and $\tau \in S_n$ is such permutation that $\eta_{\tau(1)} \underset{pt}{\geq} \eta_{\tau(2)} \underset{pt}{\geq} \dots \underset{pt}{\geq} \eta_{\tau(n)}$.

Proposition 11 Assume that $\eta,\ \xi$: $S \to \{p,\ q\text{-ROFNs}\}$, $\eta_i \equiv \eta(s_i)$, $\xi_i \equiv \xi(s_i)$, $i = 1, 2, \dots, n$ are comonotone variables on some permutation $\tau \in S_n$ with respect to partial ordering $\underset{pt}{\leq}$; g: $2^S \to [0, 1]$ is some fuzzy measure with its APC $\{Asp_\sigma\}_{\sigma \in S_n}$. Then monotone covariance between variables η and ξ is also p, q-ROFN number with the following expression

$$\mathrm{MCov}(\eta,\ \xi) = \mathrm{AsCov}_\tau(\eta,\ \xi)$$

$$= \left\langle \left(0 \vee \frac{\left[1-A_\tau^{1-u_{\eta\xi}^{p}}\right]-\left[1-A_\tau^{1-u_\xi^{p}}\right]\times\left[1-A_\tau^{1-u_\eta^{p}}\right]}{1-\left[1-A_\tau^{1-u_\xi^{p}}\right]\times\left[1-A_\tau^{1-u_\eta^{p}}\right]}\right)^{1/p},\right.$$

$$\left. 1 \wedge \left(\frac{A_\tau^{v_{\eta\xi}^{q}}}{1-\left[1-A_\tau^{v_\xi^{q}}\right]\times\left[1-A_\tau^{v_\eta^{q}}\right]}\right)^{1/q} \wedge \left(\frac{A_\tau^{1-u_{\eta\xi}^{p}}}{1-\left[1-A_\tau^{1-u_\xi^{p}}\right]\times\left[1-A_\tau^{1-u_\eta^{p}}\right]}\right)^{1/p}\right\rangle, \tag{37}$$

where

$$A_\tau^{1-u_{\eta\xi}^{p}} \equiv \prod_{k=1}^{n}\left(1-u_{\eta_{\tau(k)}}^{p}\cdot u_{\xi_{\tau(k)}}^{p}\right)^{Asp_\tau\left(s_{\tau(k)}\right)},$$

$$A_\tau^{v_{\eta\xi}^{q}} \equiv \prod_{k=1}^{n}\left(1-\left(1-v_{\eta_{\tau(k)}}^{q}\right)\cdot\left(1-v_{\xi_{\tau(k)}}^{q}\right)\right)^{Asp_\tau\left(s_{\tau(k)}\right)}, \tag{38}$$

and $\tau \in S_n$ is such a permutation that $\eta_{\tau(1)} \underset{pt}{\geq} \eta_{\tau(2)} \underset{pt}{\geq} \dots \underset{pt}{\geq} \eta_{\tau(n)}$ and $\xi_{\tau(1)} \underset{pt}{\geq} \xi_{\tau(2)} \underset{pt}{\geq} \dots \underset{pt}{\geq} \xi_{\tau(n)}$.

There are similar proofs for the monotone correlation coefficient (formulas (26) and (30)), when η and ξ are comonotone with respect to some permutation $\tau \in S_n$ under partial ordering $\underset{pt}{\leq}$. The value of MCor is also a p, q-ROFN number, and its expression is not given here because of its very large size. Note also that $\mathrm{AsPA}_\tau(\eta) = \left\langle (1 - A_\tau^{1-u_\eta^p})^{1/p},\, (A_\tau^{v_\tau^q})^{1/q} \right\rangle$.

To achieve reduction and minimization of decision-making risks, statistics based on the finite Choquet integral for p, q-ROF numbers are considered: monotone probability average, monotone variance, monotone covariance, and monotone correlation coefficient. Their analytical properties are considered. Connections between monotone and associated statistics are derived. Efficient formulas for calculating monotone statistics are also derived.

5. F-associated statistics in interactive MADM under p, q-ROF environment

We have already noted that, unlike the principle of classical statistics, which can consider samples of independent trial results for statistical analysis, this principle does not apply and does not work for expert data, due to the expert nature of the data source. Here, additive aggregations are replaced by non-additive, but monotone aggregations. For example, in MADM models, expert assessments of alternatives to attributes have interdependencies, since the relevant attributes interact in the decision process. The development of systematic statistical analysis on expert data involves the creation of aggregation tools that take into account all possible interactions of attributes as much as possible. This is not entirely possible with the monotone statistics explained in the previous section. This also applies to p, q-ROF data aggregations. In this section, one approach to solving this problem is developed for the p, q-ROF environment. The correct extension of monotone statistics to the so-called F-associated statistics will be introduced, which take into account all the interactions of attributes. The correctness of the distribution implies the following. F-associated statistics coincide with the corresponding monotone statistics if in the definition of F-associated statistics, we take the so-called second-order extreme Choquet capacities as the fuzzy measure. This class of fuzzy measures is well known to researchers and practitioners. This class is one of the main basic classes in the fuzzy measure theory (Def. 7, [35])

As we discussed in the introduction, the results of the aggregation of monotone statistics in MADM models, when interactions between attributes are observed, imply the use of non-additive aggregation tools for ranking decision alternatives. For example, with the Choquet integral type aggregations, when the index of uncertainty arising in expert evaluations of attributes is represented by some fuzzy measure. It should be noted that the monotone expectation, variance and other operators of extended aggregations built on the Choquet finite integral take into account the interaction of attribute groups with only a certain single consonant structure of attributes:

$$\left\{s_{\tau(1)}\right\},\, \left\{s_{\tau(1)}, s_{\tau(2)}\right\},\, \dots,\, \left\{s_{\tau(1)}, s_{\tau(2)}, \dots, s_{\tau(n)}\right\}, \tag{39}$$

which leads to unsatisfactory and less reliable rankings in cases of strong interactions. The authors in [8, 17, 27–34, 64, 66, 67] developed an approach whose aggregation operators take into account all $n!$ number

$$\left\langle \left\{s_{\sigma(1)}\right\},\, \left\{s_{\sigma(1)}, s_{\sigma(2)}\right\},\, \dots,\, \left\{s_{\sigma(1)}, s_{\sigma(2)}, \dots, s_{\sigma(n)}\right\} \right\rangle,\ \sigma \in S_n, \tag{40}$$

interactions between attribute groups of consonant structures. In particular, the differences in fuzzy measure values of neighboring elements of these structures,

$$g(\left\{s_{\sigma(1)}, \dots, s_{\sigma(i)}\right\}) - g(\left\{s_{\sigma(1)}, \dots, s_{\sigma(i-1)}\right\}),\ i = 1, 2, \dots, n;\ g\left\{s_{\sigma(0)}\right\} = 0,$$

as the degree of impact obtained when adding a new element $\{s_{\sigma(i)}\}$ to any set $\{s_{\sigma(1)}, ..., s_{\sigma(i-1)}\}$ of this structure.

We will extend this concept for the p, q-ROF environment.

Definition 16 Let be given a finite fuzzy measure g: $2^S \to [0, 1]$, $S = \{s_1, s_2, ..., s_n\}$ with associated probability class $\langle Asp_{\tau_i}\rangle_{i=1, ..., n!} \equiv \{Asp_\sigma\}_{\sigma\in S_n}$, some averaging operator

$$F\colon \{p,\, q\text{-ROFNs}\}^{n!} \to \{p,\, q\text{-ROFNs}\} \tag{41}$$

and two p, q-ROFN values variables: η, ξ: $S \to \{p,\, q\text{-ROFN}\}$; $\eta_i = \eta(s_i)$, $\xi_i = \xi(s_i)$, $i = 1, 2, ..., n$. Then

(1) F-Associated p, q-Rung Orthopair Fuzzy Probability Averaging (F-As-p, q-ROFPA) operator is called

$$F\text{-As-}p,\, q\text{-ROFPA}(\eta) = F(\mathrm{AsPA}_{\tau_1}(\eta), ..., \mathrm{AsPA}_{\tau_{n!}}(\eta)); \tag{42}$$

(2) F-Associated p, q-Rung Orthopair Fuzzy Variance (F-As-p, q-ROFVar) operator is called

$$F\text{-As-}p,\, q\text{-ROFVar}(\eta) = F(\mathrm{AsVar}_{\tau_1}(\eta), ..., \mathrm{AsVar}_{\tau_{n!}}(\eta)); \tag{43}$$

(3) F-Associated p, q-Rung Orthopair Fuzzy Covariance (F-As-p, q-ROFCov) operator is called

$$F\text{-As-}p,\, q\text{-ROFCov}(\eta,\, \xi) = F(\mathrm{AsCov}_{\tau_1}(\eta,\, \xi), ..., \mathrm{AsCov}_{\tau_{n!}}(\eta,\, \xi)); \tag{44}$$

(4) F-Associated p, q-Rung Orthopair Fuzzy Correlation coefficient (F-As-p, q-ROFCor) operator is called

$$F\text{-As-}p,\, q\text{-ROFCor}(\eta,\, \xi) = F(\mathrm{AsCor}_{\tau_1}(\eta,\, \xi), ..., \mathrm{AsCor}_{\tau_{n!}}(\eta,\, \xi)), \tag{45}$$

where $\{\mathrm{AsVar}_\tau\}_{\tau\in S_n}$, $\{\mathrm{AsCov}_\tau\}_{\tau\in S_n}$ and $\{\mathrm{AsCor}_\tau\}_{\tau\in S_n}$ classes are respectively associated variance, covariance and correlation coefficient classes of a fuzzy measure g.

In this article, for operator F we will consider only two values $F \in \{\max, \min\}$. For them, we will define specific new F-associated operators for the probability average, variance, covariance, and correlation coefficient.

$$\max\text{-As-}p,\, q\text{-ROFPA}(\eta) = \bigvee_{\sigma\in S_n} \mathrm{AsPA}_\sigma(\eta),$$

$$\min\text{-As-}p,\, q\text{-ROFPA}(\eta) = \bigwedge_{\sigma\in S_n} \mathrm{AsPA}_\sigma(\eta),$$

$$\max\text{-As-}p,\, q\text{-ROFVa}(\eta) = \bigvee_{\sigma\in S_n} \mathrm{AsVar}_\sigma(\eta),$$

$$\min\text{-As-}p,\, q\text{-ROFVar}(\eta) = \bigwedge_{\sigma\in S_n} \mathrm{AsVar}_\sigma(\eta),$$

$$\max\text{-As-}p,\, q\text{-ROFCov}(\eta,\, \xi) = \bigvee_{\sigma \in S_n} \text{AsCov}_{\sigma}(\eta,\, \xi),$$

$$\min\text{-As-}p,\, q\text{-ROFCov}(\eta,\, \xi) = \bigwedge_{\sigma \in S_n} \text{AsCov}_{\sigma}(\eta,\, \xi),$$

$$\max\text{-As-}p,\, q\text{-ROFCor}(\eta,\, \xi) = \bigvee_{\sigma \in S_n} \text{AsCor}_{\sigma}(\eta,\, \xi),$$

$$\min\text{-As-}p,\, q\text{-ROFCor}(\eta,\, \xi) = \bigwedge_{\sigma \in S_n} \text{AsCor}_{\sigma}(\eta,\, \xi). \tag{46}$$

Based on the mathematical induction method, it is simple to prove the proposition:

Proposition 12 Let be given some fuzzy measure g: $2^S \rightarrow [0,\, 1]$ and some p, q-ROFNs valued variables η and ξ, η, ξ: $S \rightarrow \{p,\, q\text{-ROFNs}\}$. Then values of new operators from (46) are p, q-ROFNs and have the following expressions:

$$\max\text{-As-}p,\, q\text{-ROFPA}(\eta) = \left\langle \bigvee_{\sigma \in S_n} \left(1 - A_{\sigma}^{1-\mu_{\eta}^{p}}\right)^{1/p},\, \bigwedge_{\sigma \in S_n} \left(A_{\sigma}^{v_{\eta}^{q}}\right)^{1/q} \right\rangle,$$

$$\min\text{-As-}p,\, q\text{-ROFPA}(\eta) = \left\langle \bigwedge_{\sigma \in S_n} \left(1 - A_{\sigma}^{1-\mu_{\eta}^{p}}\right)^{1/p},\, \bigvee_{\sigma \in S_n} \left(A_{\sigma}^{v_{\eta}^{q}}\right)^{1/q} \right\rangle,$$

$$\max\text{-As-}p,\, q\text{-ROFVar}(\eta) = \left\langle \bigvee_{\sigma \in S_n} \left(0 \vee \frac{\left[1 - A_{\sigma}^{1-u_{\eta}^{2p}}\right] - \left[1 - A_{\sigma}^{1-u_{\eta}^{p}}\right]^2}{1 - \left[1 - A_{\sigma}^{1-u_{\eta}^{p}}\right]^2}\right)^{1/p}, \right.$$

$$\left. \bigwedge_{\tau \in S_n} \left(1 \wedge \frac{A_{\sigma}^{1-v_{\eta}^{2q}}}{1 - \left[1 - A_{\sigma}^{1-v_{\eta}^{q}}\right]^2}\right)^{1/q} \wedge \left(\frac{A_{\sigma}^{1-u_{\eta}^{2p}}}{1 - \left[1 - A_{\sigma}^{1-u_{\eta}^{p}}\right]^2}\right)^{1/p} \right\rangle,$$

$$\min\text{-As-}p,\, q\text{-ROFVar}(\eta) = \left\langle \bigwedge_{\sigma \in S_n} \left(0 \vee \frac{\left[1 - A_{\sigma}^{1-u_{\eta}^{2p}}\right] - \left[1 - A_{\sigma}^{1-u_{\eta}^{p}}\right]^2}{1 - \left[1 - A_{\sigma}^{1-u_{\eta}^{p}}\right]^2}\right)^{1/p}, \right.$$

$$\left. \bigvee_{\sigma \in S_n} \left(1 \wedge \frac{A_{\sigma}^{1-v_{\eta}^{2q}}}{1 - \left[1 - A_{\sigma}^{1-v_{\eta}^{q}}\right]^2}\right)^{1/q} \wedge \left(\frac{A_{\sigma}^{1-u_{\eta}^{2p}}}{1 - \left[1 - A_{\sigma}^{1-u_{\eta}^{p}}\right]^2}\right)^{1/p} \right\rangle,$$

$$\max\text{-As-}p,\, q\text{-ROFCov}(\eta,\, \xi) = \left\langle \bigvee_{\sigma\in S_n} \left(0 \vee \frac{\left[1 - A_\sigma^{1-u^p_{\eta\xi}}\right] - \left[1 - A_\sigma^{1-u^p_\eta}\right] \times \left[1 - A_\sigma^{1-u^p_\xi}\right]}{1 - \left[1 - A_\sigma^{1-u^p_\eta}\right] \times \left[1 - A_\sigma^{1-u^p_\xi}\right]} \right)^{1/p}, \right.$$

$$\bigwedge_{\sigma\in S_n} \left(\left(1 \wedge \frac{A_\sigma^{v^q_{\eta\xi}}}{1 - \left[1 - A_\sigma^{v^q_\eta}\right] \times \left[1 - A_\sigma^{v^q_\xi}\right]} \right)^{1/q} \right.$$

$$\left. \left. \wedge \left(\frac{A_\sigma^{1-u^p_{\eta\xi}}}{1 - \left[1 - A_\sigma^{1-u^p_\eta}\right] \times \left[1 - A_\sigma^{1-u^p_\xi}\right]} \right)^{1/p} \right) \right\rangle$$

$$\min\text{-As-}p,\, q\text{-ROFCov}(\eta,\, \xi) = \left\langle \bigwedge_{\sigma\in S_n} \left(0 \vee \frac{\left[1 - A_\sigma^{1-u^p_{\eta\xi}}\right] - \left[1 - A_\sigma^{1-u^p_\eta}\right] \times \left[1 - A_\sigma^{1-u^p_\xi}\right]}{1 - \left[1 - A_\sigma^{1-u^p_\eta}\right] \times \left[1 - A_\sigma^{1-u^p_\xi}\right]} \right)^{1/p}, \right.$$

$$\bigvee_{\sigma\in S_n} \left(\left(1 \wedge \frac{A_\sigma^{v^q_{\eta\xi}}}{1 - \left[1 - A_\sigma^{v^q_\eta}\right] \times \left[1 - A_\sigma^{v^q_\xi}\right]} \right)^{1/q} \right.$$

$$\left. \left. \wedge \left(\frac{A_\sigma^{1-u^p_{\eta\xi}}}{1 - \left[1 - A_\sigma^{1-u^p_\eta}\right] \times \left[1 - A_\sigma^{1-u^p_\xi}\right]} \right)^{1/p} \right) \right\rangle. \tag{47}$$

Note that we do not include the expressions of the operators max-As-p, q-ROFCor and min-As-p, q-ROFCor due to their very large size.

It is not difficult to show the following connections between the new operators:

$$\mathrm{AsVar}_\tau(\eta) = \mathrm{AsCov}_\tau(\eta,\, \eta), \quad \tau \in S_n,$$

$$F\text{-As-}p,\, q\text{-ROFVar}(\eta) = F\text{-As-}p,\, q\text{-ROFCov}(\eta,\, \eta),$$

$$\mathrm{AsVar}_\tau(\lambda \cdot \eta) = \lambda^2 \cdot \mathrm{AsVar}_\tau(\eta), \quad \lambda > 0,\, \tau \in S_n,$$

$$F\text{-As-}p,\, q\text{-ROFVar}(\lambda \cdot \eta) = \lambda^2 \cdot F\text{-As-}p,\, q\text{-ROFVar}(\eta), \quad \lambda > 0,$$

$$\text{AsCor}_\tau(\eta, \eta) = e,$$

$$F\text{-As-}p,\ q\text{-ROFCor}(\eta, \eta) = e. \tag{48}$$

The proofs of these formulas are simplified if we note the following identities:

$$A_\tau^{1-u_{\eta\eta}^p} = A_\tau^{1-u_\eta^{2p}},\ A_\tau^{v_{\eta\eta}^q} = A_\tau^{v_\eta^{2q}}.$$

There are connections between monotone statistics and F-associated statistics, which is a matter of the extension correctness:

Proposition 13 If g: $2^S \to [0.1]$ is some fuzzy measure; $S = \{s_1, \ldots, s_n\}$, ξ and η, ξ, η: $S \to \{p,\ q\text{-ROFNs}\}$, are comonotone on some permutation $\tau \in S_n$ with respect to partial ordering $\underset{pt}{\leq}$, then

a) If g is the upper capacity of the second order, then

$$\max\text{-As-}p,\ q\text{-ROFPA}(\eta) = \text{ME}_g(\eta),$$

$$\max\text{-As-}p,\ q\text{-ROFVar}(\eta) = \text{MVar}_g(\eta),$$

$$\max\text{-As-}p,\ q\text{-ROFCov}(\eta, \xi) = \text{MCov}_g(\eta, \xi),$$

$$\max\text{-As-}p,\ q\text{-ROFCor}(\eta, \xi) = \text{MCor}_g(\eta, \xi). \tag{49}$$

b) If g is the lower capacity of the second order, then

$$\min\text{-As-}p,\ q\text{-ROFPA}(\eta) = \text{ME}_g(\eta),$$

$$\min\text{-As-}p,\ q\text{-ROFVar}(\eta) = \text{MVar}_g(\eta),$$

$$\min\text{-As-}p,\ q\text{-ROFCov}(\eta, \xi) = \text{MCov}_g(\eta, \xi),$$

$$\min\text{-As-}p,\ q\text{-ROFCor}(\eta, \xi) = \text{MCor}_g(\eta, \xi). \tag{50}$$

Note that if $p = q$ and we have q-ROF environment, then the proofs of the first equations of formulas (49) and (50) of this proposition are given in [17]. Since the technique for proving the remaining formulas is almost the same, we will not give the rest of the proof here.

At the end of this section, we will present binary relations for ranking MADM alternatives, which are constructed with new operators. Consider the model $\langle D, S\rangle$ of MADM, where $D = \{d_1, \ldots, d_m\}$ are possible alternatives, and $S = \{s_1, \ldots, s_n\}$ is a set of attributes. For each alternative $d_i \in D$, the corresponding row in the decision matrix $\{\eta_{ij}\}$ corresponds to some ratings, utilities, or other evaluations, and which correspond to a variable η_i with values in p, q-ROFNs:

$$\eta_i = (\eta_{i1},\, \eta_{i2},\, ...,\, \eta_{in});\; \eta_{ij} = \eta_i(s_j) \in p,\, q\text{-ROFNs}, \quad i = 1,\, 2,\, ...,\, m; \quad j = 1,\, 2,\, ...,\, n. \tag{51}$$

Definition 17 a) We say that d_i is no worse alternative than the alternative d_j according to monotone expectation if

$$d_i \underset{\text{ME}}{\geq} d_j \Leftrightarrow \text{ME}_g(\eta_i) \underset{t}{\geq} \text{ME}_g(\eta_j); \tag{52}$$

b) We say that d_i is no worse alternative than the alternative d_j according to monotone variation if

$$d_i \underset{\text{MVar}}{\geq} d_j \Leftrightarrow \text{MVar}_g(\eta_i) \underset{t}{\leq} \text{MVar}_g(\eta_j); \tag{53}$$

c) We say that d_i is no worse alternative than the alternative d_j according to the F-associated probability average, $F \in \{\max,\, \min\}$, if

$$d_i \underset{F\text{-PA}}{\geq} d_j \Leftrightarrow F\text{-As-}p,\, q\text{-ROFPA}(\eta_i) \underset{t}{\geq} F\text{-As-}p,\, q\text{-ROFPA}(\eta_j); \tag{54}$$

d) We say that d_i is no worse alternative than the alternative d_j according to the F-associated variance, $F \in \{\max,\, \min\}$, if

$$d_i \underset{F\text{-Var}}{\geq} d_j \Leftrightarrow F\text{-As-}p,\, q\text{-ROFVar}(\eta_i) \underset{t}{\leq} F\text{-As-}p,\, q\text{-ROFVar}(\eta_j). \tag{55}$$

Determining the degree of statistical independence between possible alternatives in MADM models is an essential and important problem in practical problems. A certain solution to this problem is presented in the analysis below for the p, q-ROFNs environment.

Definition 18 We say that there is a statistically high degree of dependence between the alternatives d_i and d_j, if $\text{MCor}(\eta_i,\, \eta_j) \approx e = (1,\, 0)$, that is, $u_{\text{MCor}} \to 1$ and $v_{\text{MCor}} \to 0$.

As we have already mentioned $\text{MCor}(\eta_i,\, \eta_i) = e$.

Definition 19 We say that there is a statistically high degree of independence between the alternatives d_i and d_j if $\text{MCor}(\eta_i,\, \eta_i) \approx \theta = (0,\, 1) \Leftrightarrow u_{\text{MCor}} \to 0,\, v_{\text{MCor}} \to 1$.

Note that if η_i and η_j are comonotone variables on some permutation $\tau \in S_n$ with respect to partial ordering $\underset{pt}{\leq}$ and the probability covariance for the permutation $\tau \in S_n$ is equal to θ, then we will have a high degree of independence between the alternatives d_i and d_j, since $\text{MCor}(\eta_i,\, \eta_j) = \text{AsCor}_\tau(\eta_i,\, \eta_j) = \theta$.

Based on the definitions given here, it is possible to rank all alternatives $\{d_i\}$, $i = 1,\, ...,\, m$, from high to low statistical independence in relation to the remaining alternatives. For this, the concept of a Statistical Independence Index (SII) is used, which ranks all d_i alternatives accordingly:

$$\text{SII}\,(d_i) = \frac{1}{2(m-1)} \sum_{j \neq i} \left[\text{Large}\,(v_{\text{MCor}}\,(\eta_i,\, \eta_j)) + \text{Small}\,(u_{\text{MCor}}\,(\eta_i,\, \eta_j))\right] \tag{56}$$

Where Large, Small: $[0, 1] \to [0, 1]$ are fuzzy consistency functions. In the future we will use the following specific functions: $\text{Large}(x) = (e^x - 1)/(e - 1)$ and $\text{Small}(x) = (e - e^x)/(e - 1)$.

Definition 20 We say that an alternative d_i is no worse an independent alternative than an alternative d_j according to the monotone correlation coefficient if

$$d_i \underset{\text{SII}}{\geq} d_j \Leftrightarrow \text{SII}(d_i) \geq \text{SII}(d_j). \tag{57}$$

Similarly, we introduce the relation of ordering of independence with respect to F-associated p, q-ROF correlation coefficient.

Definition 21 Let $F \in \{\min, \max\}$. We say that an alternative d_i is not a worse independent alternative than an alternative d_j according to the F-associated correlation coefficient, if

$$d_i \underset{\text{SII}_F}{\geq} d_j \Leftrightarrow \text{SII}_F(d_i) \geq \text{SII}_F(d_j), \tag{58}$$

where

$$\text{SII}_F(d_i) = \frac{1}{2(m-1)} \sum_{j \neq i} [\text{Large}(v_{F\text{-AS-}p,\, q\text{-ROFCor}}(\eta_i, \eta_j)) + \text{Small}(u_{F\text{-AS-}p,\, q\text{-ROFCor}}(\eta_i, \eta_j))]. \tag{59}$$

Note that the alternatives ranked by the indices SII and SII_F indicate their degree of statistical independence in relation to the other alternatives.

Since the aggregations obtained by the monotone statistics cannot fully reflect the degrees of interaction of attributes in MADM decision process, their extensions are introduced: F-associated probability average, F-associated variance, F-associated covariance, and F-associated correlation coefficient. In F-aggregations, they take into account all possible variants of consonant structures of attributes–that is, they combine all variants. This creates fewer risks in the rankings of alternatives obtained through relatively monotone statistics and aggregations. Extension correctness implies that if the second-order extremal Choquet capacities are taken as a fuzzy measure and the F-aggregation function is a max or min, then the F-associated statistics and monotone statistics values coincide.

6. Illustrative examples of aggregations of F-associated and monotone statistics in MADM under the p, q-ROF environment

Quite different statistics for ranking alternatives have been constructed for different practical problems in the p, q-ROFNs environments. A certain orderly form has been given to the p, q-ROFNs statistical analysis. To demonstrate the obtained results, we present a numerical example. Now let us consider a specific example of MADM a model $\langle D, S \rangle$, where the elements in the decision matrix are p, q-ROFNs numbers. Suppose we have 3 attributes $S = \{s_1, s_2, s_3\}$ and 4 possible alternatives $D = \{d_1, d_2, d_3, d_4\}$. Let the decision matrix has the following form.

Table 1. Decision-making p, q-ROF matrix

D	S		
	s_1	s_2	s_3
d_1	(0.6, 0.4)	(0.6, 0.3)	(0.8, 0.4)
d_2	(0.4, 0.7)	(0.8, 0.4)	(0.6, 0.4)
d_3	(0.7, 0.4)	(0.6, 0.3)	(0.7, 0.3)
d_1	(0.8, 0.6)	(0.9, 0.5)	(0.9, 0.4)

Note that the evaluations of alternatives d_i, $i = 1, 2, 3$ with respect to attributes S will be represented by a p, q-ROF vector $\eta_i = (\eta_{i1}, \eta_{i2}, \eta_{i3})$ from Table 1, e.g. $\eta_1 = ((0.6, 0.4), (0.6, 0.3), (0.8, 0.4))$.

In order to better imagine the process of aggregation calculations with the operators constructed in the article for ranking alternatives D, we present a MADM scheme for the given data.

Table 2. Fuzzy measure g and its associated probability class

$B \in 2^S$	$g(B)$	$Asp_{\sigma_1}(B)$	$Asp_{\sigma_2}(B)$	$Asp_{\sigma_3}(B)$	$Asp_{\sigma_4}(B)$	$Asp_{\sigma_5}(B)$	$Asp_{\sigma_6}(B)$
s_1	0.25	0.25	0.25	0.40	0.55	0.30	0.55
s_2	0.15	0.30	0.30	0.15	0.15	0.35	0.10
s_3	0.35	0.45	0.45	0.45	0.30	0.35	0.35
s_1, s_2	0.55	0.55	0.55	0.55	0.70	0.65	0.65
s_1, s_3	0.65	0.70	0.70	0.95	0.85	0.65	0.95
s_2, s_3	0.45	0.75	0.75	0.60	0.45	0.70	0.45
s_1, s_2, s_3	1.0	1.0	1.0	1.0	1.0	1.0	1.0

Table 3. Permutations

Permutation $\sigma \in S$	(1, 2, 3)	(1, 3, 2)	(2, 1, 3)	(2, 3, 1)	(3, 1, 2)	(3, 2, 1)
Notation $\sigma_i \in S_3$	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6

Calculation scheme 1.

Step 1: Create a MADM model $\langle D, S\rangle$ and a decision-making matrix;

Step 2: Form the variables with $\eta_{ij} \in p$, q-ROFNs of expert evaluations in decision matrix (Table 1); Let us fix the numerical values of the parameters p and q;

Step 3: Create an uncertainty index for the MADM model, forming a fuzzy measure g defined on the attributes S;

Step 4: Construct a class of $\{Asp_\sigma\}_{\sigma\in S_n}$-associated probabilities with the associated fuzzy measure g (here $S_n = S_3$);

Step 5: Construct a class of $\{\text{AsPA}_\sigma\}_{\sigma\in S_n}$-associated probability averages;

Step 6: Construct a class of $\{\text{AsVar}_\sigma\}_{\sigma\in S_n}$-associated variances;

Step 7: Construct a class of $\{\text{AsCov}_\sigma\}_{\sigma\in S_n}$-associated covariances;

Step 8: Construct a class of $\{\text{AsCor}_\sigma\}_{\sigma\in S_n}$-associated correlation coefficients;

Step 9: For each alternative d_i calculate $\text{ME}_g(\eta_i)$, $\text{MVar}_g(\eta_i)$-monotone statistics;

Step 10: For each pair of alternatives (d_i, d_j) calculate $\text{MCov}_g(\eta_i, \eta_j)$ and $\text{MCor}_g(\eta_i, \eta_j)$ monotone statistics;

Step 11: For each alternative d_i build F-As-p, q-ROFPA(η_i), F-As-p, q-ROFVar(η_i) statistics $(F \in \{\max, \min\}$;

Step 12: For each pair of alternatives (d_i, d_j) construct F-As-p, q-ROFCov(η_i, η_j), F-As-p, q-ROFCor(η_i, η) statistics $(F \in \{\max, \min\}$;

Step 13: For each alternative d_i calculate Statistical Independent Indexes-SII(d_i) and SII$_F(d_i)$ ($F \in \{\max, \min\}$;

Step 14: Construct binary relations $\underset{\text{ME}}{\geq}$; $\underset{\text{MVar}}{\geq}$; $\underset{F\text{-PA}}{\geq}$; $\underset{F\text{-Var}}{\geq}$; $\underset{\text{SII}}{\geq}$ and $\underset{\text{SII}_F}{\geq}$ ($F \in \{\max, \min\}$, for ranking alternatives;

Step 15: Rank the alternatives D of the MADM model using the binary relations of Step 14.

Now let us continue to implement this scheme for the data given in Table 1, when the numerical values of p and q are fixed. We proceed to Step 3. Suppose we are given some g fuzzy measure on 2^S, its values are presented in the second column of Table 2. The next 6 columns in this table are given to the associated probabilities of g. Note that since $S = \{s_1, s_2, s_3\}$, then we have 3 attributes and the number of permutations will be 3! = 6. Let us introduce the conditional notations of these permutations (Table 3).

Based on the steps 4–12 we calculate the respect values of new aggregation statistics:

Associated probability averagings and monotone expectations for $p = 4$, $q = 3$ are given in Table 4.

Table 4. Associated probability averagings and monotone expectations (formulas (23) and (34)) for $p = 4$, $q = 3$

D	$\text{AsPA}_{\sigma_1}(\eta_i)$	$\text{AsPA}_{\sigma_2}(\eta_i)$	$\text{AsPA}_{\sigma_3}(\eta_i)$	$\text{AsPA}_{\sigma_4}(\eta_i)$	$\text{AsPA}_{\sigma_5}(\eta_i)$	$\text{AsPA}_{\sigma_6}(\eta_i)$	$\text{ME}_g(\eta_i)$
$d_1 \sim \eta_1$	0.720, 0.367	0.720, 0.367	0.720, 0.383	0.689, 0.383	0.700, 0.362	0.700, 0.389	0.700, 0.389
$d_2 \sim \eta_2$	0.671, 0.460	0.671, 0.460	0.613, 0.500	0.596, 0.544	0.680, 0.473	0.575, 0.544	0.596, 0.544
$d_3 \sim \eta_3$	0.676, 0.322	0.676, 0.322	0.688, 0.337	0.688, 0.351	0.671, 0.327	0.692, 0.351	0.671, 0.327
$d_4 \sim \eta_4$	0.882, 0.473	0.882, 0.473	0.870, 0.486	0.856, 0.517	0.878, 0.488	0.856, 0.511	0.856, 0.511

Associated variances and monotone variances for $p = 4$, $q = 3$ are given in Table 5.

Table 5. Associated variances and monotone variances (formulas (25), (28), (35), (36)) for $p = 4$, $q = 3$

D	$\text{AsVar}_{\sigma_1}(\eta_i)$	$\text{AsVar}_{\sigma_2}(\eta_i)$	$\text{AsVar}_{\sigma_3}(\eta_i)$	$\text{AsVar}_{\sigma_4}(\eta_i)$	$\text{AsVar}_{\sigma_5}(\eta_i)$	$\text{AsVar}_{\sigma_6}(\eta_i)$	$\text{MVar}(\eta_i)$
$d_1 \sim \eta_1$	0.359, 0.994	0.359, 0.994	0.359, 0.994	0.349, 0.995	0.354, 0.995	0.354, 0.995	0.354, 0.995
$d_2 \sim \eta_2$	0.379, 0.993	0.379, 0.993	0.351, 0.99	0.360, 0.989	0.39, 0.992	0.335, 0.989	0.351, 0.990
$d_3 \sim \eta_3$	0.216, 0.992	0.216, 0.989	0.192, 0.999	0.194, 0.979	0.221, 0.949	0.174, 0.989	0.216, 0.989
$d_4 \sim \eta_4$	0.314, 0.997	0.314, 0.997	0.337, 0.996	0.344, 0.995	0.324, 0.996	0.344, 0.995	0.337, 0.996

Associated covariances and monotone covariances for the pairs of alternatives for $p = 4$, $q = 3$ are given in Table 6.

Table 6. Associated covariances and monotone covariances for the pairs of alternatives (Formulas (24), (29), (37), (38)) for $p = 4$, $q = 3$

$\langle d_i, d_j \rangle$	$\text{AsCov}_{\sigma_1}(\eta_i, \eta_j)$	$\text{AsCov}_{\sigma_2}(\eta_i, \eta_j)$	$\text{AsCov}_{\sigma_3}(\eta_i, \eta_j)$	$\text{AsCov}_{\sigma_4}(\eta_i, \eta_j)$	$\text{AsCov}_{\sigma_5}(\eta_i, \eta_j)$	$\text{AsCov}_{\sigma_6}(\eta_i, \eta_j)$	$\text{MCov}(\eta_i, \eta_j)$
$\langle d_1, d_2 \rangle$	0.002, 0.975	0.005, 0.996	0.001, 0.976	0.005, 0.985	0.002, 0.952	0.001, 0.938	0.001, 0.938
$\langle d_1, d_3 \rangle$	0.207, 0.999	0.207, 0.998	0.003, 0.958	0.002, 0.918	0.201, 0.999	0.005, 0.962	0.207, 0.998
$\langle d_1, d_4 \rangle$	0.234, 0.999	0.234, 0.999	0.303, 0.997	0.301, 0.997	0.227, 0.999	0.318, 0.997	0.318, 0.997
$\langle d_2, d_3 \rangle$	0.004, 0.985	0.001, 0.896	0.001, 0.976	0.003, 0.974	0.004, 0.982	0.002, 0.944	0.001, 0.976
$\langle d_2, d_4 \rangle$	0.267, 0.998	0.267, 0.998	0.288, 0.998	0.309, 0.997	0.294, 0.998	0.297, 0.997	0.309, 0.997
$\langle d_3, d_4 \rangle$	0.004, 0.985	0.001, 0.896	0.318, 0.997	0.006, 0.987	0.005, 0.993	0.003, 0.974	0.318, 0.997

Associated correlation coefficients and monotone correlation coefficients for the pairs of alternatives for $p = 4$, $q = 3$ are given in Table 7.

Table 7. Associated correlation coefficients and monotone correlation coefficients for the pairs of alternatives (Formulas (26), (36)) for $p = 4$, $q = 3$

$\langle d_i, d_j \rangle$	$\text{AsCor}_{\sigma_1}(\eta_i, \eta_j)$	$\text{AsCor}_{\sigma_2}(\eta_i, \eta_j)$	$\text{AsCor}_{\sigma_3}(\eta_i, \eta_j)$	$\text{AsCor}_{\sigma_4}(\eta_i, \eta_j)$	$\text{AsCor}_{\sigma_5}(\eta_i, \eta_j)$	$\text{AsCor}_{\sigma_6}(\eta_i, \eta_j)$	$\text{MCor}(\eta_i, \eta_j)$
$\langle d_1, d_2 \rangle$	0.006, 0.875	0.007, 0.896	0.004, 0.973	0.005, 0.925	0.006, 0.932	0.002, 0.941	0.002, 0.941
$\langle d_1, d_3 \rangle$	0.067, 0.986	0.214, 0.939	0.047, 0.858	0.021, 0.921	0.041, 0.991	0.065, 0.992	0.047, 0.858
$\langle d_1, d_4 \rangle$	0.134, 0.899	0.034, 0.792	0.103, 0.897	0.203, 0.984	0.182, 0.968	0.038 0.938	0.182, 0.968
$\langle d_2, d_3 \rangle$	0.002, 0.935	0.004, 0.796	0.003, 0.771	0.004, 0.984	0.014, 0.972	0.006, 0.947	0.014, 0.972
$\langle d_2, d_4 \rangle$	0.043, 0.923	0.036, 0.831	0.225, 0.899	0.015, 0.973	0.043, 0.996	0.012, 0.938	0.043, 0.923
$\langle d_3, d_4 \rangle$	0.023, 0.976	0.036, 0.887	0.145, 0.988	0.054, 0.996	0.015, 0.992	0.024, 0.975	0.024, 0.975

Max and Min-associated and monotone statistics for $p = 4$, $q = 3$ are given in Table 8.

Table 8. Max and Min-associated and monotone statistics (formulas (42)-(45) and (47)) $p = 4$, $q = 3$

D	max-p, q-AsPA(η_i)	min-p, q-AsPA(η_i)	$\text{ME}_g(\eta_i)$	max-p, q-AsVar(η_i)	min-p, q-AsVar(η_i)	MVar(η_i)
d_1	0.720, 0.367	0.689, 0.383	0.700, 0.389	0.359, 0.995	0.349, 0.995	0.354, 0.995
d_2	0.680, 0.473	0.575, 0.544	0.596, 0.544	0.360, 0.989	0.379, 0.993	0.351, 0.990
d_3	0.688, 0.337	0.671, 0.327	0.671, 0.327	0.221, 0.999	0.174, 0.999	0.216, 0.989
d_4	0.882, 0.473	0.856, 0.517	0.856, 0.511	0.344, 0.995	0.314, 0.997	0.337, 0.996
$\langle d_i, d_j \rangle$	max-p, q-AsCov(η_i, η_j)	min-p, q-AsCov(η_i, η_j)	MCov(η_i, η_j)	max-p, q-AsCor(η_i, η_j)	min-p, q-AsCor(η_i, η_j)	MCor(η_i, η_j)
$\langle d_1, d_2 \rangle$	0.001, 0.976	0.005, 0.985	0.001, 0.938	0.004, 0.973	0.007, 0.896	0.002, 0.941
$\langle d_1, d_3 \rangle$	0.003, 0.958	0.201, 0.999	0.307, 0.998	0.041, 0.991	0.214, 0.939	0.047, 0.858
$\langle d_1, d_4 \rangle$	0.227, 0.999	0.303, 0.997	0.318, 0.997	0.038, 0.938	0.034, 0.792	0.182, 0.968
$\langle d_2, d_3 \rangle$	0.004, 0.985	0.004, 0.982	0.001, 0.976	0.004, 0.984	0.003, 0.771	0.014, 0.972
$\langle d_2, d_4 \rangle$	0.267, 0.998	0.309, 0.997	0.309, 0.997	0.015, 0.973	0.225, 0.899	0.043, 0.923
$\langle d_3, d_4 \rangle$	0.005, 0.993	0.318, 0.997	0.318, 0.997	0.015, 0.992	0.145, 0.988	0.024, 0.975

According to Step 13, calculate indices SII and SII_F for all alternatives (Table 9).

Statistical independent indexes SII and SII_F for $p = 4$, $q = 3$ are given in Table 9.

Table 9. Statistical independent indexes SII and SII_F (formulas (56), (59)) for $p = 4$, $q = 3$

D	$\text{SII}(d_i)$	$\text{SII}_{\max}(d_i)$	$\text{SII}_{\min}(d_i)$
d_1	0.9735	0.9856	0.996
d_2	0.6667	0.8656	0.9656
d_3	0.8971	0.9767	0.9062
d_4	0.6467	0.8553	0.8793

We go to on the ending steps (14, 15) of the scheme. We rank MADM alternatives $D = \{d_1, d_2, d_3, d_4\}$ be the constructed by the order relations: $\underset{Me}{\geq}$; $\underset{\text{MVar}}{\geq}$; $\underset{F\text{-PA}}{\geq}$; $\underset{F\text{-Var}}{\geq}$; $\underset{\text{SII}}{\geq}$ and $\underset{\text{SII}_F}{\geq}$ $F \in \{\max, \min\}$.

Ranking of alternatives by the binary relations for $p = 3$, $q = 3$ are given in Table 10.

Ranking of alternatives by the binary relations for $p = 4$, $q = 3$ are given in Table 11.

Ranking of alternatives by the binary relations for $p = 5$, $q = 4$ are given in Table 12.

Table 10. Ranking of alternatives by the binary relations (formulas (52)–(55), (56)–(59)) for $p = 4$, $q = 3$

Ranking binary relations	Ranking of alternatives
$\underset{Me}{\geq}$	$d_4 \geq d_1 \geq d_3 \geq d_2$
$\underset{\text{MVar}}{\geq}$	$d_3 \geq d_1 \geq d_4 \geq d_2$
$\underset{\max\text{-PA}}{\geq}$	$d_4 \geq d_1 \geq d_3 \geq d_2$
$\underset{\min\text{-PA}}{\geq}$	$d_4 \geq d_1 \geq d_3 \geq d_2$
$\underset{\max\text{-Var}}{\geq}$	$d_3 \geq d_4 \geq d_1 \geq d_2$
$\underset{\min\text{-Var}}{\geq}$	$d_3 \geq d_4 \geq d_1 \geq d_2$
$\underset{\text{SII}}{\geq}$	$d_1 \geq d_3 \geq d_2 \geq d_4$
$\underset{\text{SII}_{\max}}{\geq}$	$d_1 \geq d_3 \geq d_2 \geq d_4$
$\underset{\text{SII}_{\min}}{\geq}$	$d_1 \geq d_2 \geq d_3 \geq d_4$

Table 11. Ranking of alternatives by the binary relations for $p = 3$, $q = 3$

Ranking binary relations	Ranking of alternatives
$\underset{Me}{\geq}$	$d_4 \geq d_1 \geq d_3 \geq d_2$
$\underset{\text{MVar}}{\geq}$	$d_1 \geq d_3 \geq d_4 \geq d_2$
$\underset{\max\text{-PA}}{\geq}$	$d_4 \geq d_1 \geq d_3 \geq d_2$
$\underset{\min\text{-PA}}{\geq}$	$d_4 \geq d_3 \geq d_1 \geq d_2$
$\underset{\max\text{-Var}}{\geq}$	$d_3 \geq d_4 \geq d_1 \geq d_2$
$\underset{\min\text{-Var}}{\geq}$	$d_3 \geq d_4 \geq d_1 \geq d_2$
$\underset{\text{SII}}{\geq}$	$d_1 \geq d_3 \geq d_2 \geq d_4$
$\underset{\text{SII}_{\max}}{\geq}$	$d_1 \geq d_3 \geq d_2 \geq d_4$
$\underset{\text{SII}_{\min}}{\geq}$	$d_1 \geq d_2 \geq d_3 \geq d_4$

Table 12. Ranking of alternatives by the binary relations for $p = 5$, $q = 4$

Ranking binary relations	Ranking of alternatives
$\underset{Me}{\geq}$	$d_4 \geq d_1 \geq d_3 \geq d_2$
$\underset{\text{MVar}}{\geq}$	$d_3 \geq d_4 \geq d_1 \geq d_2$
$\underset{\max\text{-PA}}{\geq}$	$d_4 \geq d_1 \geq d_3 \geq d_2$
$\underset{\min\text{-PA}}{\geq}$	$d_4 \geq d_1 \geq d_3 \geq d_2$
$\underset{\max\text{-Var}}{\geq}$	$d_3 \geq d_1 \geq d_4 \geq d_2$
$\underset{\min\text{-Var}}{\geq}$	$d_3 \geq d_4 \geq d_1 \geq d_2$
$\underset{\text{SII}}{\geq}$	$d_1 \geq d_3 \geq d_2 \geq d_4$
$\underset{\text{SII}_{\max}}{\geq}$	$d_1 \geq d_3 \geq d_2 \geq d_4$
$\underset{\text{SII}_{\min}}{\geq}$	$d_1 \geq d_2 \geq d_3 \geq d_4$

6.1 *Comparative analysis*

The example considers the case of three values of a pair (p, q). To compare the results obtained by ranking, it is natural to divide the new aggregation operators into three groups. 1). Averaging operators with the corresponding ranking binary relations: $\underset{Me}{\geq}$, $\underset{\text{max-PA}}{\geq}$, and $\underset{\text{min-PA}}{\geq}$. If we compare the rankings with these relations from Tables 10–12, it is clear that a kind of stability is observed with d_4-optimal alternative. However, the subsequent ranking alternatives for the relations $\underset{\text{max-PA}}{\geq}$ and $\underset{\text{min-PA}}{\geq}$ are partially different from the corresponding ranking for the relation $\underset{Me}{\geq}$. This should be due to the sufficient degree of interaction of the attributes; 2). Decision risk operators with the corresponding ranking binary relations: $\underset{\text{MVar}}{\geq}$, $\underset{\text{max-Var}}{\geq}$, and $\underset{\text{min-Var}}{\geq}$. For these relations the optimal alternative is d_3, except for one case, which is replaced by d_1 and which is the second ranking alternative in the remaining two cases; 3). Aggregation operators statistically taking into account independence with the corresponding ranking binary relations: $\underset{\text{SII}}{\geq}$, $\underset{\text{SII}_{\max}}{\geq}$, and $\underset{\text{SII}_{\min}}{\geq}$. For these relations the optimal alternative is d_1. The next ranking alternatives are d_2 and d_3. This difference must be due to the phenomenon of attributes interaction.

In conclusion, we can say that:

1. With the change of the pair (p, q) values, with the increase of their values, the rankings remained essentially unchanged, which indicates the sensitivity of our approach.

2. For three different groups of aggregation operators, three stable different optimal solutions are observed. Accordingly, for the first, second and third groups, we get alternatives d_4, d_3 and d_1. This was expected. However, there are differences in the aggregations of the second group of ranking, which is caused by the phenomenon of the interaction of attributes.

3. The large spectrum of new aggregations indicates that, depending on the decision-making mood of the decision-maker, the appropriate aggregation group should be selected. In addition, there is the possibility of selecting pessimistic (min) and optimistic (max) aggregation operators in the group.

4. As for the different results for the aggregated results of monotone and F-associated statistics. As it was said in the research part, monotone statistics take into account the phenomenon of interaction of attributes of MADM partially, while F-associated statistics take into account this phenomenon completely. Therefore, conducting this analysis in specific MADM practical problems would be of essential importance. Since this example is created only for the calculation scheme built in the article, there is no substantive side to this problem.

7. Conclusions

Obviously, the construction of aggregation tools for minimal decision-making risks or independent optimal alternative selection in interactive MADM models is an important problem. In fuzzy interactive MADM modeling, the main problem is the handling of the dual conflict phenomenon of uncertainty-imprecision caused by expert assessments. The presented study is concerned with this problem. Specifically, with the problem of systematizing statistical analysis for p, q-ROF values, which are quite common in interactive MADM models when the uncertainty index of interacting attributes is described not by an additive (probability) measure, but by a monotone (fuzzy) measure. It is known that in such cases, non-additive and nonlinear instruments and operators should be used for data aggregation. On the other hand, in order to systematize statistical analysis in interactive MADM models, the concept of “Monotone Expectation” is considered for the extension of classical statistics in the p, q-ROF environment. In the classical view, the arithmetic operations of subtraction and division are not defined in p, q-ROF values. The operations of pseudo-subtraction and pseudo-division are introduced to define the covariance and correlation coefficient. The analytical properties of the new operations were also presented. To achieve reduction and minimization of decision-making risks, statistics based on the finite integral of Choquet for p, q-ROF information are considered: monotone probability average, monotone variance, monotone covariance, and monotone correlation coefficient. Since the aggregations obtained by these statistics cannot fully reflect the degrees of interaction of attributes, their extensions are introduced: F-associated probability average, F-associated variance, F-associated covariance, and F-associated correlation coefficient. In their aggregations, they take into account all possible variants of consonant structures of attributes–that is, they combine all variants. This creates fewer risks in the rankings of alternatives obtained through relatively monotone statistics and aggregations. Extension correctness implies that if the second-order extremal Choquet capacities are taken as a fuzzy measure and the F-mean aggregation function is a max or min, then the F-associated statistics and monotone statistics values coincide. Quite different statistics for ranking MADM alternatives have been constructed for different practical problems in p, q-ROFNs environment. A certain orderly form has been given to p, q-ROFNs-statistical analysis. To illustrate the obtained results, a numerical example of a simple fuzzy interactive MADM is presented. Comparative analysis of the attribute rankings obtained with constructed statistics is performed. A limitation of the application of our new approach in interactive MADM problems of practical value is that it will be necessary to solve a rather difficult fuzzy measure identification problem. For the development of future research, we will extend the class of monotone and F-associated statistics considered for p, q-ROF and other fuzzy environments. The concept of "Monotone Expectation" will be extended in the presence of new fuzzy information. The obtained results will be illustrated in high value decision-making problems.

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Conflict of interest

The authors declare no competing financial interest.

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