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Research Article

Characterizing the Solvability of Bipolar Equations in the Unit Interval

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Abstract: Bipolar equations are the base of bipolar fuzzy relation equations, a modified version of fuzzy relation equations involving a negation operator. This paper characterizes the solvability of bipolar equations defined with an adjoint pair in the unit interval from three different perspectives: the case of a bijective negation; the case of a continuous negation; and the case of an adjoint conjunctor with continuous and injective partial mappings. Ample examples illustrate the usefulness and simplicity of the corresponding characterizations.

***Keywords*:** bipolar equation, bipolar fuzzy relation equation (BFRE), negation operator, adjoint pair, unit interval, solvability

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1. Introduction

Sanchez's pioneering works applied the concept of fuzzy relation equation to determine an unknown fuzzy relation representing the degree to which certain syndromes are associated with the clinical characteristics of some diseases [1, 2]. More specifically, the unknown relation models how the presence of a syndrome indicates the presence of a disease in a patient. In this applied context, a bipolar fuzzy relation equation (BFRE) would model not only how the presence, but also how the absence of a syndrome, are indicative of the existence of a disease in a patient.

The initial formal definition of BFRE was proposed by Freson et al. [3] in terms of a max-min composition and the standard negation, whose resolution has been deeply studied afterwards [3–5]. This also applies to other kinds of BFRE defined with particular max-T compositions and the standard negation, like max-product BFRE [6] and max-Łukasiewicz BFRE [7, 8], and also to max-product BFRE defined with the product negation [9–11]. Additionally, there exist three works which analyse the resolution of families of BFRE, namely max-archimedean BFRE with the standard negation [12], BFRE defined in residuated lattices with an involutive negation [13] and bipolar multi-adjoint relation equations defined in multi-adjoint lattices with an involutive negation [14]. From an applied perspective, BFRE have had a significant impact on the field of optimization problems, where BFRE are considered as constraints [15–18].

To date, all theoretical published studies devoted to solving BFRE impose conditions on both, the composition and the negation, mostly assuming a specific operator. The present manuscript addresses the initial steps for solving BFRE requiring hypotheses exclusively on the max-composition or exclusively on the negation operator, thus significantly widening the range of BFRE that can be solved. Note that, by default, the max-composition involved in a BFRE is assumed to be related to an adjoint conjunctor, as in fuzzy relation equations [19–21].

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Similarly to bipolar multi-adjoint relation equations [14], BFRE are equivalent to systems of bipolar sup-equations, or simply bipolar equations. Therefore, the resolution of BFRE is closely related to solving bipolar equations, since a necessary condition for the solvability of a BFRE is that all the bipolar equations that comprise it are solvable. It must be stressed that the majority of the aforementioned works directly address a unique bipolar equation under the term BFRE. In this paper, we make use of the term bipolar equation to emphasize the fact that the present study handles a single bipolar equation and not a system of such equations, for which we reserve the term BFRE.

To be precise, in this work, we will focus on characterizing the solvability of bipolar equations defined in the unit interval, analysing three independent cases:

• The case of a bijective negation.

• The case of a continuous, or equivalently surjective, negation.

• The case of an injective continuous conjunctor.

By developing a dual approach, we will show that the resulting characterizations are valid regardless of whether the unknown variables appear on the right-hand side of the adjoint conjunctor or on the left-hand side.

The relevance of such characterizations are highlighted by two main reasons. On the one hand, the imposed conditions significantly weaken the requirements assumed so far in the literature of bipolar equations. On the other hand, in all cases, the characterization of the solvability of a bipolar equation simply relies on checking whether two particular tuples are solutions of the bipolar equation.

2. Preliminaries

Firstly, concepts related to generalized logical operators will be introduced as well as some properties of these operators. Subsequently, we present mathematical analysis results on the continuity of non-increasing mappings, which will be applied afterwards.

2.1 *Generalized logical operators*

Definition 1 [22] Given two mappings $\&, \nwarrow \colon [0,1] \times [0,1] \to [0,1]$, we say that $\nwarrow$ is the right adjoint implication of $\&$ and $(\&, \nwarrow)$ is a right adjoint pair if, for any $x, y, z \in [0,1]$, the right adjoint property is satisfied, that is:

$$y \leq z \nwarrow x \quad \text{if and only if} \quad x \& y \leq z.$$

It is important to emphasize that, in the literature, the term residuated pair is frequently used instead of adjoint pair. Nevertheless, the concept of residuated pair generally presupposes commutativity in the operator $*$. As we do not require this property of commutativity in the operator $*$, we prefer to employ the term adjoint pair.

The following result shows some properties of an arbitrary right adjoint pair. In particular, a boundary condition, a monotonicity property and the uniqueness of the right adjoint implication are presented [23].

Proposition 1 Let $(\&, \nwarrow)$ be a right adjoint pair, then:

• $x \& 0 = 0$ for all $x \in [0,1]$.

• $\&$ and $\nwarrow$ are order-preserving on the second and first argument, respectively.

• The right adjoint implication of $\&$ is unique and, for all $x, z \in [0,1]$:

$$z \nwarrow x = \max\{y \in [0,1] \mid x \& y \leq z\}.$$

Example 1 The product right adjoint pair $(*, \nwarrow_P)$ is formed by the following operators:

$$x * y = x \cdot y \qquad \text{(product operator)}$$

$$z \nwarrow_P x = \begin{cases} 1 & \text{if } x \leq z \\ \frac{z}{x} & \text{if } z < x \end{cases} \qquad \text{(right adjoint implication of } *\text{).}$$

Definition 2 Let $\neg\colon [0,1] \to [0,1]$ be a mapping. We say that $\neg$ is a negation operator if it is order-reversing and $\neg(0) = 1$, $\neg(1) = 0$. Moreover, $\neg$ is an involutive negation if $\neg \circ \neg(x) = x$ for all $x \in [0,1]$.

In the absence of any ambiguity, we will consider $\neg x$ instead of $\neg(x)$. Next, we show common negation operators considered in the literature [6, 24].

Example 2 The mappings $\neg_S, \neg_Y, \neg_P\colon [0,1] \to [0,1]$, given as follows, are negation operators:

$$\neg_S x = 1 - x \qquad \text{(Standard negation)}$$

$$\neg_P x = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x > 0 \end{cases} \qquad \text{(Product negation)}$$

$$\neg_Y x = \sqrt{1 - x^2} \qquad \text{(Yager negation).}$$

2.2 *Mathematical analysis*

This section recalls some basic real analysis results, such as the Intermediate Value Theorem. Furthermore, we prove that, for a negation operator, being continuous is equivalent to being surjective. This result will be particularly useful in the subsequent sections.

Firstly, we present a well-known analysis result of continuous functions [25] (English translation in [26]), which yields to a sufficient condition for the surjectivity of a negation operator.

Theorem 1 (Intermediate Value Theorem) Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. There exists $c \in]a,b[$ such that $f(c) = y$ for each $y \in]\min\{f(a), f(b)\}, \max\{f(a), f(b)\}[$.

Corollary 1 Let $\neg\colon [0,1] \to [0,1]$ be a negation operator. If the mapping $\neg$ is continuous, then it is surjective.

Proof. Let $\neg\colon [0,1] \to [0,1]$ be a continuous negation operator. Since $\neg 0 = 1$ and $\neg 1 = 0$, applying Intermediate Value Theorem 1, we conclude that for any value $y \in]0,1[$, there exists $c \in]0,1[$ satisfying $\neg c = y$. Therefore, $\neg$ is surjective. $\square$

The subsequent stage is to determine whether the converse implication of Corollary 1 is true. That is, if a surjective negation operator is necessarily continuous, which means that for a negation operator the properties of continuity and surjectivity are equivalent.

This converse statement of Corollary 1 can be proven as a consequence of a well known result in mathematical analysis, which states that if $f\colon [a,b] \to \mathbb{R}$ is a monotonic function satisfying the Darboux's (or intermediate value) property, then it is continuous. Nevertheless, as far as we know, this result always appears in the literature as a proposed exercise [27, 28].

Therefore, for the sake of self-containment, we will derive this result as a corollary of the next lemma.

Lemma 1 If $f\colon [a,b] \to \mathbb{R}$ is a non-increasing, non-constant mapping and $[f(b), f(a)] = f([a,b])$, then f is continuous.

Proof. Let $x_0 \in [a,b]$, $\varepsilon > 0$. Consider the values $\varepsilon_1, \varepsilon_2 \in \mathbb{R}^+$ given by the following expressions:

$$\varepsilon_1 = \min\{\varepsilon, |f(x_0) - f(a)|\}$$

$$\varepsilon_2 = \min\{\varepsilon, |f(x_0) - f(b)|\}.$$

Notice that, since f is a non-constant mapping, either $\varepsilon_1 \neq 0$ or $\varepsilon_2 \neq 0$. As a consequence:

$$]f(x_0)-\varepsilon_2, f(x_0)+\varepsilon_1[\subseteq [f(b), f(a)].$$

Now, since $[f(b), f(a)] = f([a,b])$, there exist two values $c_1, c_2 \in [a,b]$ verifying:

$$f(c_1) = f(x_0)+\varepsilon_1$$

$$f(c_2) = f(x_0)-\varepsilon_2.$$

Because either $\varepsilon_1 \neq 0$ or $\varepsilon_2 \neq 0$, $f(c_2) < f(c_1)$, which means that $c_1 \neq c_2$. Moreover, from $f(c_2) < f(c_1)$ and the non-increasing monotonicity of the mapping f, we deduce that $c_1 < c_2$. In addition, since $f(c_2) \leq f(x_0) \leq f(c_1)$, as a consequence of the monotonicity of f, it holds that $x_0 \in [c_1, c_2]$. Now, we define the value $\delta > 0$ as:

$$\delta = \begin{cases} |x_0 - c_1| & \text{if} \quad x_0 = c_2, \\ |x_0 - c_2| & \text{if} \quad x_0 = c_1, \\ \min\{|x_0 - c_1|, |x_0 - c_2|\} & \text{if} \quad x_0 \neq c_1, c_2. \end{cases}$$

Due to the definition of $\delta > 0$, $x \in [c_1, c_2]$ for all $x \in]x_0 - \delta, x_0 + \delta[\cap[a,b]$. Therefore, since f is non-increasing, $f(x) \in [f(x_0)-\varepsilon_2, f(x_0)+\varepsilon_1]$, from which

$$|f(x) - f(x_0)| \leq \max\{\varepsilon_1, \varepsilon_2\} \leq \varepsilon.$$

□

As a consequence of Lemma 1, we obtain that surjectivity is sufficient for the continuity of a negation operator.

Corollary 2 Let $\neg\colon [0,1] \to [0,1]$ be a negation operator. If the mapping $\neg$ is surjective, then it is continuous.

Proof. Let $\neg\colon [0,1] \to [0,1]$ be a surjective negation operator. Since $\neg$ is a negation operator, it is $\neg(1) = 0$ and $\neg(0) = 1$, therefore, $\neg$ is a non-constant mapping. On the other hand, since $\neg$ is surjective, we deduce that $\neg([0,1]) = [0,1] = [\neg(1), \neg(0)]$. Hence, by Lemma 1, $\neg$ is a continuous mapping. □

Therefore, according to Corollaries 1 and 2, we obtain the following equivalence.

Corollary 3 Let $\neg\colon [0,1] \to [0,1]$ be a negation operator. Then, $\neg$ is continuous if and only if $\neg$ is surjective.

3. Bipolar equations

This section introduces the notion of bipolar equation and presents some technical results. Furthermore, for the particular case of the unit interval, we show the characterization of solvability for bipolar equations presented in Theorem 6 of [14], which generalizes the characterization previously introduced in [13]. As far as we know, the characterization presented in [14] is the most general characterization of solvability for bipolar equations in the literature. It must be stressed that [14] extends the characterization presented in [13] for the case of bipolar multi-adjoint sup-equations, weakening the hypothesis of the underlying adjoint pair.

As usual in the literature, bipolar equations are defined in terms of a negation operator and a right adjoint pair.

Definition 3 Let $(*, \nwarrow)$ be a right adjoint pair and $\neg$ a negation operator. A bipolar equation is of the form:

$$\bigvee_{j=1}^{m} (a_j^+ * x_j) \vee (a_j^- * \neg x_j) = b. \tag{1}$$

where $x_1, \ldots, x_m$ are unknown.

In the literature, a bipolar equation of the form (1) is occasionally referred to as a BFRE. However, from a formal perspective, a BFRE corresponds to a set of systems of bipolar equations. For this reason, we employ here the term bipolar equation for (1). Notwithstanding, solving bipolar equations plays a crucial role in the resolution of BFRE.

In what follows, we present some technical properties of bipolar equations and necessary conditions for a tuple to be a solution.

Proposition 2 If a tuple $(x_1, \ldots, x_m) \in [0,1]^m$ is a solution of (1), then $x_j \leq b \nwarrow a_j^+$ for all $j \in \{1, \ldots, m\}$.

Proof. By definition of supremum, $a_j^+ * x_j \leq b$ for all $j \in \{1, \ldots, m\}$. Due to Proposition 1, for all $j \in \{1, \ldots, m\}$, we deduce that:

$$b \nwarrow a_j^+ = \max\{x \in [0,1] \mid a_j^+ * x \leq b\}$$

which yields to $x_j \leq b \nwarrow a_j^+$. □

Lemma 2 If (1) is solvable, then $a_j^- * \neg(b \nwarrow a_j^+) \leq b$ for all $j \in \{1, \ldots, m\}$.

Proof. Let $(x_1, \ldots, x_m) \in [0,1]^m$ be a solution of (1). For all $j \in \{1, \ldots, m\}$, from Proposition 2 we obtain that $x_j \leq b \nwarrow a_j^+$. Moreover, since $\neg$ is order-reversing, applying $\neg$ in both sides of the inequality, we deduce that $\neg(b \nwarrow a_j^+) \leq \neg x_j$. In addition, due to the order-preserving monotonicity of the operator $*$, we can operate with the value a_j^- in both sides of the previous inequality and obtain that:

$$a_j^- * \neg(b \nwarrow a_j^+) \leq a_j^- * \neg x_j.$$

Now, since $(x_1, \ldots, x_m)$ is a solution of (1), it is $a_j^- * \neg x_j \leq b$. Therefore, we have proven that:

$$a_j^- * \neg(b \nwarrow a_j^+) \leq a_j^- * \neg x_j \leq b.$$

□

The subsequent result establishes a sufficient condition under which it can be asserted that $(b \nwarrow a_1^+, \ldots, b \nwarrow a_m^+)$ is a solution of (1).

Lemma 3 If (1) is solvable and there exists $k \in \{1, \ldots, m\}$ such that $a_k^+ * (b \nwarrow a_k^+) = b$, then $(b \nwarrow a_1^+, \ldots, b \nwarrow a_m^+)$ is a solution of (1).

Proof. Due to Proposition 1 and Lemma 2, we deduce that:

$$\bigvee_{j=1}^{m} (a_j^+ * (b \nwarrow a_j^+)) \vee (a_j^- * \neg(b \nwarrow a_j^+)) \leq b.$$

Moreover, since $a_k^+ * (b \nwarrow a_k^+) = b$, we conclude that $(b \nwarrow a_1^+, \ldots, b \nwarrow a_m^+)$ is a solution of (1). □

In what follows, we show a simplified version of the characterization presented in [14] for the solvability of bipolar multi-adjoint sup-equations, which has been adapted to bipolar equations defined in the unit interval. In order to present this version, we will introduce some preliminary notions.

Definition 4 A mapping $f\colon [0,1] \to [0,1]$ is a homomorphism if $f(\sup A) = \sup f(A)$ and $f(\inf A) = \inf f(A)$, for all $A \in \mathscr{P}([0,1]) \setminus \{\varnothing\}$.

Definition 5 A mapping $*\colon [0,1] \times [0,1] \to [0,1]$ is a right homomorphism if the mappings $*_x\colon [0,1] \to [0,1]$ given by $*_x(z) = x * z$ with $z \in [0,1]$ are homomorphisms.

With the aim of presenting the aforementioned characterization of solvability, the following notation is introduced:

$$g^+ = (b \nwarrow a_1^+, \ldots, b \nwarrow a_m^+)$$

$$g^- = (b \nwarrow a_1^-, \ldots, b \nwarrow a_m^-)$$

$$\neg g^- = (\neg(b \nwarrow a_1^-), \ldots, \neg(b \nwarrow a_m^-)).$$

Now, we recall a characterization of the solvability of bipolar equations in terms of g^+ and $\neg g^-$ presented in [14], for the specific case of the unit interval, an involutive negation and $*$ being a right homomorphism with right unital element 1, that is, $x * 1 = x$ for each $x \in [0,1]$.

Theorem 2 [14] Let $(*, \nwarrow)$ be a right adjoint pair such that $*$ is a right homomorphism with right unital element 1 and let $\neg\colon [0,1] \to [0,1]$ be an involutive negation. Then, (1) is solvable if and only if g^+ or $\neg g^-$ is a solution.

Theorem 2 presents a simple solvability characterization for a bipolar equation. Nevertheless, it should be noted that the hypotheses are imposed on both the right adjoint pair and the negation operator.

From now on, we will focus on characterizing the solvability of bipolar equations under weaker conditions than those established in Theorem 2. Furthermore, these conditions will be imposed exclusively either on the negation operator or on the right adjoint pair, thereby ensuring that simultaneous conditions are not imposed on both operators.

4. Characterizing the solvability in terms of the negation operator

The objective of this section is to characterize the solvability of a bipolar equation imposing conditions only on the negation operator. As it is the case in Theorem 2, we aim at stating such characterization in terms of two tuples. Following the approach previously taken in [13, 14], the idea comes from splitting the cases when the independent term is reached by a positive coefficient a_j^+ and when it is obtained by a negative coefficient a_j^-.

According to the literature on fuzzy relation equations [19–21], the tuple g^+ characterizes the solvability of such equations. In the bipolar setting, this would justify the use of g^+ to represent the case when the independent term is reached by a positive coefficient. In fact, if a solution of a bipolar equation reaches the independent term in an expression of the form $a_j^+ * x_j = b$, Proposition 1 together with Lemma 3 guarantee that g^+ is a solution of the bipolar equation.

Now, we will address the case in which the independent term is reached in a negative coefficient a_j^-. More specifically, the goal is finding a tuple $(x_1, \ldots, x_m)$ which represents the case when the independent term is obtained in an expression of the form $a_j^- * \neg x_j = b$. Clearly, the component x_j of such a tuple necessary belongs to the following set:

$$\{x \in [0,1] \mid a_j^- * \neg x = b\}$$

and therefore to the set:

$$\{x \in [0,1] \mid a_j^- * \neg x \leq b\}.$$

Moreover, if the pursued tuple is a candidate to be a solution of the bipolar equation, the same applies to the remaining components, that is, for all $k \in \{1, \ldots, m\}$, the component x_k of the tuple belongs to the set:

$$\{x \in [0,1] \mid a_k^- * \neg x \leq b\}.$$

Finally, among all possible elements of the previous sets, we will choose the best candidate which may lead to a solution. This means that the expression $a_k^+ * x$ cannot surpass the element b. Therefore, by the non-decreasing monotonicity of the operator $*$, the optimal candidate for the component x_k is given by:

$$\inf\{x \in [0,1] \mid a_k^- * \neg x \leq b\}.$$

Note that, since the unit interval is a complete lattice, it is possible to consider the infimum of any subset and therefore the above expression is well defined.

Consequently, given a bipolar equation of the form (1), we define the following tuple:

$$g_\neg^- = (b \nwarrow_\neg a_1^-, \ldots, b \nwarrow_\neg a_1^-)$$

where:

$$b \nwarrow_\neg a_j^- = \inf\{x \in [0,1] \mid a_j^- * \neg x \leq b\}$$

for all $j \in \{1, \ldots, m\}$.

Note that, under the assumptions considered in Theorem 2, if $\neg$ is an involutive negation, then $\neg g^-$ coincides with $g_\neg^-$. Taking this into consideration, it seems natural to try to characterize the solvability of a bipolar equation in terms of their corresponding tuples g^+ and $g_\neg^-$. Nevertheless, in general, it is not possible to characterize the solvability of a bipolar equation exclusively in terms of the tuples g^+ and $g_\neg^-$. The following example illustrates this fact.

Example 3 Consider the product right adjoint pair $(*, \nwarrow_P)$ introduced in Example 1 and the negation operator $\neg\colon [0,1] \to [0,1]$ given as:

$$\neg x = \begin{cases} 1 & \text{if} \quad 0 \leq x \leq \frac{1}{4}, \\ \frac{3}{4} & \text{if} \quad \frac{1}{4} < x \leq \frac{1}{2}, \\ \frac{3}{2}(1-x) & \text{if} \quad \frac{1}{2} < x \leq 1. \end{cases} \tag{2}$$

whose graph is shown in Figure 1:

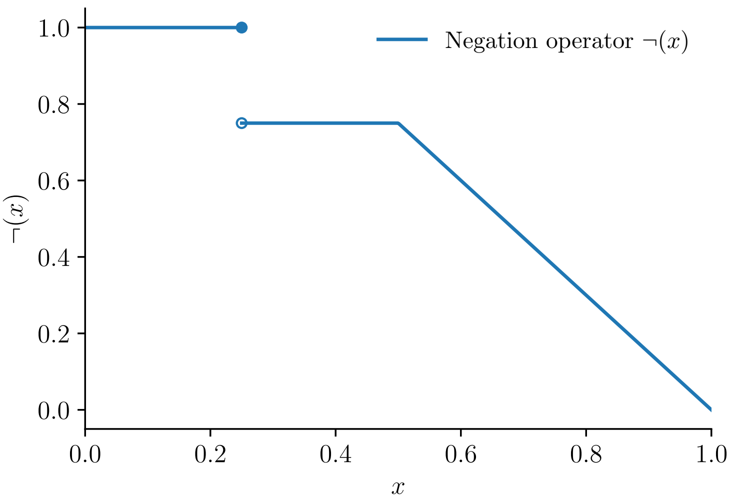

Figure 1. Graph of the negation operator $\neg$

Now, consider the following bipolar equation:

$$(0.2 * x_1) \vee (0.8 * \neg x_1) \vee (0.4 * x_2) \vee (0.1 * \neg x_2) \vee (0.5 * x_3) = 0.6. \tag{3}$$

A simple calculation yields the expression of the tuples g^+ and $g^-_\neg$:

$$g^+ = (0.6 \nwarrow_P 0.2, 0.6 \nwarrow_P 0.4, 0.6 \nwarrow_P 0.5) = (1, 1, 1)$$

$$g^-_\neg = (0.6 \nwarrow_\neg 0.8, 0.6 \nwarrow_\neg 0.1, 0.6 \nwarrow_\neg 0) = (0.25, 0, 0).$$

However, if the tuples g^+ and $g^-_\neg$ are evaluated in Equation (3), we obtain:

$$(0.2 * 1) \vee (0.8 * \neg 1) \vee (0.4 * 1) \vee (0.1 * \neg 1) \vee (0.5 * 1) = 0.5$$

$$(0.2 * 0.25) \vee (0.8 * \neg 0.25) \vee (0.4 * 0) \vee (0.1 * \neg 0) \vee (0.5 * 0) = 0.8.$$

Therefore, none of the tuples g^+ or $g^-_\neg$ is a solution of Equation (3). However, if we evaluate the tuple $(0.4, 0, 0)$ in Equation (3) we obtain that:

$$(0.2 * 0.4) \vee (0.8 * \neg 0.4) \vee (0.4 * 0) \vee (0.1 * \neg 0) \vee (0.5 * 0) = 0.6.$$

That is, the tuple $(0.4, 0, 0)$ is a solution of Equation (3).

Based on Example 3, we deduce that, in general, it is not possible to conclude that a bipolar equation is solvable if and only if some of the tuples g^+ or $g^-_\neg$ is a solution. Therefore, it requires assuming additional hypotheses on the negation operator $\neg$ to fulfil such characterization. In this sense, we will address the consideration of additional hypotheses from two different perspectives, presenting an algebraic approach and an analytical approach. The interests of this distinction relies on the fact that the algebraic approach is likely to be generalized to lattice domains different from the unit interval $[0,1]$ in the future, whilst the conditions assumed in the analytical approach are actually weaker than those in the algebraic approach, but the former only apply for real-valued domains like $[0,1]$.

In order to introduce the main results of this section, we firstly establish two interesting properties of the tuple $g^-_\neg$. In what follows, the argument $j \in \{1,\dots,m\}$ of the tuple $g^-_\neg$ is denoted as $g^-_{\neg,j}$.

Proposition 3 If a tuple $(x_1,\dots,x_m)$ is a solution of (1), then $g^-_{\neg,j} \leq x_j$ for all $j \in \{1,\dots,m\}$.

Proof. Let $(x_1,\dots,x_m)$ be a tuple satisfying that $x_k < g^-_{\neg,k}$ for some $k \in \{1,\dots,m\}$. In particular, this fact implies that $x_k \notin \{x \in [0,1] \mid a^-_k * \neg x \leq b\}$ and therefore, $a^-_k * \neg x_k > b$. Hence, we conclude that the tuple $(x_1,\dots,x_m)$ is not a solution of Equation (1). □

Proposition 4 If (1) is solvable, then $a^+_j * g^-_{\neg,j} \leq b$ for all $j \in \{1,\dots,m\}$.

Proof. Let $(x_1,\dots,x_m)$ be a solution of (1). By Proposition 3, we deduce that $g^-_{\neg,j} \leq x_j$ for all $j \in \{1,\dots,m\}$. Then, as a consequence of the monotonicity of $*$:

$$a^+_j * g^-_{\neg,j} \leq a^+_j * x_j \leq b$$

for all $j \in \{1,\dots,m\}$. □

4.1 *The case of a bijective negation, an algebraic approach*

This subsection focuses on characterizing the solvability of a bipolar equation, in terms of the negation operator, from an algebraic point of view. To this aim, the problem under study hereinafter is the solvability of bipolar equations under the hypothesis of a bijective negation.

Consider a bipolar equation of the form (1) being $\neg\colon [0,1] \to [0,1]$ a bijective negation, and let $\neg^{-1}\colon [0,1] \to [0,1]$ be the inverse mapping of the negation operator $\neg$. Notice that, since $\neg$ is bijective, then it is surjective. Therefore, as a consequence of the monotonicity of the operator $\neg$ and its bijectiviy, it follows that:

$$g^-_{\neg,j} = \inf\{x \in [0,1] \mid a^-_j * \neg x \leq b\} = \inf\{x \in [0,1] \mid \neg x \leq b \nwarrow a^-_j\}$$

$$= \inf\{x \in [0,1] \mid x \geq \neg^{-1}(b \nwarrow a^-_j)\} = \neg^{-1}(b \nwarrow a^-_j).$$

Consequently,

$$g^-_\neg = (\neg^{-1}(b \nwarrow a^-_1),\dots,\neg^{-1}(b \nwarrow a^-_m)) = \neg^{-1}(b \nwarrow a^-_1,\dots,b \nwarrow a^-_m) = \neg^{-1} g^-.$$

Thus, in this section we will adopt the notation $\neg^{-1} g^-$ to refer to the tuple $g^-_\neg$.

As stated in the following result, the solvability of a bipolar equation with bijective negation can be characterized in terms of the tuples g^+ and $\neg^{-1} g^-$.

Theorem 3 Let $\neg\colon [0,1] \to [0,1]$ be a bijective negation. Then, (1) is solvable if and only if g^+ or $\neg^{-1} g^-$ is a solution.

Proof. Suppose that Equation (1) is solvable, then there exists $(x_1,\dots,x_m) \in [0,1]^m$ a solution. Since $[0,1]$ is a chain, $a^+_k * x_k = b$ or $a^-_k * \neg x_k = b$ for some $k \in \{1,\dots,m\}$. Considering both cases:

• If $a_k^+ * x_k = b$, then by Proposition 1 and the monotonicity of the operator $*$, we deduce $a_k^+ * g_k^+ = b$. Therefore, due to Lemma 3, we conclude that g^+ is a solution of (1).

• If $a_k^- * \neg x_k = b$, by Proposition 1 and the monotonicity of the operator $*$, it holds that $a_k^- * g_k^- = b$. Moreover, since $\neg\neg^{-1} g_k^- = g_k^-$, thus $a_k^- * \neg\neg^{-1} g_k^- = b$. In addition, since (1) is solvable, applying Proposition 4, we conclude that $a_k^+ * \neg^{-1} g_k^- \leq b$. Therefore:

$$(a_k^+ * \neg^{-1} g_k^-) \vee (a_k^- * \neg\neg^{-1} g_k^-) = b.$$

On the other hand, Proposition 4 and the adjoint property, yield to:

$$\bigvee_{\substack{j=1\\ j\neq k}}^{m} (a_j^+ * \neg^{-1} g_j^-) \vee (a_j^- * \neg\neg^{-1} g_j^-) = \bigvee_{\substack{j=1\\ j\neq k}}^{m} (a_j^+ * \neg^{-1} g_j^-) \vee (a_j^- * g_j^-) \leq b.$$

Hence:

$$\bigvee_{\substack{j=1\\ j\neq k}}^{m} (a_j^+ * \neg^{-1} g_j^-) \vee (a_j^- * \neg\neg^{-1} g_j^-) \bigvee (a_k^+ * \neg^{-1} g_k^-) \vee (a_k^- * \neg\neg^{-1} g_k^-) = b.$$

That is, $\neg^{-1} g^-$ is a solution of (1). □

In accordance with Theorem 3, the determination of the solvability of bipolar equations with a bijective negation is reduced to a straightforward evaluation of the tuples g^+ and $\neg^{-1} g^-$. Clearly, characterizing the solvability of a bipolar equation in terms of g^+ and $\neg^{-1} g^-$ requires that the tuple $\neg^{-1} g^-$ be well-defined. On the one hand, this implies that the inverse of the mapping $\neg$ exists, which leads to the requirement of the negation operator being injective. On the other hand, it is necessary that the values $b \nwarrow a_j^-$ lie within the image set of the mapping $\neg$, that is, $b \nwarrow a_j^- \in \neg([0,1])$ for all $j \in \{1,\ldots,m\}$. In fact, the last condition corresponds to the surjectivity of the negation operator. The previous reasoning justifies the need for the condition of bijectivity for the establishment of Theorem 3.

Furthermore, since the inverse mapping of an involutive negation $\neg$ coincides with the same mapping $\neg$, Theorem 3 is established as a generalization of Theorem 2, in which not only a bijective negation is considered instead of an involutive one, but the imposed properties on the right adjoint pair $(*, \nwarrow)$ are also weakened. Indeed, the conditions of $*$ being a right homomorphism with right unital element 1 have been removed.

The subsequent examples are intended to provide a clearer understanding of the aforementioned points.

Example 4 Let $n \in \mathbb{N}$ and $\neg_n \colon [0,1] \to [0,1]$ defined as:

$$\neg_n(x) = \frac{1-x^n}{1+x^n}.$$

Consider the bipolar equation given by:

$$(0.2 * x_1) \vee (0.8 * \neg_n x_1) \vee (0.4 * x_2) \vee (0.1 * \neg_n x_2) \vee (0.5 * x_3) = 0.6 \tag{4}$$

which coincides with the bipolar Equation (3) studied in Example 1, except for the negation operator.

Notice that, the mapping $\neg_n$ is a bijective negation. Furthermore, the inverse mapping $\neg_n^{-1}\colon [0,1] \to [0,1]$ of $\neg_n$ is defined as the following expression:

$$\neg_n^{-1}(x) = \sqrt[n]{\frac{1-x}{1+x}}.$$

Observe that, since $\neg_n^{-1} \neq \neg_n$, $\neg_n$ is not an involutive negation, which means that we can not apply Theorem 2 to discuss the solvability of (4). However, in accordance with Theorem 3, we can make use of the tuples g^+ and $\neg^{-1}g^-$ for that purpose. Therefore, we firstly compute these two tuples:

$$g^+ = (0.6 \nwarrow_P 0.2, 0.6 \nwarrow_P 0.4, 0.6 \nwarrow_P 0.5) = (1,1,1)$$

$$\begin{aligned}\neg^{-1}g^- &= (\neg_n^{-1}(0.6 \nwarrow_P 0.8), \neg_n^{-1}(0.6 \nwarrow_P 0.1), \neg_n^{-1}(0.6 \nwarrow_P 0))\\ &= (\neg_n^{-1}(0.75), \neg_n^{-1}(1), \neg_n^{-1}(1)).\end{aligned}$$

As shown below, $\neg_n^{-1}g^-$ is a solution of (4):

$$(0.2 * \neg_n^{-1}(0.75)) \vee (0.8 * 0.75) \vee (0.4 * \neg^{-1}(1)) \vee (0.1 * 1) \vee (0.5 * \neg_n^{-1}(1)) = 0.6.$$

Consequently, we conclude that (4) is solvable.

Example 5 In this example, we consider the standard negation $\neg_S$ shown in Example 2 and the mapping

$$x * y = \begin{cases} 0 & \text{if} \quad 0 = y, \\ \dfrac{1}{4}\left[x(1-\dfrac{y}{2})+y\right] & \text{if} \quad 0 < y \leq \frac{1}{2}, \\ \dfrac{1+x}{2} & \text{if} \quad \frac{1}{2} < y \leq 1. \end{cases}$$

Following an analogous reasoning to that presented in Proposition 2.47 of [29], we deduce that, since the partial mapping $*_x\colon [0,1] \to [0,1]$ given as $*_x(z) = x * z$ is left-continuous for all $x \in [0,1]$, then the operator $*$ has a right adjoint implication $\nwarrow$ given by:

$$z \nwarrow x = \max\{y \in [0,1] \mid x * y \leq z\}.$$

Moreover, studying the previous expression, the right adjoint implication $\nwarrow$ of the operator $*$ can be rewritten as:

$$z \nwarrow x = \begin{cases} 1 & \text{if} \quad \dfrac{1+x}{2} \leq z, \\ \max\left\{0, \min\left\{\dfrac{4z-x}{1-\frac{x}{2}}, \dfrac{1}{2}\right\}\right\} & \text{otherwise.} \end{cases}$$

Therefore, $(*, \nwarrow)$ is a right adjoint pair. Now, we consider the next bipolar equation:

$$(0.2 * x_1) \vee (0.4 * \neg_S x_1) \vee (0.5 * x_2) \vee (0.3 * \neg_S x_2) \vee (0 * x_3) \vee (0.7 * \neg_S x_3) = 0.8. \tag{5}$$

Notwithstanding the involutivity of the negation $\neg_S$, it can be proven that the operator $*$ has no right unital element and $*$ is not a right infimum-morphism. In other words, the operator $*$ fails to satisfy almost all of the properties imposed on Theorem 2. Consequently, we cannot apply Theorem 2 to determine whether (5) is solvable. Nevertheless, the negation $\neg_S$ is a bijective negation, hence, due to Theorem 3, we can discuss the solvability of (5) by means of g^+ and $\neg_S^{-1} g^-$. Hence, in order to determine the solvability of (5), we firstly calculate these tuples:

$$\begin{aligned} g^+ &= (0.8 \nwarrow 0.2, 0.8 \nwarrow 0.5, 0.8 \nwarrow 0) \\ &= (1, 1, 1) \\ \neg_S^{-1} g^- &= (\neg_S^{-1}(0.8 \nwarrow 0.4), \neg_S^{-1}(0.8 \nwarrow 0.3), \neg_S^{-1}(0.8 \nwarrow 0.7)) \\ &= (\neg_S 1, \neg_S 1, \neg_S 0.5) \\ &= (0, 0, 0.5). \end{aligned}$$

Now, we compute both tuples:

$$(0.2 * 1) \vee (0.4 * \neg_S 1) \vee (0.5 * 1) \vee (0.3 * \neg_S 1) \vee (0 * 1) \vee (0.7 * \neg_S 1)$$

$$= 0.6 \vee 0 \vee 0.75 \vee 0 \vee 0.5 \vee 0 = 0.75 < 0.8$$

$$(0.2 * 0) \vee (0.4 * \neg_S 0) \vee (0.5 * 0) \vee (0.3 * \neg_S 0) \vee (0 * 0.5) \vee (0.7 * \neg_S 0.5)$$

$$= 0 \vee 0.7 \vee 0 \vee 0.65 \vee 0.125 \vee 0.256 = 0.7 < 0.8.$$

Therefore, none of the tuples g^+ or $\neg_S^{-1} g^-$ is a solution of (5). Consequently, applying Theorem 3, we conclude that (5) is not solvable.

Examples 4 and 5 show two practical cases of an explicit method of resolution of bipolar equations defined with a bijective negation, deduced from Theorem 3, which can be summarized as the following algorithm.

As aforementioned, solving a bipolar equation defined with a bijective negation is reduced to evaluating two predefined tuples, which means that the number of operations required to determine whether a bipolar equation is solvable or not, is very low or negligible. In particular, Algorithm 1 runs in $O(m)$ time, where m denotes the number of unknown variables.

Through Examples 4 and 5, we have seen the usefulness of Theorem 3 and how this generalization results in a substantial augmentation of the range of cases of bipolar equations that are susceptible to study. Moreover, it has been shown that, under the assumption of a bijective negation, the imposition of conditions on the right adjoint pair $(*, \nwarrow)$ is unnecessary for the determination of solvability of bipolar equations.

Algorithm 1: Resolution method for bipolar equations defined with bijective negations

Input: Bipolar equation with bijective negation
Output: True if the bipolar equation is solvable, **False** otherwise

Compute the tuple g^+;
Evaluate the tuple g^+ ;
if g^+ *is a solution* **then**
  return True;
else
  Compute the tuple $\neg^{-1}g^-$;
  Evaluate the tuple $\neg^{-1}g^-$;
  if $\neg^{-1}g^-$ *is a solution* **then**
    return True;
  else
    return False;
  end
end

4.2 *The case of a continuous (or equivalently surjective) negation, an analytical approach*

In contrast with Section 4.1, which handles the resolution of bipolar equations from an algebraic perspective, we here tackle the problem according to the topological structure of the unit interval, thus taking an analytical approach. To be precise, this section presents necessary conditions and a characterization for the solvability of bipolar equations defined with a continuous negation.

First, due to Corollary 3, the continuity of a negation operator is equivalent to the surjectivity of such negation operator. Consequently, it is equivalent to solve bipolar equations with a continuous negation operator and with a surjective one. The results of this section are presented in terms of a continuous negation, although, due to the equivalence between continuity and surjectivity for negation operators, the properties of continuity and surjectivity will be used throughout all the following development. Naturally, a wholly analogous development could be formulated by starting from a surjective negation.

From here on, consider fixed a continuous negation operator $\neg$, and let $\neg^{-1}$ be the mapping $\neg^{-1}\colon \mathscr{P}([0,1])\setminus\{\varnothing\} \to \mathscr{P}([0,1])\setminus\{\varnothing\}$ given as follows:

$$\neg^{-1}(A) = \{x \in [0,1] \mid \neg x \in A\}$$

for all $A \in \mathscr{P}([0,1])\setminus\{\varnothing\}$. The set $\neg^{-1}(A)$ is called the preimage set of A. Notice that, since $\neg$ is a continuous negation operator, then $\neg$ is a surjective mapping, which implies that the preimage set $\neg^{-1}(A)$ is non-empty for all $A \in \mathscr{P}([0,1])\setminus\{\varnothing\}$. Therefore, we conclude that $\neg^{-1}$ is well-defined.

Moreover, applying Proposition 1 and the facts that $\neg$ is an order-reversing mapping and $b \nwarrow a_j^- \in \neg([0,1])$:

$$g^-_{\neg,j} = \inf\{x \in [0,1] \mid a_j^- * \neg x \le b\} = \inf\{x \in [0,1] \mid \neg x \le b \nwarrow a_j^-\}$$

$$= \inf\{x \in [0,1] \mid \neg x = b \nwarrow a_j^-\} = \inf\{\neg^{-1}(\{b \nwarrow a_j^-\})\}.$$

Since the unit interval is a Fréchet topological space, the unitary sets given as $\{b \nwarrow a_j^-\}$ for any $j \in \{1,\dots,m\}$ are closed sets in $[0,1]$. Moreover, as $\neg$ is a continuous mapping, we deduce that the non-empty preimage sets $\neg^{-1}(\{b \nwarrow a_j^-\})$ are closed sets for any $j \in \{1,\dots,m\}$. Consequently, we deduce that:

$$\inf\{\neg^{-1}(\{b \nwarrow a_j^-\})\} = \min\{\neg^{-1}(\{b \nwarrow a_j^-\})\}.$$

Hence:

$$g_\neg^- = (\min\{\neg^{-1}(\{b \nwarrow a_1^-\})\},\dots,\min\{\neg^{-1}(\{b \nwarrow a_m^-\})\}).$$

We can assert then that the tuple $g_\neg^-$ is computed, in case that $\neg$ is a continuous negation, as the minimum of the preimage of the tuple g^- by $\neg$. As a result, in this subsection, we will adopt the notation $\neg_{\min}^{-1} g^-$ to refer to the tuple $g_\neg^-$.

In the subsequent result, we show a simple characterization of the solvability of a bipolar equation under the hypothesis of a continuous negation operator. This characterization is stated in terms of g^+ and $\neg_{\min}^{-1} g^-$.

Theorem 4 Let $\neg\colon [0,1] \to [0,1]$ be a continuous negation operator. Then, (1) is solvable if and only if g^+ or $\neg_{\min}^{-1} g^-$ is a solution.

Proof. Suppose that $(x_1,\dots,x_m)$ is a solution of (1). Since $[0,1]$ is a chain, $a_k^+ * x_k = b$ or $a_k^- * \neg x_k = b$ for some $k \in \{1,\dots,m\}$. We will study both scenarios, concluding that g^+ is a solution or $\neg_{\min}^{-1} g^-$ is a solution, respectively:

• If $a_k^+ * x_k = b$, due to Proposition 1 and the monotonicity of $*$, $a_k^+ * g_k^+ = b$. Therefore, by Lemma 3, g^+ is a solution of (1).

• If $a_k^- * \neg x_k = b$, due to Proposition 1 and the monotonicity of the operator $*$, it is $a_k^- * g_k^- = b$. Furthermore, the definition of $\neg_{\min}^{-1} g^-$ implies that $\neg\neg_{\min}^{-1} g_k^- = g_k^-$, hence $a_k^- * \neg\neg_{\min}^{-1} g_k^- = b$. On the other hand, due to Proposition 4, $a_j^+ * \neg_{\min}^{-1} g_j^- \leq b$ for all $j \in \{1,\dots,m\}$. Therefore, we have proven that:

$$(a_k^+ * \neg_{\min}^{-1} g_k^-) \vee (a_k^- * \neg\neg_{\min}^{-1} g_k^-) = b.$$

Moreover, by Proposition 1, $a_j^- * \neg\neg_{\min}^{-1} g_j^- = a_j^- * g_j^- \leq b$ for all $j \in \{1,\dots,m\}$. Hence:

$$\bigvee_{\substack{j=1\\ j\neq k}}^{m} (a_j^+ * \neg_{\min}^{-1} g_j^-) \vee (a_j^- * \neg\neg_{\min}^{-1} g_j^-) = \bigvee_{\substack{j=1\\ j\neq k}}^{m} (a_j^+ * \neg_{\min}^{-1} g_j^-) \vee (a_j^- * g_j^-) \leq b.$$

Therefore, we conclude that:

$$\bigvee_{\substack{j=1\\ j\neq k}}^{m} (a_j^+ * \neg_{\min}^{-1} g^-) \vee (a_j^- * \neg\neg_{\min}^{-1} g_j^-) \bigvee (a_k^+ * \neg_{\min}^{-1} g_k^-) \vee (a_k^- * \neg\neg_{\min}^{-1} g_k^-) = b.$$

That is, the tuple $\neg_{\min}^{-1} g^-$ is a solution of (1). $\square$

As a result of the equivalence stated in Corollary 3, Theorem 4 can be stated as follows for bipolar equations defined with a surjective negation operator.

Theorem 5 Let $\neg\colon [0,1] \to [0,1]$ be a surjective negation operator. Then, (1) is solvable if and only if g^+ or $\neg_{\min}^{-1} g^-$ is a solution.

Theorems 4 and 5 establish a simple characterization for bipolar equations with a continuous negation (equivalently a surjective negation). These characterizations show that the solvability of these bipolar equations is entirely determined by just two tuples g^+ and $\neg_{\min}^{-1} g^-$, which greatly simplifies the verification process.

Similarly to Section 4.1, it has been demonstrated that under certain hypotheses in the negation operator (in this case continuity or equivalently surjectivity), it is not necessary to impose additional conditions on the right adjoint pair when studying the resolution of bipolar equations.

Note that the characterization established in Theorems 4 and 5 is based on the existence of the tuple $\neg_{\min}^{-1} g^-$, that is, of the value $\min\{\neg^{-1}(\{b \nwarrow a_j^-\})\}$ for all $j \in \{1,\ldots,m\}$. As shown at the beginning of this subsection, a sufficient condition for such existence is the continuity (equivalently the surjectivivity) of the negation operator. Conversely, if the value $\min\{\neg^{-1}(\{b \nwarrow a_j^-\})\}$ exists for all $j \in \{1,\ldots,m\}$, then in particular $b \nwarrow a_j^- \in \neg([0,1])$ for all $j \in \{1,\ldots,m\}$. This fact implies that surjectivity (equivalently continuity) is not only a sufficient but a necessary condition for the existence of the tuple $\neg_{\min}^{-1} g^-$, and therefore for obtaining the characterization presented in Theorems 4 and 5.

Furthermore, it is imperative to emphasize the distinction between Theorem 3 and Theorem 5. Firstly, it should be noted that, in contrast to Theorem 3, which makes use of the property of bijectivity in the negation operator without reference to the topological structure of the unit interval, Theorem 5 requires the consideration of topological and analytical notions such as the continuity of the negation operator and the topological structure of the unit interval. Thus, Theorem 3 is presented as an algebraic characterization, which does not depend in any way on analytical or topological notions, while Theorem 5 is presented as an analytical characterization in which these conditions are necessary.

This is particularly relevant due to a possible generalization of both characterizations to bipolar equations on a latticized algebraic structure different from $[0,1]$. Indeed, this generalization would only be possible for the algebraic approach because it does not require topological or analytical notions. However, the analytical approach prevents this possible generalization to a more complex algebraic structure due to this need for analytical results stated in the unit interval.

We illustrate below the case of bipolar equations defined with a non-injective and surjective negation, in order to show the usefulness of the results obtained in Section 4.2.

Example 6 Let $(*_G, \nwarrow_G)$ be the Gödel right adjoint pair, given by the following operators:

$$x *_G y = \min\{x,y\} \qquad \text{(Gödel t-norm)}$$

$$z \nwarrow_G x = \begin{cases} 1 & \text{if } x \le z \\ z & \text{otherwise} \end{cases} \qquad \text{(Right adjoint implication of } *_G\text{)}$$

and the negation operator $\neg\colon [0,1] \to [0,1]$ given as:

$$\neg x = \begin{cases} 1 & \text{if} \quad x < \dfrac{1}{2}, \\ 2(1-x) & \text{if} \quad x \ge \dfrac{1}{2}. \end{cases}$$

It is a simple verification to prove that the mapping $\neg$ is surjective. Therefore, we can use Theorem 5 to discuss the solvability of the following bipolar equation:

$$(0.2 *_G x_1) \vee (0.8 *_G \neg x_1) \vee (0.4 *_G x_2) \vee (0.1 *_G \neg x_2) \vee (0.5 *_G x_3) = 0.6. \tag{6}$$

Notice that (6) has the same coefficients as the bipolar Equations (3) and (4) analysed in Examples 3 and 4, but the right adjoint pair and the negation operator are different.

Since $\neg 0 = 1 = \neg\frac{1}{4}$, the negation operator $\neg$ is not injective, hence we can not apply Theorem 3 to discuss the solvability of (6). However, as $\neg$ is a surjective negation, we can discuss the solvability of (6) according to Theorem 5. Therefore, in order to apply Theorem 5, we compute g^+ and $\neg_{\min}^{-1} g^-$:

$$
\begin{aligned}
g^+ &= (0.6 \nwarrow_G 0.2, 0.6 \nwarrow_G 0.4, 0.6 \nwarrow_G 0.5)\\
&= (1,1,1)\\
\neg_{\min}^{-1} g^- &= (\min\{\neg^{-1}(\{0.6 \nwarrow_G 0.8\})\}, \min\{\neg^{-1}(\{0.6 \nwarrow_G 0.1\})\}, \min\{\neg^{-1}(\{0.6 \nwarrow_G 0\})\})\\
&= (0.7,0,0).
\end{aligned}
$$

Now, we will see if g^+ or $\neg_{\min}^{-1} g^-$ is a solution of (6):

$$
\begin{aligned}
&(0.2 *_G g_1^+) \vee (0.8 *_G \neg g_1^+) \vee (0.4 *_G g_2^+) \vee (0.1 *_G \neg g_2^+) \vee (0.5 *_G g_3^+)\\
= &(0.2 *_G 1) \vee (0.8 *_G \neg 1) \vee (0.4 *_G 1) \vee (0.1 *_G \neg 1) \vee (0.5 *_G 1)\\
= &0.2 \vee 0 \vee 0.4 \vee 0 \vee 0.5 = 0.5\\
&(0.2 *_G \neg_{\min}^{-1} g_1^-) \vee (0.8 *_G \neg\neg_{\min}^{-1} g_1^-) \vee (0.4 *_G \neg_{\min}^{-1} g_2^-) \vee (0.1 *_G \neg\neg_{\min}^{-1} g_2^-) \vee (0.5 *_G \neg_{\min}^{-1} g_3^-)\\
= &(0.2 *_G 0.7) \vee (0.8 *_G 0.6) \vee (0.4 *_G 0) \vee (0.1 *_G 1) \vee (0.5 *_G 0)\\
= &0.2 \vee 0.6 \vee 0.1 \vee 0.5 = 0.6.
\end{aligned}
$$

Since $\neg_{\min}^{-1} g^-$ is a solution of Equation (6) we directly deduce that Equation (6) is solvable.

Example 6 shows a case of a bipolar equation defined with a surjective and non-injective negation operator. As a result, Theorem 3 cannot be applied. Nevertheless, paying attention to the form of the negation operator under consideration in Example 6, the injectivity only fails at the point 1, that is, the only element whose preimage by $\neg$ is not a singleton is 1. One might think that this case could be simply avoided from an applied perspective, for instance, assuming a bijective negation which coincides with $\neg$ except for a tolerance, as it is illustrated in Figure 2.

With this reasoning, the results stated in Section 4.1 could be applied to discuss the solvability of bipolar equations defined with a negation operator similar to the one considered in Example 6. However, this reasoning could be extremely complex when the injectivity of the negation operator fails in an infinite number of points. Then, the following example is presented to show the case of a bipolar equation defined with a more complex negation operator satisfying this fact.

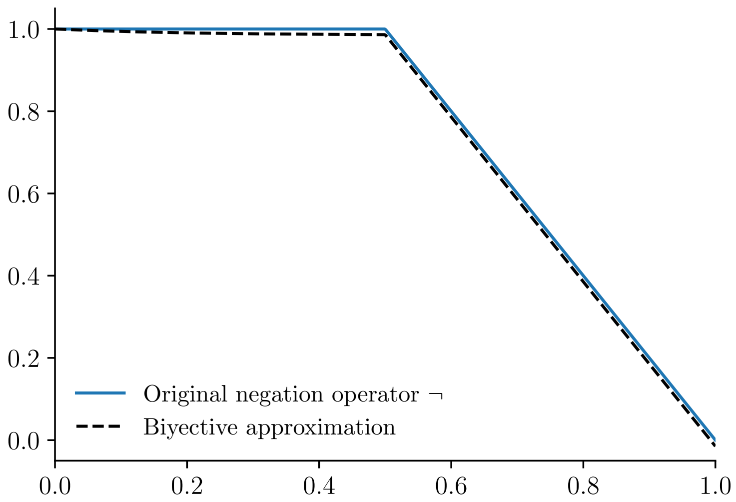

Figure 2. Bijective approximation of the negation operator

Example 7 Let, for any $n \in \mathbb{N}$:

$$a_n = 1 - \frac{1}{n} \qquad b_n = \frac{a_n + a_{n+1}}{2}.$$

It is clear that $a_n < b_n < a_{n+1}$ and $\bigcup_{n \in \mathbb{N}} [a_n, a_{n+1}] = [0, 1[$. Therefore, we define the negation operator $\neg \colon [0, 1] \to [0, 1]$ as follows:

$$\neg x = \begin{cases} \dfrac{1}{n} & \text{if} \quad x \in [a_n, b_n[, \\ \dfrac{1}{n} - 2(x - b_n) & \text{if} \quad x \in [b_n, a_{n+1}[, \\ 0 & \text{if} \quad x = 1. \end{cases}$$

It can be proven that $\neg$ is a continuous operator. Indeed, it is enough to prove that $\neg a_n = \frac{1}{n}$ and $\neg b_n = \frac{1}{n}$, which is a simple computation. Therefore, we can use Theorem 4 to discuss the solvability of a bipolar equation defined with $\neg$.

On the other hand, notice that, for any value $c_n = \frac{1}{n}$ for $n \in \mathbb{N}$, it is $\neg^{-1}(\{\frac{1}{n}\}) = [a_n, b_n]$. Therefore, there exists a countably infinite set of elements in the unit interval such that its preimage set by $\neg$ is an interval. In other words, there exist a countably infinite number of elements in the unit interval that cause the negation operator $\neg$ to be non-injective. This fact is illustrated in Figure 3, which shows the graph of an approximation of the mapping $\neg$ for $N = 300$.

Figure 4 also shows how the negation $\neg$ behaves near the value 1.

Now, we will consider the product right adjoint pair $(*, \nwarrow_P)$ and the bipolar equation given as:

$$(\frac{1}{48} * x_1) \vee (\frac{1}{6} \neg x_1) \vee (\frac{1}{30} * x_2) \vee (\frac{1}{12} * \neg x_2) \vee (\frac{1}{10} * \neg x_3) = \frac{1}{24}. \tag{7}$$

In what follows, we will study the solvability of (7). In first place, since the negation operator $\neg$ is not injective, we can not apply Theorem 3 to discuss the solvability of (7). However, we have proven that $\neg$ is a continuous negation operator, therefore according with Theorem 4, we can determine if (7) is solvable or not in terms of the tuples g^+ and $\neg_{\min}^{-1} g^-$.

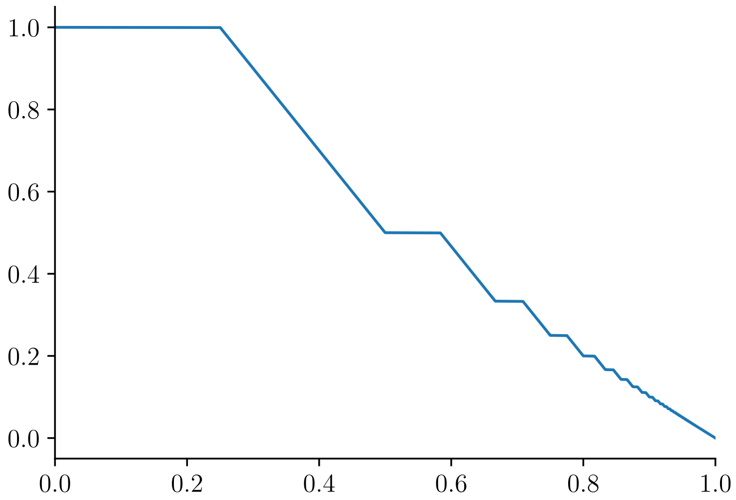

Figure 3. Global approximation of $\neg\colon [0,1] \to [0,1]$ for $N = 300$

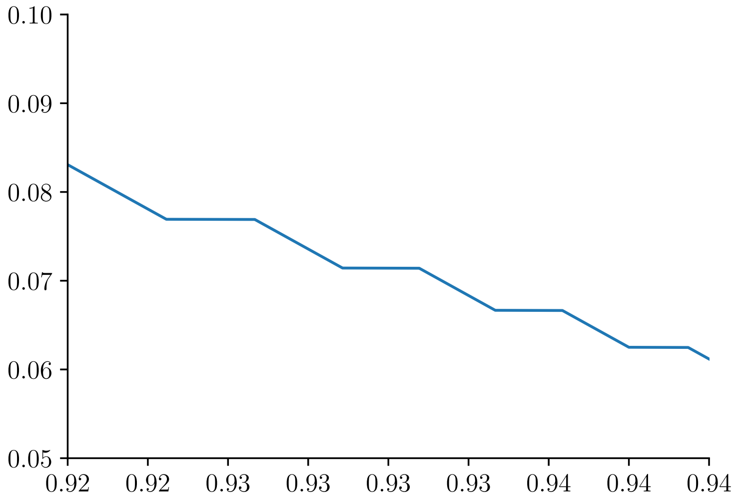

Figure 4. Zoom for $x \in [0.92, 0.94]$

Notice that, by the definition of $\neg_{\min}^{-1} g^-$, we deduce that $\min\{\neg^{-1}(\{\frac{1}{n}\})\} = a_n = 1 - \frac{1}{n}$ for all $n \in \mathbb{N}$. Hence, we conclude that:

$$\neg_{\min}^{-1} g^- = \left(\min\{\neg^{-1}(\{\frac{1}{24} \nwarrow_P \frac{1}{6}\})\}, \min\{\neg^{-1}(\{\frac{1}{24} \nwarrow_P \frac{1}{12}\})\}, \min\{\neg^{-1}(\{\frac{1}{24} \nwarrow_P \frac{1}{10}\})\}\right) = \left(\frac{3}{4}, \frac{1}{2}, \frac{5}{8}\right)$$

$$g^+ = \left(\frac{1}{24} \nwarrow_P \frac{1}{48}, \frac{1}{24} \nwarrow_P \frac{1}{30}, \frac{1}{24} \nwarrow_P 0\right) = (1, 1, 1).$$

Now, we compute both tuples on Equation (7):

$$\begin{aligned}
&(\frac{1}{48} * g_1^+) \vee (\frac{1}{6} * \neg g_1^+) \vee (\frac{1}{30} * g_2^+) \vee (\frac{1}{12} * \neg g_2^+) \vee (\frac{1}{10} * \neg g_3^+)\\
&= (\frac{1}{48} * 1) \vee (\frac{1}{6} * 0) \vee (\frac{1}{30} * 1) \vee (\frac{1}{12} * 0) \vee (\frac{1}{10} * 0)\\
&= \frac{1}{48} \vee 0 \vee \frac{1}{30} \vee 0 \vee 0 = \frac{1}{30}
\end{aligned}$$

$$\begin{aligned}
&(\frac{1}{48} * \neg_{\min}^{-1} g_1^-) \vee (\frac{1}{6} * \neg\neg_{\min}^{-1} g_1^-) \vee (\frac{1}{30} * \neg_{\min}^{-1} g_2^-) \vee (\frac{1}{12} * \neg\neg_{\min}^{-1} g_2^-) \vee (\frac{1}{10} * \neg\neg_{\min}^{-1} g_3^-)\\
&= (\frac{1}{48} * \frac{3}{4}) \vee (\frac{1}{6} * \frac{1}{4}) \vee (\frac{1}{30} * \frac{1}{2}) \vee (\frac{1}{12} * \frac{1}{2}) \vee (\frac{1}{10} * \frac{5}{12})\\
&= \frac{1}{64} \vee \frac{1}{24} \vee \frac{1}{60} \vee \frac{1}{24} \vee \frac{1}{24} = \frac{1}{24}.
\end{aligned}$$

Therefore, the tuple $\neg_{\min}^{-1} g^-$ is a solution of (7), which means that (7) is solvable.

The resolution strategy followed in Examples 6 and 7 is presented in Algorithm 2, which applies to any bipolar equation with continuous (or equivalently surjective) negation.

Like Algorithm 1, the overall time complexity of Algorithm 2 is $O(m)$.

Algorithm 2: Resolution method for bipolar equations defined with a continuous negation

Input: Bipolar equation with continuous negation
Output: **True** if the bipolar equation is solvable, **False** otherwise

```
Compute the tuple g+;
Evaluate the tuple g+ ;
if g+ is a solution then
    return True;
else
    Compute the tuple ¬min^-1 g−;
    Evaluate the tuple ¬min^-1 g−;
    if ¬min^-1 g− is a solution then
        return True;
    else
        return False;
    end
end
```

In Sections 4.1 and 4.2, it has been shown that in the case of a bijective negation, a surjective negation or a continuous negation, we can characterize the solvability of a bipolar equation in terms of the preimage of the negation operator, which in the bijective case would coincide with the inverse mapping of the negation. Moreover, in Section 4, no conditions are imposed on the right adjoint pair under consideration in a bipolar equation.

Furthermore, in Sections 4.1 and 4.2, not only have we not imposed any conditions on the right adjoint pair, but we have also weakened the hypothesis of involutivity in the negation operator considered in Theorem 2, replacing it with weaker conditions such as bijectivity, continuity or surjectivity.

Section 4 has covered a study of the solvability of bipolar equations in terms of the negation operator. In what follows, we will focus on studying the solvability of bipolar equations in terms of the right adjoint pair.

5. Characterizing the solvability in terms of the right adjoint pair

In this section, certain conditions will be imposed exclusively on the right adjoint pair to characterize the solvability of a bipolar equation. In particular, properties such as injectivity and continuity will be established in the right adjoint conjunctor $*$. It is important to emphasize that the results shown in this section only require continuity for the mappings $*_j^+\colon [0,1] \to [0,1]$ and injectivity for the mappings $*_j^-\colon [0,1] \to [0,1]$ defined as:

$$*_j^+(x) = a_j^+ * x$$

$$*_j^-(x) = a_j^- * x.$$

Nevertheless, to simplify the notation, these conditions will be directly imposed on the right partial mappings of $*$. Specifically, we will say that the right partial mapping of $*$ satisfies a certain property if it holds for all mappings of the form $*_z(x) = z * x$ with $z \in [0,1]$.

In what follows, we present a useful result that is derived from the continuity condition of the operator $*$.

Proposition 5 Let $(*, \nwarrow)$ be a right adjoint pair such that the right partial mapping of $*$ is continuous and let $a_j, b \in [0,1]$. If $a_j * (b \nwarrow a_j) < b$, then $a_j * x < b$ for all $x \in [0,1]$.

Proof. Let $(*, \nwarrow)$ be a right adjoint pair such that the right partial mapping of $*$ is continuous and let $a_j, b \in [0,1]$. Therefore, the mapping $*_j\colon [0,1] \to [0,1]$ given as

$$*_j(x) = a_j * x$$

is also continuous.

Now, suppose $b \leq a_j * x_0 = *_j(x_0)$ for some $x_0 \in [0,1]$. Due to Proposition 1, we deduce that $0 = a_j * 0 = *_j(0)$. Therefore, applying Intermediate Value Theorem 1, since $*_j$ is a continuous function, there exists $c \in [0,1]$ such that $a_j * c = b$. Hence, due to Proposition 1 and the monotonicity of $*$, we have proven that $a_j * (b \nwarrow a_j) = b$. □

Theorems 3, 4 and 5 characterize the solvability of a bipolar equation in terms of the tuples g^+ and $g^-_\neg$, where, in both cases, tuple $g^-_\neg$ is defined as a modification of tuple g^-. Building on this idea, we seek to define an operator, without imposing any condition on the negation operator, that modifies the tuple g^- and allows us to obtain conclusions about the solvability of bipolar equations.

With this purpose, the notion of pseudo-inverse is defined, which will be crucial to determine whether a bipolar equation is solvable.

Definition 6 Let $\neg\colon [0,1] \to [0,1]$ be a negation operator. A psuedo-inverse of $\neg$ is a mapping $\neg^{\mathscr{AC}}\colon [0,1] \to [0,1]$ of the form:

$$\neg^{\mathscr{AC}} x = \begin{cases} y & \text{if} \quad x \in \neg([0,1]), \\ 1 & \text{if} \quad x \notin \neg([0,1]). \end{cases}$$

where $y \in \neg^{-1}(\{x\})$ is an element of the preimage set of $x \in \neg([0,1])$.

Notice that, in Sections 4.1 and 4.2, certain arguments were made based on the preimage set of the considered negation and the imposed properties of this negation operator, in order to define the tuples $\neg^{-1}g^{-}$ and $\neg^{-1}_{\min}g^{-}$, which proved to be very useful in characterizing the solvability of (1). The underlying idea in the definition of pseudo-inverse is analogous to that in Sections 4.1 and 4.2, that is, using the preimage set of the negation under consideration to define a specific tuple that allows us to characterize the solvability of (1).

However, as no properties have been imposed on the negation operator, arguments similar to those in Sections 4.1 and 4.2 cannot be made. This is why, in Definition 6, we simply consider an element $y \in \neg^{-1}(\{x\})$ in the preimage set of $x \in \neg([0,1])$. In fact, the choice of this element $y \in \neg^{-1}(\{x\})$ can be made completely arbitrarily, this is why we have opted for the notation $\neg^{\mathscr{AC}}$, which refers to this arbitrary choice.

On the other hand, notice that, since the negation operator $\neg$ satisfies $\neg 1 = 0$, then it follows that:

$$\neg \circ \neg^{\mathscr{AC}} x = \begin{cases} x & \text{if} \quad x \in \neg([0,1]), \\ 0 & \text{if} \quad x \notin \neg([0,1]). \end{cases}$$

In other words, the mapping $\neg^{\mathscr{AC}}$ behaves as the inverse mapping of $\neg$ on the imagen set $\neg([0,1])$. Indeed, if $\neg\colon [0,1] \to [0,1]$ is a bijective negation operator, then there exists a unique pseudo-inverse $\neg^{\mathscr{AC}}$ of $\neg$ and it coincides with the inverse mapping of $\neg$.

The following result presents a property of the mapping $\neg^{\mathscr{AC}}$.

Proposition 6 Let $\neg\colon [0,1] \to [0,1]$ be a negation operator and $\neg^{\mathscr{AC}}\colon [0,1] \to [0,1]$ be a pseudo-inverse of $\neg$. Then, $a_j^- * \neg\neg^{\mathscr{AC}}(b \nwarrow a_j^-) \leq b$ for all $j \in \{1, \ldots, m\}$.

Proof. We will distinguish two cases for any $j \in \{1, \ldots, m\}$:

• If $b \nwarrow a_j^- \in \neg([0,1])$, then $\neg\neg^{\mathscr{AC}}(b \nwarrow a_j^-) = b \nwarrow a_j^-$. Thus, applying Proposition 1, we deduce that:

$$a_j^- * \neg\neg^{\mathscr{AC}}(b \nwarrow a_j^-) = a_j^- * (b \nwarrow a_j^-) \leq b.$$

• If $b \nwarrow a_j^- \notin \neg([0,1])$, then $\neg\neg^{\mathscr{AC}}(b \nwarrow a_j^-) = \neg 1 = 0$. Therefore, as a consequence of Proposition 1, we deduce that:

$$a_j^- * \neg\neg^{\mathscr{AC}}(b \nwarrow a_j^-) = a_j^- * 0 = 0 \leq b.$$

□

In order to characterize the solvability of (1), we introduce the followings tuples:

$$g^+ = (b \nwarrow a_1^+, \ldots, b \nwarrow a_m^+)$$

$$\neg^{\mathscr{AC}} g^- = (\neg^{\mathscr{AC}}(b \nwarrow a_1^-), \ldots, \neg^{\mathscr{AC}}(b \nwarrow a_m^-))$$

where $\neg^{\mathscr{AC}}$ is a pseudo-inverse of the negation operator $\neg$.

The next result presents a characterization for the solvability of (1) in terms of g^+ and $\neg^{\mathscr{AC}} g^-$.

Theorem 6 Let $(*, \nwarrow)$ be a right adjoint pair such that the right partial mapping of $*$ is continuous and injective. Let $\neg^{\mathscr{AC}}$ be a pseudo-inverse of the negation operator $\neg$. Then, (1) is solvable if and only if g^+ or $\neg^{\mathscr{AC}} g^-$ is a solution.

Proof. Suppose that $(x_1,\ldots,x_m)$ is a solution of (1). We will distinguish between two different and exclusive cases.

• If there exists $k\in\{1,\ldots,m\}$ such that $a_k^+ * g_k^+ \geq b$, due to Proposition 1, it is $a_k^+ * g_k^+ = b$. Therefore, due to Lemma 3, we conclude that g^+ is a solution of (1).

• If $a_j^+ * g_j^+ < b$ for all $j\in\{1,\ldots,m\}$, applying Proposition 5, we deduce that $a_j^+ * z < b$ for all $z\in[0,1]$, $j\in\{1,\ldots,m\}$. Consequently, $a_j^+ * \neg^{\mathscr{AC}} g_j^- < b$. Moreover, due to Proposition 6, it follows that $a_j^- * \neg\neg^{\mathscr{AC}} g_j^- \leq b$. Hence, we have proven that:

$$\bigvee_{j=1}^{m} (a_j^+ * \neg^{\mathscr{AC}} g_j^-) \vee (a_j^- * \neg\neg^{\mathscr{AC}} g_j^-) \leq b.$$

On the other hand, since $a_j^+ * z < b$ for all $z\in[0,1]$, it follows that $a_j^+ * x_j < b$ for all $j\in\{1,\ldots,m\}$.Therefore, since $[0,1]$ is a chain and $(x_1,\ldots,x_m)$ is a solution of (1), then $a_k^- * \neg x_k = b$ for some $k\in\{1,\ldots,m\}$.

Moreover, due to the injectivity of the right partial mapping of $*$ and the definition of g^-, it follows that $\neg x_k = g_k^-$. Hence, $g_k^- \in \neg([0,1])$, from which $\neg\circ\neg^{\mathscr{AC}} g_k^- = g_k^-$. Therefore, $a_k^- * \neg\neg^{\mathscr{AC}} g_k^- = b$. In addition, since $a_k^+ * z < b$ for all $z\in[0,1]$, it is $a_k^+ * \neg^{\mathscr{AC}} g_k^- < b$. So, we have proven:

$$(a_k^+ * \neg^{\mathscr{AC}} g_k^-) \vee (a_k^- * \neg\neg^{\mathscr{AC}} g_k^-) = b.$$

Thus $\neg^{\mathscr{AC}} g^-$ is a solution of (1), that is:

$$\bigvee_{\substack{j=1\\ j\neq k}}^{m} (a_j^+ * \neg^{\mathscr{AC}} g_j^-) \vee (a_j^- * \neg\neg^{\mathscr{AC}} g_j^-) \bigvee (a_k^+ * \neg^{\mathscr{AC}} g_k^-) \vee (a_k^- * \neg\neg^{\mathscr{AC}} g_k^-) = b.$$

□

Theorem 6 presents a simple solvability characterization of bipolar equations defined with a right adjoint conjuntor such that its right partial mapping is continuous and injective in terms of a pseudo-inverse of the negation under consideration. It is important to emphasize that, despite the existence of multiple pseudo-inverses for a single negation operator, the verification of solvability based on the tuples g^+ and $\neg^{\mathscr{AC}} g^-$ can be conducted employing any of these pseudo-inverses. This fact allows us to simplify the determination of the solvability of (1) by defining a pseudo-inverse at our convenience, taking into account the context and the expression of the equation under consideration.

As mentioned in the first paragraph of Section 5, the proof of Theorem 6 only applies continuity on the right partial mappings $*_j^+$ (given as $*_j^+(x) = a_j^+ * x$) and injectivity on the right partial mappings $*_j^-$ (given as $*_j^-(x) = a_j^- * x$),. Therefore, it is sufficient to require such properties to derive the characterization presented in Theorem 6. Nevertheless, for the sake of clarity and applicability, Theorem 6 is stated in terms of stronger hypotheses as a convenient simplification.

As a consequence of Theorem 6, Algorithm 3 illustrates the recipe to determine the solvability of bipolar equations defined with a right adjoint pair with continuous and injective partial mappings. Notice that, Algorithm 3 operates in linear time, specifically $O(m)$, with respect to the number of components of a solution.

The following example shows the usefulness of defining a particular pseudo-inverse that simplifies the process of determining whether a bipolar equation is solvable. In addition, an example of solving a bipolar equation using Algorithm 3 is shown.

Algorithm 3: Resolution method for bipolar equations with C-I right partial mappings

Input: Bipolar equation with C-I right adjoint pair

Output: **True** if the bipolar equation is solvable, **False** otherwise

```
1  Compute the tuple g⁺;
2  Evaluate the tuple g⁺ ;
3  if g⁺ is a solution then
4  |   return True;
5  else
6  |   Calculate a pseudo-inverse and compute the tuple ¬^{AC} g⁻;
7  |   Evaluate the tuple ¬^{AC} g⁻;
8  |   if ¬^{AC} g⁻ is a solution then
9  |   |   return True;
10 |   else
11 |   |   return False;
12 |   end
13 end
```

Example 8 Consider the product right adjoint pair $(*, \nwarrow_P)$ and the product negation $\neg_P$, both shown in Examples 1 and 2, respectively. Due to their multiple applications, bipolar equations under these operators have been extensively studied [9–11]. In this example, we will first show the resolution of a particular case of these bipolar equations.

Consider the following bipolar equation:

$$(0.2 * x_1) \vee (0.8 * \neg_P x_1) \vee (0.4 * x_2) \vee (0.1 * \neg_P x_2) \vee (0.5 * x_3) = 0.6 \tag{8}$$

whose coefficients correspond to the bipolar Equations (3), (4) and (6).

According to Theorem 6, we will define a pseudo inverse $\neg_P^{\mathscr{A}\mathscr{C}}$ of $\neg_P$ and study the solvability of (8) in terms of the tuples g^+ and $\neg_P^{\mathscr{A}\mathscr{C}} g^-$. In particular, we will study the solvability of Equation (8) by applying Algorithm 3 step by step.

Applying the Step 1 of Algorithm 3, we compute the tuple g^+:

$$g^+ = (0.6 \nwarrow_P 0.2, 0.6 \nwarrow_P 0.4, 0.6 \nwarrow_P 0.5).$$

$$= (1, 1, 1).$$

Subsequently, following Step 2, we evaluate the tuple g^+ in Equation (8):

$$(0.2 * 1) \vee (0.8 * \neg_P 1) \vee (0.4 * 1) \vee (0.1 * \neg_P 1) \vee (0.5 * 1) = 0.5$$

According to Steps 3–5, since the tuple g^+ is not a solution of Equation (8), we proceed with Step 6 of Algorithm 3, which consists of computing a pseudo-inverse of $\neg_P$ and the corresponding tuple $\neg_P^{\mathscr{A}\mathscr{C}} g^-$. For instance, let $\neg_P^{\mathscr{A}\mathscr{C}} \colon [0, 1] \to [0, 1]$ be the mapping given as follows:

$$\neg_P^{\mathscr{AC}} x = \begin{cases} 0 & \text{if } x = 1. \\ 1 & \text{if } x < 1. \end{cases}$$

It can be easily proven that $\neg_P^{\mathscr{AC}}$ is a pseudo-inverse of the negation $\neg_P$. The tuple $\neg_P^{\mathscr{AC}} g^-$ is computed as follows:

$$\begin{aligned}\neg_P^{\mathscr{AC}} g^- &= (\neg_P^{\mathscr{AC}}(0.6 \nwarrow_P 0.8), \neg_P^{\mathscr{AC}}(0.6 \nwarrow_P 0.1), \neg_P^{\mathscr{AC}}(0.6 \nwarrow 0)) \\ &= (\neg_P^{\mathscr{AC}}(0.75), \neg_P^{\mathscr{AC}}(1), \neg_P^{\mathscr{AC}}(1)) \\ &= (1, 0, 0).\end{aligned}$$

Finally, applying Step 7, we evaluate the tuple $\neg_P^{\mathscr{AC}} g^-$ in Equation (8):

$$(0.2 * 1) \vee (0.8 * \neg_P 1) \vee (0.4 * 0) \vee (0.1 * \neg_P 0) \vee (0.5 * 0) = 0.2.$$

According to Steps 8–13, since the tuple $\neg_P^{\mathscr{AC}} g^-$ is not a solution, Algorithm 3 ends and it is concluded that Equation (8) is not solvable.

The consideration of the particular pseudo-inverse $\neg_P^{\mathscr{AC}}$ in Example 8 gives rise to a new characterization of the solvability of bipolar equations defined with the product right adjoint pair and the product negation, which is derived from Theorem 6. This result can be formalized as follows.

Corollary 4 A bipolar equation defined with the product right adjoint pair and the product negation is solvable if and only if g^+ or $\neg_P^{\mathscr{AC}} g^-$ is a solution.

Proof. If $a_j^+, a_j^- > 0$ for all $j \in \{1, \ldots, m\}$, the statement is directly deduced from Theorem 6. If there exists $k \in \{1, \ldots, m\}$ such that $a_k^+ = 0$ or $a_k^- = 0$, it is enough to remove the component $(a_k^+ * x_k)$ or $(a_k^- * \neg x_k)$ and apply Theorem 6. □

The usefulness of Corollary 4 lies in the simplicity of the pseudo-inverse considered. Indeed, the use of the mapping $\neg_P^{\mathscr{AC}}$ reduces the determination of whether an equation is solvable to a simple check. This new characterization in terms of the mapping $\neg_P^{\mathscr{AC}}$ also offers significant computational advantages when defining better resolution algorithms than those currently available in the literature [10, 11].

Moreover, it can be easily proven that $z \nwarrow_P x = 1$ if and only if $x \leq z$, through which the equivalence between Corollary 4 and the characterization presented in [10] can be deduced.

Through the framework of bipolar equations with the product conjunctor and the product negation, we have shown the usefulness of defining a particular pseudo-inverse that considerably facilitates the resolution process. Furthermore, the definition of a pseudo-inverse is independent of the properties satisfied by the original negation, which facilitates, among other cases, the resolution of bipolar equations with the product operator and an arbitrary negation, the resolution of which has been a studied topic in the literature [6, 11, 30].

In Section 5, we have presented a characterization of the solvability of bipolar equations by imposing conditions only on the right adjoint pair. This approach enable the study of a bipolar equation through the right adjoint pair considered, independently of the negation operator. In addition, the conditions imposed in Theorem 2 for the right adjoint pair have been modified. That is, it is not required here that the conjunctor is a right homomorphism with right unital element 1.

6. Right and left bipolar equations

In Sections 4 and 5, we have presented different sufficient and necessary conditions, depending on the underlying algebraic structure, for the solvability of bipolar equations. In particular, these sections study bipolar equations whose unknown variables always appear on the right argument of the conjunctor, that is, we consider the supremum of expressions of the form $a*x$ or $a*\neg x$, where x is unknown. Indeed, this is the main reason why we can study these bipolar equations in terms of a right adjoint pair.

This section tackles the problem of characterizing the solvability of a dual type of bipolar equation in which the unknown variables appear on the left argument of a conjunctor. That is, we consider here the supremum of expressions of the form $x*a$ or $\neg x*a$, where x is an unknown variable.

As expected, in order to formalize this new type of bipolar equation, we introduce first a dual version of the concept of right adjoint pair, called left adjoint pair.

Definition 7 [22] Given two mappings $\&, \swarrow \colon [0,1]\times[0,1]\to[0,1]$, we say that $\swarrow$ is the left adjoint implication of $\&$ and $(\&,\swarrow)$ is a left adjoint pair if, for any $x,y,z\in[0,1]$, the left adjoint property is satisfied, that is:

$$x \le z \swarrow y \quad \text{if and only if} \quad x\&y \le z.$$

Definition 8 Let $(*,\swarrow)$ be a left adjoint pair and $\neg$ a negation operator. A left bipolar equation is of the form:

$$\bigvee_{j=1}^{m}(x_j*a_j^+)\vee(\neg x_j*a_j^-)=b \tag{9}$$

where $x_1,\dots,x_m$ are unknown.

Following the terminology of Definition 8, we could refer to bipolar equations of the form (1) as right bipolar equations.

In what follows, we will see how the results shown in Sections 4 and 5 can be adapted to characterize the solvability of a left bipolar equation of the form (9) in terms of its underlying left adjoint pair. First of all, we define the following tuples in a entirely analogous manner to Sections 4.1, 4.2 and 5:

$$g^+=(b\swarrow a_1^+,\dots,b\swarrow a_m^+)$$

$$\neg^{-1}g^-=(\neg^{-1}(b\swarrow a_1^-),\dots,\neg^{-1}(b\swarrow a_m^-))$$

$$\neg_{\min}^{-1}g^-=(\min\{\neg^{-1}(\{b\swarrow a_1^-\})\},\dots,\min\{\neg^{-1}(\{b\swarrow a_m^-\})\})$$

$$\neg^{\mathscr{AC}}g^-=(\neg^{\mathscr{AC}}(b\swarrow a_1^-),\dots,\neg^{\mathscr{AC}}(b\swarrow a_m^-)).$$

It should be noted that, the definition of some of the previous tuples requires the same additional hypotheses in the negation operator and in the left adjoint pair as those imposed in Sections 4.1, 4.2 and 5, which ensures that these tuples are well defined.

The following result presents characterizations of the solvability of a broad family of left bipolar equations, depending on different hypotheses, in terms of the tuples $g^+,\neg^{-1}g^-,\neg_{\min}^{-1}g^-,\neg^{\mathscr{AC}}g^-$.

Theorem 7 The following statements hold:

1. If $\neg$ is a bijective negation operator, then (9) is solvable if and only if g^+ or $\neg^{-1}g^-$ is a solution.
2. If $\neg$ is a continuous negation operator, then (9) is solvable if and only if g^+ or $\neg_{\min}^{-1}g^-$ is a solution.
3. If $\neg$ is a surjective negation operator, then (9) is solvable if and only if g^+ or $\neg_{\min}^{-1}g^-$ is a solution.
4. If the left partial mapping of $*$(i.e., all mappings of the form $*_z(x) = x * z$) is continuous and injective, then (9) is solvable if and only if g^+ or $\neg^{\mathscr{AC}}g^-$ is a solution.

Proof.

1. Dual to Theorem 3.
2. Dual to Theorem 4.
3. Dual to Theorem 5.
4. Dual to Theorem 6. □

7. Conclusions and future work

This work undertakes an exhaustive study on characterizing the solvability of bipolar equations defined in the unit interval. In particular, the solvability of right and left bipolar equations defined with a bijective negation, a continuous negation (equivalently surjective) or an adjoint conjunctor with continuous and injective right partial mapping, has been characterized. Furthermore, in all cases, the characterization simply relies on checking whether any of two specific tuples are solutions of the considered bipolar equation.

It must be stressed that the results presented here show that the solvability of bipolar equations can be characterized by imposing conditions exclusively either on the negation operator or on the adjoint pair under consideration, which notably expands the amount of bipolar equations within the scope of this paper. For all the above reasons, this manuscript represents a meaningful contribution in the framework of solving bipolar equations in the unit interval.

As discussed in Section 4, the hypotheses of a bijective negation and a continuous negation (equivalently surjective) are necessary to apply the characterization approach presented in Sections 4.1 and 4.2, respectively. In other words, Theorems 3 and 4 (equivalently 5) do not hold under weaker hypotheses. On the contrary, it is unknown whether the assumptions on the adjoint pair in Theorem 6 are actually necessary to hold the characterization. Hence, proving the necessity of a continuous and injective right partial mapping or weakening the assumptions taken in Section 5 constitute an appealing prospect for future work. Moreover, we will explore the extension of the results presented here to a general algebraic structure, such as complete lattices or multi-adjoint lattices. In addition, we will study new characterizations of solvability for bipolar equations in these more general algebraic structures. After analysing the case of single bipolar equations, the next natural step will be characterizing the solvability of bipolar fuzzy relation equations (BFRE), which is proposed as an additional line of future research.

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Conflict of interest

The authors declare no competing financial interest.

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