Contemporary Mathematics
http://ojs.wiserpub.com/index.php/CM/

Research Article

Inverse Weibull-Rayleigh Distribution with Properties and Applications: A Member of the Inverse Weibull-X Family

Md. Tusharuzzaman Tushar[1], Abir Ahmed[2], Yusra A. Tashkandy[3], M. E. Bakr[3], Md Jakir Hossen[4*], Anoop Kumar[5], Alexis Habineza[6], Ahmed M. Gemeay[7], Abdullah Al Noman[8], Md. Mahabubur Rahman[1*]

[1]Department of Statistics and Data Science, Islamic University, Kushtia 7003, Bangladesh
[2]Department of Information Technology, Washington University of Science & Technology, Alexandria, VA 22314, USA
[3]Department of Statistics and Operations Research, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
[4]Center for Advanced Analytics (CAA), COE for Artificial Intelligence, Faculty of Engineering & Technology, Multimedia University, Melaka 75450, Malaysia
[5]Department of Statistics, Faculty of Basic Science, Central University of Haryana, Mahendergarh 123031, India
[6]Department of Sciences, Kibogora Polytechnic University, Nyamasheke P.O. Box: 50, Rwanda
[7]Department of Mathematics, Faculty of Science, Tanta University, Tanta 31527, Egypt
[8]Department of IST, Wilmington University, New Castle, DE 19720, USA
E-mail: jakir.hossen@mmu.edu.my (M.J.H.); sagor@stat.iu.ac.bd (M.M.R.)

Received: 31 December 2025; **Revised:** 13 March 2026; **Accepted:** 20 March 2026

Abstract: This research introduces a novel class of distributions termed the inverse Weibull-X (IW-X) family, derived from the broader T-X family of distributions. We systematically develop and describe several expanded sub-models within this newly proposed family, focusing on their unique characteristics and advantages. A particular emphasis is placed on the extended model known as the Inverse Weibull-Rayleigh (IW-R) distribution. This model is explored in depth, detailing its distributional properties, inferential methods, and practical applications. We provide a comprehensive theoretical framework for the IW-R distribution, including derivations of its moments, reliability functions, and other key statistical properties. Additionally, we discuss estimation techniques for the model parameters, utilizing the maximum likelihood estimation. The practical utility of the IW-R distribution is demonstrated through application to real-life data sets. These examples illustrate the model's enhanced adaptability and robustness in fitting data across various domains. The results indicate that the IW-R distribution offers significant improvements in flexibility and accuracy over traditional models, highlighting its potential as a valuable tool for statistical analysis and inference in diverse applications.

Keywords: family of distributions, T-X family, inverse Weibull-X (IW-X) family, Inverse Weibull-Rayleigh (IW-R) distribution, reliability function, order statistics, maximum likelihood estimation (MLE)

MSC: 62E15, 62G30, 62N05

1. Introduction

The process of modeling lifetime and reliability data needs distribution systems that provide researchers with multiple options to study various hazard rate patterns, data asymmetry, and extreme value distributions. The inverse Weibull

DOI: https://doi.org/10.37256/cm.7420269513

distribution and other classical models are popular because they provide easy mathematical analysis and match the requirements of decreasing hazard functions, but these models cannot represent complex aging patterns and multiple hazard states and extreme right-tail distributions that occur in actual data. The constraints of inherent baseline distributions impact both reliability assessment activities and statistical sample collection procedures and simulation methods because the fixed baseline distributions create synthetic samples that fail to capture actual population diversity, which leads to incorrect parameter estimations and diminished effectiveness of maximum likelihood estimation and Monte Carlo simulation methods. The IW-X family and IW-R distribution allow researchers to generate authentic random samples for their studies by incorporating additional shape parameters and flexible hazard patterns, leading to improved study outcomes, enhanced simulation work, and more practical applications of their findings, which motivated us for this study.

Multiple generalized classes of distributions have been created and used to cover a range of events due to the desire to create more adaptable statistical distributions. According to the density function $y = f(x)$ of the Pearson distribution [1], each correct answer to the differential equation qualifies as a legitimate distribution of continuous probabilities. Twelve distinct solutions to the differential equation

$$dF = F(1-F)g(x)dx,$$

where $0 \leq F \leq 1$ and $g(x)$ is a non-negative function over x, were given by a set of continuous distributions provided by Burr [2]. Johnson [3] presented a normalizing transformation-based approach to producing distributions. The new univariate and multivariate Burr type II distributions developed by Fry [4] are an extension of the Logistic Regression Model.

Ramberg and Schmeiser developed the generalized lambda distribution (GLD) [5, 6] as well as Ramberg et al. [7], where the lambda distribution was initially introduced by Tukey [8]. The skew-normal family of distributions, a novel class of density functions first described by Azzalini [9], is based on a shape parameter where zero corresponds to the conventional normal density. The generalization approach addresses the shape parameter started by Marshall and Olkin [10]. Gilchrist [11] provided a method for statistical modeling with the quantile function, who also referred to sample transformation maps (STMs) and rank transformation maps (RTMs) as Q-transformations and P-transformations.

A general class of distributions known as beta-generated distributions, as introduced by Eugene et al. [12], are formed from the logit of the beta random variable. Several beta-generated distributions have previously been created, including the beta-Gumbel distribution [13], beta exponential distribution [14], beta-Weibull distribution [15], beta-gamma distribution [16], beta-Pareto distribution [17], and others. A well-known popular method for generalizing probability distribution termed a quadratic transmuted family of distributions was introduced by Shaw and Buckley [18], which has the following cdf

$$F(x) = (1+\lambda)\, G(x) - \lambda G(x)^2; \;\; x \in \mathbb{R},$$

where $\lambda \in [-1, 1]$ is the transmutation parameter and $G(x)$ is the base distribution function of any standard probability model. Some contributed research work over this family are transmuted extreme value [19], transmuted Weibull [20], transmuted Log-logistic [21], transmuted Lomax [22], etc.

An extended version of the beta-generated family of distributions provided by Jones [23], Cordeiro, and de Castro [24] of Kumaraswamy distribution [25], introduced a generalized family of distributions called Kw-G family of distributions. Several generalized distributions related to Kw-G have already been developed, like the Kumaraswamy Weibull distribution by Cordeiro et al. [26], the Kumaraswamy generalized gamma distribution by de Castro et al. [27], and the Kumaraswamy generalized half-normal distribution by Cordeiro et al. [28]. Therefore, beyond the 0 to 1 range, the produced distributions from Beta and Kumaraswamy generated families of distributions are least capable of handling the increasingly complicated data.

To get over the drawback of employing other distributions with various supports as the generator to derive new classes of distributions and increase the regions of application of the produced distributions, a fascinating technique named the T-X family of distributions for generating a continuous family of distributions was introduced by Alzaatreh et al. [29], which has the following cdf

$$F_{T-X}(x) = \int_{c}^{W(G(x))} r(t)dt = R\left[W(G(x))\right], \tag{1}$$

where $T \in [c, d]$, for $-\infty \leq c < d \leq \infty$ be a continuous transformed random variable with density and distribution functions $r(t)$ and $R(t)$ respectively as well as the transformer $W(G(x))$ be a function of the cdf $G(x)$ of any random variable X so that $W(G(x))$ satisfies the following conditions:

$$\left.\begin{array}{l} W(G(x)) \in [c, d] \\ \\ W(G(x)) \text{ is differentiable and monotonically non-decreasing} \\ \\ W(G(x)) \rightarrow c \;\; as \;\; x \rightarrow -\infty \;\; \text{and} \;\; W(G(x)) \rightarrow d \;\; \text{as} \;\; x \rightarrow \infty. \end{array}\right\}$$

Then the corresponding pdf associated with the equation (1) can be given as

$$f_{T-X}(x) = \left[\frac{d}{dx}W(G(x))\right] r\left[W(G(x))\right]. \tag{2}$$

Several distributional family, such as Weibull-X [30], logistic-X [31], exponential-X [32], Pareto-X [33], exponent power-X [34], and others, have previously been generated using the method of the T-X family.

Despite the extensive development of the family of distributions, there are still limited contributions from researchers in constructing an inverse Weibull-centered family construction under this framework that can accommodate diverse fields of data with different tail behaviors, degrees of skewness, and hazard rate shapes. This gap can limit the flexibility of the inverse Weibull-generated modeling in accommodating varying lifetime data sets. To address this gap, this paper proposes a new IW-X family of distributions.

The specific objectives of this paper are to derive the properties of the proposed family of distributions along with a method for estimation and statistical inference, to generate new probability distributions utilizing the proposed family, and to show the superior flexibility of the IW-X family with some numerical results and graphical representation after applying real-life data sets over a selected generated distribution, the IW-R distribution.

The rest of this paper is organized as follows: The derivation and distributional characteristics of the IW-X family of distributions, order statistics, parameter estimates, and simulation procedure have all been covered in Section 2. The generated IW-R distribution from the proposed family is shown in Section 3, along with all of this distribution's fundamental statistical characteristics, order statistics, estimation, simulation study, and real-life applications to demonstrate the dominating flexibility of the resulting distribution. Finally, in Section 4, some concluding remarks have been bestowed.

2. IW-X family of distributions (new)

This section provides the development of the proposed IW-X family of distributions along with some special cases of the family as well as distributional properties. The order statistics, estimation, and process of simulation of the IW-X family are further discussed.

2.1 *Development of the IW-X family*

According to the technique of the T-X family as equated in (1), $W(G(x))$ depends on the support of the random variable T. For different $W(G(x))$, we will obtain a new family of distributions like if $T \geq c$, c is non-negative. Now, with the assumed zero value of c, without loss of generality there will be four choices of $W(G(x))$ can be defined as

$$-\log(1-G(x)), \quad \frac{G(x)}{(1-G(x))}, \quad -\log(1-G^{\alpha}(x)), \quad \text{and} \quad \frac{G^{\alpha}(x)}{(1-G^{\alpha}(x))},$$

where $\alpha > 0$. However, for producing the proposed IW-X family of distributions, here we consider $W(G(x)) = -\log(1-G(x))$, then the cdf of the new family (a special case of the T-X family) of distributions using (1) will be

$$F_{IW-X}(x) = \int_{0}^{-\log(1-G(x))} r(t)dt = R\left[-\log(1-G(x))\right]. \tag{3}$$

Then the pdf of the proposed IW-X family corresponds to equation (3) will be

$$f_{IW-X}(x) = \frac{g(x)}{1-G(x)} r\left[-\log(1-G(x))\right] = h(x) r\left[-\log(1-G(x))\right], \tag{4}$$

where $h(x)$ is the hazard function for the random variable X with pdf $g(x)$ and cdf $G(x)$ as well as $r(t)$ and $R(t)$ are the transformed density and distribution functions of Inverse Weibull distribution introduced by Keller and Kamath [35] and further used by Okasha et al. [36] defined as

$$r(t) = \alpha\theta t^{-(\alpha+1)} e^{-\left(\frac{\theta}{t^{\alpha}}\right)}; \quad t \in [0, \infty),$$

$$R(t) = e^{-\left(\frac{\theta}{t^{\alpha}}\right)},$$

where $\alpha, \theta > 0$ are the shape and scale parameters, respectively. After that, with the substitution of $r(t)$ and $R(t)$ in equation (3) and after some simplification, the new IW-X family of distributions can be defined as

$$F_{IW-X}(x; \alpha, \theta) = e^{-\frac{\theta}{[-\log(1-G(x))]^{\alpha}}}; \quad x \in \mathbb{R}, \tag{5}$$

where α and θ are the positive shape and scale parameters respectively. Then, with the idea of equation (4), the pdf of the new IW-X family of distributions can be obtained by differentiating (5) with respect to x as

$$f_{IW-X}(x; \alpha, \theta) = \frac{\alpha\theta g(x)}{[1-G(x)]} \frac{e^{-\frac{\theta}{[-\log(1-G(x))]^{\alpha}}}}{[-\log(1-G(x))]^{\alpha+1}}; \quad x \in \mathbb{R}. \tag{6}$$

Now, several extended flexible distributions (for several base distributions $G(x)$) can be generated as a special case of the proposed IW-X family of distributions.

2.2 *Special cases of the IW-X family*

This subsection lists some members of the proposed IW-X family of distributions based on different base distributions, which are displayed in Table 1. The plots of density functions for these distributions are drawn by using the R-package "ggplot2" (Wickham et al. [37]) and are exhibited in Figure 1.

Table 1. List some special cases of the IW-X family of distributions

Generated Distribution	pdf	Support
Inverse Weibull-Uniform	$\frac{\alpha\theta}{1-x}\frac{e^{-\frac{\theta}{[-\log(1-x)]^{\alpha}}}}{[-\log(1-x)]^{\alpha+1}}$	$x\in(0,1)$
Inverse Weibull-Kumaraswamy	$\frac{\alpha\theta bsx^{s-1}}{1-x^{s}}\frac{e^{-\theta\left[-\log\left\{(1-x^{s})^{b}\right\}\right]^{-\alpha}}}{\left[-\log\left\{(1-x^{s})^{b}\right\}\right]^{\alpha+1}}$	$x\in(0,1)$
Inverse Weibull-Arcsine	$\frac{\alpha}{4\sqrt{(x^{2}-x)}\mathrm{ArcCos}(\sqrt{x})}\frac{e^{-\theta\left[\log\left(\frac{\pi}{2\mathrm{ArcCos}(\sqrt{x})}\right)\right]^{-\alpha}}}{\left[\log\left(\frac{\pi}{2\mathrm{ArcCos}(\sqrt{x})}\right)\right]^{\alpha+1}}$	$x\in[0,1]$
Inverse Weibull-Gompertz	$\alpha\theta b\eta\frac{e^{\left[-\theta\left\{-\log\left(e^{\eta}-\eta e^{\mathrm{bx}}\right)\right\}^{-\alpha}+\mathrm{bx}\right]}}{\left[-\log\left(e^{\eta}-\eta e^{\mathrm{bx}}\right)\right]^{\alpha+1}}$	$x\in[0,\infty)$
Inverse Weibull-Lomax	$\frac{\alpha\theta\beta}{\lambda+x}\frac{e^{-\theta\left[-\log\left\{\left(\frac{\lambda+x}{\lambda}\right)^{-\beta}\right\}\right]^{-\alpha}}}{\left[-\log\left\{\left(\frac{\lambda+x}{\lambda}\right)^{-\beta}\right\}\right]^{\alpha+1}}$	$x\in[0,\infty)$
Inverse Weibull-Exponential	$\alpha\theta\lambda\frac{e^{-\frac{\theta}{(\lambda x)^{\alpha}}}}{(\lambda x)^{\alpha+1}}$	$x\in[0,\infty)$
Inverse Weibull-Weibull	$\frac{\alpha\theta k}{x}\frac{e^{-\theta\left[\left(\frac{x}{\lambda}\right)^{k}\right]^{-\alpha}}}{\left[\left(\frac{x}{\lambda}\right)^{k}\right]^{\alpha}}$	$x\in[0,\infty)$
Inverse Weibull-Inverse Weibull	$\frac{\alpha\theta bsx^{-s-1}\left(-\log\left(1-e^{-bx^{-s}}\right)\right)^{-\alpha-1}e^{-\theta\left(-\log\left(1-e^{-bx^{-s}}\right)\right)^{-\alpha}}}{e^{bx^{-s}}-1}$	$x\in[0,\infty)$
Inverse Weibull-Pareto	$\frac{\alpha\theta\beta\mathrm{k}^{\beta}}{x^{\beta+1}\left(\frac{k}{x}\right)^{\beta}}\frac{e^{-\theta\left[-\log\left\{\left(\frac{k}{x}\right)^{\beta}\right\}\right]^{-\alpha}}}{\left[-\log\left\{\left(\frac{k}{x}\right\}^{\beta}\right\}\right]^{\alpha+1}}$	$x\in[k,\infty)$
Inverse Weibull-Burr XII	$\frac{\alpha\theta ckx^{c-1}}{x^{c}+1}\frac{e^{-\theta\left[-\log\left\{(x^{c}+1)^{-k}\right\}\right]^{-\alpha}}}{\left[-\log\left\{(x^{c}+1)^{-k}\right\}\right]^{\alpha+1}}$	$x\in\mathbf{R}^{+}$
Inverse Weibull-Gumbel	$\frac{\alpha\theta}{\beta\left(1-e^{-e^{\frac{\mu-x}{\beta}}}\right)}\frac{e^{\left[-\frac{x+\beta e^{-x}-\mu+\beta\theta\left(-e^{\frac{\mu-x}{\beta}}\right)^{-\alpha}}{\beta}\right]}}{\left(-e^{\frac{\mu-x}{\beta}}\right)^{\alpha+1}}$	$x\in\mathbf{R}$
Inverse Weibull-Logistic	$\frac{\alpha\theta}{s\left(e^{\mu/s}+e^{x/s}\right)}\frac{e^{\frac{x}{s}-\theta\left[-\log\left(1-\frac{1}{e^{\frac{\mu-x}{s}}+1}\right)\right]^{-\alpha}}}{\left[-\log\left(1-\frac{1}{e^{\frac{\mu-x}{s}}+1}\right)\right]^{\alpha+1}}$	$x\in\mathbf{R}$

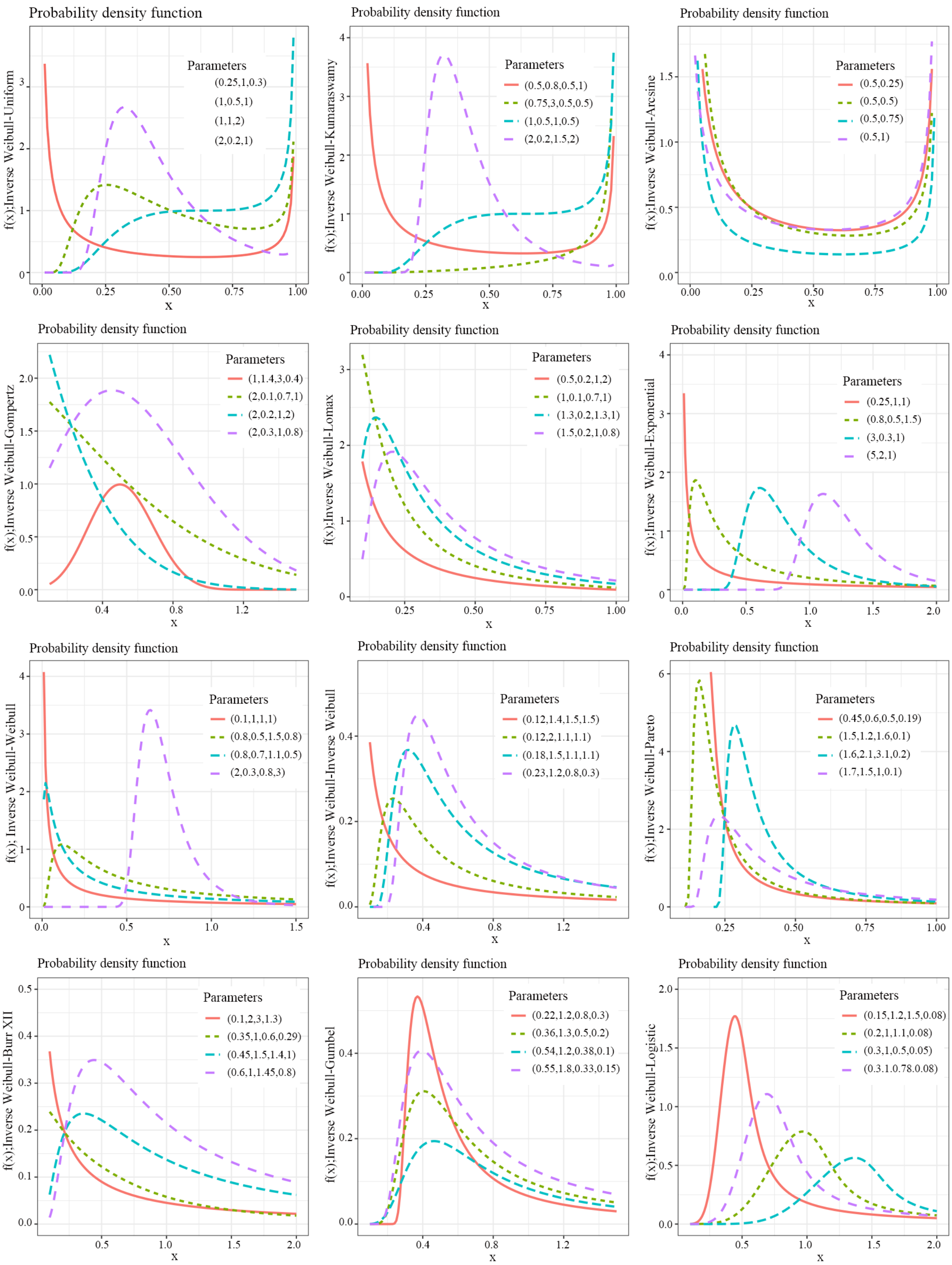

Figure 1. Plots illustrating the pdf for the members of the IW-X family

2.3 *Specific properties of the IW-X family*

The qualities or measurements that characterize the distribution of data points in a data set or population are known as the distributional properties of a distribution. Important details about the distribution of the data, the distribution's shape, and other significant characteristics are revealed by these qualities. Some specific distributional properties of the proposed IW-X family are described below:

2.3.1 *Moments*

The r^{th} moment of the IW-X family of distributions can be obtained in the context of probability-weighted moments. The probability-weighted moment (pwm) of the IW-X random variable is defined as

$$M_{r,l,s} = E\left[X^r \{G(x)\}^l \{1-G(x)\}^s\right] = \int_{-\infty}^{\infty} x^r \{G(x)\}^l \{1-G(x)\}^s g(x)dx. \tag{7}$$

We have to find the r^{th} moments of the IW-X family using the following formula

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\, dx, \tag{8}$$

where $f(x)$ is equated in equation (6). Using $f(x)$ from (6) to (8), the r^{th} moment becomes

$$\mu_r' = \int_{-\infty}^{\infty} x^r \frac{\alpha\theta g(x)}{[1-G(x)]} \frac{e^{-\frac{\theta}{[-\log(1-G(x))]^{\alpha}}}}{[-\log(1-G(x))]^{\alpha+1}} dx. \tag{9}$$

We know from the exponentiated expansion,

$$e^{-x} = \sum_{i=0}^{\infty} \frac{(-1)^i x^i}{i!}.$$

Hence the expression

$$e^{-\frac{\theta}{[-\log(1-G(x))]^{\alpha}}} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \frac{\theta^i}{[-\log(1-G(x))]^{i\alpha}}. \tag{10}$$

Then using (10), the expression

$$\frac{e^{-\frac{\theta}{[-\log(1-G(x))]^{\alpha}}}}{[-\log(1-G(x))]^{\alpha+1}} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \frac{\theta^i [-\log(1-G(x))]^{-i\alpha}}{[-\log(1-G(x))]^{\alpha+1}}$$

$$= \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \theta^i [-\log(1-G(x))]^{-\alpha(i+1)-1}. \tag{11}$$

But we know from the logarithm expansion,

$$\log(1-x) = \sum_{j=1}^{m} \frac{-x^j}{j}.$$

Using the logarithm expansion in (11) and after some simplification, the expression

$$\frac{e^{-\frac{\theta}{[-\log(1-G(x))]^{\alpha}}}}{[-\log(1-G(x))]^{\alpha+1}} = \sum_{i=0}^{\infty}\sum_{j=1}^{m} \frac{(-1)^i}{i!}\theta^i j^{\alpha(i+1)+1}\{G(x)\}^{-j\alpha(i+1)-j}. \tag{12}$$

Now, the distribution and density functions of the IW-X family of distributions can also be written in equations (13) and (14), respectively, as

$$F_{IW-X}(x;\alpha,\theta) = \sum_{i=0}^{\infty}\sum_{j=1}^{m} \frac{(-1)^i}{i!}\theta^i j^{i\alpha}\{G(x)\}^{-ij\alpha}, \tag{13}$$

and

$$f_{IW-X}(x;\alpha,\theta) = \alpha\sum_{i=0}^{\infty}\sum_{j=1}^{m} \frac{(-1)^i g(x)}{i!\{1-G(x)\}}\theta^{i+1} j^{\alpha(i+1)+1}\{G(x)\}^{-j\alpha(i+1)-j}. \tag{14}$$

After that, substitute the expression from (12) to (9), the r^{th} moment becomes

$$\mu_r' = \alpha\sum_{i=0}^{\infty}\sum_{j=1}^{m} \frac{(-1)^i}{i!}\theta^{i+1} j^{\alpha(i+1)+1}\int_{-\infty}^{\infty} x^r\ \{G(x)\}^{-j\alpha(i+1)-j}\{1-G(x)\}^{-1} g(x)dx. \tag{15}$$

Therefore, using the pwm from equation (7) and substitute in equation (15) , the r^{th} moment of the proposed IW-X family of distributions is

$$\mu_r' = \alpha\sum_{i=0}^{\infty}\sum_{j=1}^{m} \frac{(-1)^i}{i!}\theta^{i+1} j^{\alpha(i+1)+1} M_{r,-j\alpha(i+1)-j,-1}, \tag{16}$$

where,

$$M_{r,-j\alpha(i+1)-j,-1} = \int_{-\infty}^{\infty} x^r\ \{G(x)\}^{-j\alpha(i+1)-j}\{1-G(x)\}^{-1} g(x)dx.$$

Now, the moment-generating function (MGF) of the proposed IW-X family can be expressed as

$$M_X(t) = \alpha \sum_{i,r=0}^{\infty} \sum_{j=1}^{m} \frac{(-1)^i t^r}{i! r!} \theta^{i+1} j^{\alpha(i+1)+1} M_{r,-j\alpha(i+1)-j,-1}. \tag{17}$$

2.3.2 *Quantile function and median*

The inverse of a distribution function, which specifies the value of the chance variable is the quantile function. The quantile function of the proposed IW-X family of distributions can be obtained by inverting the distribution of the IW-X family as equated in equation (6) and expressed as

$$X = G_I \left[1 + e^{p(u)} \right], \tag{18}$$

where, $G_I(.) = G^{-1}(.)$, $p(u) = \left\{ -\frac{\log(u)}{\theta} \right\}^{-1/\alpha}$, and $u \sim U(0, 1)$.

Similarly, the median of the proposed family can be obtained just by the replacement of X and u in equation (18), through M and $\frac{1}{2}$ respectively, and other terms remain the same. Hence the expression of the median of the IW-X family can be expressed as

$$M = G_I \left[1 + e^{p\left(\frac{1}{2}\right)} \right]. \tag{19}$$

2.4 *Distribution of the order statistics for the IW-X family*

Order statistics are a fundamental concept in statistics that deals with the arrangement or ordering of a set of random variables or data points. Let, $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$ be the order statistics of the independent and identically distributed (iid) random variables $X_1, X_2, \ldots, X_n$. Then, the pdf of the k^{th} order statistic $X_{k:n}$ for the proposed family can be defined as follows

$$f_{X_{k:n}}(x) = \frac{n!}{(k-1)!(n-k)!} [F(x)]^{k-1} [1 - F(x)]^{n-k} f(x), \tag{20}$$

where the expanded $F(x)$ and $f(x)$ of the proposed family are equated in equations (13) and (14), respectively. Hence the equation (20) will be

$$f_{X_{k:n}}(x) = \frac{n!}{(k-1)!(n-k)!} \left[\sum_{i=0}^{\infty} \sum_{j=1}^{m} \frac{(-1)^i}{i!} \theta^i j^{i\alpha} \{G(x)\}^{ij\alpha} \right]^{k-1} \left[1 - \sum_{i=0}^{\infty} \sum_{j=1}^{m} \frac{(-1)^i}{i!} \theta^i j^{i\alpha} \{G(x)\}^{-ij\alpha} \right]^{n-k}$$

$$\times \left[\alpha \sum_{i=0}^{\infty} \sum_{j=1}^{m} \frac{(-1)^i g(x)}{i! \{1 - G(x)\}} \theta^{i+1} j^{\alpha(i+1)+1} \{G(x)\}^{-j\alpha(i+1)-j} \right]. \tag{21}$$

Now, the other order statistics of the proposed IW-X family can be get-at-able through the pdf of the k^{th} order statistic $X_{k:n}$ as equated in equation (21).

2.5 *Model parameter estimation of the IW-X family*

The estimation of parameters of the IW-X family of distributions through maximum likelihood estimation (MLE) is discussed here. Let $x_1, x_2, \ldots, x_n$ represent the random sample of size n drawn from the proposed family $\{f(.;\vartheta)|\vartheta \in \Theta\}$, with vector of parameters $\vartheta = (\alpha, \theta)'$. Then the corresponding log-likelihood function can be expressed as

$$l(\vartheta) = n\log(\alpha\theta) + \sum_{i=1}^{n} \log\left[\frac{g(x_i)}{1-G(x_i)}\right] - \sum_{i=1}^{n}\left[\frac{\theta}{\{-\log(1-G(x_i))\}^{\alpha}}\right]$$

$$-(\alpha+1)\sum_{i=1}^{n}\log[-\log(1-G(x_i))]. \tag{22}$$

Finding the model parameter values that maximize the likelihood function is the aim of MLE. Gradient-based optimization methods like gradient descent or specialized algorithms like the Newton-Raphson method are commonly employed to find the values of the parameters (ϑ) that maximize the log-likelihood function. In order to do this, take the derivatives of (22) with respect to the unknown parameters and continue as follows:

$$\frac{\partial l(\vartheta)}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\left[\frac{\theta\log\{-\log(1-G(x_i))\}}{\{-\log(1-G(x_i))\}^{\alpha}}\right] - \sum_{i=1}^{n} log[-\log(1-G(x_i))],$$

$$\frac{\partial l(\vartheta)}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^{n}\left[\frac{1}{\{-\log(1-G(x_i))\}^{\alpha}}\right].$$

Setting the above partial derivatives equal to zero and solving the nonlinear equations analytically yields the maximum likelihood estimates (MLEs) $\hat{\vartheta} = (\hat{\alpha}, \hat{\theta})'$. After that, the "bbmle" [38] package can be implemented to solve the ensuing equations numerically, as they could not be solved analytically.

2.6 *Simulation study of the IW-X family*

The procedure of simulation analysis (see details in [39]) has been carried out in this subsection for the developed models of the proposed IW-X family of distributions. The steps are explained below:

1. Using equation (18) and with the set initial values of the model parameters, random samples of several arbitrary sample sizes have to be generated first for the developed model.
2. Then, the generated model's parameters must be obtained through MLE.
3. The above process has to be repeated n times to calculate these estimators' mean and mean square error (MSE).

3. IW-R distribution (new)

This section introduces a new generalization of the Rayleigh distribution named the IW-R distribution, utilizing the proposed IW-X family of distributions. To develop it, let's consider the density and distribution functions of the Rayleigh distribution, which was named after Lord Rayleigh [40],

$$g(x) = \frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}, \quad x \in [0, \infty), \tag{23}$$

$$
G(x) = 1 - e^{-\frac{x^2}{2\sigma^2}}, \tag{24}
$$

where $\sigma \in \mathbf{R}^+$ is the scale parameter. The cdf of the IW-R distribution can be obtained by substituting $G(x)$ from equation (24) to (5), which can be given as

$$
F(x; \alpha, \theta, \sigma) = e^{-\frac{\theta}{\left(\frac{x^2}{2\sigma^2}\right)^{\alpha}}}; \quad x \in [0, \infty), \tag{25}
$$

where $\alpha, \theta \in \mathbf{R}^+$ are the shape and scale parameters respectively. Then, the corresponding pdf of the IW-R distribution can be given as

$$
f(x; \alpha, \theta, \sigma) = \frac{\alpha \theta x}{\sigma^2} \frac{e^{-\frac{\theta}{\left(\frac{x^2}{2\sigma^2}\right)^{\alpha}}}}{\left(\frac{x^2}{2\sigma^2}\right)^{\alpha+1}}; \quad x \in [0, \infty). \tag{26}
$$

Some special sub-models of the IW-R distribution are listed in the following Table 2.

Table 2. List of some special cases of the IW-R distribution

Value of Parameters	Special Sub-models
$\alpha = \frac{\alpha}{2}$ and $\sigma = \sqrt{\frac{1}{2}}$	Inverse Weibull
$\alpha = \frac{1}{2}$ and $\sigma = \sqrt{\frac{1}{2}}$	Inverse Exponential
$\alpha = 1$, $\theta = 1$, and $\sigma = \sqrt{\frac{\sigma}{2}}$	Inverse Rayleigh

Now, some possible shapes of the density and distribution functions of IW-R distribution for various chosen values of the model parameters (α, θ, and λ) are displayed in Figure 2. The derived IW-R distribution clearly can manage the various behaviors of the complicated data because the majority of the density function's forms are positively skewed and one is exponential.

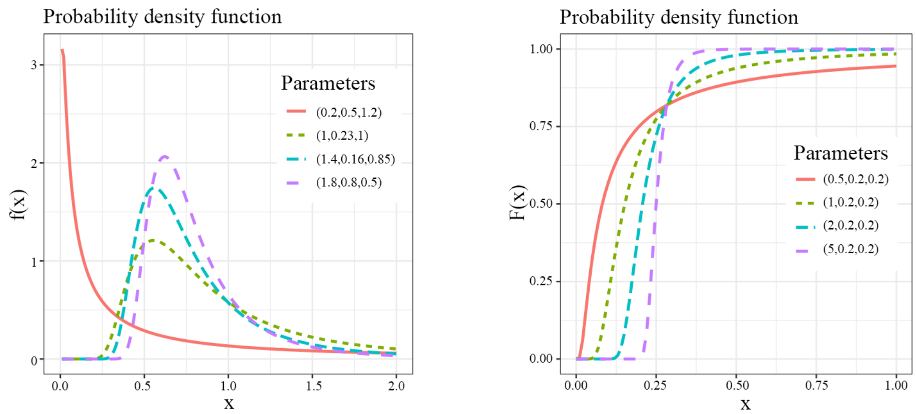

Figure 2. Plots illustrating the density and distribution functions of the IW-R distribution

3.1 *Distributional properties of the IW-R distribution*

Some basic distributional properties include moment, quantile, reliability, and hazard functions of the IW-R distribution have been discussed in this subsection.

3.1.1 *Moments*

The r^{th} moment of IW-R distribution can be expressed as follows

$$\mu_r' = 2^{\frac{r}{2}} \sigma^r \theta^{\frac{r}{2\alpha}} \Gamma\left(1 - \frac{r}{2\alpha}\right); \quad \alpha > \frac{r}{2}. \tag{27}$$

Then, the mean and variance of the IW-R distribution can be easily obtained through the r^{th} moment in (27) by substituting $r = 1, 2$ respectively, which are

$$\text{Mean} = \mu_1 = 2^{\frac{1}{2}} \sigma \theta^{\frac{1}{2\alpha}} \Gamma\left(1 - \frac{1}{2\alpha}\right); \quad \alpha > \frac{1}{2}, \tag{28}$$

and

$$\text{Variance} = \mu_2 = 2\sigma^2 \theta^{\frac{1}{\alpha}} \left[\Gamma\left(1 - \frac{1}{\alpha}\right) - \Gamma\left(1 - \frac{1}{2\alpha}\right)^2\right], \quad \alpha > 1. \tag{29}$$

Different possible values of the model parameters of the IW-R distribution are shown in Table 3, which presents the mean and variance (in parentheses) charts as well as the possible shapes demonstrated in Figure 3. It has been observed from the table and figure that with the increasing values of the shape parameter (α), both the values of the mean and variance have decreased, and they grow with the rising values of the scale parameter (σ).

Table 3. Mean and variance (in parenthesis) chart for the IW-R model

		$\sigma = 1$	$\sigma = 1.2$	$\sigma = 1.4$	$\sigma = 1.6$	$\sigma = 1.8$
$\theta = 1$	$\alpha = 1.1$	2.304 (14.976)	2.765 (21.566)	3.225 (29.353)	3.686 (38.339)	4.147 (48.523)
	$\alpha = 1.2$	2.162 (5.889)	2.594 (8.480)	3.027 (11.542)	3.459 (15.075)	3.891 (19.080)
	$\alpha = 1.3$	2.058 (3.199)	2.469 (4.606)	2.881 (6.269)	3.292 (8.188)	3.704 (10.363)
	$\alpha = 1.4$	1.978 (2.007)	2.373 (2.891)	2.769 (3.935)	3.165 (5.139)	3.560 (6.504)
	$\alpha = 1.5$	1.915 (1.372)	2.298 (1.976)	2.681 (2.689)	3.064 (3.513)	3.447 (4.446)

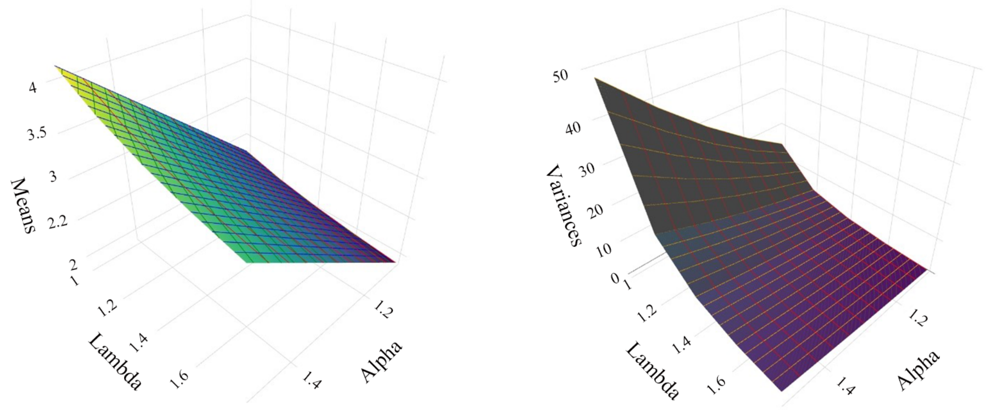

Figure 3. Plots illustrating the mean (left) and variance (right) of the IW-R distribution

Similarly, the skewness and kurtosis of the IW-R distribution can be obtained with the higher value of $r(>2)$ in (27) and expressed as

$$\text{Skewness} = \frac{\left[2\Gamma\left(1-\frac{1}{2\alpha}\right)^3 - 3\Gamma\left(1-\frac{1}{\alpha}\right)\Gamma\left(1-\frac{1}{2\alpha}\right) + \Gamma\left(1-\frac{3}{2\alpha}\right)\right]^2}{\left[\Gamma\left(1-\frac{1}{\alpha}\right) - \Gamma\left(1-\frac{1}{2\alpha}\right)^2\right]^3}; \;\; \alpha > \frac{3}{2}, \tag{30}$$

and

$$\text{Kurtosis} = \frac{6\Gamma\left(\frac{\alpha-1}{\alpha}\right)\Gamma\left(1-\frac{1}{2\alpha}\right)^2 - 4\Gamma\left(1-\frac{3}{2\alpha}\right)\Gamma\left(1-\frac{1}{2\alpha}\right)}{\left[\Gamma\left(1-\frac{1}{\alpha}\right) - \Gamma\left(1-\frac{1}{2\alpha}\right)^2\right]^2}$$

$$+ \frac{\Gamma\left(1-\frac{2}{\alpha}\right) - 3\Gamma\left(1-\frac{1}{2\alpha}\right)^4}{\left[\Gamma\left(1-\frac{1}{\alpha}\right) - \Gamma\left(1-\frac{1}{2\alpha}\right)^2\right]^2}, \;\; \alpha > 2. \tag{31}$$

Now, the moment generating function of the IW-R distribution can be expressed as

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} 2^{\frac{r}{2}} \sigma^r \theta^{\frac{r}{2\alpha}} \Gamma\left(1 - \frac{r}{2\alpha}\right); \;\; \alpha > \frac{r}{2}, \tag{32}$$

where $t \in \mathbb{R}$. From here different moments can be obtained.

3.1.2 *Quantile function and median*

The quantile function of IW-R distribution can be expressed as

$$X = \sqrt{2\sigma^2 \left[-\frac{\log(q)}{\theta}\right]^{-\frac{1}{\alpha}}}. \tag{33}$$

This function can be easily obtained either by substituting the $cdf\ G(x)$ of the Rayleigh distribution in the quantile function of the proposed family in (18) or solving the distribution function of the IW-R model equal $u(\sim$ Uniform distribution in (0,1), in terms of x. The median of the IW-R model can be obtained similarly to the quantile function or we have to just replace u with 0.5 from equation (33). Hence the median becomes

$$M = \sqrt{2\sigma^2 \left[-\frac{\log(\frac{1}{2})}{\theta}\right]^{-\frac{1}{\alpha}}} \qquad \text{Or,} \qquad M = \sqrt{2\sigma^2 \left[\frac{0.3010}{\theta}\right]^{-\frac{1}{\alpha}}}.$$

3.1.3 *Reliability analysis*

A key idea in survival analysis and reliability engineering is the reliability function, abbreviated as $R(t)$. It measures the likelihood that a system, component, or entity will continue to operate without malfunction or degradation beyond a given time point, t. Then, the reliability function of the IW-R distribution can be defined as

$$R(t) = 1 - e^{-\frac{\theta}{\left(\frac{t^2}{2\sigma^2}\right)^{\alpha}}}.$$

The hazard rate function is denoted as $h(t)$ of the IW-R distribution can be obtained by the ratio of density and reliability function of the IW-R distribution, which given as

$$h(t) = \frac{f(t)}{R(t)} = \frac{2^{\alpha+1}\alpha\theta\left(\frac{t^2}{\sigma^2}\right)^{-\alpha}}{t\left[e^{\frac{\theta}{\left(\frac{t^2}{2\sigma^2}\right)^{\alpha}}} - 1\right]}.$$

Now, the mean residual life (MRL) function of the IW-R distribution denoted by $m(t)$, which measures the expected remaining lifetime given survival until time t, can be expressed as

$$m(t) = \frac{(2\sigma^2)^{1/2}\,\theta^{1/\alpha}}{R(t)}\Gamma\left[-\frac{1}{\alpha},\ \theta\left(\frac{2\sigma^2}{t^2}\right)^{\alpha}\right], \quad t > 0.$$

Tail behavior is particularly applicable in the domain of actuarial science, where accurately modeling extreme lifetimes or rare events affects risk assessment, insurance policies, and maintenance scheduling are the crucial tasks. The IW-R distribution's flexible hazard and tail structures allow it to accommodate heavy-tailed or skewed data, enabling realistic random sample generation and improving both inference and simulation studies. To demonstrate these, here Figure 4 has been considered with various combinations of the model parameters (α, θ, and σ), which exhibit different possible forms for the reliability (monotonically declining), hazard rate (reducing, increasing and decreasing) and mean residual life (esclating, dropping and growing) functions. These results indicate that the proposed IW-R distribution can capture different behaviours of the datasets of diverse reliability scenarios.

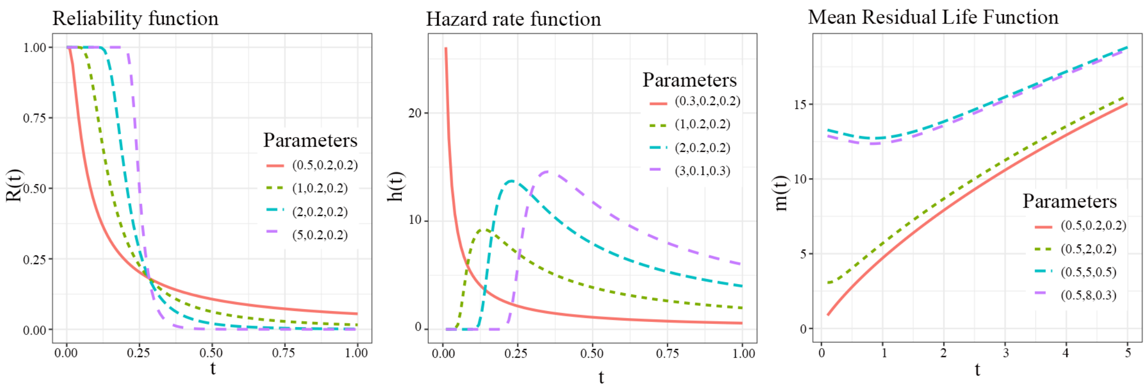

Figure 4. Plots illustrating the reliability, hazard rate, and mean residual life functions of the IW-R distribution

3.2 *Distribution of the order statistics for the IW-R distribution*

By the definition of the pdf of the k^{th} order statistic for the IW-R distribution and with the assumed random variables, the density of lowest and highest order statistics $X_{1:n}$ and $X_{n:n}$ respectively, can be given as

$$f_{X_{1:n}}(x) = \frac{n\alpha\theta x}{\sigma^2} \frac{e^{-\frac{\theta}{\left(\frac{x^2}{2\sigma^2}\right)^{\alpha}}}}{\left(\frac{x^2}{2\sigma^2}\right)^{\alpha+1}} \left[1 - e^{-\frac{\theta}{\left(\frac{x^2}{2\sigma^2}\right)^{\alpha}}}\right]^{n-1},$$

and

$$f_{X_{n:n}}(x) = \frac{n\alpha\theta x}{\sigma^2} \frac{\left[e^{-\frac{\theta}{\left(\frac{x^2}{2\sigma^2}\right)^{\alpha}}}\right]^{n}}{\left(\frac{x^2}{2\sigma^2}\right)^{\alpha+1}}.$$

Then, the density function of the k^{th} order statistic for the proposed IW-R distribution can be obtained by using the following formula

$$E(X_{k:n}^r) = \int_0^{\infty} x_k^r \, f_{X_{k:n}}(x)\, dx,$$

where,

$$f_{X_{k:n}}(x) = \frac{n!}{(k-1)!(n-k)!}\left[\frac{\alpha\theta x}{\sigma^2}\frac{e^{-\frac{\theta}{\left(\frac{x^2}{2\sigma^2}\right)^{\alpha}}}}{\left(\frac{x^2}{2\sigma^2}\right)^{\alpha+1}}\right]\left[e^{-\frac{\theta}{\left(\frac{x^2}{2\sigma^2}\right)^{\alpha}}}\right]^{k-1}\left[1-e^{-\frac{\theta}{\left(\frac{x^2}{2\sigma^2}\right)^{\alpha}}}\right]^{n-k}.$$

3.3 *Model parameter estimation of the IW-R distribution*

Considering the proposed IW-X family's estimation method (MLE), the log-likelihood function of the generated IW-R distribution can be expressed as follows

$$l(\vartheta) = n\log(\alpha) + n\log(\theta) - 2n\log(\sigma) + \sum_{i=1}^{n}\log(x_i)$$

$$-(\alpha+1)\sum_{i=1}^{n}\log\left(\frac{x_i^2}{2\sigma^2}\right) - \theta\sum_{i=1}^{n}\left(\frac{x_i^2}{2\sigma^2}\right)^{-\alpha}, \tag{34}$$

where the parameter vector is $\vartheta = (\alpha, \theta, \sigma)'$. According to the rules of MLE, we have to take derivatives (34) with the respective parameters α, θ, and σ for maximizing $l(\vartheta)$. Hence, proceed as follows.

$$\frac{\partial l(\vartheta)}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\log\left(\frac{x_i^2}{2\sigma^2}\right) - \sum_{i=1}^{n}\left[2^{\alpha}\theta\log(2)\left(\frac{x_i^2}{\sigma^2}\right)^{-\alpha} - 2^{\alpha}\theta\left(\frac{x_i^2}{\sigma^2}\right)^{-\alpha}\log\left(\frac{x_i^2}{\sigma^2}\right)\right],$$

$$\frac{\partial l(\vartheta)}{\partial\theta} = \frac{n}{\theta} - \sum_{i=1}^{n}\left(\frac{x_i^2}{2\sigma^2}\right)^{-\alpha},$$

and

$$\frac{\partial l(\vartheta)}{\partial\sigma} = \frac{2(\alpha+1)n}{\sigma} - \frac{2n}{\sigma} - \sum_{i=1}^{n}\left[\frac{2^{\alpha+1}\alpha\theta x_i^2\left(\frac{x_i^2}{\sigma^2}\right)^{-\alpha-1}}{\sigma^3}\right].$$

However, it can be challenging and occasionally impossible to solve the three nonlinear equations analytically after setting $\frac{\partial l(\vartheta)}{\partial\alpha} = 0$, $\frac{\partial l(\vartheta)}{\partial\theta} = 0$, and $\frac{\partial l(\vartheta)}{\partial\sigma} = 0$,. Additionally, there is a possibility of invalid estimated values of the parameters that can exceed the support of the parameters. As a result, we can apply logarithmic reparametrization here to ensure that the estimated parameters remain within their bounds, i.e.,

$$\alpha = \exp(\alpha'), \quad \theta = \exp(\theta'), \quad \lambda = \exp(\sigma'),$$

where $(\alpha', \theta', \sigma')$ are the unconstrained real numbers and $\alpha' = \log\alpha$, $\theta' = \log\theta$, and $\sigma' = \log\sigma$. Now, one can use the R package to solve the nonlinear equations numerically quickly, as discussed in the estimation part of the IW-X family, where the estimates of the original parameters were then recovered by back-transformation. The maximum likelihood estimates (MLEs) for the IW-R distribution are therefore $\hat{\vartheta} = \left(\hat{\alpha}, \hat{\theta}, \hat{\sigma}\right)'$ of $\vartheta = (\alpha, \theta, \sigma)'$. Also, as $n \to \infty$, the asymptotic distribution of the MLE is given by

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\theta} \\ \hat{\sigma} \end{pmatrix} \sim N \left[\begin{pmatrix} \alpha \\ \theta \\ \sigma \end{pmatrix}, \begin{pmatrix} \hat{V}_{11} & \hat{V}_{12} & \hat{V}_{13} \\ \hat{V}_{21} & \hat{V}_{22} & \hat{V}_{23} \\ \hat{V}_{31} & \hat{V}_{32} & \hat{V}_{33} \end{pmatrix} \right].$$

Here, V is produced by inverting the Hessian matrix and represents the asymptotic variance-covariance matrix (see Appendix) of the estimates $\hat{\alpha}$, $\hat{\theta}$, and $\hat{\sigma}$. Then, for these three parameters, an estimated $100(1-\alpha)\%$ two-sided confidence interval is provided as

$$\hat{\alpha} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V}_{11}}, \quad \hat{\theta} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V}_{22}}, \quad \text{and} \quad \hat{\sigma} \pm Z_{\frac{\alpha}{2}} \sqrt{\hat{V}_{33}},$$

where Z_{α} is the α percentile of the standard normal distribution.

3.4 *Simulation study of the IW-R distribution*

This subsection has carried out the Monte Carlo simulation analysis for the IW-R. distribution in accordance with the method of simulation analysis for the IW-X family. With different sample sizes (50, 100, 200, and 500), random samples have been produced using the equation (33) for the initial values of the parameters $\alpha = 1$, $\theta = 0.80$, and $\sigma = 0.50$. After obtaining the MLEs, the procedure was repeated $n = 1000$ times. The computed results for mean and mean square error (MSE) for these 1000 times values are then provided in Table 4, which demonstrates that the estimated values of the parameters are almost identical to the values of the original parameters and that the MSE reduces with increasing sample sizes, displayed in Figure 5. Therefore, from the Monte Carlo simulation results, it can be confirmed that the MLE estimators for the generated IW-R distribution are consistent and efficient.

Table 4. Descriptive statistics for the chosen data sets

Sample size	Parameter	True value	Estimate	MSE
50	α	1.00	1.028	0.014
	θ	0.80	0.675	0.091
	σ	0.50	0.570	0.016
100	α	1.00	1.016	0.006
	θ	0.80	0.637	0.080
	σ	0.50	0.580	0.013
200	α	1.00	1.004	0.003
	θ	0.80	0.616	0.063
	σ	0.50	0.583	0.011
500	α	1.00	1.004	0.001
	θ	0.80	0.613	0.049
	σ	0.50	0.578	0.009

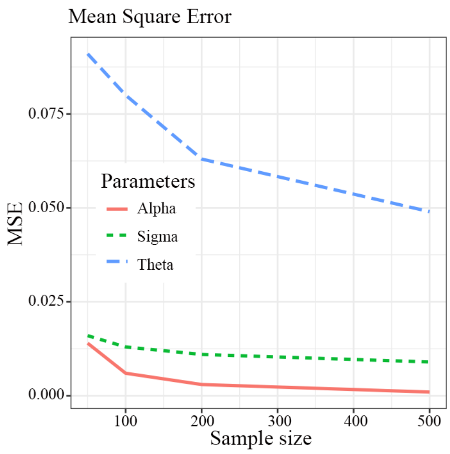

Figure 5. Plot illustrating the MSE of parameters vs sample size for the IW-R distribution

3.5 *Applications of the IW-R distribution*

This section explores the performance of the IW-R distribution by applying four real-life data sets to the IW-R model as well as some related comparable models, including Weibull-Rayleigh (W-R) from the Weibull-X family, Logistic-Exponential (L-E) from the Logistic-X family, Exponential-Rayleigh (E-R) from the Exponential-X family, Pareto-Exponential (P-E) from the Pareto-X family, Exponent Power-Weibull (EP-W) from the Exponent Power-X family, Transmuted Rayleigh (T-R), Weibull (W), and Rayleigh (R). The sources of data sets are now discussed below:

Carbon Fiber Data: We are utilizing an actual data set that Bader and Priest [41] utilized for their data analysis and recently used by Joshi and Kumar [42]. The information provided is the strength measured in GPA for single carbon fibers with gauge lengths of 10 mm and 63 samples. The data set is:

1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.

Relief Times Data: The second data set is taken from Gross and Clark [43] and further used by Afify et al. [44], which gives the relief times of 20 patients receiving an analgesic. The data set is:

1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2.

Taxes Revenue Data: This data set was taken from Nassar and Nada [45] and further utilized by Owoloko et al. [46], which represents the monthly real taxes revenue in Egypt between January 2006 and November 2010 (in 1000 million Egyptian pounds). The data set is listed below:

5.9, 20.4, 14.9, 16.2, 17.2, 7.8, 6.1, 9.2, 10.2, 9.6, 13.3, 8.5, 21.6, 18.5, 5.1, 6.7, 17, 8.6, 9.7, 39.2, 35.7, 15.7, 9.7, 10, 4.1, 36, 8.5, 8, 9.2, 26.2, 21.9, 16.7, 21.3, 35.4, 14.3, 8.5, 10.6, 19.1, 20.5, 7.1, 7.7, 18.1, 16.5, 11.9, 7, 8.6, 12.5, 10.3, 11.2, 6.1, 8.4, 11, 11.6, 11.9, 5.2, 6.8, 8.9, 7.1, 10.8.

Wind Speed Data: This data set was taken from Seguro and Lambert [47], which represents the wind speed per hour of a day illustrated in time series formate.

3.3, 3.8, 4.2, 3.3, 2.8, 3.0, 4.0, 2.7, 5.2, 6.7, 6.8, 6.8, 5.7, 8.5, 8.9, 9.3, 6.5, 4.2, 4.3, 3.7, 4.0, 2.8, 3.7, 3.3.

The descriptive statistics of all four data sets are displayed in Table 5. From the tabulated value of skewness for these data sets, it is obvious that all four data sets are skewed to the right.

Table 5. Descriptive statistics for the chosen data sets

Data Sets	Min	Q_1	Median	Q_3	Max	Mean	Skewness	Kurtosis
Carbon Fiber	1.901	2.530	2.986	3.415	5.020	3.050	0.640	0.191
Relief Times	1.100	1.475	1.700	2.050	4.100	1.900	1.592	2.347
Taxes Revenue	4.100	8.450	10.600	16.850	39.200	13.490	1.568	2.079
Wind Speed	2.700	3.300	4.100	6.550	9.300	4.896	0.835	−0.6397

The values of MLE parameters and LL for the IW-R model, as well as the comparable models, are given in Table 6. The Akaike Information Criterion (AIC), the Second Order Estimate of the Akaike Information Criterion (AICc), the Bayesian Information Criterion (BIC), log-likelihood (LL), -2log–likelihood (-2LL), and Kolmogorov-Smirnov (KS) statistic, as well as the graphical illustration of the probability-probability (P-P) plot, Empirical Cumulative Distribution Function (ECDF), and Empirical Probability Distribution Function (EPDF) (see [48, 49] for details) for the chosen data sets, have been considered here to select the superior model and eventually displayed in Table 7 and Figures 6 and 7, respectively.

Table 6. Estimated parameter values with log-likelihood (LL) for different models

Data sets	Distribution	Parameter estimates	LL
Carbon Fiber	IW-R	$\alpha = 2.739,\ \theta = 1.244,\ \sigma = 1.844$	−59.291
	W-R	$a = 2.513,\ k = 10.925$	−62.967
	L-E	$\lambda = 4.434,\ b = 0.226$	−58.621
	E-R	$a = 0.103,\ b = 2.855 \times 10^{-8}$	−94.822
	P-E	l=1.500, a=1.385	−130.887
	EP-W	$a = 0.015,\ b = 3.768,\ \lambda = 0.800$	−59.450
	T-R	$\sigma = 1.818,\ \lambda = -1.000$	−76.818
	W	$a = 3.305,\ b = 5.027$	−62.967
	R	$\sigma = 2.200$	−94.822
Relief Times	IW-R	$\alpha = 2.009,\ \theta = 0.120,\ \sigma = 1.873$	−15.409
	W-R	$a = 1.394,\ k = 4.537$	−20.586
	L-E	$\lambda = 5.890,\ b = 0.571$	−16.477
	E-R	$a = 0.245,\ b = 2.242 \times 10^{-8}$	−22.479
	P-E	$l = 1.500,\ a = 1.332,\ \sigma = 6.411$	−31.556
	EP-W	$a = 0.254,\ b = 2.215,\ \lambda = 0.822$	−18.471
	T-R	$\sigma = 1.164,\ \lambda = -1.000$	−19.105
	W	$a = 2.130,\ b = 2.787$	−20.586
	R	$\sigma = 1.428$	−22.479
Taxes Revenue	IW-R	$\alpha = 1.123,\ \theta = 2.837,\ \sigma = 4.069$	−188.940
	W-R	$a = 0.920,\ k = 234.324$	−197.291
	L-E	$\lambda = 3.365,\ b = 0.088$	−190.632
	E-R	$a = 0.002,\ b = 1.867$	−194.953
	P-E	l=1.500, a=0.914	−212.547
	EP-W	$a = 0.030,\ b = 1.406,\ \lambda = 0.768$	−193.169
	T-R	$\sigma = 13.131,\ \lambda = 0.642$	−195.711
	W	$a = 15.305,\ b = 1.840$	−197.291
	R	$\sigma = 11.084$	−197.711

Table 6. (cont.)

Data sets	Distribution	Parameter estimates	LL
Wind Speed	IW-R	$\alpha = 1.653,\ \theta = 1.125,\ \sigma = 2.595$	-45.791
	W-R	$a = 1.320,\ k = 30.573$	-49.453
	L-E	$\lambda = 4.434,\ b = 0.226$	-47.880
	E-R	$a = 0.036,\ b = 1.289 \times 10^{-8}$	-50.903
	P-E	l=2626.900, aa=1.299	-60.450
	EP-W	$a = 0.054,\ b = 1.951,\ \lambda = 0.617$	-48.333
	T-R	$\sigma = 3.033,\ \lambda = -1.000$	-48.531
	W	a=5.530, b=2.641	-49.453
	R	$\sigma = 3.734$	-50.903

Table 7. Estimation of the criteria for choosing models

Data sets	Distribution	$-2LL$	AIC	AICc	BIC	KS	A	C-vM
Carbon Fiber	IW-R	118.582	124.582	124.982	131.059	0.083	3.646	0.204
	W-R	125.934	129.933	130.130	134.251	0.888	102.441	2.746
	L-E	117.242	121.242	121.439	125.560	0.565	27.152	1.419
	E-R	189.644	193.645	193.842	197.963	0.347	12.234	0.745
	P-E	261.774	265.775	265.971	270.092	0.507	11.350	0.697
	EP-W	118.900	124.899	125.299	131.376	0.095	4.006	0.215
	T-R	153.636	157.636	157.833	161.954	0.227	9.269	0.530
	W	125.934	129.933	130.130	134.251	0.084	4.833	0.263
	R	189.644	191.645	191.709	193.804	0.347	12.232	0.745
Relief Times	IW-R	30.818	36.817	38.317	39.805	0.083	1.615	0.147
	W-R	41.172	45.173	45.879	47.164	0.528	9.487	0.943
	L-E	32.954	36.953	37.659	38.945	0.111	1.676	0.181
	E-R	44.958	48.958	49.663	50.949	0.207	3.193	0.334
	P-E	63.132	67.133	67.839	69.124	0.444	4.176	0.461
	EP-W	36.942	42.943	44.443	45.930	0.165	2.609	0.280
	T-R	38.210	42.209	42.915	44.201	0.163	2.683	0.283
	W	41.172	45.173	45.879	47.164	0.185	3.037	0.323
	R	44.958	46.958	47.180	47.953	0.207	3.193	0.334
Taxes Revenue	IW-R	377.880	383.879	384.316	390.112	0.048	0.744	0.039
	W-R	394.582	398.581	398.795	402.736	0.825	21.168	0.707
	L-E	381.264	385.263	385.478	389.418	0.072	1.298	0.078
	E-R	389.906	393.905	394.120	398.060	0.102	1.749	0.097
	P-E	425.093	429.093	429.308	433.248	0.391	4.570	0.294
	EP-W	386.338	392.337	392.774	398.570	0.120	1.745	0.105
	T-R	391.422	395.422	395.636	399.577	0.130	1.890	0.111
	W	394.582	398.581	398.795	402.736	0.143	2.066	0.121
	R	395.422	397.422	397.492	399.499	0.172	2.436	0.147

Table 7. (cont.)

Data sets	Distribution	$-2LL$	AIC	AICc	BIC	KS	A	C-vM
Wind Speed	IW-R	91.582	97.583	98.783	101.117	0.111	1.510	0.128
	W-R	98.906	102.906	103.478	105.262	0.812	14.362	0.837
	L-E	95.760	99.760	100.331	102.116	0.154	1.667	0.157
	E-R	101.806	105.806	106.377	108.162	0.188	1.718	0.157
	P-E	120.899	124.899	125.471	127.255	0.443	3.037	0.307
	EP-W	96.666	102.667	103.867	106.201	0.204	1.966	0.187
	T-R	97.062	101.062	101.633	103.418	0.223	2.224	0.211
	W	98.906	102.906	103.478	105.262	0.223	2.027	0.193
	R	101.806	103.806	103.988	104.984	0.188	1.718	0.157

After examining all of the above analyses from Tables 6 and 7, it is clear that the results of the goodness of fit for the IW-R model are evident to have all the qualities of being the best-fitted model compared to the selected models for the above applied data sets. Except for wind speed data, for the remaining three data sets, the proposed model's some criterion values are slightly bigger than the L-E model, a larger portion of criterion and test statistics values show better outcomes than both L-E and other models. This can also be visually illustrated by the P-P plots from Figures 6–8, where all four data sets' empirical lines are significantly closer to the IW-R model's line as compared to the other three selected models. Therefore, the graphical illustration also agrees with the decision that the IW-R model captured the studied data sets more effectively than the competing models.

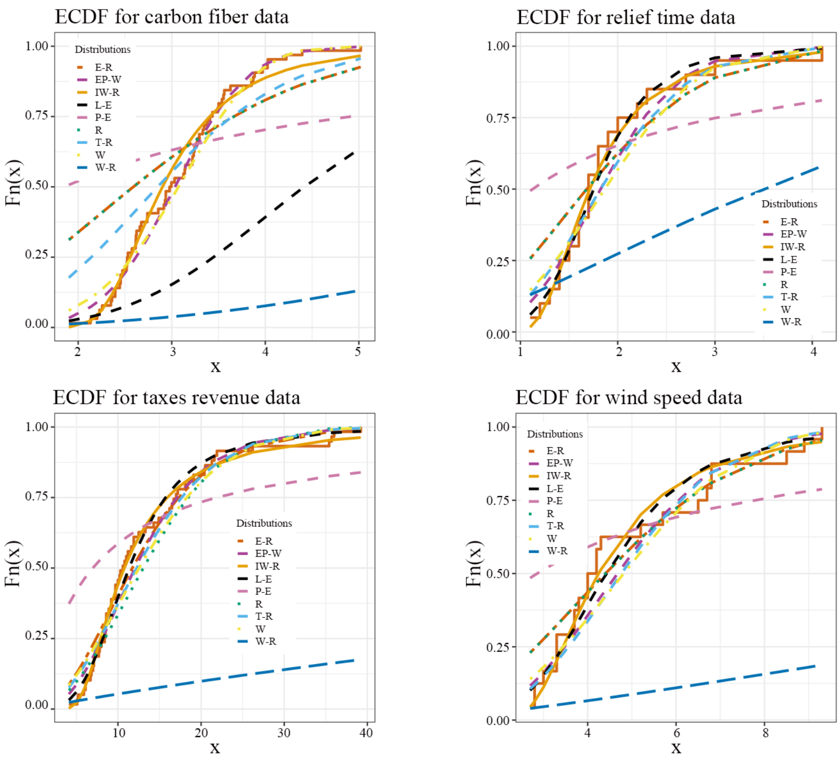

Figure 6. Plots illustrating the ECDF for four different data sets

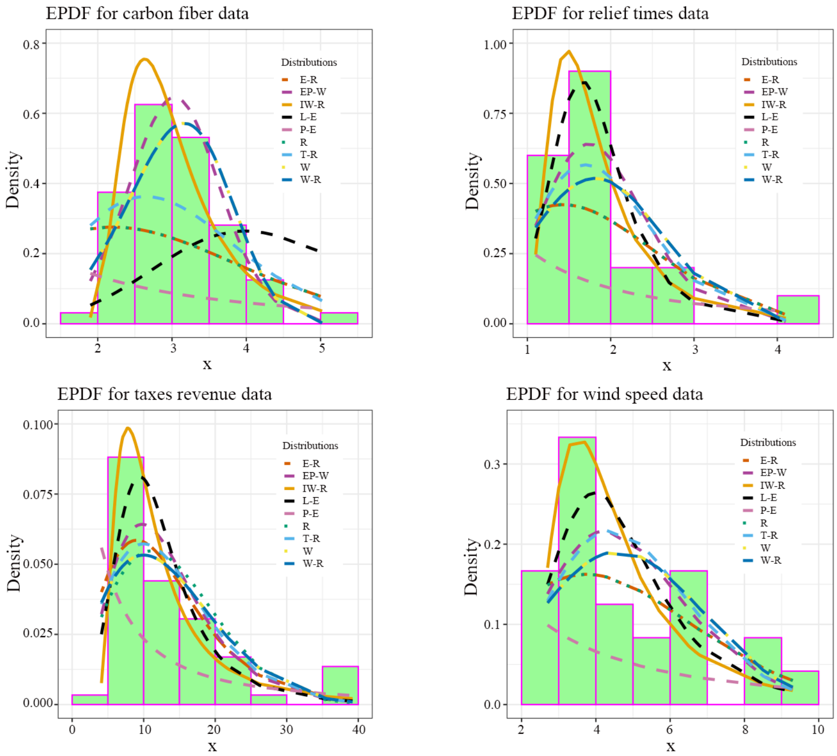

Figure 7. Plots illustrating the EPDF for four different data sets

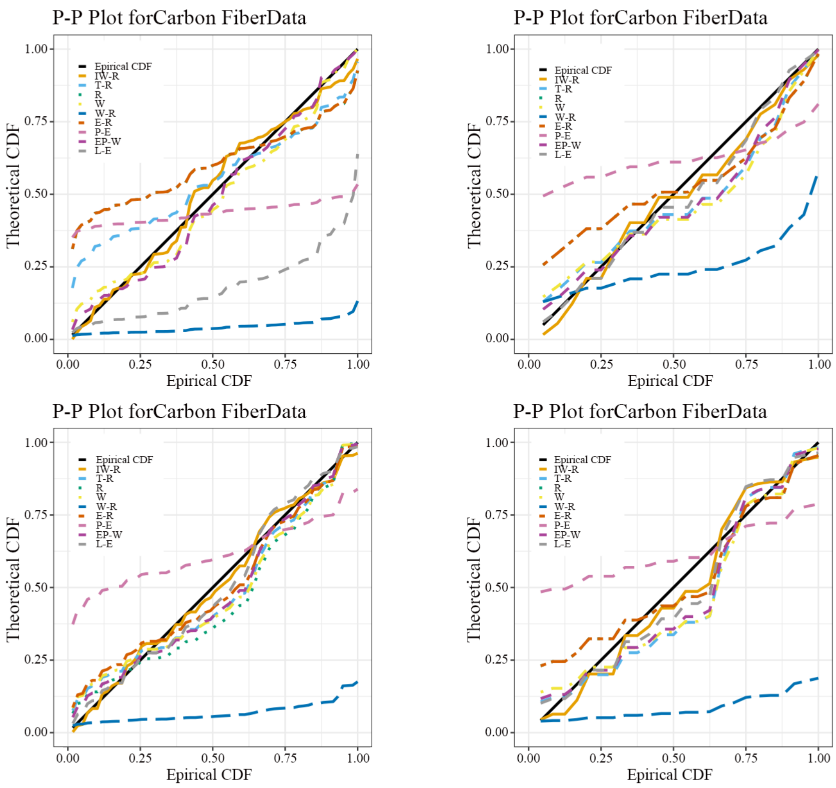

Figure 8. Plots illustrating the P–P plots for four different lifetime data sets

4. Concluding remarks

Through this study, a new framework has been developed from which several extended IW-based models can be generated. The proposed family has presented several distributional properties and characteristics suitable for modeling varying complex phenomena. Within this framework, the IW-R model has been investigated with suitable statistical properties along with estimation and simulation studies. Four real-world applications demonstrate the significance of the IW-X family and the modeling capability of the IW-R model. Therefore, the proposed technique makes a valuable contribution to the analysis and fitting of real-world data from various fields, including medicine, engineering, economics, and so on.

Acknowledgement

Authors have worked equally to write and review the manuscript.

Conflict of interest

The authors declare no competing financial interest.

References

[1] Pearson K. X. Contributions to the mathematical theory of evolution. II. Skew variation in homogeneous material. *Philosophical Transactions of the Royal Society of London A*. 1895; 186: 343–414. Available from: https://doi.org/10.1098/rsta.1895.0010.

[2] Burr IW. Cumulative frequency functions. *The Annals of Mathematical Statistics*. 1942; 13(2): 215–232. Available from: https://doi.org/10.1214/aoms/1177731607.

[3] Johnson NL. Systems of frequency curves generated by methods of translation. *Biometrika*. 1949; 36(1–2): 149–176. Available from: https://doi.org/10.1093/biomet/36.1-2.149.

[4] Fry TRL. *Univariate and Multivariate Burr Distributions: A Survey*. Monash University; 1989. Available from: https://doi.org/10.22004/ag.econ.266976.

[5] Ramberg JS, Schmeiser BW. An approximate method for generating symmetric random variables. *Communications of the ACM*. 1972; 15(11): 987–990. Available from: https://doi.org/10.1145/355606.361888.

[6] Ramberg JS, Schmeiser BW. An approximate method for generating asymmetric random variables. *Communications of the ACM*. 1974; 17(2): 78–82. Available from: https://doi.org/10.1145/360827.36084.

[7] Ramberg JS, Dudewicz EJ, Tadikamalla PR, Mykytka EF. A probability distribution and its uses in fitting data. *Technometrics*. 1979; 21(2): 201–214. Available from: https://doi.org/10.1080/00401706.1979.10489750.

[8] Tukey JW. *The Practical Relationship between the Common Transformations of Percentages of Counts and Amounts*. Princeton University; 1960.

[9] Azzalini A. A class of distributions which includes the normal ones. *Scandinavian Journal of Statistics*. 1985; 12(2): 171–178.

[10] Marshall AW, Olkin I. A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika*. 1997; 84(3): 641–652. Available from: https://doi.org/10.1093/biomet/84.3.641.

[11] Gilchrist W. *Statistical Modelling with Quantile Functions*. CRC Press; 2000. Available from: https://doi.org/10.1201/9781420035919.

[12] Eugene N, Lee C, Famoye F. Beta-normal distribution and its applications. *Communications in Statistics - Theory and Methods*. 2002; 31(4): 497–512. Available from: https://doi.org/10.1081/STA-120003130.

[13] Nadarajah S, Kotz S. The beta Gumbel distribution. *Mathematical Problems in Engineering*. 2004; 2004(4): 323–332. Available from: https://doi.org/10.1155/S1024123X04403068.

[14] Nadarajah S, Kotz S. The beta exponential distribution. *Reliability Engineering and System Safety*. 2006; 91(6): 689–697. Available from: https://doi.org/10.1016/j.ress.2005.05.008.

[15] Lee C, Famoye F, Olumolade O. Beta-Weibull distribution: some properties and applications to censored data. *Journal of Modern Applied Statistical Methods*. 2007; 6(1): 17–28. Available from: https://doi.org/10.22237/jmasm/1177992960.

[16] Kong L, Lee C, Sepanski JH. On the properties of beta-gamma distribution. *Journal of Modern Applied Statistical Methods*. 2007; 6(1): 18–27. Available from: https://doi.org/10.22237/jmasm/1177993020.

[17] Akinsete A, Famoye F, Lee C. The beta-Pareto distribution. *Statistics*. 2008; 42(6): 547–563. Available from: https://doi.org/10.1080/02331880801983876.

[18] Shaw WT, Buckley IR. *The alchemy of probability distributions: beyond Gram-Charlier and Cornish-Fisher expansions, and skew-normal or kurtotic-normal distributions*. 2007. Available from: https://library.wolfram.com/infocenter/Articles/6670/alchemy.pdf [Accessed 3 March 2026].

[19] Aryal GR, Tsokos CP. On the transmuted extreme value distribution with application. *Nonlinear Analysis: Theory, Methods and Applications*. 2009; 71(12): e1401–e1407. Available from: https://doi.org/10.1016/j.na.2009.01.168.

[20] Aryal GR, Tsokos CP. Transmuted Weibull distribution: a generalization of the Weibull probability distribution. *European Journal of Pure and Applied Mathematics*. 2011; 4(2): 89–102.

[21] Aryal GR. Transmuted log-logistic distribution. *Journal of Statistics Applications and Probability*. 2013; 2(1): 11–24.

[22] Ashour SK, Eltehiwy MA. Transmuted Lomax distribution. *American Journal of Applied Mathematics and Statistics*. 2013; 1(6): 121–127. Available from: https://doi.org/10.12691/ajams-1-6-3.

[23] Jones MC. Kumaraswamy's distribution: a beta-type distribution with some tractability advantages. *Statistical Methodology*. 2009; 6(1): 70–81. Available from: https://doi.org/10.1016/j.stamet.2008.04.001.

[24] Cordeiro GM, de Castro M. A new family of generalized distributions. *Journal of Statistical Computation and Simulation*. 2011; 81(7): 883–898. Available from: https://doi.org/10.1080/00949650903530745.

[25] Kumaraswamy P. A generalized probability density function for double-bounded random processes. *Journal of Hydrology*. 1980; 46(1–2): 79–88. Available from: https://doi.org/10.1016/0022-1694(80)90036-0.

[26] Cordeiro GM, Ortega EMM, Nadarajah S. The Kumaraswamy Weibull distribution with application to failure data. *Journal of the Franklin Institute*. 2010; 347(8): 1399–1429. Available from: https://doi.org/10.1016/j.jfranklin.2010.06.010.

[27] De Castro MAR, Ortega EMM, Cordeiro GM. The Kumaraswamy generalized gamma distribution with application in survival analysis. *Statistical Methodology*. 2011; 8(5): 411–433. Available from: https://doi.org/10.1016/j.stamet.2011.04.001.

[28] Cordeiro GM, Pescim RR, Ortega EMM. The Kumaraswamy generalized half-normal distribution for skewed positive data. *Journal of Data Science*. 2012; 10(2): 195–224. Available from: https://doi.org/10.6339/JDS.2012.10(2).1010.

[29] Alzaatreh A, Lee C, Famoye F. A new method for generating families of continuous distributions. *Metron*. 2013; 71(1): 63–79. Available from: https://doi.org/10.1007/s40300-013-0007-y.
31

[30] Alzaatreh A, Ghosh I. On the Weibull-X family of distributions. *Journal of Statistical Theory and Applications*. 2015; 14(2): 169–183. Available from: https://doi.org/10.2991/jsta.2015.14.2.5.

[31] Tahir MH, Cordeiro GM, Alzaatreh A, Mansoor M, Zubair M. The logistic-X family of distributions and its applications. *Communications in Statistics-Theory and Methods*. 2016; 45(24): 7326–7349. Available from: https://doi.org/10.1080/03610926.2014.980516.
32

[32] Huo X, Khosa SK, Ahmad Z, Almaspoor Z, Ilyas M, Aamir M. A new lifetime exponential-X family of distributions with applications to reliability data. *Mathematical Problems in Engineering*. 2020; 2020(1): 1–16. Available from: https://doi.org/10.1155/2020/1316345.

[33] Rana MS, Shahbaz SH, Shahbaz MQ, Rahman MM. Pareto-Weibull distribution with properties and applications: a member of Pareto-X family. *Pakistan Journal of Statistics and Operation Research*. 2022; 18(1): 121–132. Available from: https://doi.org/10.18187/pjsor.v18i1.3821.

[34] Shah Z, Khan DM, Khan I, Ahmad B, Jeridi M, Al-Marzouki S. A novel flexible exponent power-X family of distributions with applications to COVID-19 mortality rate in Mexico and Canada. *Scientific Reports*. 2024; 14(1): 8992. Available from: https://doi.org/10.1038/s41598-024-59720-1.

[35] Keller AZ, Kamath ARR, Perera UD. Reliability analysis of CNC machine tools. *Reliability Engineering*. 1982; 3(6): 449–473. Available from: https://doi.org/10.1016/0143-8174(82)90036-1.

[36] Okasha HM, El-Baz AH, Tarabia AMK, Basheer AM. Extended inverse Weibull distribution with reliability application. *Journal of the Egyptian Mathematical Society*. 2017; 25(3): 343–349. Available from: https://doi.org/10.1016/j.joems.2017.02.006.

[37] Wickham H. *ggplot2: Elegant Graphics for Data Analysis*. Springer; 2016. Available from: https://doi.org/10.1007/978-3-319-24277-4.

[38] Bolker B. *Package 'bbmle': Tools for General Maximum Likelihood Estimation*. Available from: https://cran.r-project.org/package=bbmle [Accessed 3 March 2026].

[39] Morris TP, White IR, Crowther MJ. Using simulation studies to evaluate statistical methods. *Statistics in Medicine*. 2019; 38(11): 2074–2102. Available from: https://doi.org/10.1002/sim.8086.

[40] Pearson K. The problem of the random walk. *Nature*. 1905; 72(1867): 342. Available from: https://doi.org/10.1038/072294b0.

[41] Bader MG, Priest AM. Statistical aspects of fibre and bundle strength in hybrid composites. *Progress in Science and Engineering of Composites*. 1982: 1129–1136. Available from: https://doi.org/10.5281/zenodo.15402866.

[42] Joshi RK, Kumar V. Poisson inverse Weibull distribution with theory and applications. *International Journal of Statistics and Systems*. 2021; 16(1): 1–16.

[43] Gross AJ, Clark VA. *Survival Distributions: Reliability Applications in the Biomedical Sciences*. John Wiley and Sons; 1976.

[44] Afify AZ, Altun E, Alizadeh M, Ozel G, Hamedani GG. The odd exponentiated half-logistic-G family: properties, characterizations and applications. *Chilean Journal of Statistics*. 2017; 8(2): 65–91.

[45] Nassar MM, Nada NK. The beta generalized Pareto distribution. *Journal of Statistics: Advances in Theory and Applications*. 2011; 6(1–2): 1–17.

[46] Owoloko EA, Oguntunde PE, Adejumo AO. Performance rating of the transmuted exponential distribution: an analytical approach. *SpringerPlus*. 2015; 4: 1–15. Available from: https://doi.org/10.1186/s40064-015-1590-6.

[47] Seguro JV, Lambert TW. Modern estimation of the parameters of the Weibull wind speed distribution for wind energy analysis. *Journal of Wind Engineering and Industrial Aerodynamics*. 2000; 85(1): 75–84. Available from: https://doi.org/10.1016/S0167-6105(99)00122-1.

[48] Brewer MJ, Butler A, Cooksley SL. The relative performance of AIC, AICC and BIC in the presence of unobserved heterogeneity. *Methods in Ecology and Evolution*. 2016; 7(6): 679–692. Available from: https://doi.org/10.1111/2041-210X.12541.

[49] Evans DL, Drew JH, Leemis LM. The distribution of the Kolmogorov-Smirnov, Cramer-von Mises, and Anderson-Darling test statistics for exponential populations with estimated parameters. *Computational Probability Applications*. 2017: 165–190. Available from: https://doi.org/10.1007/978-3-319-43317-2_13.

Appendix

The Hessian matrix is given as

$$H=\begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix},$$

where the variance-covariance matrix V is obtained by

$$V=\begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix}=\begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \\ H_{31} & H_{32} & H_{33} \end{pmatrix}^{-1},$$

with the elements of the Hessian matrix obtained as

$$H_{11}=\sum_{i=1}^{n}\left\{2^{\alpha}\theta\log(2)^{2}\left(\frac{x_i^2}{\sigma^2}\right)^{-\alpha}+2^{\alpha}\theta\left(\frac{x_i^2}{\sigma^2}\right)^{-\alpha}\times\log\left(\frac{x_i^2}{\sigma^2}\right)^{2}-2^{\alpha+1}\theta\log(2)\left(\frac{x_i^2}{\sigma^2}\right)^{-\alpha}\log\left(\frac{x_i^2}{\sigma^2}\right)\right\}+\frac{n}{\alpha^2},$$

$$H_{12}=2^{\alpha}\sum_{i=1}^{n}\left\{\log(2)\left(\frac{x_i^2}{\sigma^2}\right)^{-\alpha}-\left(\frac{x_i^2}{\sigma^2}\right)^{-\alpha}\log\left(\frac{x_i^2}{\sigma^2}\right)\right\},$$

$$H_{13}=\sum_{i=1}^{n}\left\{\frac{2^{\alpha+1}\theta x_i^2\left(\frac{x_i^2}{\sigma^2}\right)^{-(\alpha+1)}}{\sigma^3}+\frac{2^{\alpha+1}\alpha\theta\log(2)x_i^2\left(\frac{x_i^2}{\sigma^2}\right)^{-(\alpha+1)}}{\sigma^3}-\frac{2^{\alpha+1}\alpha\theta x_i^2\left(\frac{x_i^2}{\sigma^2}\right)^{-(\alpha+1)}\log\left(\frac{x_i^2}{\sigma^2}\right)}{\sigma^3}\right\}-\frac{2n}{\sigma},$$

$$H_{22}=\frac{n}{\theta^2},$$

$$H_{23}=\frac{2^{\alpha+1}\alpha}{\sigma^3}\sum_{i=1}^{n}x_i^2\left(\frac{x_i^2}{\sigma^2}\right)^{-(\alpha+1)}, \text{ and}$$

$$H_{33}=2^{\alpha}\sum_{i=1}^{n}\theta\left(\frac{4(\alpha+1)\alpha x_i^4\left(\frac{x_i^2}{\sigma^2}\right)^{-(\alpha+2)}}{\sigma^6}-\frac{6\alpha x_i^2\left(\frac{x_i^2}{\sigma^2}\right)^{-(\alpha+1)}}{\sigma^4}\right)+\frac{2(\alpha+1)n}{\sigma^2}-\frac{2n}{\sigma^2}.$$