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Research Article

Transmission Dynamics of Dengue Disease Using Control Strategies and Real-Data

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Abstract: Dengue remains the most rapid spreading vector-borne disease globally. It is severe public health challenge in tropical and subtropical region especially in Pakistan. In this work, a fractional mathematical model has been analyzed to study the dynamics of dengue epidemic, using the Atangana-Baleanu Fractional Derivative (A-BFD) to capture memory and hereditary effects. The developed model incorporates interactions among host and vector populations employing non-singular and non-local properties of the Atangana-Baleanu (A-B) derivative. It also represents the progression of dengue disease more accurately. The existence and uniqueness of solutions has been proved through the application of fixed-point theory within an appropriate functional framework. To ensure the practical relevance, the model has been applied to the actual data of dengue cases occurred in Pakistan during June 2023–December 2025 and is proved that the fractional-order structure is calibrated to the real-world situations. A phase-wise fitting strategy has been adopted across three epidemic phases which demonstrate the superior capability of the fractional Atangana-Baleanu Caputo (ABC) framework in capturing complex multi wave epidemic dynamics. The findings of this work highlight that the Atangana-Baleanu derivative is very helpful in modeling, analysis and control of dengue epidemics.

***Keywords*:** epidemic model, Atangana-Baleanu Fractional Derivative (A-BFD), existence and uniqueness, parameter estimation, controlling strategies

MSC: 26A33, 34A08, 03C65

Abbreviation

SVIR	Susceptible-Vaccinated-Infected-Recovered
DFE	Disease Free Equilibrium
EE	Endemic Equilibrium
U-H	Ulam-Hyers
ANNs	Artificial Neural Networks
MSE	Mean Square Error
RMSE	Root Mean Square Error

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1. Introduction

The fractional derivatives appeared in the world of science during a discussion letter of two notorious mathematicians of 17th century l'Hospital and Leibniz. Later, many mathematicians contributed by their work to this important development. Specially, in the past five decades numerous research articles using Caputo fractional derivative [1] in practical fields as well as in theoretical mathematics came to existence. Never the less, the definition of Caputo fractional derivative has noteworthy weakness of having singular kernel. It is worth noticing that, this issue was successfully handled by Atangana-Baleanu [2] by giving a non-singular kernal having non-local behavior in their definition of Atangana-Baleanu Fractional Derivative (A-BFD).

The A-BFD is crucial in modeling complex systems where traditional derivatives fall short, particularly in capturing memory and hereditary properties. Unlike classical derivatives, it provides a more accurate description of non-local processes, making it invaluable in fields like physics, biology, and engineering. Its non-singular kernel offers a significant advantage, ensuring better stability and convergence in numerical simulations. This derivative enhances our ability to solve real-world problems with greater precision and insight.

Recent advancements in fractional calculus are contributing significantly in many disciplines. Pongsumpun et al. [3] considered a mathematical model of dengue disease with vaccination having seven classes in which authors discussed the disease qualitatively. Ramzan et al. [4] proposed a compartmental model of nine classes on the monkeypox, in which they applied the Caputo-Fabrizio fractional derivative and analysed the model in different way. Abbas et al. [5] discussed the malaria disease with system of differential equations in a very efficient way their analysis includes the global stability and optimal control of the disease. Machine learning tools have been applied on a fractional order time derivative, of a COVID-19, mathematical model by Zanib et al. [6]. Lamwong and Pongsumpun [7] considered a dengue disease model with fractional-order ABC derivative then analyzed the system thoroughly by calculating the global behaviour of equilibrium points, parameter estimation and performed the optimal control. Authors [8] introduced a powerful numerical method for stochastic infectious disease model. Authors [9] have been studied liver transplantation dynamical system by using fractional order derivatives. A fractional order derivative dengue disease reinfection model using Atangana-Baleanu has been studied by Lamwong and Pongsumpun [10]. A hand, foot and mouth disease is discussed by Lamwong and Pongsumpun [11] using a fractional form of mathematical model with ABC-derivative. A seasonal factor is invloved to study the dengue disease with Atangana-Baleanu Caputo (ABC) fractional mathematical model by Lamwong and Pongsumpun [12]. With the help of real data fractional form of rabies disease model with ABC derivative is discussed by considering saturation incidence rate [13] and standard incidence rate [14]. Tajadodi et al. successfully handled optimal control problems using A-BFD [15]. Joshi et al. [16] described calcium dynamics in neurons through a mathematical model involving ABC derivative. This operator explored some interesting phenomena about calcium dynamics.

Solution of optimal control problems involving fractional derivatives has been obtained by Jafari et al. [17] using A-BFD together with the mittag-leffer kernel.

Fractional difference operator, Atangana-Baleanu in the Liouville-Caputo form, has been utilized by Farman et al. [18] to analyze Bovine Brucellosis disease. Forecasts obtained from the model certainly help health care agencies to take preventive actions. Transmission mechanism of hemorrhagic fever caused by Marburg virus has been studied through the fractional model by Singh et al. [19]. They suggested, through the analysis, that newly recovered individuals should protect themselves to interact with the infected bats as they are the main contributors in spreading the disease. Olayiwola and Oluwafemi [20] used the Caputo derivative of fractional order to investigate co-infection model of COVID-19 and Tuberculosis recently. The mentioned authors [21] have studied hereditary and antimicrobial factor which is shaping extracellular bacteria dynamics in an in host mathematical model of tuberculosis. Authors [22] studied tuberculosis dynamics with awareness and vaccination by using fractional calculus. Researchers [23] used fractional calculus techniques to study a mathematical model of tuberculosis disease. Abiodun and Olayiwola [24] have analyzed the tuberculosis infection dynamics by using the Caputo fractional-order models.

Burgers' equations, using A-BFD, has been investigated by Baleanu et al. [25] Authors use Laplace variation iteration approach for the demonstration of mathematical solutions of described model. Modified A-BFD in the sense of Caputo was utilized by Bhatter et al. to examine the early detection of human liver damaging for better treatment [26]. Olayiwola et al.

[27] have used A-BFD to investigate the booster vaccine effect on new COVID-19 variant management employees system. Authors have discussed in detail about the validation of models using real data [28, 29]. Sharma and Vats have studied the Sobolev-type fractional order system for the existence and uniqueness solution [30]. Delayed integro differential equation having fractional order has been studied for the existence and uniqueness of its solution by Sharma and Vats [31]. For the study of fractional integro differential equations one may go through [32–35].

Motivated from the above ideas, we developed a mathematical model consisting of seven classes expressing human and mosquito population. In the modified version, the exposed class in human population has been included which was missing in Boulaaras et al. work [36]. In the present work, we are interested to analyze a quickly spreading viral disease dengue using idea of non-local differential operator.

2. Basic definitions

Definition 1 Let f be the element of $\in H^1(a,b)$ $for\ a<b$ and $\omega \in [0,1)$. The fractional derivative having Mittag Leffler kernel, in Caputo sense, has the form [2]

$$ {}^{ABC}_{a}D^{\omega}_{t} f(t) = -\frac{\Re(\omega)}{\omega-1}\int_{a}^{t} f'(x) E_{\omega}\left(-\omega\frac{(t-x)^{\omega}}{1-\omega}\right)dx $$

where the function $\Re(\omega)$ describes the normalization satisfying $\Re(0)=\Re(1)=1$ as in case of Caputo and Fabrizio [37]. Real world problems can be easily handled using the above definition. This is the modern definition of fractional order derivative which avoids the singular power law kernel. E_{ω} is the Mittag Leffler function acting here as a kernel producing non singular and non local memory effects, with $E_{\omega}(x)=\sum_{l=0}^{\infty}\frac{x^{l}}{\Gamma(\omega l+1)}$. However, when $\omega=0$ the original function cannot be recovered except it vanishes at the origin. To overcome this issue the following definition (in Riemann sense) is proposed [2].

Definition 2 ${}^{ABR}_{a}D^{\omega}_{t} f(t) = -\frac{\Re(\omega)}{\omega-1}\frac{d}{dt}\int_{a}^{t} f(x) E_{\omega}\left(-\omega\frac{(t-x)^{\omega}}{1-\omega}\right)dx$ is known as Atangana-Baleanu fractional derivative of order ω. This definition is given by Abdon Atangana and Dumitru Baleanu. Here the kernel is non-local. Kernel is the form of Mittag Leffler function which produces the smooth memory effects by generalizing the exponential function.

Remark 1 We observe that when $f(t)=C;\ C\in\mathbb{R}$ ${}^{ABC}_{a}D^{\omega}_{t} f(t)$ in definition 1 yields zero.

Definition 3 The integral operator which will act as the inverse operator of the Atangana-Baleanu fractional derivative in the sense of Caputo, definition 1, is defined as under [38] ${}^{AB}_{a}I^{\omega}_{t} f(t) = \frac{1-\omega}{\Re(\omega)} f(t) + \frac{\omega}{\Re(\omega)\Gamma(\omega)}\int_{a}^{t} f(x)(t-x)^{\omega-1}dx$.

Remark 2 It is noted that the initial function and ordinary for $\omega=0$ and $\omega=1$, respectively.

Remark 3 The following property is important to mention [38]: $\left({}^{AB}_{a}I^{\omega}_{t} \circ {}^{ABC}_{a}D^{\omega}_{t}\right) f(t) = f(t) = \left({}^{ABC}_{a}D^{\omega}_{t} \circ {}^{AB}_{a}I^{\omega}_{t}\right) f(t)$ if $f(0)=0$.

3. Model derivation

Using the classical differentiation, numerous mathematical models have been developed for analyzing the transmission pattern of dengue. Recently Boulaaras et al. [36] developed a sophisticated mathematical model assuming different classes such as vaccination, reinfection and carrier class. Although this model was very good and gives the deeper insight of dengue infection but the exposed class of vectors is missing in this work, as suggested by Abidemi and Bazilah Aziz [39], which we include in the present study.

Let N_h be the whole humans at any time t where as N_v be the total vector population at t. Human population is divided into five subclasses: susceptible (S_{h1}), partially immune sensitive (S_{h2}), infected (I_h), carrier (I_{hA}) and recovered (R_h), while vectors are partitioned into three subclasses: susceptible (S_v), exposed (E_v) and infectious vectors (I_v).

The primary mode for transmission of dengue infection occurs through biting of a vector. It depends upon the host's susceptibility, his (her) status of infection, vector biting rate and the probability of transmission between hosts and vectors. The infection rates for susceptible hosts and vectors are $\frac{\alpha_{h1}b}{N_h}I_v$, $\frac{\alpha_{h2}b}{N_h}I_v$, and $\frac{\beta_v b}{N_h}(I_h+I_{hA})$, respectively. We assume that the probability of virus transfer from susceptible host is greater than partially immune sensitive hosts i.e. $\alpha_{h1} > \alpha_{h2}$. The probability of transferring of virus from dengue patients to susceptible vectors is represented as β_v. Fertility and mortality rates of human population and vectors are considered equal represented by the parameters γ_1 and γ_2, respectively. Susceptible people are vaccinated with the rate p where as immune persons looses their immunity with the rate v. The following system results from the aforementioned presumptions:

$$\begin{aligned}
\frac{dS_{h1}}{dt} &= \gamma_1 N_h - \frac{\alpha_{h1}b}{N_h}S_{h1}I_v - pS_{h1} - \gamma_1 S_{h1},\\
\frac{dS_{h2}}{dt} &= vR_h - \frac{\alpha_{h2}b}{N_h}S_{h2}I_v - \gamma_1 S_{h2},\\
\frac{dI_h}{dt} &= (1-\phi)\frac{\alpha_{h1}b}{N_h}S_{h1}I_v + (1-\phi)\frac{\alpha_{h2}b}{N_h}S_{h2}I_v - (\gamma_1+\Lambda+\xi_h)I_h,\\
\frac{dI_{hA}}{dt} &= \phi\frac{\alpha_{h1}b}{N_h}S_{h1}I_v + \phi\frac{\alpha_{h2}b}{N_h}S_{h2}I_v - (\gamma_1+\xi_h)I_{hA},\\
\frac{dR_h}{dt} &= pS_{h1} + \xi_h(I_h+I_{hA}) + \Lambda I_h - (v+\gamma_1)R_h,\\
\frac{dS_v}{dt} &= \gamma_2 N_v - \frac{\beta_v b}{N_h}(I_h+I_{hA})S_v - \gamma_2 S_v,\\
\frac{dE_v}{dt} &= \frac{\beta_v b}{N_h}(I_h+I_{hA})S_v - \eta_v E_v - \gamma_2 E_v,\\
\frac{dI_v}{dt} &= \eta_v E_v - \gamma_2 I_v,
\end{aligned} \tag{1}$$

with initial conditions, $S_{h1}(0)\geq 0, S_{h2}(0)\geq 0, I_h(0)\geq 0, I_{hA}(0)\geq 0, R_h(0)\geq 0, S_v(0)\geq 0, E_v(0)\geq 0, I_v(0)\geq 0$. Flow diagram of the model is given in Figure 1.

Proposition 1 The feasible region for the fractional system is

$$\Delta = \{S_{h1}(t), S_{h2}(t), I_h(t), I_{hA}(t), R_h(t), S_v(t), E_v(t), I_v(t) \in R_+^7, 0\leq N_h(t)\leq M_1, 0\leq N_v(t)\leq M_2.\}$$

Proof. Add first five equations of system 1, we have

$$\frac{dN_h}{dt} = \gamma_1 N_h - \gamma_1(S_{h1}+S_{h2}+I_h+I_{hA}+R_h),$$

$$\frac{dN_h}{dt} = 0,$$

hence N_h is less than or equal to a constant say M_1.

Also last three equations of system 1, we have

$$\frac{dN_v}{dt} = \gamma_2 N_v - \gamma_2(S_v + E_v + I_v),$$

$$\frac{dN_v}{dt} = 0,$$

hence N_v is less than or equal to a constant say M_2. □

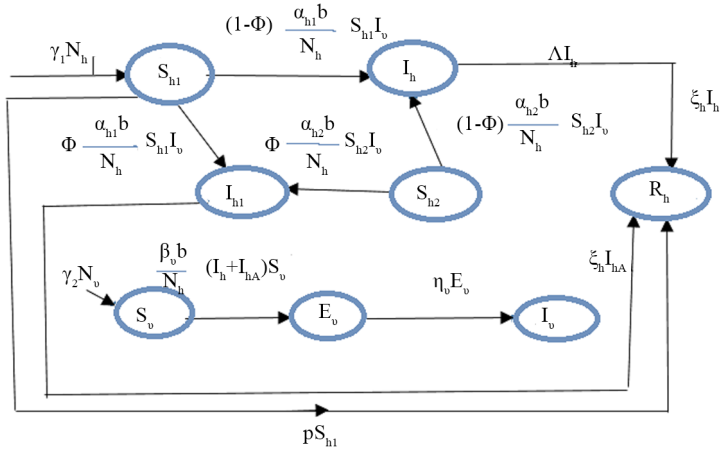


Figure 1. Flow diagram

4. Existence of solutions for proposed model

It can be observed that problem under consideration here is nonlinear as well as nonlocal, it is possible that no current approach can offer precise solutions to these problems. Nonetheless, a crucial mathematical idea for the existence of outcomes helps ensure that, sometimes the precise answer will exist. This section will provide a detailed existence of our system's solution. This will be made possible by the famous Banach fixed-point theorem.

Let $\mathbb{P} = \mathbb{K}(s) \times \mathbb{K}(s)$ where $\mathbb{K}(s)$ is a Banach space containing the functions with domain interval s having the property that values are continues. Moreover, the norm is defined as

$$\|S_{h1}, S_{h2}, I_h, I_{hA}, R_h, S_v, E_v, I_v\| = \|S_{h1}\| + \|S_{h2}\| + \|I_h\| + \|I_{hA}\| + \|R_h\| + \|S_v\| + \|E_v\| + \|I_v\|.$$

Here, $\|S_{h1}\| = sup\{|S_{h1}(t)| : t \in s\}$, $\|S_{h2}\| = sup\{|S_{h2}(t)| : t \in s\}$, $\|I_h\| = sup\{|I_h(t)| : t \in s\}$, $\|I_{hA}\| = sup\{|I_{hA}(t)| : t \in s\}$, $\|R_h\| = sup\{|R_h(t)| : t \in s\}$, $\|E_v\| = sup\{|E_v(t)| : t \in s\}$ and $\|I_v\| = sup\{|I_v(t)| : t \in s\}$.

Let us redefine the classical model of Dengue disease by replacing the usual derivative by the non commensurate ordered derivative:

$$
\begin{aligned}
{}_0^{ABC}D_t^{\omega} S_{h1} &= \gamma_1 N_h - \frac{\alpha_{h1} b}{N_h} S_{h1} I_v - p S_{h1} - \gamma_1 S_{h1},\\
{}_0^{ABC}D_t^{\omega} S_{h2} &= v R_h - \frac{\alpha_{h2} b}{N_h} S_{h2} I_v - \gamma_1 S_{h2},\\
{}_0^{ABC}D_t^{\omega} I_h &= (1-\phi)\frac{\alpha_{h1} b}{N_h} S_{h1} I_v + (1-\phi)\frac{\alpha_{h2} b}{N_h} S_{h2} I_v - (\gamma_1 + \Lambda + \xi_h) I_h,\\
{}_0^{ABC}D_t^{\omega} I_{hA} &= \phi \frac{\alpha_{h1} b}{N_h} S_{h1} I_v + \phi \frac{\alpha_{h2} b}{N_h} S_{h2} I_v - (\gamma_1 + \xi_h) I_{hA},\\
{}_0^{ABC}D_t^{\omega} R_h &= p S_{h1} + \xi_h (I_h + I_{hA}) + \Lambda I_h - (v + \gamma_1) R_h,\\
{}_0^{ABC}D_t^{\omega} S_v &= \gamma_2 N_v - \frac{\beta_v b}{N_h} (I_h + I_{hA}) S_v - \gamma_2 S_v,\\
{}_0^{ABC}D_t^{\omega} E_v &= \frac{\beta_v b}{N_h} (I_h + I_{hA}) S_v - \eta_v E_v - \gamma_2 E_v,\\
{}_0^{ABC}D_t^{\omega} I_v &= \eta_v E_v - \gamma_2 I_v.
\end{aligned} \tag{2}
$$

With initial conditions $S_{h1}(0) = (S_{h1})_0, S_{h2}(0) = (S_{h2})_0,\ I_h(0) = (I_h)_0,\ I_{hA}(0) = (I_{hA})_0, R_h(0) = (R_h)_0,\ S_v(0) = (S_v)_0,\ E_v(0) = (E_v)_0,\ I_v(0) = (I_v)_0$.

The corresponding Volterra type integral equations with the Atangana-Baleanu fractional integral for the above system (2) are given below. Thus

$$
\begin{aligned}
S_{h1}(t) - S_{h1}(0) &= \frac{1-\omega}{\Re(\omega)} \left\{ \gamma_1 N_h - \frac{\alpha_{h1} b}{N_h} S_{h1}(t) I_v(t) - p S_{h1}(t) - \gamma_1 S_{h1}(t) \right\}\\
&\quad + \frac{\omega}{\Gamma(\omega)\Re(\omega)} \int_0^t (t-y)^{\omega-1} \left\{ \gamma_1 N_h - \frac{\alpha_{h1} b}{N_h} S_{h1}(y) I_v(y) - p S_{h1}(y) - \gamma_1 S_{h1}(y) \right\} dy,\\
S_{h2}(t) - S_{h2}(0) &= \frac{1-\omega}{\Re(\omega)} \left\{ v R_h(t) - \frac{\alpha_{h2} b}{N_h} S_{h2}(t) I_v(t) - \gamma_1 S_{h2}(t) \right\}\\
&\quad + \frac{\omega}{\Gamma(\omega)\Re(\omega)} \int_0^t (t-y)^{\omega-1} \left\{ v R_h(y) - \frac{\alpha_{h2} b}{N_h} S_{h2}(y) I_v(y) - \gamma_1 S_{h2}(y) \right\} dy,\\
I_h(t) - I_h(0) &= \frac{1-\omega}{\Re(\omega)} \left\{ (1-\phi)\frac{\alpha_{h1} b}{N_h} S_{h1}(t) I_v(t) + (1-\phi)\frac{\alpha_{h2} b}{N_h} S_{h2}(t) I_v(t) - (\gamma_1 + \Lambda + \xi_h) I_h(t) \right\}
\end{aligned}
$$

$$+\frac{\omega}{\Gamma(\omega)\Re(\omega)}\int_0^t (t-y)^{\omega-1}$$

$$\left\{(1-\phi)\frac{\alpha_{h1}b}{N_h}S_{h1}(y)I_v(y)+(1-\phi)\frac{\alpha_{h2}b}{N_h}S_{h2}(y)I_v(y)-(\gamma_1+\Lambda+\xi_h)I_h(y)\right\}dy,$$

$$I_{hA}(t)-I_{hA}(0)=\frac{1-\omega}{\Re(\omega)}\left\{\phi\frac{\alpha_{h1}b}{N_h}S_{h1}(t)I_v(t)+\phi\frac{\alpha_{h2}b}{N_h}S_{h2}(t)I_v(t)-(\gamma_1+\xi_h)I_{hA}(t)\right\}$$

$$+\frac{\omega}{\Gamma(\omega)\Re(\omega)}\int_0^t (t-y)^{\omega-1}\left\{\phi\frac{\alpha_{h1}b}{N_h}S_{h1}(y)I_v(y)+\phi\frac{\alpha_{h2}b}{N_h}S_{h2}(y)I_v(y)-(\gamma_1+\xi_h)I_{hA}(y)\right\}dy,$$

$$R_h(t)-R_h(0)=\frac{1-\omega}{\Re(\omega)}\{pS_{h1}(t)+\xi_h(I_h(t)+I_{hA}(t))+\Lambda I_h(t)-(v+\gamma_1)R_h(t)\}$$

$$+\frac{\omega}{\Gamma(\omega)\Re(\omega)}\int_0^t (t-y)^{\omega-1}\{pS_{h1}(y)+\xi_h(I_h(y)+I_{hA}(y))+\Lambda I_h(y)-(v+\gamma_1)R_h(y)\}dy,$$

$$S_v(t)-S_v(0)=\frac{1-\omega}{\Re(\omega)}\left\{\gamma_2 N_v-\frac{\beta_v b}{N_h}(I_h(t)+I_{hA}(t))S_v(t)-\gamma_2 S_v(t)\right\}$$

$$+\frac{\omega}{\Gamma(\omega)\Re(\omega)}\int_0^t (t-y)^{\omega-1}\left\{\gamma_2 N_v-\frac{\beta_v b}{N_h}(I_h(y)+I_{hA}(y))S_v(y)-\gamma_2 S_v(y)\right\}dy,$$

$$E_v(t)-E_v(0)=\frac{1-\omega}{\Re(\omega)}\left\{\frac{\beta_v b}{N_h}(I_h(t)+I_{hA}(t))S_v(t)-\eta_v E_v(t)-\gamma_2 E_v(t)\right\}$$

$$+\frac{\omega}{\Gamma(\omega)\Re(\omega)}\int_0^t (t-y)^{\omega-1}\left\{\frac{\beta_v b}{N_h}(I_h(y)+I_{hA}(y))S_v(y)-\eta_v E_v(y)-\gamma_2 E_v(y)\right\}dy,$$

$$I_v(t)-I_v(0)=\frac{1-\omega}{\Re(\omega)}\{\eta_v E_v(t)-\gamma_2 I_v(t)\}+\frac{\omega}{\Gamma(\omega)\Re(\omega)}\int_0^t (t-y)^{\omega-1}\{\eta_v E_v(y)-\gamma_2 I_v(y)\}dy.$$

For simplification, we can write

$$K_1(t,S_{h1})=\gamma_1 N_h-\frac{\alpha_{h1}b}{N_h}S_{h1}I_v-pS_{h1}-\gamma_1 S_{h1},$$

$$K_2(t,S_{h2})=vR_h-\frac{\alpha_{h2}b}{N_h}S_{h2}I_v-\gamma_1 S_{h2},$$

$$K_3(t,I_h)=(1-\phi)\frac{\alpha_{h1}b}{N_h}S_{h1}I_v+(1-\phi)\frac{\alpha_{h2}b}{N_h}S_{h2}I_v-(\gamma_1+\Lambda+\xi_h)I_h,$$

$$K_4(t,I_{hA}) = \phi\frac{\alpha_{h1}b}{N_h}S_{h1}I_v + \phi\frac{\alpha_{h2}b}{N_h}S_{h2}I_v - (\gamma_1+\xi_h)I_{hA},$$

$$K_5(t,R_h) = pS_{h1} + \xi_h(I_h+I_{hA}) + \Lambda I_h - (v+\gamma_1)R_h,$$

$$K_6(t,S_v) = \gamma_2 N_v - \frac{\beta_v b}{N_h}(I_h+I_{hA})S_v - \gamma_2 S_v,$$

$$K_7(t,E_v) = \frac{\beta_v b}{N_h}(I_h+I_{hA})S_v - \eta_v E_v - \gamma_2 E_v,$$

$$K_8(t,I_v) = \eta_v E_v - \gamma_2 I_v. \tag{3}$$

Theorem 1 The kernels $K_1,\ K_2,\ K_3,\ K_4,\ K_5,\ K_6,\ K_7$ and K_8 satisfy the Lipschitz condition and are contractions if the following inequality holds:

$$\begin{gathered}
0 \le \beta_1 < 1,\\
0 \le \beta_2 < 1,\\
0 \le \beta_3 < 1,\\
0 \le \beta_4 < 1,\\
0 \le \beta_5 < 1,\\
0 \le \beta_6 < 1,\\
0 \le \beta_7 < 1,\\
0 \le \beta_8 < 1.
\end{gathered} \tag{4}$$

Proof. Let us start the kernel $K_1(t,S_{h1}) = \gamma_1 N_h - \frac{\alpha_{h1}b}{N_h}S_{h1}I_v - pS_{h1} - \gamma_1 S_{h1}$. For S_{h1} and $\tilde{S}_{h1}$ we have:

$$\begin{aligned}
\left\|K_1(t,S_{h1}) - K_1(t,\tilde{S}_{h1})\right\| &= \left\| -\frac{\alpha_{h1}b}{N_h}S_{h1}I_v + \frac{\alpha_{h1}b}{N_h}\tilde{S}_{h1}I_v - pS_{h1} + p\tilde{S}_{h1} \right\|\\
&\le \left(\alpha_{h1}b\left\|I_v\right\| + p\right)\left\|S_{h1} - \tilde{S}_{h1}\right\|.
\end{aligned} \tag{5}$$

Taking $\beta_1 = (\alpha_{h1} b h_1 + p)$, where $a_1 = max_{t\in I} \|S_{h1}(t)\|$, $b_1 = max_{t\in I} \|S_{h2}(t)\|$, $c_1 = max_{t\in I} \|I_h(t)\|$, $d_1 = max_{t\in I} \|I_{hA}(t)\|$, $e_1 = max_{t\in I} \|R_h(t)\|$, $f_1 = max_{t\in I} \|S_v(t)\|$, $g_1 = max_{t\in I} \|E_v(t)\|$, $h_1 = max_{t\in I} \|I_v(t)\|$ are bounded function, then we get

$$\left\|K_1(t, S_{h1}) - K_1(t, \tilde{S}_{h1})\right\| \leq \beta_1 \left\|S_{h1}(t) - \tilde{S}_{h1}(t)\right\|. \tag{6}$$

(6) shows that Lipschitz condition is true for K_1, and if $0 \leq \beta_1 < 1$, then K_1 is a contraction as well. The same is true for rest of the kernels, this can be seen as follows:

$$\begin{aligned}
&\left\|K_2(t, S_{h2}) - K_2(t, \tilde{S}_{h2})\right\| \leq \beta_2 \left\|S_{h2}(t) - \tilde{S}_{h2}(t)\right\|,\\
&\left\|K_3(t, S_v) - K_3(t, \tilde{S}_v)\right\| \leq \beta_3 \left\|S_v(t) - \tilde{S}_v(t)\right\|,\\
&\left\|K_4(t, I_h) - K_4(t, \tilde{I}_h)\right\| \leq \beta_4 \left\|I_h(t) - \tilde{I}_h(t)\right\|,\\
&\left\|K_5(t, I_{hA}) - K_5(t, \tilde{I}_{hA})\right\| \leq \beta_5 \left\|I_{hA}(t) - \tilde{I}_{hA}(t)\right\|,\\
&\left\|K_6(t, R_h) - K_6(t, \tilde{R}_h)\right\| \leq \beta_6 \left\|R_h(t) - \tilde{R}_h(t)\right\|,\\
&\left\|K_7(t, E_v) - K_7(t, \tilde{E}_v)\right\| \leq \beta_7 \left\|E_v(t) - \tilde{E}_v(t)\right\|,\\
&\left\|K_8(t, I_v) - K_8(t, \tilde{I}_v)\right\| \leq \beta_8 \left\|I_v(t) - \tilde{I}_v(t)\right\|,
\end{aligned} \tag{7}$$

where $\beta_2 = (\alpha_{h2} b h_1 + \gamma_1), \beta_3 = \gamma_1 + \Lambda + \xi_h, \beta_4 = \gamma_1 + \xi_h, \beta_5 = \nu + \gamma_1, \beta_6 = \beta_v b(c_1 + d_1), \beta_7 = \eta_v + \gamma_2, \beta_8 = \gamma_2$. The numerical values of $\beta_1, ..., \beta_8$ are calculated as given in Table 1.

Table 1. Numerical Values of $\beta_1, ..., \beta_8$

Contraction Parameters	Phase I	Phase II	Phase III
β_1	0.00504173	0.04515161	0.04689103
β_2	0.27101839	0.07737868	0.08952473
β_3	0.06110979	0.8041662	0.26775875
β_4	0.05104206	0.50416624	0.05104275
β_5	0.01104206	0.03966041	0.3010419
β_6	0.49082063	0.31192848	0.56846289
β_7	0.5851057	0.55896804	0.97999948
β_8	0.48510464	0.28498527	0.48999965

We can rewrite system (4) by taking the kernels in the model:

$$S_{h1}(t) = S_{h1}(0) + \frac{1-\omega}{\Re(\omega)}K_1(t, S_{h1}) + \frac{\omega}{\Gamma(\omega)\Re(\omega)}\int_0^t (t-y)^{\omega-1} K_1(y, S_{h1})dy,$$

$$S_{h2}(t) = S_{h2}(0) + \frac{1-\omega}{\Re(\omega)}K_2(t, S_{h2}) + \frac{\omega}{\Gamma(\omega)\Re(\omega)}\int_0^t (t-y)^{\omega-1} K_2(y, S_{h2})dy,$$

$$I_h(t) = I_h(0) + \frac{1-\omega}{\Re(\omega)}K_3(t, I_h) + \frac{\omega}{\Gamma(\omega)\Re(\omega)}\int_0^t (t-y)^{\omega-1} K_3(y, I_h)dy,$$

$$I_{hA}(t) = I_{hA}(0) + \frac{1-\omega}{\Re(\omega)}K_4(t, I_{hA}) + \frac{\omega}{\Gamma(\omega)\Re(\omega)}\int_0^t (t-y)^{\omega-1} K_4(y, I_{hA})dy,$$

$$R_h(t) = R_h(0) + \frac{1-\omega}{\Re(\omega)}K_5(t, R_h) + \frac{\omega}{\Gamma(\omega)\Re(\omega)}\int_0^t (t-y)^{\omega-1} K_5(y, R_h)dy,$$

$$S_v(t) = S_v(0) + \frac{1-\omega}{\Re(\omega)}K_6(t, S_v) + \frac{\omega}{\Gamma(\omega)\Re(\omega)}\int_0^t (t-y)^{\omega-1} K_6(y, S_v)dy,$$

$$E_v(t) = E_v(0) + \frac{1-\omega}{\Re(\omega)}K_7(t, E_v) + \frac{\omega}{\Gamma(\omega)\Re(\omega)}\int_0^t (t-y)^{\omega-1} K_7(y, E_v)dy,$$

$$I_v(t) = I_v(0) + \frac{1-\omega}{\Re(\omega)}K_8(t, I_v) + \frac{\omega}{\Gamma(\omega)\Re(\omega)}\int_0^t (t-y)^{\omega-1} K_8(y, I_v)dy. \tag{8}$$

Now we are able to write the recursive formula as:

$$(S_{h1})_n(t) = \frac{1-\omega}{\Re(\omega)}K_1(t, (S_{h1})_{n-1}) + \frac{\omega}{\Gamma(\omega)\Re(\omega)}\int_0^t \frac{1}{(t-y)^{1-\omega}} K_1(y, (S_{h1})_{n-1})dy,$$

$$(S_{h2})_n(t) = \frac{1-\omega}{\Re(\omega)}K_2(t, (S_{h2})_{n-1}) + \frac{\omega}{\Gamma(\omega)\Re(\omega)}\int_0^t (t-y)^{\omega-1} K_2(y, (S_{h2})_{n-1})dy,$$

$$(I_h)_n(t) = \frac{1-\omega}{\Re(\omega)}K_3(t, (I_h)_{n-1}) + \frac{\omega}{\Gamma(\omega)\Re(\omega)}\int_0^t (t-y)^{\omega-1} K_3(y, (I_h)_{n-1})dy,$$

$$(I_{hA})_n(t) = \frac{1-\omega}{\Re(\omega)}K_4(t, (I_{hA})_{n-1}) + \frac{\omega}{\Gamma(\omega)\Re(\omega)}\int_0^t (t-y)^{\omega-1} K_4(y, (I_{hA})_{n-1})dy,$$

$$(R_h)_n(t) = \frac{1-\omega}{\Re(\omega)}K_5(t, (R_h)_{n-1}) + \frac{\omega}{\Gamma(\omega)\Re(\omega)}\int_0^t (t-y)^{\omega-1} K_5(y, (R_h)_{n-1})dy,$$

$$(S_v)_n(t) = \frac{1-\omega}{\Re(\omega)}K_6(t, (S_v)_{n-1}) + \frac{\omega}{\Gamma(\omega)\Re(\omega)}\int_0^t (t-y)^{\omega-1} K_6(y, (S_v)_{n-1})dy,$$

$$(E_v)_n(t) = \frac{1-\omega}{\Re(\omega)} K_7(t,(E_v)_{n-1}) + \frac{\omega}{\Gamma(\omega)\Re(\omega)} \int_0^t (t-y)^{\omega-1} K_7(y,(E_v)_{n-1}) dy,$$

$$(I_v)_n(t) = \frac{1-\omega}{\Re(\omega)} K_8(t,(I_v)_{n-1}) + \frac{\omega}{\Gamma(\omega)\Re(\omega)} \int_0^t (t-y)^{\omega-1} K_8(y,(I_v)_{n-1}) dy. \tag{9}$$

Together with initial conditions given by:

$$(S_{h1})_0(t) = S_{h1}(0),$$

$$(S_{h2})_0(t) = S_{h2}(0),$$

$$(I_h)_0(t) = I_h(0),$$

$$(I_{hA})_0(t) = I_{hA}(0),$$

$$(R_h)_0(t) = R_h(0),$$

$$(S_v)_0(t) = S_v(0),$$

$$(E_v)_0(t) = E_v(0),$$

$$(I_v)_0(t) = I_v(0). \tag{10}$$

Now, we obtain the difference between among adjacent terms for all the variables

$$\gamma_i(t) = (S_{h1})_i(t) - (S_{h1})_{i-1}(t) = \frac{1-\omega}{\Re(\omega)} \left\{ K_1(t,(S_{h1})_{i-1}) - K_1(t,(S_{h1})_{i-2}) \right\}$$

$$+ \frac{\omega}{\Gamma(\omega)\Re(\omega)} \int_0^t (t-y)^{\omega-1} \left\{ K_1(y,(S_{h1})_{i-1}) - K_1(y,(S_{h1})_{i-2}) \right\} dy,$$

$$\phi_i(t) = (S_{h2})_i - (S_{h2})_{i-1} = \frac{1-\omega}{\Re(\omega)} \left\{ K_2(t,(S_{h2})_{i-1}) - K_2(t,(S_{h2})_{i-2}) \right\}$$

$$+ \frac{\omega}{\Gamma(\omega)\Re(\omega)} \int_0^t (t-y)^{\omega-1} \left\{ K_2(y,(S_{h2})_{i-1}) - K_2(y,(S_{h2})_{i-2}) \right\} dy,$$

$$\psi_i(t) = (I_h)_n(t) - (I_h)_{i-1}(t) = \frac{1-\omega}{\Re(\omega)} \left\{ K_3(t,(I_h)_{i-1}) - K_3(t,(I_h)_{i-2}) \right\}$$

$$+\frac{\omega}{\Gamma(\omega)\Re(\omega)}\int_0^t (t-\eta)^{\omega-1}\left\{K_3(\eta,(I_h)_{i-1})-K_3(\eta,(I_h)_{i-2})\right\}d\eta,$$

$$\varepsilon_i(t)=(I_{hA})_i(t)-(I_{hA})_{i-1}(t)=\frac{1-\omega}{\Re(\omega)}\left\{K_4(t,(I_{hA})_{i-1})-K_4(t,(I_{hA})_{i-2})\right\}$$

$$+\frac{\omega}{\Gamma(\omega)\Re(\omega)}\int_0^t (t-y)^{\omega-1}\left\{K_4(y,(I_{hA})_{i-1})-K_4(y,(I_{hA})_{i-2})\right\}dy,$$

$$\lambda_i(t)=(R_h)_i(t)-(R_h)_{i-1}(t)=\frac{1-\omega}{\Re(\omega)}\left\{K_5(t,(R_h)_{i-1})-K_5(t,(R_h)_{i-2})\right\}$$

$$+\frac{\omega}{\Gamma(\omega)\Re(\omega)}\int_0^t (t-y)^{\omega-1}\left\{K_5(y,(R_h)_{i-1})-K_5(y,(R_h)_{i-2})\right\}dy,$$

$$\chi_i(t)=(S_v)_i(t)-(S_v)_{i-1}(t)=\frac{1-\omega}{\Re(\omega)}\left\{K_6(t,(S_v)_{i-1})-K_6(t,(S_v)_{i-2})\right\}$$

$$+\frac{\omega}{\Gamma(\omega)\Re(\omega)}\int_0^t (t-\eta)^{\omega-1}\left\{K_6(\eta,(S_v)_{i-1})-K_6(\eta,(S_v)_{i-2})\right\}d\eta,$$

$$\sigma_i(t)=(E_v)_i(t)-(E_v)_{i-1}(t)=\frac{1-\omega}{\Re(\omega)}\left\{K_7(t,(E_v)_{i-1})-K_7(t,(E_v)_{i-2})\right\}$$

$$+\frac{\omega}{\Gamma(\omega)\Re(\omega)}\int_0^t (t-y)^{\omega-1}\left\{K_7(y,(E_v)_{i-1})-K_7(y,(E_v)_{i-2})\right\}dy,$$

$$\theta_i(t)=(I_v)_i(t)-(I_v)_{i-1}(t)=\frac{1-\omega}{\Re(\omega)}\left\{K_8(t,(I_v)_{i-1})-K_8(t,(I_v)_{i-2})\right\}$$

$$+\frac{\omega}{\Gamma(\omega)\Re(\omega)}\int_0^t (t-y)^{\omega-1}\left\{K_8(y,(I_v)_{i-1})-K_8(y,(I_v)_{i-2})\right\}dy. \tag{11}$$

Noteworthy is the fact that:

$$(S_{h1})_n(t)=\sum_{i=1}^{n}\gamma_i(t),$$

$$(S_{h2})_n(t)=\sum_{i=1}^{n}\phi_i(t),$$

$$(I_h)_n(t)=\sum_{i=1}^{n}\psi_i(t),$$

$$(I_{hA})_n(t) = \sum_{i=1}^{n} \varepsilon_i(t),$$

$$(R_h)_n(t) = \sum_{i=1}^{n} \lambda_i(t),$$

$$(S_v)_n(t) = \sum_{i=1}^{n} \chi_i(t),$$

$$(E_v)_n(t) = \sum_{i=1}^{n} \sigma_i(t),$$

$$(I_v)_n(t) = \sum_{i=1}^{n} \theta_i(t). \tag{12}$$

Let us apply the norm on both sides of the first expression appearing in (12) and using the triangular inequality, we have

$$\begin{aligned} \|\gamma_n(t)\| &= \left\|(S_{h1})_n(t) - (S_{h1})_{n-1}(t)\right\| \\ &\leq \frac{1-\omega}{\Re(\omega)} \left\|K_1(t,(S_{h1})_{n-1}) - K_1(t,(S_{h1})_{n-2})\right\| \\ &\quad + \frac{\omega}{\Gamma(\omega)\Re(\omega)} \left\| \int_0^t (t-\eta)^{\omega-1} K_1(\eta,(S_{h1})_{n-1}) - K_1(\eta,(S_{h1})_{n-2}) \right\| d\eta, \end{aligned} \tag{13}$$

As K_1 holds the Lipschitz condition, we get,

$$\begin{aligned} \left\|(S_{h1})_n(t) - (S_{h1})_{n-1}(t)\right\| &\leq \frac{1}{\Re(\omega)} \beta_1 (1-\omega) \left\|(S_{h1})_{n-1} - (S_{h1})_{n-2}\right\| \\ &\quad + \frac{\omega}{\Gamma(\omega)\Re(\omega)} \beta_1 \int_0^t (t-\eta)^{\omega-1} \left\|(S_{h1})_{n-1} - (S_{h1})_{n-2}\right\| d\eta, \end{aligned} \tag{14}$$

furthermore,

$$\|\gamma_n(t)\| \leq \frac{\beta_1}{\Re(\omega)} \left[\|\gamma_{n-1}(t)\| (1-\omega) + \frac{\omega}{\Gamma(\omega)} \int_0^t (t-y)^{\omega-1} \|\gamma_{n-1}(y)\| dy \right]. \tag{15}$$

Following the same manner we obtain the results given below:

$$\|\phi_n(t)\| \leq \frac{1-\omega}{\Re(\omega)}\beta_2 \|\phi_{n-1}(t)\| + \frac{\omega}{\Re(\omega)\Gamma(\omega)}\beta_2 \int_0^t (t-y)^{\omega-1} \|\phi_{n-1}(y)\| dy,$$

$$\|\psi_n(t)\| \leq \frac{1-\omega}{\Re(\omega)}\beta_3 \|\psi_{n-1}(t)\| + \beta_3 \int_0^t \frac{\omega}{\Gamma(\omega)\Re(\omega)} (t-y)^{\omega-1} \|\psi_{n-1}(y)\| dy,$$

$$\|\varepsilon_n(t)\| \leq \frac{1-\omega}{\Re(\omega)}\beta_4 \|\varepsilon_{n-1}(t)\| + \beta_4 \int_0^t \frac{\omega}{\Gamma(\omega)\Re(\omega)} (t-y)^{\omega-1} \|\varepsilon_{n-1}(y)\| dy,$$

$$\|\lambda_n(t)\| \leq \frac{1-\omega}{\Re(\omega)}\beta_5 \|\lambda_{n-1}(t)\| + \frac{\omega}{B(\omega)\Gamma(\omega)}\beta_5 \int_0^t (t-y)^{\omega-1} \|\lambda_{n-1}(y)\| dy,$$

$$\|\chi_n(t)\| \leq \frac{1-\omega}{\Re(\omega)}\beta_6 \|\chi_{n-1}(t)\| + \frac{\omega}{\Re(\omega)\Gamma(\omega)}\beta_6 \int_0^t (t-y)^{\omega-1} \|\chi_{n-1}(y)\| dy,$$

$$\|\sigma_n(t)\| \leq \frac{1-\omega}{\Re(\omega)}\beta_7 \|\sigma_{n-1}(t)\| + \frac{\omega}{B(\omega)\Gamma(\omega)}\beta_7 \int_0^t (t-y)^{\omega-1} \|\sigma_{n-1}(y)\| dy,$$

$$\|\theta_n(t)\| \leq \frac{1-\omega}{\Re(\omega)}\beta_8 \|\theta_{n-1}(t)\| + \frac{\omega}{B(\omega)\Gamma(\omega)}\beta_8 \int_0^t (t-y)^{\omega-1} \|\theta_{n-1}(y)\| dy. \tag{16}$$

□

Let's provide a new theorem in light of the aforementioned findings.

Theorem 2 There exists a unique solution to the Dengue spread model (2) provided that t_{max} satisfies:

$$\frac{1}{\Re(\omega)}\left[(1-\omega)\beta_i + \frac{t_{max}^{\omega}}{\Gamma(\omega)}\right]\beta_i < 1, for\ i = 1,2,...,8. \tag{17}$$

Proof. We know that $S_{h1}(t)$, $S_{h2}(t)$, $I_h(t)$, $I_{hA}(t)$, $R_h(t)$, $E_v(t)$, $S_v(t)$ and $I_v(t)$ are bounded together with their kernels fulfills the Lipschitzian property. So from the inequality (17), we get the following relations:

$$\|\gamma_n(t)\| \leq \|S_{h1}(0)\| \frac{1}{\Re(\omega)}\left[(1-\omega)\beta_1 + \frac{t_{max}^{\omega}}{\Gamma(\omega)}\right]\beta_1^n,$$

$$\|\phi_n(t)\| \leq \|S_{h2}(0)\| \frac{1}{\Re(\omega)}\left[(1-\omega)\beta_2 + \frac{t_{max}^{\omega}}{\Gamma(\omega)}\right]\beta_2^n,$$

$$\|\psi_n(t)\| \leq \|I_h(0)\| \left[(1-\omega)\beta_3 + \frac{t_{max}^{\omega}}{\Gamma(\omega)}\right]\frac{1}{\Re(\omega)}\beta_3^n,$$

$$\|\varepsilon_n(t)\| \leq \|I_{hA}(0)\| \frac{1}{\Re(\omega)}\left[(1-\omega)\beta_4 + \frac{t_{max}^{\omega}}{\Gamma(\omega)}\right]\beta_4^n,$$

$$\|\lambda_n(t)\| \leq \frac{1}{\mathfrak{K}(\omega)} \|R_h(0)\| \left[(1-\omega)\beta_5 + \frac{t_{max}^{\omega}}{\Gamma(\omega)}\right] \beta_5^n,$$

$$\|\chi_n(t)\| \leq \|S_v(0)\| \frac{1}{\mathfrak{K}(\omega)} \left[(1-\omega)\beta_6 + \frac{t_{max}^{\omega}}{\Gamma(\omega)}\right] \beta_6^n,$$

$$\|\sigma_n(t)\| \leq \|E_v(0)\| \frac{1}{\mathfrak{K}(\omega)} \left[(1-\omega)\beta_7 + \frac{t_{max}^{\omega}}{\Gamma(\omega)}\right] \beta_7^n,$$

$$\|\theta_n(t)\| \leq \|I_v(0)\| \frac{1}{\mathfrak{K}(\omega)} \left[(1-\omega)\beta_8 + \frac{t_{max}^{\omega}}{\Gamma(\omega)}\right] \beta_8^n. \tag{18}$$

Thus (14) exists and is a smooth function. For the sake of proving that above mentioned functions provide solution to the problem, we let:

$$S_{h1}(t) - S_{h1}(0) = (S_{h1})_n(t) - b_n(t),$$

$$S_{h2}(t) - S_{h2}(0) = (S_{h2})_n(t) - c_n(t),$$

$$I_h(t) - I_h(0) = (I_h)_n(t) - d_n(t),$$

$$I_{hA}(t) - I_{hA}(0) = (I_{hA})_n(t) - e_n(t),$$

$$R_h(t) - R_h(0) = (R_h)_n(t) - f_n(t),$$

$$S_v(t) - S_v(0) = (S_v)_n(t) - g_n(t),$$

$$E_v(t) - E_v(0) = (E_v)_n(t) - h_n(t),$$

$$I_v(t) - I_v(0) = (I_v)_n(t) - i_n(t). \tag{19}$$

We are to show that $\underset{n\to\infty}{\ell im} \| b_n(t)\| = 0$. For this,

$$\|b_n(t)\| \leq \frac{(1-\omega)}{\mathfrak{K}(\omega)} \left\|K_1(t, S_{h1}) - K_1(t, (S_{h1})_{n-1})\right\|$$

$$+ \int_0^t \frac{\omega}{\Gamma(\omega)\mathfrak{K}(\omega)} (t-y)^{\omega-1} \left\|K_1(t, S_{h1}) - K_1(t, (S_{h1})_{n-1})\right\| dy$$

$$\leq \frac{1-\omega}{\mathfrak{K}(\omega)} \beta_1 \left\|S_{h1} - (S_{h1})_{n-1}\right\| + \frac{t^{\omega}}{\mathfrak{K}(\omega)\Gamma(\omega)} \beta_1 \left\|S_{h1} - (S_{h1})_{n-1}\right\|. \tag{20}$$

Repeating this process recursively, we obtain

$$\|b_n(t)\| \leq \left[\frac{t^{\omega}}{\Gamma(\omega)} + (1-\omega)\right]^{n+1} \frac{\beta_1^n M}{\Re(\omega)}. \tag{21}$$

Then at t_{max} we have

$$\|b_n(t)\| \leq \frac{1}{\Re(\omega)} \left[(1-\omega) + \frac{t_{max}^{\omega}}{\Gamma(\omega)}\right]^{n+1} \beta_1^n M. \tag{22}$$

For n tends to infinity, we obtain the required result. □

4.1 *Uniqueness of the special solution*

Next we show the uniqueness of solutions for our system. Assuming on contrary, $(S_{h1})_1(t)$, $(S_{h2})_1(t)$, $(I_h)_1(t), (I_{hA})_1(t)$, $(R_h)_1(t), (S_v)_1(t)$, $(E_v)_1(t)$,and $(I_v)_1(t)$ be solutions of (2). Then

$$\begin{aligned}\|(S_{h1})(t) - (S_{h1})_1(t)\| \leq{}& \frac{1-\omega}{\Re(\omega)} \{K_1(t, S_{h1}) - K_1(t, (S_{h1})_1)\} \\ &+ \int_0^t \frac{\omega}{\Gamma(\omega)\Re(\omega)} (t-y)^{\omega-1} \{K_1(t, S_{h1}) - K_1(t, (S_{h1})_1)\} dy\end{aligned} \tag{23}$$

With the application of norm to (23), we get

$$\begin{aligned}\|(S_{h1})(t) - (S_{h1})_1(t)\| \leq{}& \frac{1-\omega}{\Re(\omega)} \|K_1(t, S_{h1}) - K_1(t, (S_{h1})_1)\| \\ &+ \int_0^t \frac{\omega}{\Gamma(\omega)\Re(\omega)} (t-y)^{\omega-1} \|K_1(t, S_{h1}) - K_1(t, (S_{h1})_1)\| dy\end{aligned} \tag{24}$$

Using the fact that kernel satisfy the Lipschitz condition, we get

$$\begin{aligned}\|(S_{h1})(t) - (S_{h1})_1(t)\| \leq{}& \frac{1-\omega}{\Re(\omega)} \beta_1 \|(S_{h1})(t) - (S_{h1})_1(t)\| \\ &+ \frac{\beta_1 t^{\omega}}{\Gamma(\omega)\Re(\omega)} \|(S_{h1})(t) - (S_{h1})_1(t)\|.\end{aligned} \tag{25}$$

This gives

$$\|(S_{h1})(t) - (S_{h1})_1(t)\| \left(1 - \frac{1-\omega}{\Re(\omega)} \beta_1 + \frac{\beta_1 t^{\omega}}{\Gamma(\omega)\Re(\omega)}\right) \leq 0, \tag{26}$$

$$\|(S_{h1})(t)-(S_{h1})_1(t)\|=0\longrightarrow(S_{h1})(t)=(S_{h1})_1(t). \tag{27}$$

So the equation has a unique solution. It is clear that we can show the same results for other solutions of $S_{h2}(t)$, $I_h(t)$, $I_{hA}(t)$, $R_h(t)$, $E_v(t)$, $S_v(t)$ and $I_v(t)$.

5. Numerical scheme

A novel numerical approach has been developed recently by Toufik and Atangana [40] to tackle some particular problems of fractional derivatives with non-local and non-singular kernel (7). It is clear from their study that their approach is quite accurate and converges to the exact solutions rapidly. Let's look at the following non-linear fractional ordinary equation to illustrate their approach:

$$\left\{\begin{array}{l} {}_0^{ABC}D_t^{\omega}y(t)=f(t,y(t)), \\ y(0)=y_{0.} \end{array}\right\} \tag{28}$$

The equivalency of fractional integral for above mentioned initial value problem can be expressed as

$$x(t)-x(0)=\frac{1-\omega}{\Re(\omega)}f(t,x(t))+\frac{\omega}{B(\omega)\Gamma(\omega)}\int_0^t f(\eta,x(\eta))(t-\eta)^{\omega-1}d\eta. \tag{29}$$

For the point t_{n+1}, where n is nonnegative integer, the above integral equation is written can be written as

$$x(t_{n+1})-x(0)=\frac{1-\omega}{\Re(\omega)}f(t_n,x(t_n))+\frac{\omega}{\Gamma(\omega)\Re(\omega)}\int_0^{t_{n+1}} f(\eta,x(\eta))(t_{n+1}-\eta)^{\omega-1}d\eta. \tag{30}$$

The following expression of $f(y,x(y))$ is obtained, by using two-step Lagrange polynomial interpolation, in the interval $[t_i,t_{i+1}]$:

$$\begin{aligned} p_m(\eta)&=f(\eta,x(\eta)) \\ &=\frac{\eta-t_{q-1}}{t_q-t_{q-1}}f(t_q,x(t_q))-\frac{\eta-t_q}{t_q-t_{q-1}}f(t_{q-1},x(t_{q-1})) \\ &=\frac{f(t_q,x(t_q))}{h}(\eta-t_{q-1})-\frac{f(t_{q-1},x(t_{q-1}))}{h}(\eta-t_q) \\ &\simeq\frac{f(t_q,x_q)}{h}(\eta-t_{q-1})-\frac{f(t_{q-1},x_{q-1})}{h}(\eta-t_q). \end{aligned} \tag{31}$$

The following results from reexamining eq. (30) using the Lagrange polynomial interpolation for $f(\eta,x(\eta))$:

$$x_{q+1} = x_0 + \frac{1-\omega}{\Re(\omega)} f(t_q, x(t_q)) + \frac{\omega}{\Gamma(\omega)\Re(\omega)} \sum_{n=0}^{q} \frac{f(t_n, x_n)}{h} \int_0^{t_{q+1}} (\eta - t_{n-1}) (t_{q+1} - \eta)^{\omega-1} d\eta$$

$$- \sum_{n=0}^{q} \frac{f(t_{n-1}, x_{n-1})}{h} \int_{t_n}^{t_{q+1}} \frac{\omega}{\Gamma(\omega)\Re(\omega)} (\eta - t_n)(t_{q+1} - \eta)^{\omega-1} d\eta. \tag{32}$$

The following equality results from computing the integral expression in the sum given above:

$$x_{q+1} = x_0 + \frac{1-\omega}{\Re(\omega)} f(t_q, x(t_q)) + \frac{\omega}{\Re(\omega)} \sum_{n=0}^{q} \left(\frac{h^\omega f(t_n, x_n)}{\Gamma(\omega+2)} \beta_1 - \frac{h^\omega f(t_{n-1}, x_{n-1})}{\Gamma(\omega+2)} \beta_2 \right) + R_q$$

where

$$\beta_1 = (q+1-n)^\omega (m-n+2+\omega) - (q-n)^\omega (q-n+2(1+\omega)),$$

$$\beta_2 = (q+1-n)^{\omega+1} - (q-n)^\omega (q-n+1+\omega)$$

and remainder term R_q^ω is given as

$$R_q^\omega = \sum_{k=0}^{q} \int_{t_k}^{t_{k-1}} \frac{\omega}{\Gamma(\omega)\Re(\omega)} \frac{(\eta - t_k)(\eta - t_{k-1})}{2!} \frac{\partial^2}{\partial \eta^2} [f(\eta, x(\eta))]_{\eta = \varepsilon_\eta} (t_{q+1} - \eta)^{\omega-1} d\eta. \tag{33}$$

5.1 *Numerical scheme for the dengue model*

Let's look at the provided Dengue spread model (2). The application of Atana-Baleanu fractional integral on both sides gives the following form of model with $K_i,\ i = 1,2,3,4,5,6,7,8$ kernels:

$$S_{h1}(t) = S_{h1}(0) \frac{1-\omega}{\Re(\omega)} K_1(t, S_{h1}(t)) + \frac{\omega}{\Gamma(\omega)\Re(\omega)} \int_0^t (t-y)^{\omega-1} K_1(y, S_{h1}(y)) dy,$$

$$S_{h2}(t) = S_{h2}(0) \frac{1-\omega}{\Re(\omega)} K_2(t, S_{h2}(t)) + \frac{\omega}{\Gamma(\omega)\Re(\omega)} \int_0^t (t-y)^{\omega-1} K_1(y, S_{h2}(y)) dy,$$

$$I_h(t) = I_h(0) \frac{1-\omega}{\Re(\omega)} K_3(t, I_h(t)) + \frac{\omega}{\Gamma(\omega)\Re(\omega)} \int_0^t (t-y)^{\omega-1} K_1(y, I_h(y)) dy,$$

$$I_{hA}(t) = I_{hA}(0) \frac{1-\omega}{\Re(\omega)} K_4(t, I_{hA}(t)) + \frac{\omega}{\Gamma(\omega)\Re(\omega)} \int_0^t (t-y)^{\omega-1} K_1(y, I_{hA}(y)) dy,$$

$$R_h(t) = R_h(0)\frac{1-\omega}{\Re(\omega)}K_5(t,R_h(t)) + \frac{\omega}{\Gamma(\omega)\Re(\omega)}\int_0^t (t-y)^{\omega-1}K_1(y,R_h(y))dy,$$

$$S_v(t) = S_v(0)\frac{1-\omega}{\Re(\omega)}K_6(t,S_v(t)) + \frac{\omega}{\Gamma(\omega)\Re(\omega)}\int_0^t (t-y)^{\omega-1}K_1(y,S_v(y))dy,$$

$$E_v(t) = E_v(0)\frac{1-\omega}{\Re(\omega)}K_7(t,E_v(t)) + \frac{\omega}{\Gamma(\omega)\Re(\omega)}\int_0^t (t-y)^{\omega-1}K_1(y,E_v(y))dy,$$

$$I_v(t) = I_v(0)\frac{1-\omega}{\Re(\omega)}K_8(t,I_v(t)) + \frac{\omega}{\Gamma(\omega)\Re(\omega)}\int_0^t (t-y)^{\omega-1}K_1(y,I_v(y))dy. \tag{34}$$

Moreover, the initial state

$$\begin{aligned} &S_{h1}(0) = S_{h1_0}, S_{h2}(0) = S_{h2_0},\ I_h(0) = I_{h_0},\\ &I_{hA}(0) = I_{hA_0},\ R_h(0) = R_{h_0},\\ &S_v(0) = S_{v_0},\ E_v(0) = E_{v_0},\ I_v(0) = I_{v_0}. \end{aligned} \tag{35}$$

The system above can now use a new numerical scheme applied at a given point $t = t_{n+1}$.

$$\begin{aligned}
(S_{h1})_{q+1} &= (S_{h1})_0\frac{1-\omega}{\Re(\omega)}K_1(t_q,S_{h1}(t_q))\\
&+\frac{\omega}{\Re(\omega)}\sum_{k=0}^{q}\left(\frac{h^\omega K_1(t_k,S_k)}{\Gamma(\omega+2)}\left((q-k+1)^\omega(q-k+2+\omega)-(q-k)^\omega(q-k+2+2\omega)\right)\right)\\
&-\frac{\omega}{\Re(\omega)}\sum_{k=0}^{q}\left(\frac{h^\omega K_1(t_{k-1},S_{k-1})}{\Gamma(\omega+2)}\left((i+1-k)^{\omega+1}-(i-k)^\omega(q-k+1+\omega)\right)\right)+{}^1R_q^\omega,\\
(S_{h2})_{q+1} &= (S_{h2})_0\frac{1-\omega}{\Re(\omega)}K_2(t_q,S_{h2}(t_q))\\
&+\frac{\omega}{\Re(\omega)}\sum_{k=0}^{q}\left(\frac{h^\omega K_2(t_k,S_k)}{\Gamma(\omega+2)}\left((q-k+1)^\omega(q-k+2+\omega)-(q-k)^\omega(q-k+2+2\omega)\right)\right)\\
&-\frac{\omega}{\Re(\omega)}\sum_{k=0}^{q}\left(\frac{h^\omega K_2(t_{k-1},S_{k-1})}{\Gamma(\omega+2)}\left((q-k+1)^{\omega+1}-(q-k)^\omega(q-k+1+\omega)\right)\right)+{}^2R_q^\omega\\
(I_h)_{q+1} &= (I_h)_0\frac{1-\omega}{\Re(\omega)}K_3(t_q,I_h(t_q))
\end{aligned}$$

$$+\frac{\omega}{\Re(\omega)}\sum_{k=0}^{q}\left(\frac{h^{\omega}K_3(t_k,S_k)}{\Gamma(\omega+2)}\left((q-k+1)^{\omega}(q-k+2+\omega)-(q-k)^{\omega}(q-k+2+2\omega)\right)\right)$$

$$-\frac{\omega}{\Re(\omega)}\left(\frac{h^{\omega}K_3(t_{k-1},S_{k-1})}{\Gamma(\omega+2)}\left((q-k+1)^{\omega+1}-(q-k)^{\omega}(q-k+1+\omega)\right)\right)+^{3}R_q^{\omega}$$

$$(I_{hA})_{q+1}=(I_{hA})_0\frac{1-\omega}{\Re(\omega)}K_4(t_q,S_{h1}(t_q))$$

$$+\frac{\omega}{\Re(\omega)}\sum_{k=0}^{q}\left(\frac{h^{\omega}K_4(t_k,S_k)}{\Gamma(\omega+2)}\left((q-k+1)^{\omega}(q-k+2+\omega)-(q-k)^{\omega}(q-k+2+2\omega)\right)\right)$$

$$-\frac{\omega}{\Re(\omega)}\sum_{k=0}^{q}\left(\frac{h^{\omega}K_4(t_{k-1},S_{k-1})}{\Gamma(\omega+2)}\left((q-k+1)^{\omega+1}-(q-k)^{\omega}(q-k+1+\omega)\right)\right)+^{4}R_q^{\omega}$$

$$(R_h)_{q+1}=(R_h)_0\frac{1-\omega}{\Re(\omega)}K_5(t_q,R_h(t_q))$$

$$+\frac{\omega}{\Re(\omega)}\sum_{k=0}^{q}\left(\frac{h^{\omega}K_5(t_k,S_k)}{\Gamma(\omega+2)}\left((q-k+1)^{\omega}(m-k+2+\omega)-(m-k)^{\omega}(q-k+2+2\omega)\right)\right)$$

$$-\frac{\omega}{\Re(\omega)}\left(\frac{h^{\omega}K_5(t_{k-1},S_{k-1})}{\Gamma(\omega+2)}\left((q-k+1)^{\omega+1}-(q-k)^{\omega}(q-k+1+\omega)\right)\right)+^{5}R_q^{\omega} \tag{36}$$

$$(S_v)_{q+1}=(S_v)_0\frac{1-\omega}{\Re(\omega)}K_6(t_q,S_v(t_q))$$

$$+\frac{\omega}{\Re(\omega)}\sum_{k=0}^{q}\left(\frac{h^{\omega}K_6(t_k,S_k)}{\Gamma(\omega+2)}\left((q-k+1)^{\omega}(q-k+2+\omega)-(q-k)^{\omega}(q-k+2+2\omega)\right)\right)$$

$$-\frac{\omega}{\Re(\omega)}\left(\frac{h^{\omega}K_6(t_{k-1},S_{k-1})}{\Gamma(\omega+2)}\left((q-k+1)^{\omega+1}-(q-k)^{\omega}(q-k+1+\omega)\right)\right)+^{6}R_q^{\omega}$$

$$(E_v)_{q+1}=(E_v)_0\frac{1-\omega}{\Re(\omega)}K_7(t_q,E_v(t_q))$$

$$+\frac{\omega}{\Re(\omega)}\sum_{k=0}^{q}\left(\frac{h^{\omega}K_7(t_k,S_k)}{\Gamma(\omega+2)}\left((q-k+1)^{\omega}(q-k+2+\omega)-(q-k)^{\omega}(q-k+2+2\omega)\right)\right)$$

$$-\frac{\omega}{\Re(\omega)}\left(\frac{h^{\omega}K_7(t_{k-1},S_{k-1})}{\Gamma(\omega+2)}\left((q-k+1)^{\omega+1}-(q-k)^{\omega}(q-k+1+\omega)\right)\right)+^{7}R_q^{\omega}$$

$$(I_v)_{q+1} = (I_v)_0 \frac{1-\omega}{\Re(\omega)} K_8(t_q, I_v(t_q))$$

$$+ \frac{\omega}{\Re(\omega)} \sum_{k=0}^{q} \left(\frac{h^\omega K_8(t_k, S_k)}{\Gamma(\omega+2)} \left((q-k+1)^\omega (q-k+2+\omega) - (q-k)^\omega (q-k+2+2\omega) \right) \right)$$

$$- \frac{\omega}{\Re(\omega)} \sum_{k=0}^{q} \left(\frac{h^\omega K_8(t_{k-1}, S_{k-1})}{\Gamma(\omega+2)} \left((q-k+1)^{\omega+1} - (q-k)^\omega (q-k+1+\omega) \right) \right) + {}^8R_q^\omega \tag{37}$$

where ${}^lR_n^\omega$, $l = 1,2,3,4,5,6,7,8$ are values of remaining terms

$${}^1R_q^\omega = \frac{\omega}{\Gamma(\omega)\Re(\omega)} \sum_{k=0}^{q} \int_{t_k}^{t_{k-1}} \frac{(\eta - t_k)(\eta - t_{k-1})}{2!} \frac{\partial^2}{\partial \eta^2} [K_1(\eta, S_{h1}(\eta))]_{\eta = \varepsilon_\eta} (t_{q+1} - \eta)^{\omega-1} d\eta,$$

$${}^2R_q^\omega = \frac{\omega}{\Gamma(\omega)\Re(\omega)} \sum_{k=0}^{q} \int_{t_k}^{t_{k-1}} \frac{(\eta - t_k)(\eta - t_{k-1})}{2!} \frac{\partial^2}{\partial \eta^2} [K_2(\eta, S_{h2}(\eta))]_{\eta = \varepsilon_\eta} (t_{q+1} - \eta)^{\omega-1} d\eta,$$

$${}^3R_q^\omega = \frac{\omega}{\Gamma(\omega)\Re(\omega)} \sum_{k=0}^{q} \int_{t_k}^{t_{k-1}} \frac{(\eta - t_k)(\eta - t_{k-1})}{2!} \frac{\partial^2}{\partial \eta^2} [K_3(\eta, I_h(\eta))]_{\eta = \varepsilon_\eta} (t_{q+1} - \eta)^{\omega-1} d\eta,$$

$${}^4R_q^\omega = \frac{\omega}{\Gamma(\omega)\Re(\omega)} \sum_{k=0}^{q} \int_{t_k}^{t_{k-1}} \frac{(\eta - t_k)(\eta - t_{k-1})}{2!} \frac{\partial^2}{\partial \eta^2} [K_4(\eta, I_{hA}(\eta))]_{\eta = \varepsilon_\eta} (t_{q+1} - \eta)^{\omega-1} d\eta,$$

$${}^5R_q^\omega = \frac{\omega}{\Gamma(\omega)\Re(\omega)} \sum_{k=0}^{n} \int_{t_k}^{t_{k-1}} \frac{(\eta - t_k)(\eta - t_{k-1})}{2!} \frac{\partial^2}{\partial \eta^2} [K_5(\eta, R_h(\eta))]_{\eta = \varepsilon_\eta} (t_{q+1} - \eta)^{\omega-1} d\eta,$$

$${}^6R_q^\omega = \sum_{k=0}^{q} \int_{t_k}^{t_{k-1}} \frac{\omega}{\Gamma(\omega)\Re(\omega)} \frac{(\eta - t_k)(\eta - t_{k-1})}{2!} \frac{\partial^2}{\partial \eta^2} [K_6(\eta, S_v(\eta))]_{\eta = \varepsilon_\eta} (t_{q+1} - \eta)^{\omega-1} d\eta,$$

$${}^7R_q^\omega = \frac{\omega}{\Gamma(\omega)\Re(\omega)} \sum_{k=0}^{q} \int_{t_k}^{t_{k-1}} \frac{(\eta - t_k)(\eta - t_{k-1})}{2!} \frac{\partial^2}{\partial \eta^2} [K_7(\eta, E_v(\eta))]_{\eta = \varepsilon_\eta} (t_{q+1} - \eta)^{\omega-1} d\eta,$$

$${}^8R_q^\omega = \frac{\omega}{\Gamma(\omega)\Re(\omega)} \sum_{k=0}^{n} \int_{t_k}^{t_{k-1}} \frac{(\eta - t_k)(\eta - t_{k-1})}{2!} \frac{\partial^2}{\partial \eta^2} [K_8(\eta, I_v(\eta))]_{\eta = \varepsilon_\eta} (t_{q+1} - \eta)^{\omega-1} d\eta. \tag{38}$$

6. Equilibria and stability through R_0

Examining the constant solutions of the model helps us to identify the conditions through which we can eradicate the disease or it is restricted to certain level. Basic reproduction number act as a threshold parameter. It is defined as the number of secondary infections produced by a single infectious individual during its course of infection. After simple calculations the disease free equilibrium is obtained as $(S_{h1}, S_{h2}, I_h, I_{hA}, R_h, S_v, E_v, I_v) = \left(\frac{\gamma_1}{p+\gamma_1}, \frac{vp}{(v+\gamma_1)(p+\gamma_1)}, 0, 0, \frac{\gamma_1 p}{(v+\gamma_1)(p+\gamma_1)}, K_2, 0, 0 \right)$.

The application of next generation matrix procedure developed by Driessche and Watmough [41] gives the expression of basic reproduction number. In order to get the value of this threshold parameter as follows:

$$F = \begin{bmatrix} (1-\phi)\frac{\alpha_{h1}b}{N_h}S_{h1}I_v + (1-\phi)\frac{\alpha_{h2}b}{N_h}S_{h2}I_v \\ \phi\frac{\alpha_{h1}b}{N_h}S_{h1}I_v + \phi\frac{\alpha_{h2}b}{N_h}S_{h2}I_v \\ 0 \\ \frac{\beta_v b}{N_h}(I_h + I_{hA})S_v \\ 0 \end{bmatrix}, V = \begin{bmatrix} (\gamma_1+\Lambda+\xi_h)I_h \\ (\gamma_1+\xi_h)I_{hA} \\ -pS_{h1} - \xi_h(I_h+I_{hA}) - \Lambda I_h + (v+\gamma_1)R_h \\ \eta_v E_v + \gamma_2 E_v \\ -\eta_v E_v + \gamma_2 I_v \end{bmatrix}$$

$$F_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & b(1-\phi)\left(\alpha_{h1}\frac{\gamma_1}{p+\gamma_1} + \alpha_{2h}\frac{vp}{(v+\gamma_1)(p+\gamma_1)}\right) \\ 0 & 0 & 0 & 0 & b\phi\left(\alpha_{h1}\frac{\gamma_1}{p+\gamma_1} + \alpha_{2h}\frac{vp}{(v+\gamma_1)(p+\gamma_1)}\right) \\ 0 & 0 & 0 & 0 & 0 \\ b\beta_v & b\beta_v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, V_1 = \begin{bmatrix} C & 0 & 0 & 0 & 0 \\ 0 & D & 0 & 0 & 0 \\ -\xi_h - \Lambda & -\xi_h & E & 0 & 0 \\ 0 & 0 & 0 & F & 0 \\ 0 & 0 & 0 & -\eta_v & \gamma_2 \end{bmatrix},$$

where $= \gamma_1 + \Lambda + \xi_h, D = (\gamma_1 + \xi_h), E = (v+\gamma_1), F = \eta_v + \gamma_2$.

$$F_1 V_1^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & A \\ 0 & 0 & 0 & 0 & B \\ 0 & 0 & 0 & 0 & 0 \\ b\beta_v & b\beta_v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{C} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{D} & 0 & 0 & 0 \\ \frac{1}{CE}(\Lambda+\xi_h) & \frac{\xi_h}{DE} & \frac{1}{E} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{F} & 0 \\ 0 & 0 & 0 & \frac{1}{F\gamma_2}\eta_v & \frac{1}{\gamma_2} \end{bmatrix}.$$

The spectral radius of the matrix FV^{-1}, which is the basic reproduction has the form

$$R_0 = \sqrt{\frac{b\beta_v\eta_v(\alpha_{h1}\gamma_1(v+\gamma_1) + \alpha_{2h}vp)b(\gamma_1+\xi_h+\phi\Lambda)}{(\gamma_1+\xi_h)\gamma_2(\gamma_1+\Lambda+\xi_h)(\eta_v+\gamma_2)(v+\gamma_1)(p+\gamma_1)}}.$$

$$R_0 = \sqrt{\frac{b^2\beta_v\eta_v\alpha_{h1}\gamma_1(\gamma_1+\xi_h+\phi\Lambda)}{(\gamma_1+\xi_h)\gamma_2(\gamma_1+\Lambda+\xi_h)(\eta_v+\gamma_2)(p+\gamma_1)} + \frac{b^2\beta_v\eta_v\alpha_{2h}vp(\gamma_1+\xi_h+\phi\Lambda)}{(\gamma_1+\xi_h)\gamma_2(\gamma_1+\Lambda+\xi_h)(\eta_v+\gamma_2)(v+\gamma_1)(p+\gamma_1)}}.$$

According to the Theorem 2 stated in [41], the local stability of the disease free equilibrium is guaranteed. Now, we discuss the Lyapunov stability of the disease free equilibrium. Consider the following Lyapunov function $L = a_1I_h + a_2I_{hA} + a_3E_v + a_4I_v$.

Taking the time derivative we get,

$$L' = \frac{b\beta_v}{(\gamma_1+\Lambda+\xi_h)}I_h' + \frac{b\beta_v}{(\gamma_1+\xi_h)}I_{hA}' + E_v' + I_v'$$

$$L' = \frac{b\beta_v}{(\gamma_1+\Lambda+\xi_h)}\left(b(1-\phi)(\alpha_{h1}+\alpha_{h2})\frac{S_{h2}I_v}{N_h} - (\gamma_1+\Lambda+\xi_h)I_h\right) + \left(b\phi(\alpha_{h1}+\alpha_{h2})\frac{S_{h2}I_v}{N_h}\right)$$

$$-\frac{b\beta_v}{(\gamma_1+\xi_h)}\left((\gamma_1+\xi_h)I_{hA}\right)+\left(\frac{\beta_v b}{N_h}(I_h+I_{hA})S_v-\eta_v E_v-\gamma_2 E_v\right)+\left(\eta_v E_v-\gamma_2 I_v\right).$$

$$L' \leq \frac{b\beta_v}{(\gamma_1+\Lambda+\xi_h)} b(1-\phi)\left(\alpha_{h1}+\alpha_{h2}\right) I_v + \frac{b\beta_v}{(\gamma_1+\xi_h)} b\phi\left(\alpha_{h1}+\alpha_{h2}\right) I_v - \gamma_2 E_v - \gamma_2 I_v.$$

$$L' \leq \left(R_1 - 1\right) I_v,$$

where, $R_1 = \frac{b\beta_v}{\gamma_2(\gamma_1+\Lambda+\xi_h)} b(1-\phi)\left(\alpha_{h1}+\alpha_{h2}\right) + \frac{b\beta_v}{\gamma_2(\gamma_1+\xi_h)} b\phi\left(\alpha_{h1}+\alpha_{h2}\right)$.

We can see that, whenever $R_1 < 1$ then $L^{'} < 0$. More over after some simplification we can get $R_0 < R_1$, hence $L^{'} < 0$ holds also for $R_0 < 1$. Hence, we have the stability of Disease Free Equilibrium (DFE).

Let $E^* = (S^*_{h1}, S^*_{h2}, I^*_h, I^*_{hA}, R^*_h, S^*_v, E^*_v, I^*_v)$ be the endemic equilibrium which can be obtained from the following equations

$$I^*_v = \frac{\eta_v}{\gamma_2} E^*_v, E^*_v = \frac{\beta_v b}{(\eta_v+\gamma_2)N^*_h}(I^*_h + I^*_{hA})S^*_v, S^*_v = \frac{\gamma_2 N^*_v N^*_h}{\beta_v b(I^*_h + I^*_{hA}) + \gamma_2}, S^*_{h1} = \frac{\gamma_1 N^{2*}_h}{\left(\alpha_{h1} b I^*_v + (p+\gamma_1)N^*_h\right)}$$

$$S^*_{h2} = \frac{\nu R^*_h N^*_h}{\left(\alpha_{h2} b I^*_v + \gamma_1 N^*_h\right)}, I^*_h = \frac{(1-\phi)\alpha_{h1} b S^*_{h1} I^*_v + (1-\phi)\alpha_{h2} b S^*_{h2} I^*_v}{N^*_h(\gamma_1+\Lambda+\xi_h)}, I^*_{hA} = \frac{\phi\alpha_{h1} b S^*_{h1} I^*_v + \phi\alpha_{h2} b S^*_{h2} I^*_v}{(\gamma_1+\xi_h)N^*_h},$$

$$R^*_h = \frac{p S^*_{h1} + \xi_h(I^*_h + I^*_{hA}) + \Lambda I^*_h}{(\nu+\gamma_1)}.$$

7. Estimation of parameters by fitting the model with real data

We have taken the monthly real cases of dengue, occurred in Pakistan during 2023–2025. It has been taken from [42] and is depicting in Figure 2. Dengue cases occurred occurred in three waves during 2023–2025. In first wave relatively low epidemic is seen with almost stable transmission except sudden increase in May 2024. The second wave describes explosive epidemic peak having maximum number of cases in October 2024. Secondary peak can be seen in October 2025 showing seasonal transmission. In Figure 3, we have plotted the solution of infectious host along with real data curve. We can see that for fractional order 0.95, both real curve and solution curve, obtained from the system, are in good agreement with each other.

In order to demonstrate the importance of fractional order in comparison of the integer order, we compute the root mean square error for the measurement of discrepancies among predicted values and actual data. Obtained smaller Root Mean Square Error (RMSE) values express that considered model is the better choice for the disease prediction in any region. Thus obtained results are displayed in Table 2.

Table 2. Root Mean Square Error (RMSE)for different fractional orders

	Phase I					Phase II					Phase III				
RMSE	674	641	716	730	738	2136	1959	4260	7294	2004	1897	1542	1545	3754	3902

From Table 2, it is clear that the fractional order model is more better than classical one. It is also verified through Figure 4. Considering Mean Absolute Percentage Error (MAPE), the estimated values are shown in Figure 5. It is observed that the model data deviates almost 18% from the original values in phase I, 6.13% in phase II and 1.22% in phase III. Thus overall MAPE is $\sim$ 8.8113%. The numerical values against the parameters of our considered model are given in Table **??**.

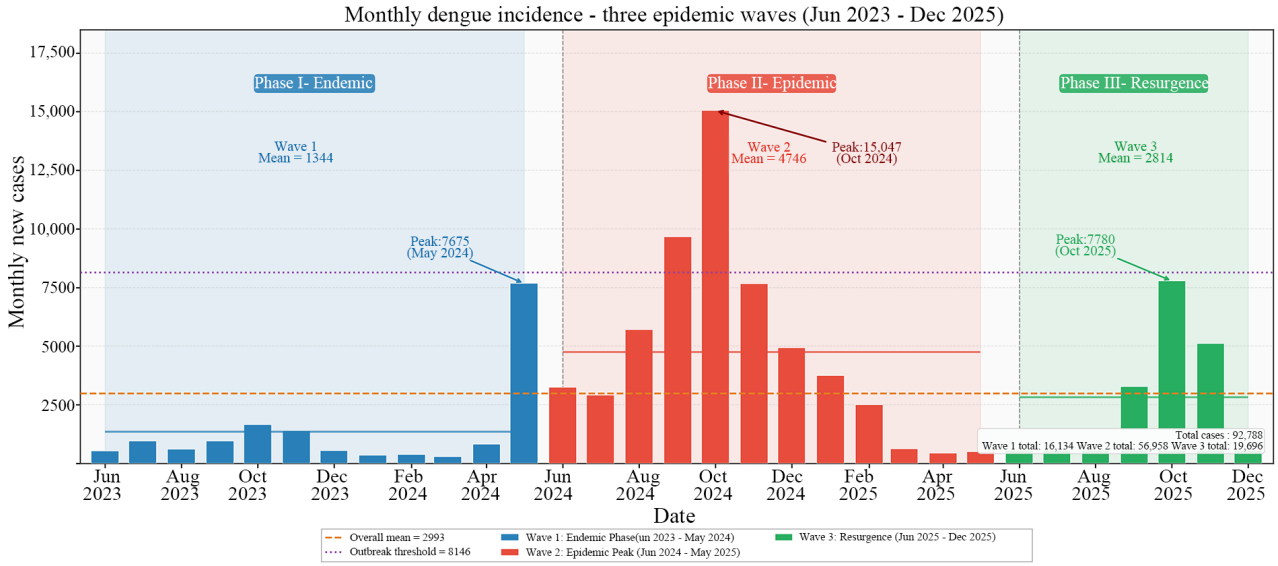


Figure 2. Actual data of dengue occurred in Pakistan during 2023–2025

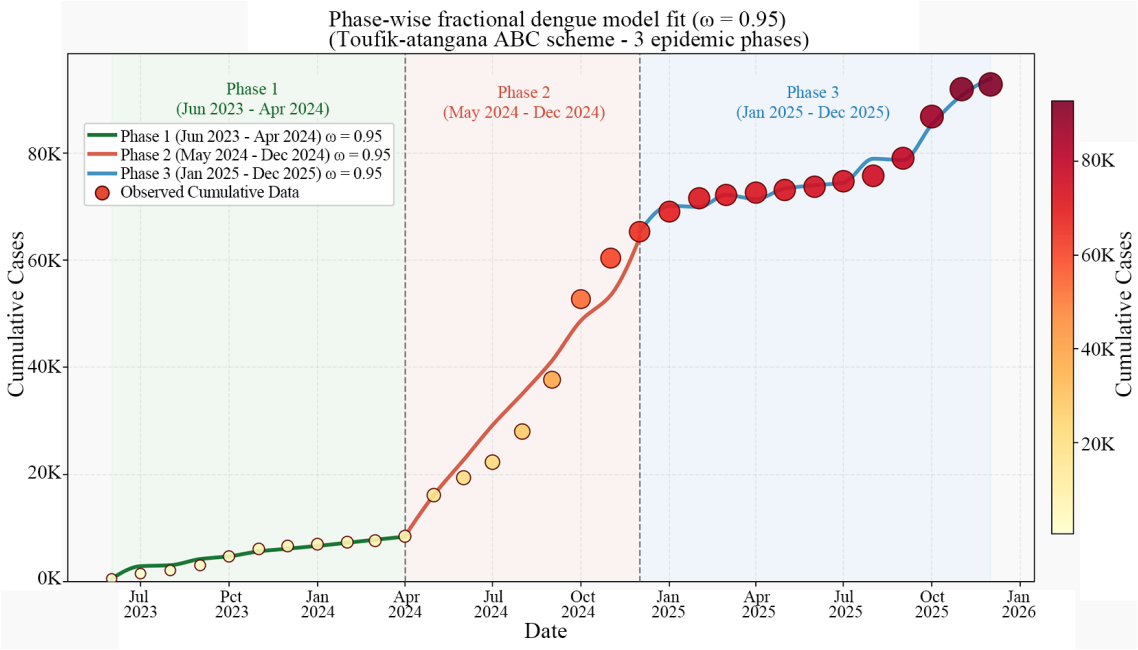


Figure 3. Actual and estimated number of cases

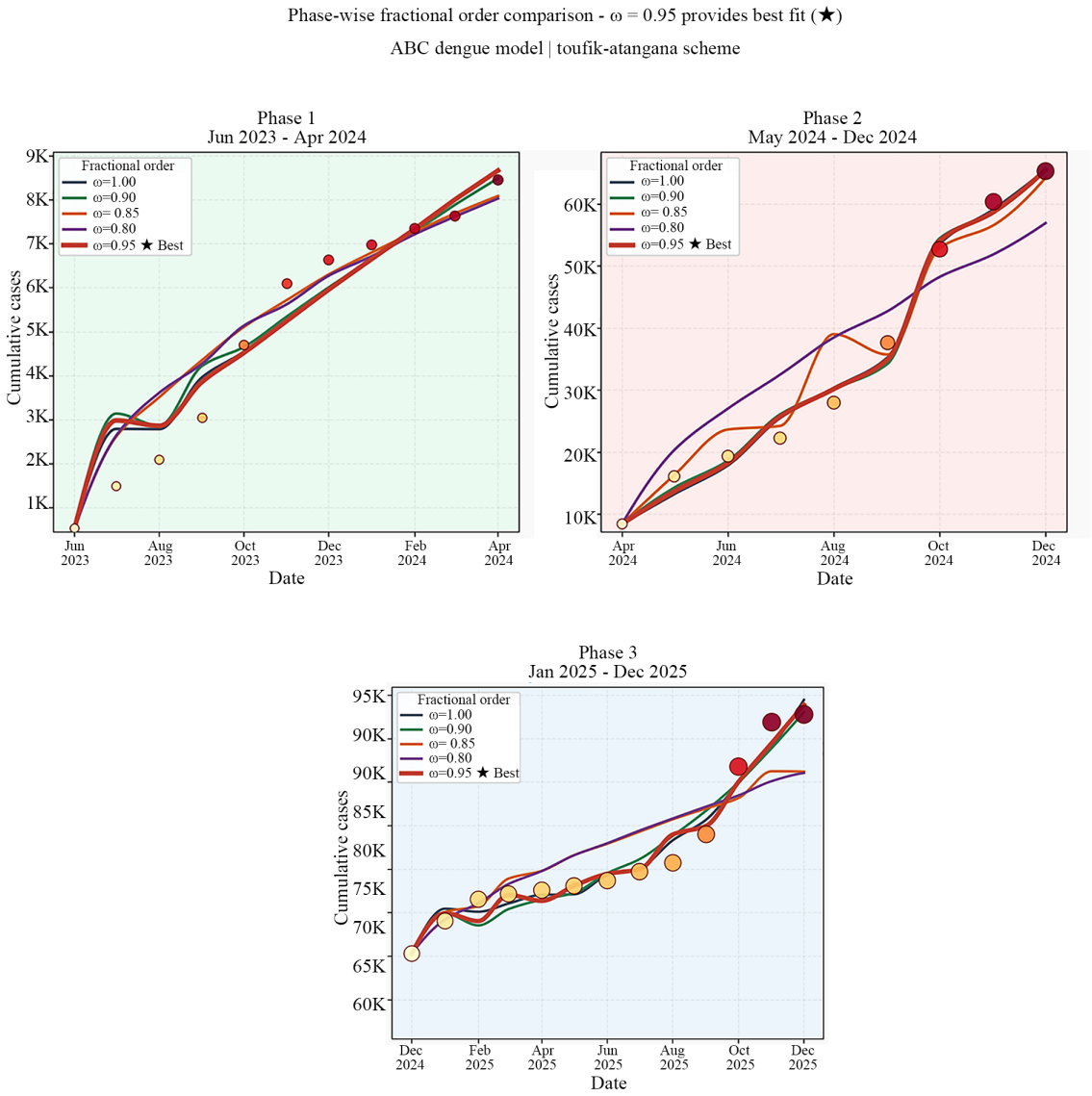


Figure 4. Cumulative dengue cases in Pakistan versus estimated cases

Table 3. Estimated values of parameters

Parameters	Values	Description	Source	Units
γ_1	[0.001, 0.004]	Fertility rate of humans	Fitted	Month^{-1}
α_{h1}	[0.01, 0.492]	Probability of virus transfer From susceptible host	Fitted	Dimensionless
b	0.29	Mosquito biting rate	Fitted	Month^{-1}
p	0.049	Vaccination rate of susceptible humans	Fitted	Month^{-1}
ν	[0.01, 0.3]	Rate at which the people looses immunity	Fitted	Month^{-1}

Table 3. (cont.)

Parameters	Values	Description	Source	Units
α_{h2}	[0.703, 0.9]	Probability of virus transfer From Partially immune susceptible host	Fitted	Dimensionless
ϕ	[0.1, 0.9]	The fraction at which susceptible humans Become carrier after getting infection	Fitted	Dimensionless
Λ	[0.01, 0.3]	Treatment rate of recovery of Infected humans	Fitted	$Month^{-1}$
ξ_h	[0.05, 0.5]	Recovery rate of humans without Proper clinical treatment	Fitted	$Month^{-1}$
γ_2	[0.285, 0.49]	Mortality rate of vectors	Fitted	$Month^{-1}$
β_v	[0.764, 0.9]	Transferring rate from Infected people to mosquitoes	Fitted	$Month^{-1}$
η_v	[0.1, 0.49]	Transfer rate of exposed vectors to The infectious class	Fitted	$Month^{-1}$

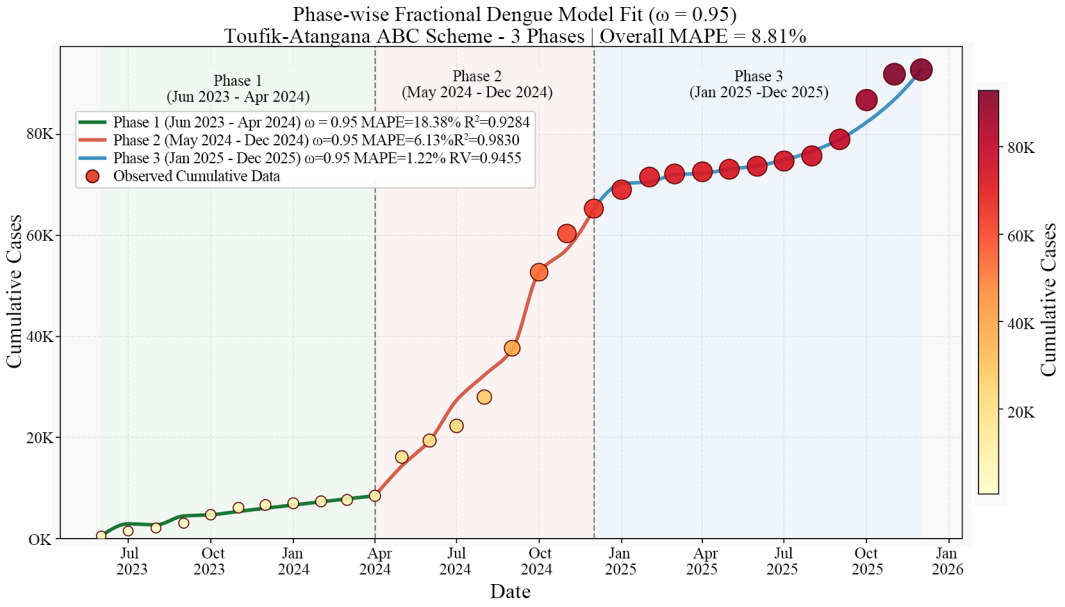


Figure 5. Cumulative dengue cases in Pakistan versus estimated cases using MAPE

7.1 *Sensitivity analysis*

We perform the sensitivity analysis to analyze the extent to which the changes in parameter values influence the model outputs. This process helps us to identify the key contributors in model's prediction. The formula for calculation of local sensitivity indices is [43]

$$\Gamma_{\delta}^{R_0} = \frac{\partial R_0}{\partial \delta} \times \frac{\delta}{R_0}.$$

Global sensitivity indices has been calculated through partial rank correlation coefficient and the corresponding values are given in Table 4. Sensitivity indecies have displayed graphically in Figure 6.

Table 4. Sensitivity indices of R_0

Parameter	Local sensitivity index	Global sensitivity index
b	1	1
p	0.0162838334	0.0162838334
ν	0.0078097141	0.0135114987
α_{h2}	0.4920119887	0.4919726877
ϕ	0.1091549296	0.1051050937
Λ	−0.0700358219	−0.0627049927
ξ_h	−0.426090627	−0.4248286657
γ_2	−0.7838827839	−0.7822349265
β_v	0.5	0.5
η_v	0.2838827839	0.292012823

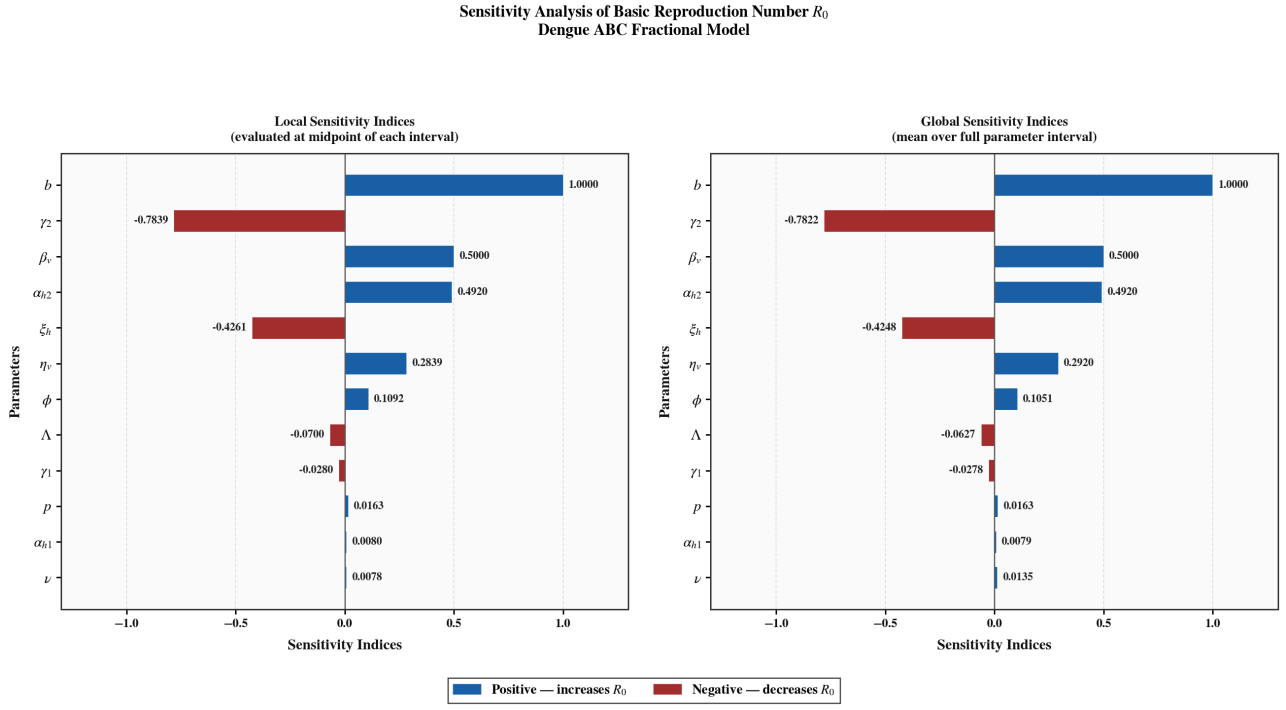


Figure 6. Bar chart of sensitivity indices

From the sensitivity indices, it is observed that the most influential parameter is the mosquito biting rate. Reduction in this rate by 10%. The basic reproduction number will be reduced almost 10%. Mosquito death rate is the second sensitive parameter. Increasing the killing rate of mosquitoes by 10%, R_0 will decrease almost 8%. Since mosquito biting rate and their killing rate are two top sensitive indices values, therefore vector control should be kept on priority basis.

8. Conclusion

In this study, we considered an extended dengue disease model with the inclusion of fractional differentiation. By using the fixed point theory we have shown that system of solution exists and is unique under some conditions. The numerical scheme, introduced by Toufik and Atangana, is used to solve the system numerically. We applied the model on actual number of dengue cases appeared in Pakistan from June2023 to December 2025 and estimated the parameters by exploring that the fractional order 0.95 best fit the model with actual data. The incorporation of ABC fractional derivative with order 0.95 proved significant in capturing the memory dependent and non-local transmission dynamics. The comparison of different fractional orders yield that the value 0.95 has the lowest RMSE and highest R^2 across all three phases showing that this order is optimal for the given dataset. The impact of parameters on the basic reproduction number has been observed through the sensitivity analysis. It has been seen that two parameters, mosquito biting rate and killing rate, impose significant impact in the reduction of dengue cases. Control measures can be applied such as by sleeping under the bed nets treated with insecticides, wearing long-sleeve shirts, long pants and socks to minimize the exposure of body. Regularly clean the flower pots, birdbaths, buckets and other things in which stagnant water can accumulate. Applying chemical larvicides may also kill larva before its development into adults. Spraying of adulticides would be beneficial to reduce adult mosquitoes in the areas during outbreak. The model discussed in this study explores good estimate with the actual data for the fractional order but there are limitations when parameters have been estimated. Since the fractional order models usually sensitive to the historical data, so limited actual data gives the values of estimated parameters which could best describe short term epidemiological pattern rather than long term. In this work, the four serotypes has not been discussed. Different sertypes normally results more severe disease and different transmission patterns. The seasonal factors such as temperature, rainfall, humidity and breeding conditions may also be included to get the more precise results.

Author contributions

S.A and M.Z wrote the initial draft, investigation. T.H and F.B included methodology theory. N.F and K.S edited revised and included explanation. T.A updated the last version. All the authors have contributed equally to this manuscript and approved the last version.

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Disclosure

The data used to support the findings of this study is available within this manuscript.

Conflict of interest

The authors declare no competing financial interest.

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