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On the Minimum and Maximum Complementary Geometric-Arithmetic Index of Connected Graphs

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Abstract: Consider a graph G whose collection of edges is represented by E. For any vertex $w \in V(G)$, let d_w denote the number of edges incident to w, referred to as its degree. The graph invariant known as the complementary geometric-arithmetic index associated with G is defined as $cGA(G) = \sum_{vu \in E} \sqrt{1-(d_v/d_u)^2}$, provided that $d_v \leq d_u$. This paper provides some bounds on this index. The graphs maximizing/minimizing this index among all fixed-order (molecular) trees are also characterized. Only regular graphs minimize cGA among connected n-order graphs for every $n \geq 3$. A computer-based approach is employed to exhaustively search the graphs maximizing cGA among connected n-order graphs for $5 \leq n \leq 10$. Based on these computational findings, a conjecture is proposed, and two structural properties of the extremal graphs are provided.

***Keywords*:** complementary geometric-arithmetic index, extremal problem, bound

MSC: 05C05, 05C07, 05C09

1. Introduction

Chemical graph theory constitutes a focused area within mathematical chemistry that employs graph-theoretic methods to describe and investigate molecular systems (see, for instance, [1]). Within this setting, a chemical compound is abstracted as a graph in which atoms correspond to vertices, while the connections between them, namely chemical bonds, are represented by edges, thereby allowing molecular characteristics to be examined through precise mathematical methods.

Most of the terminology and fundamental notions employed in this work concerning general graph theory and chemical graph theory are taken from [2, 3] and [4, 5], respectively.

One of the fundamental concerns in chemical graph theory involves the development and application of quantitative graph-based measures, commonly referred to as molecular descriptors (see [6, 7]), which play a crucial role in cheminformatics applications such as quantitative structure-property relationships (see [8]). Here, we are concerned

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with a particular class of molecular descriptors, namely topological indices. For an overview of recent progress and practical uses of topological indices within chemical research, the interested reader may consult [1, 9].

Within the broad classification of topological indices, those constructed from vertex degree information form a particularly important class (see [10–12]) and hold a prominent position owing to their computational simplicity and their good empirical correlation with many molecular properties.

Degree-based topological indices formulated through a geometrical perspective have garnered considerable attention and continue to be the focus of intensive investigation. Among these, the most prominent and pioneering example is the Sombor index, introduced by Gutman in [13]. A comprehensive discussion of this index is available in the corresponding survey article [14] and in some of the recent studies [15, 16]. The elliptic Sombor index is another example of the topological index developed via a geometrical approach (see [17]).

Recently, in [18], a new way of interpreting geometrical degree-based topological indices is proposed by introducing the notions of the angle of an edge and its complement. This formulation leads the authors of [18] to define a whole class of topological indices, which they call complementary topological indices. In this work, our focus is on the complementary geometric-arithmetic index, a graph invariant that is introduced for a graph G by the following definition:

$$cGA(G) = \sum_{vu\in E} \frac{\sqrt{|(d_v)^2 - (d_u)^2|}}{\max\{d_v, d_u\}}, \tag{1}$$

where E denotes the collection of edges of the graph G, while d_u and d_v correspond to the vertex degrees of u and v, respectively. Equation (1) can be rewritten as

$$cGA(G) = \sum_{vu\in E} \sqrt{1 - \left(\frac{d_v}{d_u}\right)^2}, \text{ provided that } d_u \geq d_v. \tag{2}$$

We remark here that the complementary geometric-arithmetic index is a variant of the geometric-arithmetic index (see [19]) defined as:

$$GA(G) = \sum_{vu\in E} \frac{2\sqrt{d_u d_v}}{d_u + d_v}.$$

The formula for cGA(G) is obtained from the one concerning GA(G) by replacing d_u and d_v with $d_u + d_v$ and $d_u - d_v$, respectively, provided that $d_u \geq d_v$.

Throughout this article, all graphs are assumed to be finite and without loops or multiple edges. A graph containing n vertices is called an n-order graph. Any vertex whose degree equals one is termed a pendent vertex, and an edge adjacent to such a vertex is called a pendent edge. Graphs whose maximum vertex degree does not exceed four are commonly referred to as molecular graphs. In particular, a tree satisfying this degree constraint is known as a molecular tree.

This paper establishes several bounds for the index cGA and characterizes the graphs that maximize or minimize its value among all fixed-order (molecular) trees. A computational approach is further employed to perform an exhaustive search for graphs that maximize the index cGA within the set of connected n-order graphs, for $5 \leq n \leq 10$. The insights obtained from these computations lead to the formulation of a conjecture.

2. Bounds

Since $0 \leq \sqrt{1-r^2} < 1$ for any positive real number $r \leq 1$, from Equation (2) the observation given below follows.

Observation 1 Let G be a graph containing m edges. Then

$$0 \leq cGA(G) < m,$$

where $cGA(G) = 0$ iff each component of G has all vertices of identical degree.

In the following proposition, we improve the upper bound given in Observation 1.

Proposition 1 Let G be a graph with m edges, largest vertex degree Δ, and smallest vertex degree δ satisfying $\delta \geq 1$. Then,

$$cGA(G) \leq m\sqrt{1-\left(\frac{\delta}{\Delta}\right)^2}. \tag{3}$$

Equality in (3) holds iff either the all vertices of the graph G have identical degrees or $\{d_x, d_y\} = \{\delta, \Delta\}\ \forall\, xy \in E$.

Proof. Let $vu \in E$ such that $d_u \geq d_v$. Then,

$$\frac{d_v}{d_u} \geq \frac{\delta}{\Delta}, \quad \text{which yields} \quad \sqrt{1-\left(\frac{d_v}{d_u}\right)^2} \leq \sqrt{1-\left(\frac{\delta}{\Delta}\right)^2}.$$

So,

$$cGA(G) = \sum_{vu \in E} \sqrt{1-\left(\frac{d_v}{d_u}\right)^2} \leq m\sqrt{1-\left(\frac{\delta}{\Delta}\right)^2},$$

where the right equality holds iff $d_u = \Delta$ and $d_v = \delta\ \forall\, vu \in E$. $\square$

Proposition 2 If G is a graph of size m, then

$$cGA(G) \leq \sqrt{m\,(m - MMSD(G))}, \tag{4}$$

where $MMSD(G)$ is the min-max sdeg index of G (for the definition of the max-min sdeg index, see [20]), defined as

$$MMSD(G) = \sum_{vu \in E} \left(\frac{\min\{d_v, d_u\}}{\max\{d_v, d_u\}}\right)^2.$$

The equality in (4) holds iff there is a real number α' provided that

$$\frac{\min\{d_v, d_u\}}{\max\{d_v, d_u\}} = \alpha',$$

$\forall\ vu \in E$.

Proof. Here,

$$\sum_{vu\in E}\frac{\sqrt{|(d_v)^2-(d_u)^2|}}{\max\{d_v,d_u\}} \leq \sqrt{m\sum_{vu\in E}\frac{|(d_v)^2-(d_u)^2|}{(\max\{d_v,d_u\})^2}}$$

$$= \sqrt{m\sum_{vu\in E}\left(1-\left(\frac{\min\{d_v,d_u\}}{\max\{d_v,d_u\}}\right)^2\right)}. \tag{5}$$

The equality in (5) holds iff there is a real number α provided that

$$\frac{\sqrt{|(d_v)^2-(d_u)^2|}}{\max\{d_v,d_u\}} = \alpha \quad \forall\ vu \in E,$$

that is, iff

$$\frac{\min\{d_v,d_u\}}{\max\{d_v,d_u\}} = \sqrt{1-\alpha^2} := \alpha',$$

$\forall\ vu \in E$. □

Proposition 3 For a graph G having m edges, the following inequality holds:

$$cGA(G) \geq m - MMSD(G),$$

and equality is achieved iff d_v is the same $\forall\ v \in V(G)$.

Proof. Let $vu \in E$ such that $d_u \geq d_v$. Then, $\frac{d_v}{d_u} \in (0,1]$, and hence

$$\sqrt{1-\left(\frac{d_v}{d_u}\right)^2} \geq 1-\left(\frac{d_v}{d_u}\right)^2,$$

with equality occurring iff $\frac{d_v}{d_u} = 1$. Summing over all edges the above inequality yields the desired conclusion. □

3. Extremal (molecular) trees

In this section, we determine graphs that minimize/maximize cGA among all n-order (molecular) trees. We first deal with trees, and subsequently consider molecular trees.

Theorem 1 Let T be a tree having order $n \geq 4$.

(i) $cGA(T)$ is uniquely maximized by the star S_n. Its value is

$$cGA(S_n) = \sqrt{n(n-2)}.$$

(ii) $cGA(T)$ is uniquely minimized by the path P_n and $cGA(P_n) = \sqrt{3}$.

Proof. (i). Let Δ denote the largest vertex degree of T. Proposition 1 confirms

$$cGA(T) \leq (n-1)\sqrt{1-\left(\frac{1}{\Delta}\right)^2},$$

with equality occurring iff each edge of the graph is adjacent to one pendent vertex and one vertex of degree Δ. However,

$$\sqrt{1-\left(\frac{1}{\Delta}\right)^2} \leq \sqrt{1-\left(\frac{1}{n-1}\right)^2},$$

with equality occurring iff $\Delta = n-1$.

(ii). Let E_p represent the collection of pendent edges in the tree T. Then

$$cGA(T) = \sum_{wx\in E_p;\ d_x=1} \frac{\sqrt{(d_w)^2-1}}{d_w} + \sum_{vu\in E\setminus E_p} \frac{\sqrt{|(d_v)^2-(d_u)^2|}}{\max\{d_v, d_u\}}. \tag{6}$$

Here,

$$\sum_{vu\in E\setminus E_p} \frac{\sqrt{|(d_v)^2-(d_u)^2|}}{\max\{d_v, d_u\}} \geq 0,$$

with equality occurring iff $d_u = d_v \ \forall \ vu \in E \setminus E_p$. Also,

$$\sum_{wx\in E_p;\ d_x=1} \frac{\sqrt{(d_w)^2-1}}{d_w} \geq \frac{\sqrt{3}}{2}\,|E_p| \geq \sqrt{3},$$

where the both equalities holds at the same time iff $d_w = 2$ and $|E_p| = 2$. Now, the required conclusion follows from (6). ☐

Theorem 1 implies that the path graph P_n uniquely minimizes cGA over the class of all n-order molecular trees $\forall$ $n \geq 4$. In what follows, we determine graphs that maximize cGA over the aforementioned class of molecular trees. For this, we define four classes of n-order molecular trees. Denote by $\mathscr{T}_{0,n}$ the class of n-order molecular trees for which $x_{1,2} = x_{2,2} = x_{3,3} = x_{3,4} = x_{4,4} = 0$, $x_{1,3} = 2$ and $x_{2,3} = 1$, provided that $4(n_4+1) = n \geq 8$. Let $\mathscr{T}_{1,n}$ denote the class of n-order molecular trees for which each of $x_{1,2}, x_{1,3}, x_{2,2}, x_{2,3}, x_{3,3}, x_{3,4}$ and $x_{4,4}$ is zero, provided that $4n_4+1 = n \geq 5$. Let $\mathscr{T}_{2,n}$ represents the set of molecular trees having n vertices for which $x_{2,2} = x_{4,4} = 0$, $x_{1,2} = 1$ and $n_3 = 0$, provided that $4n_4+2 = n \geq 6$. Finally, denote by $\mathscr{T}_{3,n}$ the class of n-order molecular trees for which $x_{1,2} = x_{2,2} = x_{3,3} = x_{2,3} = x_{4,4} = 0$, $x_{1,3} = 2$ and $x_{3,4} = 1$, such that $4n_4+3 = n \geq 7$.

The formula for the complementary geometric-arithmetic index of an n-order tree T can be written as

$$cGA(T) = \sum_{1\leq i\leq j\leq n-1} \frac{\sqrt{|j^2-i^2|}}{\max\{j,i\}}\, x_{j,i}. \tag{7}$$

Theorem 2 Let T be a molecular tree on n vertices, where $n \geq 5$. Then the following assertions are valid:

(i) If $n \equiv 1 \pmod 4$, then

$$cGA(T) \leq \left(\frac{\sqrt{3}}{4}+\frac{\sqrt{15}}{8}\right)n+\frac{3\sqrt{15}-10\sqrt{3}}{8},$$

with equality occurring iff $T \in \mathscr{T}_{1,n}$.

(ii) If $n \equiv 0 \pmod 4$, then

$$cGA(T) \leq \left(\frac{\sqrt{3}}{4}+\frac{\sqrt{15}}{8}\right)n+\frac{16\sqrt{2}-18\sqrt{3}+4\sqrt{5}-3\sqrt{15}}{12},$$

with equality occurring iff $T \in \mathscr{T}_{0,n}$.

(iii) If $n \equiv 3 \pmod 4$, then

$$cGA(T) \leq \left(\frac{\sqrt{3}}{4}+\frac{\sqrt{15}}{8}\right)n+\frac{32\sqrt{2}-42\sqrt{3}+6\sqrt{7}-3\sqrt{15}}{24},$$

with equality occurring iff $T \in \mathscr{T}_{3,n}$.

(iv) If $n \equiv 2 \pmod 4$, then

$$cGA(T) \leq \left(\frac{\sqrt{3}}{4}+\frac{\sqrt{15}}{8}\right)n-\frac{\sqrt{3}}{2},$$

with equality occurring iff $T \in \mathscr{T}_{2,n}$.

Proof. Evidently, if $T \in \cup_{i=0}^{3}\mathscr{T}_{i,n}$, then $cGA(T)$ is equal to one of the bounds given in parts (i)–(iv). Let T^* be a graph maximizing cGA among molecular trees with order n such that $n \geq 5$. Then,

$$cGA(T) \leq cGA(T^*). \tag{8}$$

For T^*, the following are valid:

$$\sum_{i=1}^{4} n_i = n, \tag{9}$$

$$\sum_{i=1}^{4} i \cdot n_i = 2(n-1), \tag{10}$$

$$\sum_{\substack{1 \leq i \leq 4 \\ i \neq j}} x_{j,i} + 2x_{j,j} = j \cdot n_j \tag{11}$$

for $j = 1, 2, 3, 4$. We solve the system of Eqs. (9)–(11) for n_1, n_2, n_3 n_4, $x_{1,4}$ and $x_{2,4}$, and then substitute the values of $x_{1,4}$ and $x_{2,4}$ in (7) to obtain:

$$\begin{aligned} cGA(T^*) = {} & \left(\frac{\sqrt{3}}{4} + \frac{\sqrt{15}}{8}\right) n + \frac{3\sqrt{15}}{8} - \frac{5\sqrt{3}}{4} + \left(\frac{3\sqrt{3}}{4} - \frac{3\sqrt{15}}{8}\right) x_{1,2} \\ & + \left(-\frac{7\sqrt{\frac{5}{3}}}{8} + \frac{2\sqrt{2}}{3} + \frac{1}{4\sqrt{3}}\right) x_{1,3} + \left(-\frac{\sqrt{3}}{4} - \frac{\sqrt{15}}{8}\right) x_{2,2} \\ & + \left(-\frac{\sqrt{\frac{5}{3}}}{8} - \frac{5}{4\sqrt{3}} + \frac{\sqrt{5}}{3}\right) x_{2,3} + \left(\frac{\sqrt{\frac{5}{3}}}{8} - \frac{7}{4\sqrt{3}}\right) x_{3,3} \\ & + \left(\frac{\sqrt{\frac{5}{3}}}{4} - \frac{2}{\sqrt{3}} + \frac{\sqrt{7}}{4}\right) x_{3,4} + \left(\frac{\sqrt{15}}{8} - \frac{3\sqrt{3}}{4}\right) x_{4,4}. \end{aligned} \tag{12}$$

We now rewrite (12), expressing the coefficients of $x_{j,i}$ in approximate form.

$$cGA(T^*) \approx \left(\frac{\sqrt{3}}{4} + \frac{\sqrt{15}}{8}\right) n + \frac{3\sqrt{15}}{8} - \frac{5\sqrt{3}}{4} + \Gamma_{cGA}(T^*), \tag{13}$$

where

$$\begin{aligned} \Gamma_{cGA}(T^*) = {} & -0.1533x_{1,2} - 0.0425x_{1,3} - 0.9171x_{2,2} - 0.1377x_{2,3} \\ & - 0.8490x_{3,3} - 0.1705x_{3,4} - 0.8149x_{4,4}. \end{aligned} \tag{14}$$

Since $\Gamma_{cGA}(T^*) \leq 0$, by the definition of T^*, the quantity $\Gamma_{cGA}(T^*)$ must be maximum provided that that at least one graph exists corresponding to that maximum value. If $n \equiv 1 \pmod 4$, then we have $\Gamma_{cGA}(T^*) = 0$ iff $T^* \in \mathscr{T}_{1,n}$. Hence, from (8) and (12), it follows that

$$cGA(T) \leq \left(\frac{\sqrt{3}}{4} + \frac{\sqrt{15}}{8}\right) n + \frac{3\sqrt{15}}{8} - \frac{5\sqrt{3}}{4},$$

with equality occurring iff $T \in \mathscr{T}_{1,n}$. The proof of part (i) is thus now completed.

If $n_3 \geq 2$, then (14) gives

$$\Gamma_{cGA}(T^*) < 2\left(-\frac{7\sqrt{\frac{5}{3}}}{8}+\frac{2\sqrt{2}}{3}+\frac{1}{4\sqrt{3}}\right)+\left(\frac{\sqrt{\frac{5}{3}}}{4}-\frac{2}{\sqrt{3}}+\frac{\sqrt{7}}{4}\right)\approx -0.2555,$$

which contradicts the definition of T^* because the class $\cup_{i=0}^{3}\mathscr{T}_{i,n}$ contains at least one tree whose Γ_{cGA}-value is at least -0.2555. Hence, $n_3 \in \{0,1\}$.

Now, we assume that $n \equiv 0 \pmod 4$. Then, we have $n_3 = 1$. Hence, $m_{1,3}+m_{2,3}+m_{3,4} = 3$. Keeping in mind (14), we deduce that $m_{1,3} = 2$. Hence, $m_{2,3}+m_{4,3} = 1$, which yields either $(m_{2,3}, m_{4,3}) = (1,0)$ or $(m_{2,3}, m_{4,3}) = (0,1)$. Since $n \equiv 0 \pmod 4$, we have $(m_{2,3}, m_{4,3}) = (1,0)$ and hence (14) gives

$$\Gamma_{cGA}(T^*) = 2\left(-\frac{7\sqrt{\frac{5}{3}}}{8}+\frac{2\sqrt{2}}{3}+\frac{1}{4\sqrt{3}}\right)+\left(-\frac{\sqrt{\frac{5}{3}}}{8}-\frac{5}{4\sqrt{3}}+\frac{\sqrt{5}}{3}\right)$$

and $T^* \in \mathscr{T}_{0,n}$. Hence, from (8) and (12), the conclusion of part (ii) follows.

Next, we assume that $n \equiv 3 \pmod 4$. Then, by the proof of part (ii), we have $n_3 = 1$, $m_{1,3} = 2$ and $m_{3,4} = 1$. Thus, (14) gives

$$\Gamma_{cGA}(T^*) = 2\left(-\frac{7\sqrt{\frac{5}{3}}}{8}+\frac{2\sqrt{2}}{3}+\frac{1}{4\sqrt{3}}\right)+\left(\frac{\sqrt{\frac{5}{3}}}{4}-\frac{2}{\sqrt{3}}+\frac{\sqrt{7}}{4}\right)$$

and $T^* \in \mathscr{T}_{3,n}$. Hence, from (8) and (12), the conclusion of part (iii) follows.

Finally, we consider the case where $n \equiv 2 \pmod 4$. Then, we have $n_3 = 0$, and hence (14) yields

$$\Gamma_{cGA}(T^*) = \left(\frac{3\sqrt{3}}{4}-\frac{3\sqrt{15}}{8}\right)x_{1,2}+\left(-\frac{\sqrt{3}}{4}-\frac{\sqrt{15}}{8}\right)x_{2,2}+\left(\frac{\sqrt{15}}{8}-\frac{3\sqrt{3}}{4}\right)x_{4,4}.$$

and $T^* \in \mathscr{T}_{2,n}$. Hence, from (8) and (12), the conclusion of part (iv) follows. □

4. Extremal connected graphs

By Observation 1, only the regular graphs minimize cGA among all connected n-order graphs $\forall\, n \geq 3$. Determining graphs maximizing cGA among connected n-order graphs seems not to be an easy task. We employed a computer-based approach to exhaustively search graphs maximizing cGA among the aforesaid graph types, and $5 \leq n \leq 10$. Figure 1 shows the extremal graphs obtained from this computer-based approach. Based on these graphs, we pose the following conjecture.

Conjecture 1 Only the graph $K^{-}_{\lfloor n/2 \rfloor} + \overline{K}_{n-\lfloor n/2 \rfloor}$ maximizes *cGA* in the collection of connected n-order graphs $\forall$ $n \geq 6$, where the graph $K^{-}_{\lfloor n/2 \rfloor}$ is constructed from the $\lfloor n/2 \rfloor$-order complete graph $K_{\lfloor n/2 \rfloor}$ by removing an edge, $\overline{K}_{n-\lfloor n/2 \rfloor}$ represents the complement of $K_{n-\lfloor n/2 \rfloor}$ and "+" denotes the join operation.

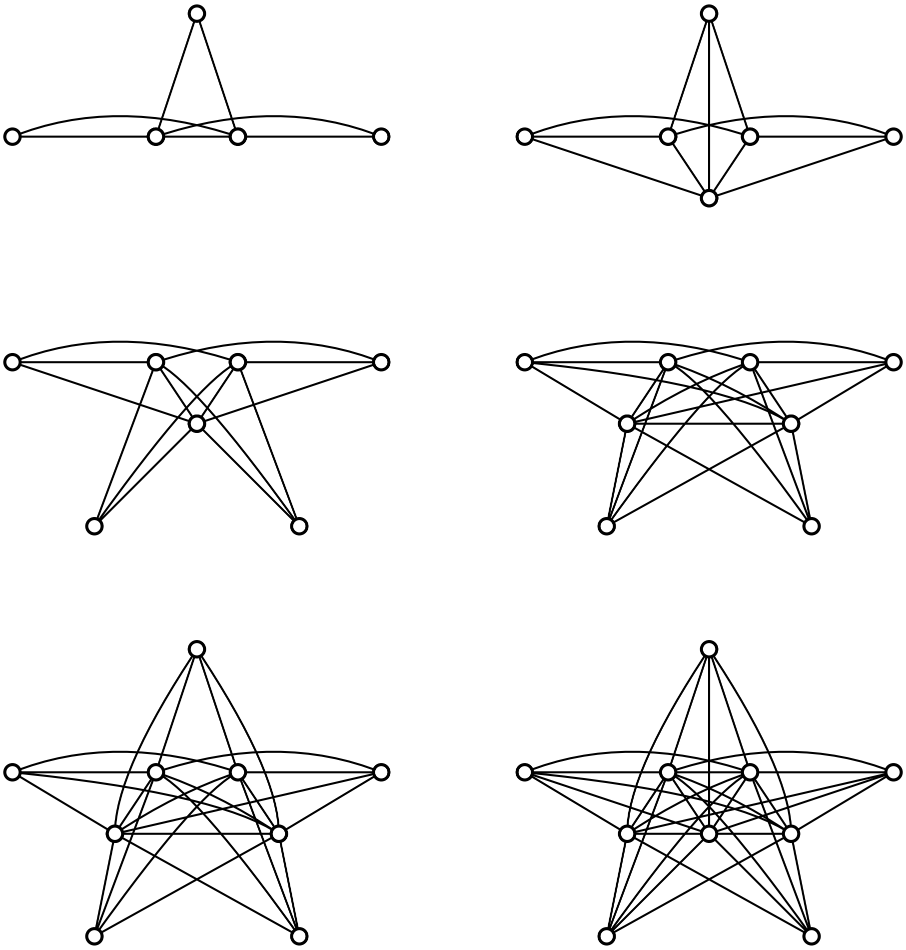

Figure 1. Graphs maximizing cGA among connected n-order graphs for $5 \leq n \leq 10$

Based on the structures of extremal graphs depicted in Figure 1, we now provide two properties of such graphs.

Theorem 3 Let G be a graph that attains the maximum value of cGA among all connected graphs with $n \geq 5$ vertices. Then, G does not contain adjacent vertices of minimum degree.

Proof. Let δ and Δ denote the minimum degree and maximum degree of G, respectively. For the sake of contradiction, assume that $v, w \in V(G)$ are adjacent vertices with degree δ. Then, $\delta \geq 2$ as $n \geq 5$. Let G' be the graph formed from G by deleting vw. Throughout the remainder of the proof, for each vertex $s \in V(G') = V(G)$, we denote by d_s its degree in the original graph G. Since

$$\sqrt{1-\left(\frac{\delta-1}{d_u}\right)^2}-\sqrt{1-\left(\frac{\delta}{d_u}\right)^2} \geq \sqrt{1-\left(\frac{\delta-1}{\Delta}\right)^2}-\sqrt{1-\left(\frac{\delta}{\Delta}\right)^2}$$

for every vertex $u \in V(G)$, it holds that

$$cGA(G')-cGA(G)=\sum_{y\in N(v)\setminus\{w\}}\left[\sqrt{1-\left(\frac{\delta-1}{d_y}\right)^2}-\sqrt{1-\left(\frac{\delta}{d_y}\right)^2}\right]$$

$$+\sum_{z\in N(w)\backslash\{v\}}\left[\sqrt{1-\left(\frac{\delta-1}{d_z}\right)^2}-\sqrt{1-\left(\frac{\delta}{d_z}\right)^2}\right]$$

$$\geq 2(\delta-1)\left[\sqrt{1-\left(\frac{\delta-1}{\Delta}\right)^2}-\sqrt{1-\left(\frac{\delta}{\Delta}\right)^2}\right]>0,$$

which yields a contradiction to the extremal property of G. □

Theorem 4 Let G be a graph that attains the maximum value of cGA among all connected graphs with $n \geq 5$ vertices. Then, G does not contain nonadjacent vertices of maximum degree.

Proof. Let δ and Δ denote the minimum degree and maximum degree of G, respectively. For the sake of contradiction, assume that $v, w \in V(G)$ are nonadjacent vertices with degree Δ. Then, $\Delta \geq 2$ as $n \geq 5$. Let G' be the graph formed from G by adding vw. Throughout the remainder of the proof, for each vertex $s \in V(G') = V(G)$, we denote by d_s its degree in the original graph G. Since

$$\sqrt{1-\left(\frac{d_u}{\Delta+1}\right)^2}-\sqrt{1-\left(\frac{d_u}{\Delta}\right)^2}\geq\sqrt{1-\left(\frac{\delta}{\Delta+1}\right)^2}-\sqrt{1-\left(\frac{\delta}{\Delta}\right)^2}$$

for every vertex $u \in V(G)$, it holds that

$$cGA(G')-cGA(G)=\sum_{y\in N(v)\backslash\{w\}}\left[\sqrt{1-\left(\frac{d_y}{\Delta+1}\right)^2}-\sqrt{1-\left(\frac{d_y}{\Delta}\right)^2}\right]$$

$$+\sum_{z\in N(w)\backslash\{v\}}\left[\sqrt{1-\left(\frac{d_z}{\Delta+1}\right)^2}-\sqrt{1-\left(\frac{d_z}{\Delta}\right)^2}\right]$$

$$\geq 2(\Delta-1)\left[\sqrt{1-\left(\frac{\delta}{\Delta+1}\right)^2}-\sqrt{1-\left(\frac{\delta}{\Delta}\right)^2}\right]>0,$$

which yields a contradiction to the extremal property of G. □

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Data availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Use of generative-AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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