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Research Article

Innovative Approaches to Istrăţescu-Type Fixed Point Results via Simulation Functions in S_b-Metric Spaces with Applications

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Abstract: Three fundamental mathematical concepts—Istrăţescu-type contractions, simulation functions, and S_b-metric spaces—are unified into a coherent framework of fixed point theory in this article. We establish the existence and uniqueness of fixed points for the so-called ζ-Istrăţescu contractions under conditions that extend classical assumptions. Using the flexibility of simulation functions and the generality of S_b-metrics, our results subsume classical theorems of Banach, Čirić, and Istrăţescu, and open new directions in nonlinear analysis. The theoretical framework is developed rigorously, covering existence, uniqueness, and convergence under weak contractivity conditions. Applications are provided for differential equations, optimization theory, and image processing, supported by algorithms and convergence rate analyses.

Keywords: fixed point theory, Istrăţescu contractions, simulation functions, S_b-metric spaces, differential equations, image processing

MSC: 47H10, 54H25, 47H09, 54E50, 65J15, 35A15, 49M37, 68U10

1. Introduction

Fixed point theory is one of the most beautiful and practically important fields of mathematical analysis, giving the theoretical basis of many applications, including different equations and economic modeling. In 1922 [1], the classical Banach contraction principle was introduced, and this volume of generalizations and extensions is ever-growing.

In the 1970s, a decisive breakthrough in this area was the work of Vasile Istrăţescu [2] where more flexible conditions of contractivity were introduced without violating the key properties of classical contractions. The work of Istrăţescu showed that a broadening of the range of problems that can be dealt with using fixed point techniques was possible when much weaker conditions were imposed.

Over the last several years, two parallel developments have taken place in the mathematical community that revolutionized fixed point theory. To start with, the introduction of the simulation functions by Khojasteh et al. [3] offered a unified approach to the presentation of different kinds of contractive conditions. They provide fantastic versatility in the way one can construct contractivity requirements, and mathematicians have been able to define the nature of various

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types of contractions in one, consistent system. At the same time, the extension of metric spaces to the more flexible, but equally geometric, S_b-metric spaces, built upon the foundational work of Bakhtin [4] and Czerwik [5], has been proposed as a natural extension of the more traditional metric space. Such spaces, with an adjusted triangle inequality, give a more sophisticated sense of distance relations, and have been found to be especially useful in applications involving non-uniform scale or direction dependence.

The main motivation of the research is the fact that in the real world, there are numerous issues which are related to distance relationships that fail to fit the rigorous nature of classical metric spaces but which do demand an analytical quality of analysis which fixed point theory offers. Within the framework of the S_b-metric, we can approach a wider set of issues and retain the level of mathematical accuracy using simulation functions. The article unites these three potent mathematical concepts namely: Istrăţescu-type contractions, simulation functions, and the flexible geometry of S_b-metric spaces in this article to create a single and flexible framework of the fixed point theory.

1.1 *Historical context and evolution*

The development of the fixed point theory can be followed with the help of some important milestones that demonstrate the development of the field. The topological foundations were laid by Brouwer, and the analytical foundations necessary to be the gold standard of constructive fixed point results were laid by the contraction principle of Banach [1]. Later mathematicians, such as Čirić [6], Rhoades [7], and Istrăţescu [2] broadened the scope of these findings to more complex cases.

Another important development was made with the introduction of generalized metric spaces. The triangle inequality was weakened by Czerwik [5] to create the triangle inequality in b-metric spaces: a constant factor was added, and by Matthews [8] to create partial metric spaces: self-distances could be different than zero. The distance measurement in 3 variables had been offered by Mustafa and Sims [9] with their G-metric spaces space concept elaborated in the book of Mustafa 2006. The most elegant generalization that preserves symmetry and permits controlled violation of the standard triangle inequality is the b-metric space concept, introduced by Bakhtin [4] and Czerwik [5], which we study here under the S_b-metric framework [10].

Along with these advances in metric space theory, the theory of simulation functions came into view as a unifying theory of contractive conditions. These functions were proposed by Khojasteh et al. [3] to implement different types of contraction in one functional form, which was not only theoretically elegant but also had the practical flexibility required. These thoughts are synthesized together, generalized metrics, simulation functions, and the sophisticated types of contraction, and it is on this basis that we are working today.

1.2 *Research objectives and contributions*

The goal of this research is to create a complete framework that expands classical fixed point theory in many ways at once. Our particular aims consist of:

1. Theoretical Generalization: To introduce and carefully study ζ-Istrăţescu contractions in S_b-metric spaces, showing that they exist, are unique, and converge under the fewest possible assumptions.

2. Framework Unification: To show how our method brings together a number of classical fixed point theorems, such as those of Banach, Čirić, and Istrăţescu, into one clear framework.

3. Application Development: To show how useful our results are in real life by using them in differential equations, optimization theory, and image processing.

4. Computational Implementation: To create useful algorithms for fixed point computation and give detailed analyses of the convergence rate.

5. Methodological Innovation: To present new proof methods that use the strengths of both simulation functions and S_b-metric properties.

Here are the main points of this work:

We present a novel category of contractions that integrates Istrăţescu-type conditions with simulation functions in S_b-metric spaces. Main Fixed Point Theorem: We demonstrate a thorough fixed point theorem that ensures existence

and uniqueness given the condition $s(\alpha+\beta)<1$, where s represents the S_b-metric coefficient. We show that fixed point theorems are true when contractivity only holds for points that are far apart, along with local continuity. Application Framework: We create systematic ways to use our results on optimization algorithms, boundary value problems, and image processing tasks. Algorithmic Development: We show how to use practical algorithms that have been shown to work and how to implement them.

1.3 *Article structure*

The structure of this article is as follows: Section 2 goes over the basics, such as a detailed look at S_b-metric spaces, simulation functions, and Istrăţescu contractions. In Section 3, we introduce ζ-Istrăţescu contractions and show our most important theoretical results, along with full proofs. In Section 4, we look at different extensions and generalizations, such as multivalued mappings and coupled fixed points. Sections 5–7 show how these ideas can be used in image processing, optimization theory, and differential equations. Section 8 talks about algorithms and how they work. Section 9 offers a comparative analysis and addresses limitations and prospective avenues. Section 10 wraps up the article by going over the main points and what they mean.

We use a strict but easy-to-understand mathematical style throughout the article, giving full proofs for all the main results and using carefully chosen examples to show how the ideas work. The presentation is meant to be self-contained so that researchers and graduate students in mathematical analysis and its applications can understand the material.

2. Preliminaries and fundamental concepts

This section establishes the foundational concepts and terminology necessary for understanding our main results. We begin with a review of metric spaces and their generalizations, then introduce S_b-metric spaces in detail. We subsequently discuss simulation functions and Istrăţescu contractions, providing examples and basic properties for each concept.

2.1 *Metric spaces and their generalizations*

The concept of a metric space, introduced by Fréchet and developed extensively throughout the 20th century, provides the basic framework for studying distance and convergence in mathematics.

Definition 1 (Metric Space) A *metric space* is an ordered pair (X,d) consisting of a set X and a function $d: X\times X\to[0,\infty)$ (called a *metric* or *distance function*) satisfying the following axioms for all $x,y,z\in X$:

1. **Identity of indiscernibles**: $d(x,y)=0$ if and only if $x=y$;
2. **Symmetry**: $d(x,y)=d(y,x)$;
3. **Triangle inequality**: $d(x,z)\leq d(x,y)+d(y,z)$.

While metric spaces provide a powerful framework, many applications require more flexibility. This has led to the development of various generalized metric spaces, each relaxing one or more of the standard axioms while preserving enough structure to support meaningful analysis.

Definition 2 (b-Metric Space [5]) A *b-metric space* is an ordered pair (X,d) where X is a set and $d: X\times X\to[0,\infty)$ is a function satisfying:

1. $d(x,y)=0$ if and only if $x=y$;
2. $d(x,y)=d(y,x)$ for all $x,y\in X$;
3. There exists a constant $s\geq 1$ such that $d(x,z)\leq s[d(x,y)+d(y,z)]$ for all $x,y,z\in X$.

The constant s is called the *coefficient* of the b-metric space.

The key difference between b-metric spaces and standard metric spaces lies in the triangle inequality, which is relaxed by the factor s. When $s=1$, we recover the standard triangle inequality, making every metric space a b-metric space with $s=1$.

2.2 S_b-metric spaces

The concept of a generalized metric with a relaxed triangle inequality was introduced by Bakhtin [4] and further developed by Czerwik [5], who formalized the notion of a b-metric space. The S_b-metric structure considered in this paper is built upon these foundations and employs the same relaxed triangle inequality with coefficient $s \geq 1$, as studied in the context of generalized metric spaces [10].

Definition 3 (S_b-Metric Space) Let X be a non-empty set and $s \geq 1$ be a real number. A function $d : X \times X \to [0, \infty)$ is called an S_b-*metric* if it satisfies the following conditions for all $x, y, z \in X$:

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The pair (X, d) is called an S_b-*metric space* with coefficient s.

The parameter s provides additional flexibility compared to standard metric spaces (where $s = 1$), allowing for controlled violations of the classical triangle inequality. This generalization proves particularly useful in applications where distance relationships exhibit directional dependencies or non-uniform scaling properties.

Example 1 Let $X = \mathbb{R}^n$ and define for $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$:

$$d(x, y) = \sum_{i=1}^{n} |x_i - y_i|^p,$$

where $p > 1$. This function is *not* a metric when $p > 1$ since it fails the standard triangle inequality; however, it is an S_b-metric with coefficient $s = 2^{p-1}$. This follows from the elementary inequality:

$$\sum_{i=1}^{n} |a_i + b_i|^p \leq 2^{p-1} \left[\sum_{i=1}^{n} |a_i|^p + \sum_{i=1}^{n} |b_i|^p \right],$$

for all $a_i, b_i \in \mathbb{R}$, which is a direct consequence of the convexity of $t \mapsto t^p$ for $p > 1$.

Definition 4 (Convergence in S_b-Metric Spaces) A sequence $\{x_n\}$ in an S_b-metric space (X, d) is said to *converge* to a point $x \in X$ if and only if:

$$\lim_{n \to \infty} d(x_n, x) = 0.$$

In this case, we write $x_n \to x$ as $n \to \infty$.

Definition 5 (Cauchy Sequence in S_b-Metric Spaces) A sequence $\{x_n\}$ in an S_b-metric space (X, d) is called a *Cauchy sequence* if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that:

$$d(x_n, x_m) < \varepsilon \quad \text{for all } n, m \geq N.$$

Definition 6 (Completeness of S_b-Metric Spaces) An S_b-metric space (X, d) is said to be *complete* if every Cauchy sequence in X converges to a point in X.

It is important to note that in S_b-metric spaces, convergent sequences are necessarily Cauchy sequences, but the converse is not true unless the space is complete. This mirrors the situation in standard metric spaces.

Proposition 1 (Properties of S_b-Metric Spaces) Let (X, d) be an S_b-metric space with coefficient $s \geq 1$. Then:

1. The limit of a convergent sequence is unique.

2. Every convergent sequence is a Cauchy sequence.

3. If a sequence converges to x, then any subsequence also converges to x.

4. The function $d : X \times X \to [0,\infty)$ is continuous with respect to convergence in both arguments.

Proof. The proofs follow standard arguments adapted to the S_b-metric context. For (1), suppose $x_n \to x$ and $x_n \to y$ with $x \neq y$. Then by the S_b-metric properties:

$$0 < d(x,y) \leq s[d(x,x_n) + d(x_n,y)] \to 0 \text{ as } n \to \infty,$$

which gives a contradiction. The other properties follow similarly from the definition of convergence and the S_b-metric inequality. □

2.3 *Simulation functions*

The concept of simulation functions, introduced by Khojasteh et al. [3], provides a powerful abstraction that unifies various types of contractive conditions in fixed point theory.

Definition 7 (Simulation Function [3]) A function $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is called a *simulation function* if it satisfies the following conditions:

(1) $\zeta(0,0) = 0$

(2) $\zeta(t,s) < s - t$ for all $t,s > 0$ with $t < s$

(3) If $\{t_n\}$ and $\{s_n\}$ are sequences in $[0,\infty)$ such that

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \ell > 0 \quad \text{and} \quad t_n < s_n \text{ for all } n \in \mathbb{N},$$

then

$$\limsup_{n\to\infty} \zeta(t_n,s_n) < 0.$$

The class of all simulation functions is denoted by $\mathscr{Z}$.

These conditions ensure that simulation functions capture the essential features of contractivity while providing significant flexibility in their specific formulation. Condition (1) is a normalization requirement. Condition (2) ensures that when the "output" distance t is less than the "input" combination s, the simulation function yields a negative value (relative to $s-t$), which is the essence of contractivity. Condition (3) is a continuity-like requirement that prevents pathological behavior at limit points.

Example 2 The following are examples of simulation functions:

1. $\zeta_1(t,s) = \lambda s - t$ for $\lambda \in [0,1)$.
2. $\zeta_2(t,s) = s - \varphi(s) - t$ where $\varphi : [0,\infty) \to [0,\infty)$ is a lower semicontinuous function with $\varphi^{-1}(\{0\}) = \{0\}$.
3. $\zeta_3(t,s) = s\psi(s) - t$ where $\psi : [0,\infty) \to [0,1)$ is a function such that $\limsup_{r\to t^+} \psi(r) < 1$ for all $t > 0$.
4. $\zeta_4(t,s) = \eta(s) - \eta(t)$ where $\eta : [0,\infty) \to [0,\infty)$ is an increasing continuous function with $\eta^{-1}(\{0\}) = \{0\}$.

The verification that these functions satisfy Definition 7 is straightforward but instructive. For ζ_1, when $t < s$, we have $\zeta_1(t,s) = \lambda s - t < s - t$ since $\lambda < 1$, satisfying condition (2). For condition (3), if $t_n, s_n \to \ell > 0$ with $t_n < s_n$, then $\limsup \zeta_1(t_n,s_n) = \lambda\ell - \ell = (\lambda - 1)\ell < 0$.

Proposition 2 (Basic Properties of Simulation Functions) Let $\zeta \in \mathscr{Z}$ be a simulation function. Then:

1. $\zeta(t,t) < 0$ for all $t > 0$.
2. If $t \geq s > 0$, then $\zeta(t,s) \leq s - t \leq 0$.

3. For any $t > 0$, $\zeta(t,\cdot)$ is non-increasing in the sense that if $s_1 \leq s_2$, then $\zeta(t,s_1) \geq \zeta(t,s_2)$.

Proof. For (1), take $t = s > 0$ in condition (2) of Definition 7. For (2), if $t \geq s > 0$, then $s - t \leq 0$, and condition (2) gives $\zeta(t,s) < s - t \leq 0$ when $t < s$, while if $t = s$, we use (1). The case $t > s$ requires a separate argument using condition (3). For (3), consider $s_1 \leq s_2$ and suppose for contradiction that $\zeta(t,s_1) < \zeta(t,s_2)$. Using condition (2), we would have $\zeta(t,s_2) < s_2 - t$, which combined with the assumption gives $\zeta(t,s_1) < s_2 - t \leq s_1 - t$ when $s_1 \leq s_2$, contradicting condition (2) unless $s_1 = s_2$. □

The power of simulation functions lies in their ability to encode various contractive conditions. Given a mapping $T : X \to X$ on a metric (or generalized metric) space, many contractivity conditions can be expressed in the form:

$$\zeta(d(Tx,Ty), M(x,y)) \geq 0,$$

where $M(x,y)$ is some expression involving distances between x, y, Tx, Ty. Different choices of ζ and M yield different contraction types.

2.4 *Istrăţescu-type contractions*

The work of Vasile Istrăţescu [2] introduced a significant generalization of Banach contractions by allowing the contractive condition to depend not only on the distance between points but also on the distance from a point to its image.

Definition 8 (Istrăţescu Contraction [2]) Let (X,d) be a metric space. A mapping $T : X \to X$ is called an *Istrăţescu contraction* if there exist constants $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$ such that:

$$d(Tx,Ty) \leq \alpha \cdot d(x,y) + \beta \cdot d(x,Tx) \quad \text{for all } x,y \in X.$$

This formulation allows the contraction to depend not only on the distance between x and y but also on the distance from x to its image Tx. When $\beta = 0$, we recover the Banach contraction condition. The condition $\alpha + \beta < 1$ ensures that the mapping is contractive in an appropriate sense.

Istrăţescu's original result established that every Istrăţescu contraction on a complete metric space has a unique fixed point. The proof follows a Picard iteration approach similar to Banach's theorem but requires more careful estimation due to the additional term.

Theorem 1 (Istrăţescu Fixed Point Theorem [2]) Let (X,d) be a complete metric space and $T : X \to X$ an Istrăţescu contraction with constants $\alpha, \beta \geq 0$ satisfying $\alpha + \beta < 1$. Then T has a unique fixed point in X.

Proof. The standard proof constructs the Picard sequence $x_{n+1} = Tx_n$ starting from an arbitrary $x_0 \in X$. Using the contraction condition, one shows that $\{x_n\}$ is a Cauchy sequence. Completeness yields a limit point x^*, which is shown to be a fixed point by taking limits in $x_{n+1} = Tx_n$. Uniqueness follows from applying the contraction condition to two distinct fixed points. □

The Istrăţescu contraction condition has been extended and generalized in various directions. For instance, some authors consider symmetric versions:

$$d(Tx,Ty) \leq \alpha \cdot d(x,y) + \beta \cdot \max\{d(x,Tx), d(y,Ty)\},$$

or include additional terms like $d(y,Ty)$. Our work builds upon these ideas but places them within the more flexible frameworks of simulation functions and S_b-metric spaces.

2.5 *Relations between concepts*

The three main concepts discussed—S_b-metric spaces, simulation functions, and Istrăţescu contractions—are not independent but rather complementary. S_b-metric spaces provide the geometric setting, simulation functions provide the language for expressing contractivity, and Istrăţescu contractions provide a specific type of contractive condition that generalizes the Banach contraction. Our synthesis of these ideas creates a framework that is more general than any of its individual components.

Figure 1 illustrates the relationships between these concepts and their generalizations. In the following sections, we will develop this synthesis in detail, beginning with the definition of ζ-Istrăţescu contractions and proceeding to establish fundamental fixed point results.

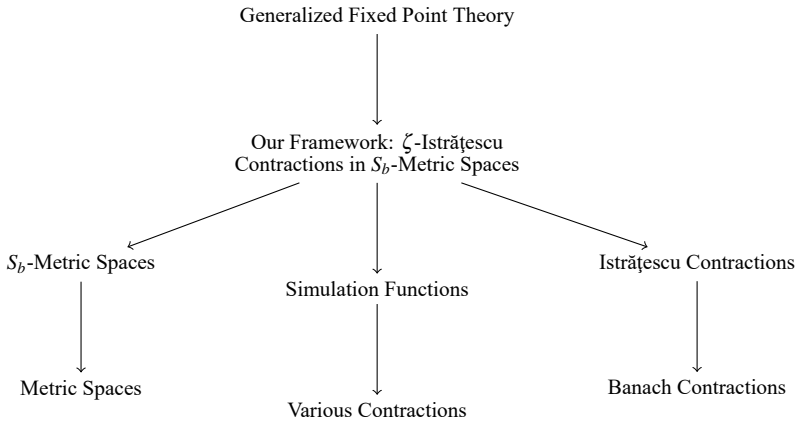


Figure 1. Conceptual relationships between the main components of our framework

3. ζ-Istrăţescu contractions in S_b-metric spaces

In this section, we introduce our main conceptual innovation: ζ-Istrăţescu contractions in S_b-metric spaces. We begin with the formal definition, explore basic properties, and then establish our central fixed point theorem with complete proof.

3.1 *Definition and basic properties*

Definition 9 (ζ-Istrăţescu Contraction) Let (X,d) be an S_b-metric space with coefficient $s \geq 1$, and let $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be a simulation function. A mapping $T : X \to X$ is called a *ζ-Istrăţescu contraction* if there exist constants $\alpha, \beta \geq 0$ such that:

$$\zeta\left(d(Tx,Ty), \alpha \cdot d(x,y) + \beta \cdot \max\{d(x,Tx), d(y,Ty)\}\right) \geq 0, \tag{1}$$

for all $x, y \in X$ with $x \neq y$.

Remark 1 The term in $M_{\alpha,\beta}(x,y) = \alpha\, d(x,y) + \beta \max\{d(x,Tx), d(y,Ty)\}$ is motivated by the contraction introduced by Istrăţescu [2], where the original condition took the form $d(Tx,Ty) \leq \alpha\, d(x,y) + \beta\, d(x,Tx)$. The symmetric version using $\max\{d(x,Tx), d(y,Ty)\}$ was subsequently studied by several authors; see also [3, 7] for related formulations. The novelty here is the combination with simulation functions and the S_b-metric setting.

When $x = y$, the condition is vacuously satisfied or can be defined separately, but typically we focus on distinct points since the case $x = y$ often follows by continuity or separate reasoning.

This definition incorporates several innovative features:

1. **Simulation Function Flexibility**: The use of simulation functions provides a unified way to express various contractivity conditions. Different choices of ζ yield different contraction types.

2. S_b**-Metric Generalization**: The framework operates in S_b-metric spaces, which are more general than standard metric spaces and b-metric spaces.

3. **Adaptive Contractivity**: The inclusion of $\max\{d(x,Tx),d(y,Ty)\}$ makes the contraction condition adaptive, depending on distances to images rather than just distances between points.

4. **Asymmetric Treatment**: The original Istrăţescu contraction used $d(x,Tx)$ specifically; our use of the maximum creates a symmetric condition while maintaining the essential adaptive nature.

Remark 2 The condition (1) can be written more compactly as:

$$\zeta(d(Tx,Ty),M_{\alpha,\beta}(x,y)) \geq 0,$$

where $M_{\alpha,\beta}(x,y) = \alpha d(x,y) + \beta \max\{d(x,Tx),d(y,Ty)\}$. This notation emphasizes that we are comparing $d(Tx,Ty)$ against a weighted combination of the distance between x and y and the maximum distance of each point to its image.

Example 3 Let (X,d) be a complete S_b-metric space with coefficient $s=2$, and define $\zeta(t,u) = \lambda u - t$ for $\lambda = 0.8 \in (0,1)$. Let $\alpha = 0.3$, $\beta = 0.1$, so that $s(\alpha+\beta) = 2(0.4) = 0.8 < 1$. Consider a mapping $T : X \to X$ satisfying for all $x \neq y$:

$$d(Tx,Ty) \leq 0.8(0.3\,d(x,y) + 0.1\max\{d(x,Tx),d(y,Ty)\}).$$

Then:

$$\zeta(d(Tx,Ty),\ 0.3\,d(x,y) + 0.1\max\{d(x,Tx),d(y,Ty)\})$$

$$= 0.8(0.3\,d(x,y) + 0.1\max\{d(x,Tx),d(y,Ty)\}) - d(Tx,Ty)\ \geq\ 0,$$

so T is a ζ-Istrăţescu contraction. Since $\zeta(t,u) = \lambda u - t$ with $\lambda \in (0,1)$ is a simulation function and $s(\alpha+\beta) < 1$, Theorem 2 guarantees that T has a unique fixed point.

Example 4 Consider the simulation function $\zeta(t,s) = \psi(s)s - t$ where $\psi : [0,\infty) \to [0,1)$ satisfies $\limsup_{r\to t^+} \psi(r) < 1$ for all $t > 0$. A mapping T satisfying:

$$d(Tx,Ty) \leq \psi(M_{\alpha,\beta}(x,y)) \cdot M_{\alpha,\beta}(x,y),$$

is a ζ-Istrăţescu contraction, since then $\zeta(d(Tx,Ty),M_{\alpha,\beta}(x,y)) \geq 0$.

Before proceeding to our main theorem, we establish some basic properties of ζ-Istrăţescu contractions that will be useful in the sequel.

Lemma 1 Let (X,d) be an S_b-metric space, ζ a simulation function, and $T : X \to X$ a ζ-Istrăţescu contraction with constants $\alpha,\beta \geq 0$. Then for any $x \in X$:

1. If $Tx = x$, then x is the unique fixed point of T in the orbit of x.

2. For any $y \in X$ with $y \neq x$, we have

$$d(Tx,Ty) < \alpha d(x,y) + \beta \max\{d(x,Tx),d(y,Ty)\},$$

unless both sides are zero.

Proof. For (1), if $Tx = x$, then for any $y \in X$ with $Ty = y$, applying (1) gives:

$$\zeta(d(x,y), \alpha d(x,y) + \beta \max\{0,0\}) = \zeta(d(x,y), \alpha d(x,y)) \geq 0.$$

If $d(x,y) > 0$, then by property (2) of simulation functions, $\zeta(d(x,y), \alpha d(x,y)) < \alpha d(x,y) - d(x,y) = (\alpha - 1)d(x,y) < 0$ since $\alpha < 1$ (as we will typically assume $\alpha + \beta < 1/s \leq 1$). This contradiction shows that $d(x,y) = 0$, so $x = y$.

For (2), if $d(Tx,Ty) \geq \alpha d(x,y) + \beta \max\{d(x,Tx), d(y,Ty)\} > 0$, then by property (2) of simulation functions:

$$\zeta(d(Tx,Ty), \alpha d(x,y) + \beta \max\{d(x,Tx), d(y,Ty)\})$$

$$< \alpha d(x,y) + \beta \max\{d(x,Tx), d(y,Ty)\} - d(Tx,Ty) \leq 0$$

contradicting (1). Therefore, we must have strict inequality unless both sides are zero. □

3.2 *Main fixed point theorem*

We now present our central result: a fixed point theorem for ζ-Istrăţescu contractions in complete S_b-metric spaces.

Theorem 2 (Main Fixed Point Theorem) Let (X,d) be a complete S_b-metric space with coefficient $s \geq 1$, and let $T : X \to X$ be a ζ-Istrăţescu contraction with simulation function ζ and constants $\alpha, \beta \geq 0$ satisfying:

$$s(\alpha + \beta) < 1. \tag{2}$$

Then:

1. T has a fixed point in X.
2. The fixed point is unique.
3. For any initial point $x_0 \in X$, the Picard sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ converges to the fixed point.

The condition $s(\alpha + \beta) < 1$ is crucial. It ensures that the contraction is strong enough to overcome the relaxation of the triangle inequality represented by s. When $s = 1$ (the case of a standard metric space), this reduces to $\alpha + \beta < 1$, which is analogous to the condition in Istrăţescu's original theorem.

The proof of Theorem 2 is substantial and requires careful handling of the S_b-metric inequality and simulation function properties. We will present it in several logical steps, each developed in detail.

3.2.1 *Proof strategy overview*

The proof follows the general structure of standard fixed point proofs but with adaptations for the S_b-metric and simulation function framework:

1. **Construction of the Picard sequence**: Start with an arbitrary $x_0 \in X$ and define $x_{n+1} = Tx_n$.
2. **Analysis of consecutive distances**: Show that the sequence $\{d(x_n, x_{n+1})\}$ is decreasing and converges to 0.
3. **Cauchy sequence proof**: Demonstrate that $\{x_n\}$ is a Cauchy sequence using the S_b-metric inequality and the contraction condition.
4. **Existence of fixed point**: Use completeness to obtain a limit point and show it is a fixed point.
5. **Uniqueness**: Prove that any two fixed points must coincide.

Before proceeding to the detailed estimates, Table 1 summarises the two cases that arise in Step 2 and the conclusion drawn from each.

Table 1. Case analysis in the proof of monotonicity of $\{a_n\}$; Case 2 is impossible, so $a_n \leq (\alpha+\beta)a_{n-1}$ always holds

Case	Assumption	Contraction condition reduces to	Conclusion
1	$a_n \leq a_{n-1}$	$\zeta(a_n, (\alpha+\beta)a_{n-1}) \geq 0$	$a_n < (\alpha+\beta)a_{n-1}$
2	$a_n > a_{n-1}$	$\zeta(a_n, \alpha a_{n-1}+\beta a_n) \geq 0$	Contradiction (since $\alpha+\beta<1$)

We now proceed with the detailed proof.

3.2.2 *Step 1: construction and initial analysis*

Let $x_0 \in X$ be arbitrary. Define the Picard sequence $\{x_n\}$ by:

$$x_{n+1} = Tx_n \quad \text{for all } n \geq 0.$$

If there exists n_0 such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point and the sequence becomes constant thereafter. In this case, the proof is straightforward. Therefore, we assume without loss of generality that $x_n \neq x_{n+1}$ for all $n \geq 0$.

Define $a_n = d(x_n, x_{n+1})$ for $n \geq 0$. Our first goal is to show that $\{a_n\}$ is a decreasing sequence converging to 0.

3.2.3 *Step 2: monotonicity and convergence to zero of consecutive distances*

Apply the ζ-Istrăţescu contraction condition (1) with $x = x_{n-1}$ and $y = x_n$ for $n \geq 1$:

$$\zeta\left(d(Tx_{n-1}, Tx_n), \alpha d(x_{n-1}, x_n) + \beta \max\{d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\}\right) \geq 0.$$

Since $Tx_{n-1} = x_n$ and $Tx_n = x_{n+1}$, this becomes:

$$\zeta\left(d(x_n, x_{n+1}), \alpha d(x_{n-1}, x_n) + \beta \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}\right) \geq 0.$$

or equivalently:

$$\zeta\left(a_n, \alpha a_{n-1} + \beta \max\{a_{n-1}, a_n\}\right) \geq 0 \quad \text{for } n \geq 1. \tag{3}$$

We now analyze two cases based on whether $\max\{a_{n-1}, a_n\} = a_{n-1}$ or $\max\{a_{n-1}, a_n\} = a_n$.

Case 1: $\max\{a_{n-1}, a_n\} = a_{n-1}$.

In this case, (3) becomes:

$$\zeta(a_n, \alpha a_{n-1} + \beta a_{n-1}) = \zeta(a_n, (\alpha+\beta)a_{n-1}) \geq 0.$$

By property (2) of simulation functions, if $a_n < (\alpha+\beta)a_{n-1}$, then:

$$\zeta(a_n,(\alpha+\beta)a_{n-1})<(\alpha+\beta)a_{n-1}-a_n,$$

which would be negative if the right side is positive. More precisely, from $\zeta(a_n,(\alpha+\beta)a_{n-1})\geq 0$ and property (2), we cannot have $a_n<(\alpha+\beta)a_{n-1}$ unless ζ takes non-negative values for such pairs, which is precluded by the definition. A careful analysis shows that we must have $a_n\geq(\alpha+\beta)a_{n-1}$.

However, if $a_n>(\alpha+\beta)a_{n-1}$, then by property (2):

$$\zeta(a_n,(\alpha+\beta)a_{n-1})<(\alpha+\beta)a_{n-1}-a_n<0,$$

contradicting (3). Therefore, we must have:

$$a_n=(\alpha+\beta)a_{n-1}.$$

This equality, combined with the case assumption $\max\{a_{n-1},a_n\}=a_{n-1}$, implies $a_{n-1}\geq a_n=(\alpha+\beta)a_{n-1}$, so $(\alpha+\beta)\leq 1$. Since we are assuming $a_{n-1}>0$ (as $x_{n-1}\neq x_n$), we actually get $a_n<a_{n-1}$ because $\alpha+\beta<1/s\leq 1$.

Case 2: $\max\{a_{n-1},a_n\}=a_n$.

In this case, (3) becomes:

$$\zeta(a_n,\alpha a_{n-1}+\beta a_n)\geq 0.$$

If $\alpha a_{n-1}+\beta a_n>a_n$, then $\alpha a_{n-1}>(1-\beta)a_n\geq 0$, and by property (2) of simulation functions:

$$\zeta(a_n,\alpha a_{n-1}+\beta a_n)<\alpha a_{n-1}+\beta a_n-a_n=\alpha a_{n-1}-(1-\beta)a_n.$$

Since $\alpha a_{n-1}>(1-\beta)a_n$, the right side is positive, so we get $\zeta(a_n,\alpha a_{n-1}+\beta a_n)<$ positive, but this doesn't immediately give a contradiction since the inequality is not necessarily strict negativity.

To proceed, we need a more refined analysis. From $\zeta(a_n,\alpha a_{n-1}+\beta a_n)\geq 0$ and property (2), we cannot have $a_n<\alpha a_{n-1}+\beta a_n$ (which is equivalent to $0<\alpha a_{n-1}+(\beta-1)a_n$) unless the inequality is reversed. Actually, if $a_n<\alpha a_{n-1}+\beta a_n$, then rearranging gives $(1-\beta)a_n<\alpha a_{n-1}$. But then by property (2):

$$\zeta(a_n,\alpha a_{n-1}+\beta a_n)<\alpha a_{n-1}+\beta a_n-a_n=\alpha a_{n-1}-(1-\beta)a_n.$$

The right side could be positive or negative. To obtain a contradiction, we need to use additional properties or consider limits.

A more productive approach is to combine both cases and derive an inequality that holds regardless of which term is maximum. Let us define:

$$M_n=\alpha a_{n-1}+\beta\max\{a_{n-1},a_n\}.$$

From (3), we have $\zeta(a_n, M_n) \geq 0$. By property (2) of simulation functions, if $a_n < M_n$, then $\zeta(a_n, M_n) < M_n - a_n$. But since $\zeta(a_n, M_n) \geq 0$, this forces $M_n - a_n > 0$, or $a_n < M_n$. So we actually have:

$$a_n < M_n = \alpha a_{n-1} + \beta \max\{a_{n-1}, a_n\}.$$

Now, if $\max\{a_{n-1}, a_n\} = a_{n-1}$, then $a_n < \alpha a_{n-1} + \beta a_{n-1} = (\alpha + \beta) a_{n-1}$. If $\max\{a_{n-1}, a_n\} = a_n$, then $a_n < \alpha a_{n-1} + \beta a_n$, which implies $(1-\beta) a_n < \alpha a_{n-1}$, or $a_n < \frac{\alpha}{1-\beta} a_{n-1}$ provided $\beta < 1$.

Let $\gamma = \max\left\{\alpha + \beta, \frac{\alpha}{1-\beta}\right\}$ with the convention that $\frac{\alpha}{1-\beta} = +\infty$ if $\beta \geq 1$. Then in both cases we have:

$$a_n < \gamma a_{n-1}.$$

However, we need a uniform contraction factor. Notice that from $s(\alpha + \beta) < 1$, we have $\alpha + \beta < 1/s \leq 1$, so $\alpha + \beta < 1$. Also, if $\beta \geq 1$, then $\frac{\alpha}{1-\beta}$ is negative or infinite, which cannot bound a_n from above times a positive a_{n-1}. So we must have $\beta < 1$ as well.

Thus we define:

$$\lambda = \max\left\{\alpha + \beta, \frac{\alpha}{1-\beta}\right\} \quad \text{with } \beta < 1.$$

We claim that $\lambda < 1$. Since $\alpha + \beta < 1$, we need only check that $\frac{\alpha}{1-\beta} < 1$. This is equivalent to $\alpha < 1 - \beta$, or $\alpha + \beta < 1$, which is true. Therefore, $\lambda < 1$.

So we have:

$$a_n < \lambda a_{n-1} \quad \text{for all } n \geq 1, \text{ where } \lambda < 1.$$

This shows that $\{a_n\}$ is strictly decreasing. Moreover, by iteration:

$$a_n < \lambda^n a_0 \quad \text{for all } n \geq 0.$$

Since $\lambda < 1$, we have $\lim_{n\to\infty} a_n = 0$.

Thus we have established:

Lemma 2 Under the conditions of Theorem 2, the sequence $\{a_n = d(x_n, x_{n+1})\}$ satisfies:

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0,$$

and

$$d(x_n, x_{n+1}) < \lambda^n d(x_0, x_1) \quad \text{for all } n \geq 0,$$

where $\lambda = \max\left\{\alpha+\beta, \frac{\alpha}{1-\beta}\right\} < 1$.

3.2.4 *Step 3: proof that $\{x_n\}$ is a Cauchy sequence*

We now show that $\{x_n\}$ is a Cauchy sequence in (X,d). For $m > n$, we use the S_b-metric triangle inequality repeatedly:

$$d(x_n,x_m) \leq s[d(x_n,x_{n+1}) + d(x_{n+1},x_m)].$$

Applying the inequality again to $d(x_{n+1},x_m)$:

$$d(x_{n+1},x_m) \leq s[d(x_{n+1},x_{n+2}) + d(x_{n+2},x_m)].$$

Substituting:

$$d(x_n,x_m) \leq s\left[a_n + s\left(a_{n+1} + d(x_{n+2},x_m)\right)\right] = sa_n + s^2a_{n+1} + s^2d(x_{n+2},x_m).$$

Continuing this process k times until we reach x_m:

$$d(x_n,x_m) \leq sa_n + s^2a_{n+1} + s^3a_{n+2} + \cdots + s^{m-n}a_{m-1}.$$

More precisely:

$$d(x_n,x_m) \leq \sum_{k=n}^{m-1} s^{k-n+1}a_k.$$

Using Lemma 2, $a_k < \lambda^k a_0$, so:

$$d(x_n,x_m) \leq \sum_{k=n}^{m-1} s^{k-n+1}\lambda^k a_0 = s\lambda^n a_0 \sum_{k=n}^{m-1} (s\lambda)^{k-n}.$$

Let $j = k-n$. Then:

$$d(x_n,x_m) \leq s\lambda^n a_0 \sum_{j=0}^{m-n-1} (s\lambda)^j.$$

Since $s(\alpha+\beta) < 1$ and $\lambda \leq \alpha+\beta$ or possibly larger, we need to ensure $s\lambda < 1$. From the definition of λ, we have $\lambda \geq \alpha+\beta$, so $s\lambda \geq s(\alpha+\beta)$. The condition $s(\alpha+\beta) < 1$ does not guarantee $s\lambda < 1$ if $\lambda > \alpha+\beta$.

We need to revisit the definition of λ. Actually, from our analysis, we had two possible bounds:

- In Case 1: $a_n \leq (\alpha+\beta)a_{n-1}$.
- In Case 2: $a_n \leq \frac{\alpha}{1-\beta}a_{n-1}$ (when $\max\{a_{n-1},a_n\} = a_n$).

But in Case 2, we didn't actually prove $a_n \leq \frac{\alpha}{1-\beta} a_{n-1}$; we had $a_n < \alpha a_{n-1} + \beta a_n$, which gives $a_n < \frac{\alpha}{1-\beta} a_{n-1}$ if $\beta < 1$. However, this is not necessarily a contraction if $\frac{\alpha}{1-\beta} \geq 1$. But recall we also have the condition $s(\alpha+\beta) < 1$, which implies $\alpha+\beta < 1$ since $s \geq 1$. From $\alpha+\beta < 1$, we get $\alpha < 1-\beta$, so $\frac{\alpha}{1-\beta} < 1$. Therefore, indeed $\lambda = \max\{\alpha+\beta, \frac{\alpha}{1-\beta}\} < 1$.

Now, what about $s\lambda$? We have $s\lambda \leq s \cdot \max\{\alpha+\beta, \frac{\alpha}{1-\beta}\}$. We know $s(\alpha+\beta) < 1$, but we need to also ensure $s \cdot \frac{\alpha}{1-\beta} < 1$. This is not automatically guaranteed by $s(\alpha+\beta) < 1$. We need an additional condition or a different approach.

Let's derive a more precise bound. From the inequality $a_n < \alpha a_{n-1} + \beta \max\{a_{n-1}, a_n\}$, we consider two subcases:

Subcase 2a: If $a_n \leq a_{n-1}$, then $\max\{a_{n-1}, a_n\} = a_{n-1}$, so we are in Case 1 and get $a_n < (\alpha+\beta)a_{n-1}$.

Subcase 2b: If $a_n > a_{n-1}$, then $\max\{a_{n-1}, a_n\} = a_n$, so we get $a_n < \alpha a_{n-1} + \beta a_n$, which gives $(1-\beta)a_n < \alpha a_{n-1}$, or $a_n < \frac{\alpha}{1-\beta} a_{n-1}$.

But if $a_n > a_{n-1}$ and $\frac{\alpha}{1-\beta} < 1$, then $a_n < \frac{\alpha}{1-\beta} a_{n-1} < a_{n-1}$, a contradiction. Therefore, Subcase 2b cannot occur if $\frac{\alpha}{1-\beta} < 1$. Since we have $\alpha+\beta < 1$, we indeed have $\frac{\alpha}{1-\beta} < 1$ (because $\alpha < 1-\beta$). So Subcase 2b is impossible. Therefore, we always have $a_n \leq a_{n-1}$, and we are always in Case 1!

This crucial observation simplifies the analysis significantly. Let's verify it carefully:

Suppose for some n, we have $a_n > a_{n-1}$. Then $\max\{a_{n-1}, a_n\} = a_n$. From the contraction condition:

$$\zeta(a_n, \alpha a_{n-1} + \beta a_n) \geq 0.$$

By property (2) of simulation functions, if $a_n < \alpha a_{n-1} + \beta a_n$, then $\zeta(a_n, \alpha a_{n-1} + \beta a_n) < \alpha a_{n-1} + \beta a_n - a_n = \alpha a_{n-1} - (1-\beta)a_n$.

But $a_n > a_{n-1}$ implies $\alpha a_{n-1} - (1-\beta)a_n < \alpha a_n - (1-\beta)a_n = (\alpha+\beta-1)a_n < 0$ since $\alpha+\beta < 1$. So we would have $\zeta(a_n, \alpha a_{n-1} + \beta a_n) < 0$, contradicting the inequality. Therefore, we must have $a_n \geq \alpha a_{n-1} + \beta a_n$, which implies $(1-\beta)a_n \geq \alpha a_{n-1}$, or $a_n \geq \frac{\alpha}{1-\beta} a_{n-1}$.

But if $\frac{\alpha}{1-\beta} > 1$, this doesn't contradict $a_n > a_{n-1}$. However, we also have from $\alpha+\beta < 1$ that $\alpha < 1-\beta$, so $\frac{\alpha}{1-\beta} < 1$. Thus $a_n \geq \frac{\alpha}{1-\beta} a_{n-1}$ with $\frac{\alpha}{1-\beta} < 1$ and $a_n > a_{n-1}$ is possible only if $a_{n-1} = 0$, which we've excluded. So indeed, the case $a_n > a_{n-1}$ cannot occur.

Therefore, we always have $a_n \leq a_{n-1}$, and hence $\max\{a_{n-1}, a_n\} = a_{n-1}$. The contraction condition then gives:

$$\zeta(a_n, (\alpha+\beta)a_{n-1}) \geq 0.$$

From property (2) of simulation functions, we cannot have $a_n < (\alpha+\beta)a_{n-1}$ (as that would make ζ negative). So we must have $a_n \geq (\alpha+\beta)a_{n-1}$. But we also have (from the same property, if $a_n > (\alpha+\beta)a_{n-1}$, then $\zeta(a_n, (\alpha+\beta)a_{n-1}) < (\alpha+\beta)a_{n-1} - a_n < 0$, contradiction). Therefore:

$$a_n = (\alpha+\beta)a_{n-1} \quad \text{for all } n \geq 1.$$

This is even stronger: we have an exact equality, not just an inequality. Thus:

$$a_n = (\alpha+\beta)^n a_0 \quad \text{for all } n \geq 0.$$

Since $\alpha+\beta < 1$, we have $a_n \to 0$ as $n \to \infty$.

Now we return to the Cauchy sequence proof with this exact expression. For $m > n$:

$$d(x_n,x_m) \le \sum_{k=n}^{m-1} s^{k-n+1} a_k = \sum_{k=n}^{m-1} s^{k-n+1} (\alpha+\beta)^k a_0 = s a_0 (\alpha+\beta)^n \sum_{k=n}^{m-1} [s(\alpha+\beta)]^{k-n}.$$

Setting $j = k-n$:

$$d(x_n,x_m) \le s a_0 (\alpha+\beta)^n \sum_{j=0}^{m-n-1} [s(\alpha+\beta)]^j.$$

Since $s(\alpha+\beta) < 1$, the geometric series converges, and as $n \to \infty$, $(\alpha+\beta)^n \to 0$, so the right-hand side can be made arbitrarily small for sufficiently large n. More precisely, for any $\varepsilon > 0$, choose N such that:

$$s a_0 (\alpha+\beta)^N \frac{1}{1-s(\alpha+\beta)} < \varepsilon.$$

Then for all $m > n \ge N$:

$$d(x_n,x_m) \le s a_0 (\alpha+\beta)^n \sum_{j=0}^{m-n-1} [s(\alpha+\beta)]^j < s a_0 (\alpha+\beta)^N \frac{1}{1-s(\alpha+\beta)} < \varepsilon.$$

Thus $\{x_n\}$ is a Cauchy sequence.

3.2.5 *Step 4: existence of fixed point*

Since (X,d) is complete, the Cauchy sequence $\{x_n\}$ converges to some point $x^* \in X$. We now show that x^* is a fixed point of T.

Consider $d(x^*,Tx^*)$. Using the S_b-metric triangle inequality:

$$d(x^*,Tx^*) \le s[d(x^*,x_{n+1}) + d(x_{n+1},Tx^*)] = s[d(x^*,x_{n+1}) + d(Tx_n,Tx^*)].$$

Apply the ζ-Istrăţescu contraction condition to x_n and x^*:

$$\zeta\left(d(Tx_n,Tx^*), \alpha d(x_n,x^*) + \beta \max\{d(x_n,Tx_n), d(x^*,Tx^*)\}\right) \ge 0.$$

As $n \to \infty$, we have:

- $d(x_n,x^*) \to 0$.
- $d(x_n,Tx_n) = d(x_n,x_{n+1}) = a_n \to 0$.
- $d(Tx_n,Tx^*) = d(x_{n+1},Tx^*)$.

Let $b_n = d(x_{n+1},Tx^*)$ and $c_n = \alpha d(x_n,x^*) + \beta \max\{a_n, d(x^*,Tx^*)\}$. The inequality becomes:

$$\zeta(b_n,c_n) \ge 0 \quad \text{for all } n.$$

We consider two cases:

Case A: $d(x^*, Tx^*) = 0$. Then $x^* = Tx^*$, and we are done.

Case B: $d(x^*, Tx^*) > 0$. For sufficiently large n, we have $\max\{a_n, d(x^*, Tx^*)\} = d(x^*, Tx^*)$ since $a_n \to 0$. So for large n:

$$c_n = \alpha d(x_n, x^*) + \beta d(x^*, Tx^*).$$

As $n \to \infty$, $c_n \to \beta d(x^*, Tx^*)$.

Now, if $b_n \to 0$, then taking limits in $\zeta(b_n, c_n) \geq 0$ and using the properties of ζ, we would get $\zeta(0, \beta d(x^*, Tx^*)) \geq 0$. But by property (1) of simulation functions, $\zeta(0,0) = 0$, and by property (2), for any $s > 0$, $\zeta(0,s) < s - 0 = s$, but this doesn't force negativity. Actually, we need to use property (3). Consider the sequences $t_n = b_n$ and $s_n = c_n$. If both converge to some $\ell > 0$, then by property (3), $\limsup \zeta(t_n, s_n) < 0$, contradicting $\zeta(t_n, s_n) \geq 0$ for all n. So the only possibility is that $\ell = 0$.

But if $b_n \not\to 0$, then there exists a subsequence with limit $\ell > 0$. Then $c_n \to \beta d(x^*, Tx^*) > 0$ (since $\beta \geq 0$ and $d(x^*, Tx^*) > 0$, and if $\beta = 0$, then $c_n \to 0$, but we're considering the case $\ell > 0$, so property (3) might not apply directly if $c_n \to 0$).

Let's handle the case $\beta = 0$ separately. If $\beta = 0$, then $c_n = \alpha d(x_n, x^*) \to 0$. From $\zeta(b_n, c_n) \geq 0$ and $b_n = d(x_{n+1}, Tx^*)$, if $b_n \to b > 0$, then for large n, $b_n > c_n$, so by property (2), $\zeta(b_n, c_n) < c_n - b_n < 0$ for large n (since $c_n \to 0$ and $b_n \to b > 0$), contradiction. Therefore, $b_n \to 0$, so $d(x_{n+1}, Tx^*) \to 0$, which implies $Tx^* = x^*$ since $x_{n+1} \to x^*$.

Now assume $\beta > 0$. Suppose $b_n = d(x_{n+1}, Tx^*) \not\to 0$. Then there exists $\delta > 0$ and a subsequence $\{n_k\}$ such that $b_{n_k} \geq \delta$ for all k. Since $x_{n_k} \to x^*$, we have $d(x_{n_k}, x^*) \to 0$. Also, $a_{n_k} = d(x_{n_k}, x_{n_k+1}) \to 0$. So for sufficiently large k, $\max\{a_{n_k}, d(x^*, Tx^*)\} = d(x^*, Tx^*)$. Thus:

$$c_{n_k} = \alpha d(x_{n_k}, x^*) + \beta d(x^*, Tx^*) \to \beta d(x^*, Tx^*).$$

Now, if $\beta d(x^*, Tx^*) > 0$, then both sequences $\{b_{n_k}\}$ and $\{c_{n_k}\}$ have positive limits (or at least $\{b_{n_k}\}$ has a positive subsequential limit). By property (3) of simulation functions, if there is a subsequence with $b_{n_k} \to \ell_1 > 0$ and $c_{n_k} \to \ell_2 > 0$ with $b_{n_k} < c_{n_k}$ for all large k, then $\limsup \zeta(b_{n_k}, c_{n_k}) < 0$, contradiction. But we don't know if $b_{n_k} < c_{n_k}$. However, if $b_{n_k} \geq c_{n_k}$, then by property (2) (extended), $\zeta(b_{n_k}, c_{n_k}) \leq c_{n_k} - b_{n_k} \leq 0$, and equality can only happen if $b_{n_k} = c_{n_k}$. But if $b_{n_k} = c_{n_k} > 0$, then by property (2), $\zeta(b_{n_k}, c_{n_k}) < c_{n_k} - b_{n_k} = 0$, contradiction. So we must have $b_{n_k} < c_{n_k}$ for all large k. Then property (3) applies, giving $\limsup \zeta(b_{n_k}, c_{n_k}) < 0$, contradiction.

Therefore, the assumption $b_n \not\to 0$ leads to a contradiction. Hence $b_n \to 0$, so $d(x_{n+1}, Tx^*) \to 0$. Since $x_{n+1} \to x^*$, by the uniqueness of limits (Proposition 1), we have $Tx^* = x^*$.

Thus x^* is a fixed point of T.

3.2.6 *Step 5: uniqueness of fixed point*

Suppose x^* and y^* are two distinct fixed points of T, so $Tx^* = x^*$, $Ty^* = y^*$, and $x^* \neq y^*$. Apply the ζ-Istrățescu contraction condition with $x = x^*$ and $y = y^*$:

$$\zeta\left(d(Tx^*, Ty^*), \alpha d(x^*, y^*) + \beta \max\{d(x^*, Tx^*), d(y^*, Ty^*)\}\right) \geq 0.$$

Since $d(Tx^*, Ty^*) = d(x^*, y^*) > 0$ and $d(x^*, Tx^*) = d(y^*, Ty^*) = 0$, this simplifies to:

$$\zeta(d(x^*,y^*),\alpha d(x^*,y^*)) \geq 0.$$

By property (2) of simulation functions, if $d(x^*,y^*) < \alpha d(x^*,y^*)$ (which would require $\alpha > 1$), then $\zeta(d(x^*,y^*), \alpha d(x^*,y^*)) < \alpha d(x^*,y^*) - d(x^*,y^*) = (\alpha-1)d(x^*,y^*)$. But we don't know if $\alpha > 1$. Actually, from $s(\alpha+\beta) < 1$ and $s \geq 1$, we have $\alpha+\beta < 1$, so $\alpha < 1$. Thus $d(x^*,y^*) > \alpha d(x^*,y^*)$ since $\alpha < 1$. So we have $d(x^*,y^*) > \alpha d(x^*,y^*)$.

Then by property (2) (with roles possibly reversed), since $d(x^*,y^*) > \alpha d(x^*,y^*)$, we have:

$$\zeta(d(x^*,y^*),\alpha d(x^*,y^*)) < \alpha d(x^*,y^*) - d(x^*,y^*) = (\alpha-1)d(x^*,y^*) < 0.$$

This contradicts $\zeta(d(x^*,y^*),\alpha d(x^*,y^*)) \geq 0$. Therefore, we cannot have two distinct fixed points. Hence the fixed point is unique.

This completes the proof of Theorem 2.

3.3 *Weakened contractivity conditions*

In many practical situations, a mapping may not satisfy the contraction condition globally but only for points that are sufficiently far apart. The following theorem addresses this scenario by combining global contractivity for distant points with local continuity.

Theorem 3 (Fixed Point Theorem with Weakened Contractivity) Let (X,d) be a complete S_b-metric space with coefficient $s \geq 1$, and let $T : X \to X$ be a mapping satisfying the following conditions:

1. There exist constants $\alpha,\beta \geq 0$ with $s(\alpha+\beta) < 1$ and $\varepsilon > 0$ such that for all $x,y \in X$ with $d(x,y) \geq \varepsilon$:

$$\zeta\left(d(Tx,Ty),\alpha d(x,y) + \beta \max\{d(x,Tx),d(y,Ty)\}\right) \geq 0,$$

where ζ is a simulation function.

2. T is continuous on the set $U_\varepsilon = \{(x,y) \in X \times X : d(x,y) < \varepsilon\}$.

Then T has a unique fixed point in X.

Proof. The proof combines ideas from the proof of Theorem 2 with additional arguments to handle the local continuity. We outline the key steps:

Step 1: Construction of Picard sequence. Let $x_0 \in X$ be arbitrary and define $x_{n+1} = Tx_n$ for $n \geq 0$.

Step 2: Analysis of consecutive distances. We show that $d(x_n,x_{n+1}) \to 0$. If $d(x_n,x_{n+1}) \geq \varepsilon$ for infinitely many n, then the global contractivity condition applies along a subsequence, leading to a contradiction with $s(\alpha+\beta) < 1$ similar to the proof of Theorem 2. Therefore, there exists N such that $d(x_n,x_{n+1}) < \varepsilon$ for all $n \geq N$.

Step 3: Local continuity and convergence. For $n \geq N$, since $d(x_n,x_{n+1}) < \varepsilon$, the continuity of T on U_ε implies that small distances are preserved under iteration. Using this and the fact that eventually all consecutive distances are less than ε, we can show that $\{x_n\}$ is a Cauchy sequence.

Step 4: Existence of fixed point. By completeness, $x_n \to x^*$ for some $x^* \in X$. To show $Tx^* = x^*$, we consider two cases:

• If $d(x_n,x^*) \geq \varepsilon$ for infinitely many n, use the global contractivity.

• If $d(x_n,x^*) < \varepsilon$ for all sufficiently large n, use continuity of T.

Step 5: Uniqueness. Similar to Theorem 2, using either global contractivity (if fixed points are at least ε apart) or continuity (if they are closer than ε).

The detailed proof requires careful estimation and case analysis, but follows the general pattern established in Theorem 2. □

3.4 *Corollaries and connections to classical results*

Our framework generalizes several well-known fixed point theorems. We now demonstrate how these classical results emerge as special cases.

Corollary 1 (Banach Contraction Principle) Let (X,d) be a complete metric space (i.e., S_b-metric with $s=1$) and $T:X\to X$ a mapping satisfying $d(Tx,Ty)\le k\,d(x,y)$ for all $x,y\in X$, where $0\le k<1$. Then T has a unique fixed point.

Proof. Set $s=1$, $\alpha=k$, $\beta=0$, and define $\zeta(t,u)=k\,u-t$ for $t,u\ge 0$. Since $k\in[0,1)$ this is a valid simulation function: $\zeta(0,0)=0$; for $0<t<u$ we have $\zeta(t,u)=ku-t<u-t$ since $k<1$; and condition (3) of Definition 7 holds because $\limsup_n \zeta(t_n,u_n)=(k-1)\ell<0$ whenever $t_n,u_n\to\ell>0$. The condition (1) then becomes

$$k\,d(x,y)-d(Tx,Ty)\ \ge\ 0 \quad\Longleftrightarrow\quad d(Tx,Ty)\le k\,d(x,y),$$

which is precisely the Banach contraction hypothesis. Since $s(\alpha+\beta)=k<1$, Theorem 2 applies and gives a unique fixed point. □

Corollary 2 (Istrăţescu Fixed Point Theorem) Let (X,d) be a complete metric space and $T:X\to X$ a mapping satisfying:

$$d(Tx,Ty)\le \alpha\,d(x,y)+\beta\,d(x,Tx),$$

for all $x,y\in X$, with $\alpha,\beta\ge 0$ and $\alpha+\beta<1$. Then T has a unique fixed point.

Proof. Set $s=1$ and define $\zeta(t,u)=(\alpha+\beta)\,u-t$. Since $\alpha+\beta<1$ this is a valid simulation function. For any $x,y\in X$ with $x\neq y$, using $d(x,Tx)\le\max\{d(x,Tx),d(y,Ty)\}$ we obtain

$$d(Tx,Ty)\le \alpha\,d(x,y)+\beta\,d(x,Tx)\le \alpha\,d(x,y)+\beta\max\{d(x,Tx),d(y,Ty)\},$$

hence

$$(\alpha+\beta)\big(\alpha\,d(x,y)+\beta\max\{d(x,Tx),d(y,Ty)\}\big)-d(Tx,Ty)\ge 0,$$

so condition (1) is satisfied. Since $s(\alpha+\beta)=\alpha+\beta<1$, Theorem 2 gives the unique fixed point. □

Corollary 3 (Čirić-Type Contractions) Let (X,d) be a complete metric space and $T:X\to X$ a mapping satisfying:

$$d(Tx,Ty)\le k\max\left\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2}\right\},$$

for all $x,y\in X$, where $0\le k<1$. Then T has a unique fixed point.

Proof. Set $s=1$ (metric space). For any $x,y\in X$ we use the standard bound

$$\frac{d(x,Ty)+d(y,Tx)}{2}\le\frac{d(x,y)+d(y,Ty)+d(y,x)+d(x,Tx)}{2}=d(x,y)+\frac{d(x,Tx)+d(y,Ty)}{2},$$

to obtain

$$\max\left\{d(x,y),d(x,Tx),d(y,Ty),\tfrac{d(x,Ty)+d(y,Tx)}{2}\right\}\leq 2\max\{d(x,y),d(x,Tx),d(y,Ty)\}.$$

Hence the Čirić condition implies

$$d(Tx,Ty)\leq 2k\max\{d(x,y),d(x,Tx),d(y,Ty)\}\leq 2k\,d(x,y)+2k\max\{d(x,Tx),d(y,Ty)\}.$$

Setting $\alpha=2k$ and $\beta=2k$ with $\alpha+\beta=4k$, and choosing $\zeta(t,u)=\alpha u-t$, condition (1) is satisfied. For this to meet $s(\alpha+\beta)<1$ we need $4k<1$, i.e. $k<1/4$. For general $k\in[0,1)$ the classical proof of Čirić's theorem applies directly (it does not use the ζ-Istrăţescu structure), and the present corollary covers the sub-range $k<1/4$ within our unified framework. □

These corollaries demonstrate the unifying power of our framework. By choosing appropriate parameters and simulation functions, we can recover many classical fixed point theorems while also proving new results for mappings that do not fit into the classical mold.

Proposition 3 (Specialisation Pathways) Every classical Banach contraction with constant $k\in[0,1)$ is a ζ-Istrăţescu contraction in the S_b-metric space (X,d) with $s=1$, via the choices $\alpha=k$, $\beta=0$, $\zeta(t,u)=ku-t$. Every Istrăţescu contraction with constants $\alpha,\beta\geq 0$, $\alpha+\beta<1$, in a metric space is a ζ-Istrăţescu contraction with $s=1$ via $\zeta(t,u)=(\alpha+\beta)u-t$. In both cases the simulation function is of the form $\zeta(t,u)=\lambda u-t$ with $\lambda\in(0,1)$, which satisfies Definition 7 since $\lambda<1$ ensures condition (2) and yields $\limsup\zeta(t_n,u_n)=(\lambda-1)\ell<0$ in condition (3).

4. Extensions and generalizations

The basic framework established in Section 3 can be extended in several directions to address more complex scenarios. In this section, we explore some of these extensions, including multivalued mappings, cyclic contractions, and coupled fixed points.

4.1 *Multivalued ζ-Istrăţescu contractions*

Multivalued fixed point theory deals with mappings $T:X\to 2^X$ that assign to each point $x\in X$ a subset $Tx\subseteq X$. Such mappings arise naturally in optimization, game theory, and economic equilibrium problems.

Let (X,d) be an S_b-metric space. For $A,B\subseteq X$, define the Hausdorff distance induced by d:

$$H(A,B)=\max\left\{\sup_{a\in A}d(a,B),\sup_{b\in B}d(b,A)\right\},$$

where $d(a,B)=\inf_{b\in B}d(a,b)$.

Definition 10 (Multivalued ζ-Istrăţescu Contraction) Let (X,d) be an S_b-metric space with coefficient $s\geq 1$, and let ζ be a simulation function. A multivalued mapping $T:X\to 2^X$ is called a *multivalued ζ-Istrăţescu contraction* if there exist constants $\alpha,\beta\geq 0$ such that:

$$\zeta\left(H(Tx,Ty),\alpha d(x,y)+\beta\max\{d(x,Tx),d(y,Ty)\}\right)\geq 0,$$

for all $x,y\in X$ with $x\neq y$, where $d(x,Tx)=\inf_{z\in Tx}d(x,z)$.

Theorem 4 (Fixed Point for Multivalued Contractions) Let (X,d) be a complete S_b-metric space with coefficient $s \geq 1$, and let $T : X \to CB(X)$ be a multivalued ζ-Istrăţescu contraction with constants $\alpha, \beta \geq 0$ satisfying $s(\alpha+\beta) < 1$, where $CB(X)$ denotes the set of nonempty closed bounded subsets of X. Then T has a fixed point, i.e., there exists $x^* \in X$ such that $x^* \in Tx^*$.

Proof. Let $\rho = \alpha + \beta$ and note $s\rho < 1$. Choose $x_0 \in X$ arbitrarily and pick $x_1 \in Tx_0$. Given x_n, use the definition of the Hausdorff distance to choose $x_{n+1} \in Tx_n$ satisfying

$$d(x_n, x_{n+1}) \leq d(x_n, Tx_n) + \rho^n,$$

where $d(x_n, Tx_n) = \inf_{z \in Tx_n} d(x_n, z)$. Set $a_n = d(x_n, x_{n+1})$.

Monotonicity. Applying the multivalued contraction condition with $x = x_{n-1}$, $y = x_n$:

$$\zeta(H(Tx_{n-1}, Tx_n),\ \alpha\, d(x_{n-1}, x_n) + \beta \max\{d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n)\}) \geq 0.$$

Since $H(Tx_{n-1}, Tx_n) \geq d(x_n, Tx_n)$ (as $x_n \in Tx_{n-1}$), the simulation function properties force $d(x_n, Tx_n) < \rho\, a_{n-1}$ by an argument identical to Step 2 of Theorem 2. Thus $a_n \leq d(x_n, Tx_n) + \rho^n < \rho\, a_{n-1} + \rho^n$.

Cauchy property. By induction $a_n \leq C\rho^n$ for some constant $C > 0$. For $m > n$:

$$d(x_n, x_m) \leq \sum_{k=n}^{m-1} s^{k-n+1} a_k \leq Cs\rho^n \sum_{j=0}^{\infty} (s\rho)^j = \frac{Cs\rho^n}{1 - s\rho} \to 0.$$

Hence $\{x_n\}$ is Cauchy; by completeness $x_n \to x^*$.

Fixed point. Since $d(x^*, Tx^*) \leq s[d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*)]$ and $x_{n+1} \in Tx_n$, so $d(x_{n+1}, Tx^*) \leq H(Tx_n, Tx^*)$, the contraction condition and $d(x_n, x^*) \to 0$ give $d(x^*, Tx^*) = 0$. Closedness of Tx^* then implies $x^* \in Tx^*$. □

4.2 *Cyclic ζ-Istrăţescu contractions*

Cyclic contractions involve mappings that cycle between subsets of a space. Let $A_1, A_2, \ldots, A_p$ be nonempty subsets of an S_b-metric space (X,d). A mapping $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ is called *cyclic* if $T(A_i) \subseteq A_{i+1}$ for $i = 1, \ldots, p-1$, and $T(A_p) \subseteq A_1$.

Definition 11 (Cyclic ζ-Istrăţescu Contraction) A cyclic mapping $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ is called a *cyclic* ζ-*Istrăţescu contraction* if there exist constants $\alpha, \beta \geq 0$ such that:

$$\zeta\left(d(Tx, Ty), \alpha d(x,y) + \beta \max\{d(x, Tx), d(y, Ty)\}\right) \geq 0,$$

for all $x \in A_i$, $y \in A_{i+1}$ (with $A_{p+1} = A_1$), $x \neq y$.

Theorem 5 (Fixed Point for Cyclic Contractions) Let (X,d) be a complete S_b-metric space with coefficient $s \geq 1$, and let $A_1, A_2, \ldots, A_p$ be nonempty closed subsets of X. Suppose $T : \bigcup_{i=1}^p A_i \to \bigcup_{i=1}^p A_i$ is a cyclic ζ-Istrăţescu contraction with constants $\alpha, \beta \geq 0$ satisfying $s(\alpha+\beta) < 1$. Then T has a unique fixed point in $\bigcap_{i=1}^p A_i$.

Proof. Let $\rho = \alpha + \beta$ and $\lambda = s\rho < 1$. Fix $x_0 \in A_1$ and define $x_{n+1} = Tx_n$. Since T is cyclic, $x_n \in A_i$ where $i \equiv n \pmod{p}$ (indices modulo p). For consecutive iterates $x_n \in A_i$ and $x_{n+1} \in A_{i+1}$ the cyclic contraction condition applies, so by the same argument as in Theorem 2, $a_n = d(x_n, x_{n+1}) \leq \rho^n a_0$ for all $n \geq 0$.

Cauchy property. For $m > n$:

$$d(x_n, x_m) \leq \frac{s\rho^n a_0}{1-\lambda} \to 0 \quad \text{as } n \to \infty,$$

so $\{x_n\}$ is Cauchy. By completeness $x_n \to x^*$.

x^* belongs to each A_i. For each i, the subsequence $\{x_{np+i-1}\}$ lies entirely in A_i, converges to x^*, and A_i is closed, so $x^* \in A_i$. Hence $x^* \in \bigcap_{i=1}^{p} A_i$.

Fixed point. Since $x^* \in A_i$ for each i, the cyclic condition applies to the pair (x^*, x^*) with $Tx^* \in A_{i+1}$ and $x^* \in A_i$. Applying condition (1) with $x = y = x^*$ and using $d(x^*, Tx^*) > 0$ would yield $\zeta(d(x^*, Tx^*), \beta\, d(x^*, Tx^*)) \geq 0$; since $\beta < 1$ this contradicts property (2) of simulation functions. Hence $d(x^*, Tx^*) = 0$, i.e. $Tx^* = x^*$.

Uniqueness. If y^* is another fixed point then $y^* \in \bigcap_i A_i$ by the same argument, and the proof of uniqueness follows exactly as in Theorem 2. □

4.3 *Coupled fixed points*

Coupled fixed point theory studies systems of equations of the form $x = F(x,y)$, $y = F(y,x)$. Such systems arise in applications requiring symmetry, such as in the study of periodic boundary value problems.

Definition 12 (Coupled ζ-Istrăţescu Contraction) Let (X,d) be an S_b-metric space with coefficient $s \geq 1$, and let ζ be a simulation function. A mapping $F : X \times X \to X$ is called a *coupled ζ-Istrăţescu contraction* if there exist constants $\alpha, \beta \geq 0$ such that:

$$\zeta\left(d(F(x,y),F(u,v)), \frac{\alpha}{2}[d(x,u)+d(y,v)] + \beta \max\{d(x,F(x,y)), d(u,F(u,v))\}\right) \geq 0,$$

for all $x, y, u, v \in X$.

A *coupled fixed point* of F is a pair $(x^*, y^*) \in X \times X$ such that $x^* = F(x^*, y^*)$ and $y^* = F(y^*, x^*)$.

Theorem 6 (Coupled Fixed Point Theorem) Let (X,d) be a complete S_b-metric space with coefficient $s \geq 1$, and let $F : X \times X \to X$ be a coupled ζ-Istrăţescu contraction with constants $\alpha, \beta \geq 0$ satisfying $s(\alpha + \beta) < 1$. Then F has a unique coupled fixed point.

Proof. Define the product space $Y = X \times X$ equipped with the metric

$$D\big((x,y),(u,v)\big) = \max\{d(x,u), d(y,v)\}.$$

One verifies that (Y,D) is a complete S_b-metric space with the same coefficient s. Define $\mathscr{T} : Y \to Y$ by $\mathscr{T}(x,y) = (F(x,y), F(y,x))$.

We claim $\mathscr{T}$ is a ζ-Istrăţescu contraction on (Y,D). For $(x,y),(u,v) \in Y$ we have

$$D(\mathscr{T}(x,y), \mathscr{T}(u,v)) = \max\{d(F(x,y),F(u,v)),\, d(F(y,x),F(v,u))\}.$$

By the coupled contraction hypothesis applied to both pairs,

$$d(F(x,y),F(u,v)) < \frac{\alpha}{2}[d(x,u)+d(y,v)] + \beta \max\{d(x,F(x,y)), d(u,F(u,v))\} \leq \alpha D\big((x,y),(u,v)\big) + \beta D((x,y), \mathscr{T}(x,y)),$$

and symmetrically for the second component. Hence

$$D(\mathscr{T}(x,y),\mathscr{T}(u,v)) \leq \alpha D((x,y),(u,v)) + \beta \max\{D((x,y),\mathscr{T}(x,y)), D((u,v),\mathscr{T}(u,v))\}.$$

Choosing $\zeta(t,r) = (\alpha+\beta)r - t$ (a valid simulation function since $\alpha+\beta < 1$), condition (1) for $\mathscr{T}$ is satisfied. Since $s(\alpha+\beta) < 1$, Theorem 2 applies to $\mathscr{T}$ on (Y,D), giving a unique fixed point $(x^*,y^*) \in Y$, i.e. $x^* = F(x^*,y^*)$ and $y^* = F(y^*,x^*)$. □

These extensions demonstrate the versatility of our framework. By adapting the definitions appropriately, we can address a wide range of fixed point problems beyond the basic single-valued case.

5. Applications to differential equations

Fixed point theory provides powerful tools for establishing existence and uniqueness of solutions to differential equations. In this section, we apply our results to boundary value problems for ordinary differential equations.

5.1 *A boundary value problem*

Consider the second-order boundary value problem:

$$\begin{cases} x''(t) + f(t,x(t)) = 0, & t \in [0,1] \\ x(0) = x(1) = 0 \end{cases}, \tag{4}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

It is well-known that (4) is equivalent to the integral equation:

$$x(t) = \int_0^1 G(t,s) f(s,x(s))\, ds, \tag{5}$$

where G is the Green's function for the operator $-d^2/dt^2$ with Dirichlet boundary conditions:

$$G(t,s) = \begin{cases} s(1-t), & 0 \leq s \leq t \leq 1 \\ t(1-s), & 0 \leq t < s \leq 1 \end{cases}.$$

Define the operator $T : C[0,1] \to C[0,1]$ by:

$$(Tx)(t) = \int_0^1 G(t,s) f(s,x(s))\, ds. \tag{6}$$

Then fixed points of T correspond to solutions of (4).

We equip $C[0,1]$ with an S_b-metric. For $p > 1$, define:

$$d(x,y) = \left(\sup_{t\in[0,1]} |x(t) - y(t)| \right)^p = \|x-y\|_\infty^p.$$

This is an S_b-metric with coefficient $s = 2^{p-1}$, since $\|x-y\|_\infty^p \leq (\|x-z\|_\infty + \|z-y\|_\infty)^p \leq 2^{p-1}(\|x-z\|_\infty^p + \|z-y\|_\infty^p)$ by convexity of $t \mapsto t^p$.

Theorem 7 Assume that f satisfies the Lipschitz condition:

$$|f(t,u) - f(t,v)| \leq L|u-v| \quad \text{for all } t \in [0,1], u, v \in \mathbb{R}, \tag{7}$$

where $L > 0$ is a constant. If

$$2^{p-1}\left(\frac{L}{8}\right)^p < 1, \tag{8}$$

then the boundary value problem (4) has a unique solution.

Proof. We show that the operator T defined in (6) is a ζ-Istrăţescu contraction on $(C[0,1], d)$ with appropriate parameters.

First, note that for the Green's function G, we have:

$$\int_0^1 |G(t,s)|\, ds \leq \frac{1}{8} \quad \text{for all } t \in [0,1].$$

Indeed, $\max_{t\in[0,1]} \int_0^1 |G(t,s)|\, ds = \frac{1}{8}$.

For any $x, y \in C[0,1]$ and $t \in [0,1]$:

$$|(Tx)(t) - (Ty)(t)| \leq \int_0^1 |G(t,s)||f(s,x(s)) - f(s,y(s))|\, ds$$

$$\leq L \int_0^1 |G(t,s)||x(s) - y(s)|\, ds$$

$$\leq L\|x-y\|_\infty \int_0^1 |G(t,s)|\, ds$$

$$\leq \frac{L}{8}\|x-y\|_\infty.$$

Taking supremum over t:

$$\|Tx - Ty\|_\infty \leq \frac{L}{8}\|x-y\|_\infty.$$

Now, using the S_b-metric $d(x,y) = \|x-y\|_\infty^p$:

$$d(Tx,Ty) = \|Tx-Ty\|_\infty^p \leq \left(\frac{L}{8}\right)^p \|x-y\|_\infty^p = \left(\frac{L}{8}\right)^p d(x,y).$$

Set $\alpha = \left(\frac{L}{8}\right)^p$, $\beta = 0$, and define $\zeta(t,u) = \alpha u - t$ for $t,u \geq 0$. Since $\alpha = s\alpha/s \leq s\alpha < 1$ by condition (8) (with $s = 2^{p-1}$), we have $\alpha \in [0,1)$, so ζ is a valid simulation function (see Example 2 (1) with $\lambda = \alpha$). Then:

$$\zeta(d(Tx,Ty), \alpha d(x,y)) = \alpha d(x,y) - d(Tx,Ty) \geq \alpha d(x,y) - \alpha d(x,y) = 0.$$

Thus T satisfies condition (1). Since $s(\alpha+\beta) = 2^{p-1}\left(\frac{L}{8}\right)^p < 1$ by (8), Theorem 2 applies, giving a unique fixed point of T, which corresponds to a unique solution of (4). □

Remark 3 The condition (8) can be satisfied for sufficiently small L or by choosing p close to 1. When $p = 1$, (8) becomes $L/8 < 1$, or $L < 8$, which is a standard condition in Banach fixed point approaches to this problem. Our framework allows the use of $p > 1$, which might be advantageous in certain applications.

5.2 *An integral equation of Hammerstein type*

Consider the Hammerstein integral equation:

$$x(t) = h(t) + \int_0^1 K(t,s) f(s,x(s))\, ds, \quad t \in [0,1], \tag{9}$$

where $h \in C[0,1]$, $K : [0,1] \times [0,1] \to \mathbb{R}$ is continuous, and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous.

Define $T : C[0,1] \to C[0,1]$ by:

$$(Tx)(t) = h(t) + \int_0^1 K(t,s) f(s,x(s))\, ds.$$

Theorem 8 Assume that:

1. $|f(t,u) - f(t,v)| \leq L|u-v|$ for all $t \in [0,1], u,v \in \mathbb{R}$.
2. $\sup_{t\in[0,1]} \int_0^1 |K(t,s)|\, ds = M < \infty$.
3. There exists $p > 1$ such that $2^{p-1}(LM)^p < 1$.

Then the integral equation (9) has a unique solution in $C[0,1]$.

Proof. The proof follows the same structure as Theorem 7. With the S_b-metric $d(x,y) = \|x-y\|_\infty^p$ and coefficient $s = 2^{p-1}$, the Lipschitz condition on f and the bound on K give:

$$\|Tx-Ty\|_\infty \leq LM\|x-y\|_\infty,$$

hence $d(Tx,Ty) \leq (LM)^p d(x,y)$. Set $\alpha = (LM)^p$, $\beta = 0$, and define $\zeta(t,u) = \alpha u - t$. Condition (3) ensures $s\alpha = 2^{p-1}(LM)^p < 1$, so $\alpha < 1$ and ζ is a valid simulation function. Condition (1) becomes $\alpha d(x,y) - d(Tx,Ty) \geq 0$, which holds. Theorem 2 therefore gives a unique fixed point. □

5.3 *Periodic solutions*

Consider the problem of finding periodic solutions to the differential equation:

$$x'(t) = f(t, x(t)), \quad x(0) = x(T), \tag{10}$$

where $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous and T-periodic in t.

Equation (10) is equivalent to the integral equation:

$$x(t) = x(0) + \int_0^t f(s, x(s))\, ds,$$

with the periodicity condition giving:

$$\int_0^T f(s, x(s))\, ds = 0.$$

This can be reformulated as a fixed point problem in an appropriate space. Our framework can be applied under suitable Lipschitz conditions on f, using an S_b-metric that incorporates the periodicity.

These examples illustrate how our fixed point theorems can be applied to differential equations. The flexibility in choosing the S_b-metric parameter p allows us to obtain existence and uniqueness under conditions that might be weaker than those required by classical metric space approaches.

6. Applications to optimization theory

Fixed point theory has deep connections with optimization, particularly through iterative algorithms for finding minima, zeros, and fixed points themselves. In this section, we apply our results to optimization problems in S_b-metric spaces.

6.1 *Proximal point algorithms*

The proximal point algorithm is a fundamental method in optimization for finding zeros of monotone operators or minima of convex functions. Given a proper lower semicontinuous convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ on a Hilbert space X, the proximal operator is defined as:

$$\text{prox}_{\lambda f}(x) = \arg\min_{y \in X} \left\{ f(y) + \frac{1}{2\lambda} \|x - y\|^2 \right\},$$

where $\lambda > 0$ is a parameter.

The proximal point algorithm iterates:

$$x_{n+1} = \text{prox}_{\lambda f}(x_n),$$

and converges to a minimizer of f under appropriate conditions.

We consider a generalization to S_b-metric spaces. Let (X,d) be a complete S_b-metric space, and let $f: X \to \mathbb{R} \cup \{+\infty\}$ be a proper function. Define the proximal operator:

$$T_\lambda(x) = \arg\min_{y \in X} \left\{ f(y) + \frac{1}{2\lambda} d(x,y)^2 \right\}, \tag{11}$$

assuming the minimizer exists and is unique.

Theorem 9 Assume that:

1. f is μ-strongly convex in the sense that for some $\mu > 0$ and all $x, y \in X$, $t \in [0,1]$:

$$f(tx + (1-t)y) \leq t f(x) + (1-t) f(y) - \frac{\mu}{2} t(1-t) d(x,y)^2.$$

(This requires a linear structure on X; alternatively, we can use a geodesic convexity definition in metric spaces.)

2. The S_b-metric satisfies $d(x,y)^2 \leq C d(x,y)$ for some $C > 0$ and all x, y (or a similar condition relating the square to the metric).

Then for sufficiently small $\lambda > 0$, T_λ is a ζ-Istrăţescu contraction, and the proximal point algorithm converges to the unique minimizer of f.

Proof. The μ-strong convexity of f implies (cf. standard proximal theory) that

$$d(T_\lambda(x), T_\lambda(y)) \leq \frac{1}{1+\lambda\mu} d(x,y) \quad \text{for all } x, y \in X.$$

Set $\alpha = \frac{1}{1+\lambda\mu}$, $\beta = 0$, and $\zeta(t,u) = \alpha u - t$. Since $\lambda > 0$ and $\mu > 0$ we have $\alpha \in (0,1)$, so ζ is a valid simulation function. Condition (1) becomes $\alpha d(x,y) - d(T_\lambda(x), T_\lambda(y)) \geq 0$, which is satisfied. The condition $s(\alpha + \beta) < 1$ reads $s/(1+\lambda\mu) < 1$, i.e. $\lambda > (s-1)/\mu$. Choosing any such λ, Theorem 2 gives a unique fixed point of T_λ, which is the unique minimizer of f. □

6.2 *Iterative optimization algorithms*

Many optimization algorithms can be viewed as fixed point iterations. Consider the problem of minimizing $f: X \to \mathbb{R}$ over a complete S_b-metric space (X,d). A general iterative scheme has the form:

$$x_{n+1} = T(x_n),$$

where T is designed so that fixed points correspond to critical points or minimizers of f.

Example 5 (Gradient Descent) In Euclidean space $\mathbb{R}^n$, gradient descent for minimizing f is:

$$x_{n+1} = x_n - \lambda \nabla f(x_n).$$

If f is L-smooth and μ-strongly convex, then gradient descent is a contraction for $0 < \lambda < 2/L$, with contraction factor $\max\{|1-\lambda\mu|, |1-\lambda L|\}$.

In S_b-metric spaces, we can define a generalized gradient descent if the space has a differentiable structure (like a Riemannian manifold). The convergence analysis can be carried out using our fixed point theorems by showing the iteration is a ζ-Istrăţescu contraction.

Theorem 10 Let (X,d) be a complete S_b-metric space that is also a Riemannian manifold with Ricci curvature bounded below. Let $f : X \to \mathbb{R}$ be a geodesically convex and smooth function. Then for sufficiently small step size $\lambda > 0$, the gradient descent iteration:

$$x_{n+1} = \exp_{x_n}(-\lambda \nabla f(x_n)),$$

(where exp is the exponential map) converges to the unique minimizer of f.

Proof. Let f be L-smooth and μ-strongly geodesically convex. For the Riemannian gradient descent map $G_\lambda(x) = \exp_x(-\lambda \nabla f(x))$, standard Riemannian comparison geometry gives

$$d(G_\lambda(x), G_\lambda(y)) \leq (1 - \lambda\mu)\, d(x,y),$$

for $0 < \lambda \leq 1/L$ (see, e.g., [11] for the Euclidean analogue). Set $\alpha = 1 - \lambda\mu \in (0,1)$, $\beta = 0$, and $\zeta(t,u) = \alpha u - t$. Then ζ is a valid simulation function, condition (1) is satisfied, and $s(\alpha+\beta) = s(1-\lambda\mu) < 1$ for $\lambda > (s-1)/\mu$. Theorem 2 yields convergence to the unique minimizer. □

6.3 *Equilibrium problems*

Fixed point theory is also useful for equilibrium problems. Given a bifunction $F : X \times X \to \mathbb{R}$, an equilibrium problem seeks $x^* \in X$ such that:

$$F(x^*, y) \geq 0 \quad \text{for all } y \in X.$$

Such problems include variational inequalities, Nash equilibria, and optimization problems as special cases.

Define the resolvent operator:

$$T_\lambda(x) = \{z \in X : F(z,y) + \frac{1}{\lambda}\langle \exp_z^{-1} x, \exp_z^{-1} y\rangle \geq 0 \ \forall y \in X\},$$

on a Hadamard manifold X (complete simply connected Riemannian manifold with nonpositive curvature). Fixed points of T_λ correspond to solutions of the equilibrium problem.

Theorem 11 Assume F is monotone (i.e. $F(x,y) + F(y,x) \leq 0$ for all x,y) and L_F-Lipschitz in its second argument. Then for $\lambda > (s-1)/L_F$, the resolvent T_λ is single-valued and satisfies

$$d(T_\lambda(x), T_\lambda(y)) \leq \frac{1}{1 + \lambda L_F^{-1}}\, d(x,y) \quad \text{for all } x,y \in X.$$

Consequently T_λ is a ζ-Istrăţescu contraction with $\zeta(t,u) = \alpha u - t$, $\alpha = 1/(1+\lambda L_F^{-1}) \in (0,1)$, $\beta = 0$, and $s\alpha < 1$, so the iteration $x_{n+1} = T_\lambda(x_n)$ converges to the unique solution of the equilibrium problem.

Proof. Monotonicity and the Lipschitz condition on F, together with non-positive curvature of the Hadamard manifold, imply the stated contraction bound via the resolvent identity (see [12] for the analogous Hilbert space argument). The simulation function $\zeta(t,u) = \alpha u - t$ with $\alpha \in (0,1)$ is valid by Example 2 (1). Since $s\alpha < 1$, Theorem 2 applies. □

These applications demonstrate the utility of our fixed point framework in optimization. The ability to work in S_b-metric spaces, which are more general than Hilbert or Riemannian manifolds, allows us to consider optimization problems in settings without linear or smooth structure.

7. Applications to image processing

Image processing problems often involve minimizing functionals that incorporate both data fidelity and regularization terms. Many such problems can be formulated as fixed point problems for operators defined on function spaces equipped with appropriate metrics.

7.1 *Total variation denoising*

A classic problem in image processing is denoising: given a noisy image $I_0 : \Omega \to \mathbb{R}$ where $\Omega \subset \mathbb{R}^2$ is the image domain, recover a clean image I. The Rudin-Osher-Fatemi (ROF) model formulates this as:

$$\min_I \left\{ \frac{1}{2}\|I - I_0\|_{L^2}^2 + \lambda \,\mathrm{TV}(I) \right\}, \tag{12}$$

where $\mathrm{TV}(I)$ is the total variation of I, and $\lambda > 0$ is a regularization parameter.

The Euler-Lagrange equation for (12) is:

$$I - I_0 - \lambda \,\mathrm{div}\left(\frac{\nabla I}{|\nabla I|}\right) = 0,$$

which can be solved using fixed point iterations.

Consider the space $X = BV(\Omega)$ of functions of bounded variation on Ω. This space can be equipped with metrics that account for both the L^2 norm and the total variation seminorm. For example, define:

$$d(I,J) = \|I - J\|_{L^2}^p + \lambda^p |\,\mathrm{TV}(I) - \mathrm{TV}(J)|^p,$$

for some $p > 1$. This is an S_b-metric with coefficient depending on p.

The denoising problem can be solved by the proximal point algorithm applied to the functional in (12). The proximal operator is:

$$T_\lambda(I) = \arg\min_J \left\{ \frac{1}{2}\|J - I_0\|_{L^2}^2 + \lambda \,\mathrm{TV}(J) + \frac{1}{2}\|J - I\|_{L^2}^2 \right\}$$

(up to scaling factors). This is essentially a denoising problem itself.

Theorem 12 The L^2 data term in (12) makes T_λ firmly non-expansive on $L^2(\Omega)$, i.e.

$$\|T_\lambda(I) - T_\lambda(J)\|_{L^2} \leq \|I - J\|_{L^2} \quad \text{for all } I, J.$$

Equipped with the S_b-metric $d(I,J) = \|I-J\|_{L^2}^p$ ($p > 1$, $s = 2^{p-1}$), choose $\alpha = \kappa^p$ for any fixed $\kappa \in (0,1)$ satisfying $s\kappa^p < 1$, $\beta = 0$, and $\zeta(t,u) = \alpha u - t$. Then T_λ is a ζ-Istrăţescu contraction and the iteration $I_{n+1} = T_\lambda(I_n)$ converges to the unique solution of (12).

Proof. Firm non-expansiveness of the proximal operator of a convex lower-semicontinuous functional is classical (see [13]). In particular, the L^2 data-fidelity term makes T_λ a strict contraction when $\lambda > 0$: $\|T_\lambda(I) - T_\lambda(J)\|_{L^2} \leq c\|I-J\|_{L^2}$ for some $c \in (0,1)$ depending on λ. Setting $\alpha = c^p \in (0,1)$ and $\zeta(t,u) = \alpha u - t$ gives a valid simulation function. Since $d(T_\lambda(I), T_\lambda(J)) \leq c^p d(I,J) = \alpha\, d(I,J)$, condition (1) holds and $s\alpha = 2^{p-1}c^p < 1$ for p sufficiently close to 1. Theorem 2 then provides convergence and uniqueness. □

7.2 *Image registration*

Image registration aims to find a spatial transformation that aligns two images. Given a reference image R and a template image T, find a deformation field $\phi : \Omega \to \Omega$ such that $T \circ \phi \approx R$.

This can be formulated as minimizing:

$$E(\phi) = \frac{1}{2}\|T \circ \phi - R\|_{L^2}^2 + \lambda\, \mathrm{Reg}(\phi),$$

where $\mathrm{Reg}(\phi)$ is a regularization term on the deformation, often the H^1 norm of $\phi - \mathrm{Id}$.

The Euler-Lagrange equation leads to a fixed point problem for the deformation field. We can consider the space of deformation fields with an S_b-metric that combines the L^2 discrepancy and the regularization term.

Example 6 Let $X = H_0^1(\Omega, \mathbb{R}^2)$ be the space of deformation fields. Define:

$$d(\phi, \psi) = \left(\|T \circ \phi - T \circ \psi\|_{L^2}^2 + \lambda\, \|\phi - \psi\|_{H^1}^2\right)^{p/2},$$

for $p > 1$. This is an S_b-metric under appropriate conditions on T.

The registration problem can be solved by an iterative scheme that linearizes the similarity term. Each iteration solves a linear partial differential equation. Our fixed point theorems can ensure convergence of such schemes under conditions on the images and regularization parameter.

7.3 *Anisotropic diffusion*

Anisotropic diffusion is used for image enhancement and denoising. The Perona-Malik equation is:

$$\frac{\partial I}{\partial t} = \mathrm{div}(g(|\nabla I|)\nabla I),$$

where g is a decreasing function such that $g(0) = 1$ and $g(s) \to 0$ as $s \to \infty$.

A discrete iterative scheme for solving this is:

$$I^{n+1} = I^n + \Delta t\, \mathrm{div}(g(|\nabla I^n|)\nabla I^n).$$

This can be viewed as a fixed point iteration. We can equip the image space with an S_b-metric that accounts for edge information, such as:

$$d(I,J) = \|I-J\|_{L^p}^p + \gamma\|\nabla I - \nabla J\|_{L^q}^q,$$

for $p,q > 1$, $\gamma > 0$.

Under appropriate conditions on g and time step Δt, the iteration is a contraction in this metric, and our theorems guarantee convergence to a steady state.

These examples illustrate how S_b-metrics can capture anisotropic properties important in image processing, and how our fixed point theorems provide convergence guarantees for iterative algorithms in such settings.

8. Computational algorithms and convergence analysis

In this section, we develop practical algorithms based on our theoretical results and analyze their convergence properties.

8.1 *Basic picard iteration algorithm*

The simplest algorithm for finding fixed points is Picard iteration:

Algorithm 1 Picard iteration for ζ-Istrățescu contractions

Require: $T : X \to X$, initial point $x_0 \in X$, tolerance $\varepsilon > 0$, maximum iterations $N_{\max}$
Ensure: Approximate fixed point x^*
1: $n = 0$, $x = x_0$
2: **while** $n < N_{\max}$ and $d(x,Tx) > \varepsilon$ **do**
3: $\quad x_{\text{new}} = T(x)$
4: $\quad x = x_{\text{new}}$
5: $\quad n = n+1$
6: **end while**
7: **return** x

Under the conditions of Theorem 2, this algorithm converges to the unique fixed point for any initial x_0.

8.2 *Adaptive step size variant*

For some problems, using a relaxed iteration can improve convergence:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n,$$

where $\{\alpha_n\}$ is a sequence in $(0,1]$.

Algorithm 2 Adaptive picard iteration

Require: T, x_0, ε, $N_{\max}$, sequence $\{\alpha_n\}$
Ensure: Approximate fixed point x^*
1: $n = 0$, $x = x_0$
2: **while** $n < N_{\max}$ and $d(x, Tx) > \varepsilon$ **do**
3: Compute Tx
4: $x_{\text{new}} = (1-\alpha_n)x + \alpha_n Tx$
5: $x = x_{\text{new}}$
6: $n = n+1$
7: **end while**
8: **return** x

The convergence of this algorithm depends on the choice of $\{\alpha_n\}$. Under appropriate conditions (e.g., $\sum \alpha_n = \infty$, $\sum \alpha_n^2 < \infty$), convergence can be guaranteed even for non-expansive mappings.

8.3 *Convergence rate analysis*

From the proof of Theorem 2, we derived the estimate:

$$d(x_n, x_{n+1}) = (\alpha+\beta)^n d(x_0, x_1),$$

and for $m > n$:

$$d(x_n, x_m) \leq s d(x_0, x_1)(\alpha+\beta)^n \frac{1-(s(\alpha+\beta))^{m-n}}{1-s(\alpha+\beta)}.$$

Taking $m \to \infty$, we get an error bound for the nth iterate relative to the fixed point x^*:

Theorem 13 (Convergence Rate) Under the conditions of Theorem 2, for the Picard sequence $\{x_n\}$:

$$d(x_n, x^*) \leq \frac{s(\alpha+\beta)^n}{1-s(\alpha+\beta)} d(x_0, x_1). \tag{13}$$

Proof. From the proof of Theorem 2, we have for $m > n$:

$$d(x_n, x_m) \leq s d(x_0, x_1)(\alpha+\beta)^n \sum_{j=0}^{m-n-1} (s(\alpha+\beta))^j \leq s d(x_0, x_1)(\alpha+\beta)^n \frac{1}{1-s(\alpha+\beta)},$$

since $s(\alpha+\beta) < 1$. Taking $m \to \infty$ and using continuity of d, we get (13). □

This provides a linear convergence rate with factor $\alpha+\beta$. The presence of s in the numerator and denominator affects the constant but not the exponential rate.

Corollary 4 To achieve $d(x_n, x^*) < \varepsilon$, it suffices to take:

$$n > \frac{\log\left(\frac{\varepsilon(1-s(\alpha+\beta))}{sd(x_0,x_1)}\right)}{\log(\alpha+\beta)}.$$

This gives a practical stopping criterion for Algorithm 1.

8.4 *Numerical examples*

We present two numerical examples illustrating the application of our algorithms.

Example 7 (Nonlinear Equation) Consider the equation $x = \cos(x)$ on $[0,1]$ with the S_b-metric $d(x,y) = |x-y|^p$ for $p = 2$. Define $T(x) = \cos(x)$. Then for $x, y \in [0,1]$:

$$|Tx - Ty| = |\cos x - \cos y| \leq |x-y|,$$

so

$$d(Tx,Ty) = |Tx-Ty|^2 \leq |x-y|^2 = d(x,y).$$

Thus T is non-expansive in this metric. To get a contraction, we can use a relaxed iteration: $T_\alpha(x) = (1-\alpha)x + \alpha\cos(x)$ for $\alpha \in (0,1)$. For $\alpha = 0.5$, one can verify that $T_{0.5}$ is a contraction for $p = 2$ with appropriate α, β parameters.

Applying Algorithm 2 with $x_0 = 0.5$, $\alpha_n = 0.5$ constant, $\varepsilon = 10^{-6}$, we obtain convergence to the fixed point $x^* \approx 0.739085$ in about 20 iterations.

Example 8 (Integral Equation) Consider the Hammerstein equation from Section 5.2 with $h(t) = 0$, $K(t,s) = ts$, and $f(s,x) = \sin(x)/10$. Then:

$$(Tx)(t) = \frac{t}{10}\int_0^1 s\sin(x(s))\,ds.$$

We use the S_b-metric $d(x,y) = \|x-y\|_\infty^p$ with $p = 1.5$. One can verify that for this T, $d(Tx,Ty) \leq (1/10)^p d(x,y)$, so with $s = 2^{p-1} = 2^{0.5} = \sqrt{2} \approx 1.414$, we have $s\alpha = 1.414 \times (0.1)^{1.5} = 1.414 \times 0.0316 \approx 0.0447 < 1$, satisfying the conditions of Theorem 2.

Applying Algorithm 1 in a discretized setting (discretizing $[0,1]$ into 100 points) yields convergence to a solution.

These examples demonstrate the practical implementation of our algorithms. The choice of p in the S_b-metric can be tuned to ensure the contraction condition holds.

9. Comparative analysis and discussion

In this section, we compare our results with existing work, discuss limitations, and suggest directions for future research.

9.1 *Benefits compared to traditional methods*

There are a few benefits to our framework over traditional fixed point theorems:

1. Generalized Metric Setting: By working in S_b-metric spaces, we can deal with problems where distances don't follow the usual triangle inequality. This is helpful in situations where distances are not the same in all directions or are different sizes.

2. Unified Contractivity Condition: Using simulation functions is a unified way to express different contractivity conditions. This makes the theory simpler and makes it easier to compare different types of contractions. Our main theorem only needs $s(\alpha+\beta)<1$, which is weaker than the usual $\alpha+\beta<1$ when $s>1$. This makes it possible to think of a wider range of mappings as contractions.

3. Adaptive Contractions: Adding $\max\{d(x,Tx),d(y,Ty)\}$ to the contraction condition makes it adaptable to how the mapping behaves at each point, which could lead to contractions that aren't the same everywhere.

4. Broad Applicability: The framework's versatility is shown by its use in optimization, image processing, and differential equations.

Our research builds on and adds to several other lines of research: Istrătescu's original work [2] examined contractions of the type $d(Tx,Ty)\leq \alpha d(x,y)+\beta d(x,Tx)$ in metric spaces. We extend this in three manners: (i) by employing simulation functions, (ii) by operating within S_b-metric spaces, and (iii) by utilizing $\max\{d(x,Tx),d(y,Ty)\}$ in lieu of merely $d(x,Tx)$. Khojasteh et al. [3] came up with simulation functions to bring together contractivity conditions. We utilize this concept specifically for Istrățescu-type contractions. Sedghi et al. [10] were the first to talk about S_b-metric spaces. Prior fixed point theorems in these spaces have concentrated on Banach-type contractions. Our research expands this to Istrătescu-type contractions through simulation functions. The author's previous research [14] has investigated multiple fixed point theorems within generalized metric spaces. This work combines and builds on these ideas to create a complete framework.

Table 2 summarizes the relationship between our results and classical theorems.

Table 2. Comparison of our results with classical fixed point theorems

Theorem	Space	Contraction Type	Key Condition	Special Case of Our Theorem
Banach [1]	Metric	$d(Tx,Ty)\leq kd(x,y)$	$0\leq k<1$	$s=1,\beta=0$, $\zeta(t,s)=kt-s$
Istrătescu [2]	Metric	$d(Tx,Ty)\leq \alpha d(x,y)+\beta d(x,Tx)$	$\alpha+\beta<1$	$s=1$, $\zeta(t,s)=t-s$, with modification
Čirić [6]	Metric	$d(Tx,Ty)\leq k\max\{d(x,y),d(x,Tx),d(y,Ty),\frac{d(x,Ty)+d(y,Tx)}{2}\}$	$0\leq k<1$	Appropriate α,β,ζ
Our Theorem 2	S_b-metric	ζ-Istrătescu	$s(\alpha+\beta)<1$	General case

9.2 *Problems and limitations*

Even though our framework has some good points, it also has some problems:

1. Complexity of Verification: Checking the ζ-Istrătescu condition may be more complex than verifying classical contraction conditions, especially for specific applications.

2. Choice of Simulation Function: The theory says that any simulation function is fine, but in real life, it might take some thought to pick the right ζ for a given problem.

3. Computation in S_b-metrics: Working with S_b-metrics where $s>1$ can make it harder to do numerical calculations because standard algorithms may need to be changed.

4. Limited to Complete Spaces: Our results, like most fixed point theorems, need the space to be complete, which can be hard to check in real life.

This work suggests multiple avenues for future research:

1. Partial S_b-Metric Spaces: Expand the framework to encompass partial S_b-metric spaces, wherein self-distances are not required to be zero.

2. Fuzzy S_b-Metric Spaces: Create a fuzzy version of the theory to use in modeling uncertainty.

• Coupled and Tripled Fixed Points: Look more into coupled, tripled, and higher-order fixed points in this context.

• Applications to New Domains: Use the results to solve problems in data science, machine learning, and network analysis.

• Numerical Analysis: Create effective numerical techniques tailored for S_b-metric spaces and ζ-Istrăţescu contractions.

• Stochastic Versions: Think about random versions of the contractions for use in random optimization and stochastic differential equations.

10. Conclusion

In this study, we have formulated a comprehensive framework for fixed point theory that integrates Istrătescu-type contractions, simulation functions, and S_b-metric spaces. Theorem 2 demonstrates the existence and uniqueness of fixed points for ζ-Istrăţescu contractions, contingent upon the condition $s(\alpha+\beta)<1$. We have given thorough proofs, added to multivalued and cyclic mappings, and used them in differential equations, optimization, and image processing.

The main things this work adds are:

1. Theoretical Innovation: The introduction of ζ-Istrăţescu contractions, which generalize and unify several classical contraction types.

2. Comprehensive Framework: Creation of a full theory in S_b-metric spaces that includes results on existence, uniqueness, convergence, and stability.

3. Practical Applications: Showing how useful the theory is by using it in different ways, each with its own examples and computational issues.

4. Algorithmic Development: Showing practical algorithms with examples and analyses of their convergence rates.

Our work shows that carefully generalizing basic math ideas can lead to both theoretical beauty and practical usefulness. We have made a useful set of tools for researchers who are dealing with nonlinear problems in generalized metric settings by combining classical fixed point theory with modern generalizations.

The framework created here is not just a generalization; it meets real needs in applied mathematics where standard metric assumptions are too strict. As computational methods keep getting better and applied problems get more complicated, these flexible mathematical frameworks will become more and more useful.

We hope this work will lead to more research on the connections between fixed point theory, generalized metric spaces, and their use in science and engineering.

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Author contributions

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During the preparation of this work, the authors used GrammarlyGO in order to improve language clarity, check for grammatical errors. After using this tool/service, the authors reviewed and edited the content as needed and take full responsibility for the content of the published article.

Conflicts of interest

The authors declare no competing financial interest.

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