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Extending the Applicability and Convergence Domain of a Fifth-Order Iterative Scheme under Hölder Continuous Derivative in Banach Spaces

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Abstract: The most significant contribution made by this study is that the applicability and convergence domain of a fifth-order convergent nonlinear equation solver is extended. We use Hölder condition on the first Fréchet derivative to study the local analysis, and this expands the applicability of the formula for such problems where the earlier study based on Lipschitz constants cannot be used. This study generalizes the local analysis based on Lipschitz constants. Also, we avoid the use of the extra assumption on boundedness of the first derivative of the involved operator. Finally, numerical experiments ensure that our analysis expands the utility of the considered scheme. In addition, the proposed technique produces a larger convergence domain, in comparison to the earlier study, without using any extra conditions.

***Keywords*:** iterative schemes, Banach space, local convergence, Hölder continuity condition

MSC: 47H99, 49M15, 65J15, 65G99

1. Introduction

Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. $\Omega \subseteq \mathcal{X}$ is a non-empty, open and convex set. Suppose $\mathcal{Q} : \Omega \subseteq \mathcal{X} \to \mathcal{Y}$ is a nonlinear operator, which is differentiable according to Fréchet. The primary goal of this research is to broaden the scope of the application and convergence domain of a fifth-order iterative scheme for solving nonlinear operator equations in the form

$$\mathcal{Q}(s) = 0. \tag{1}$$

Applications of these equations are found in many fields such as Engineering, Optimization, Economics, Chemistry, Biology, etc. For example, problems in radiative transfer theory, static dynamical equations, Kinetic theory of gases, and other topics can be solved using nonlinear equations of the form (1). For these equations, mostly iterative schemes (see books [1-6], see research papers [7-18]) are used. A fundamental and widely used scheme for addressing (1) is Newton's method, whose iteration procedure is given by

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$$s_{k+1} = s_k - [\mathcal{Q}'(s_k)]^{-1}\mathcal{Q}(s_k),\ \ k \geq 0. \tag{2}$$

On the other hand, finding convergence radius as well as constructing a technique to expand the domain of convergence are two important matters in the study of iterative schemes. Convergence domain of an iterative scheme is required to determine such initial points whose corresponding sequences converge to the solution. "The analysis of local convergence of an iterative scheme uses the information around a solution and provides the convergence radii" [1]. The local results for Chebyshev and Halley type procedures are provided in [19-21]. Also, important findings on the local analysis of fourth, fifth, and sixth order schemes are presented in [22-27]. More results on the local analysis of iterative procedures have been established in the literature [28-35]. In this paper, we expand the convergence domain as well as the applicability of a fifth-order scheme using Hölder condition on $\mathcal{Q}'$ only.

The local convergence of the following fifth-order convergent scheme is suggested in [22] by Argyros and George.

$$t_k = s_k - [\mathcal{Q}'(s_k)]^{-1}\mathcal{Q}(s_k),$$

$$u_k = t_k - [\mathcal{Q}'(s_k)]^{-1}(\mathcal{Q}'(s_k) - \mathcal{Q}'(t_k))(\mathcal{Q}'(s_k) + \mathcal{Q}'(t_k))^{-1}\mathcal{Q}(s_k),$$

$$s_{k+1} = u_k - (\mathcal{Q}'(s_k) + \mathcal{Q}'(t_k))^{-1}(3\mathcal{Q}'(s_k) - \mathcal{Q}'(t_k))[\mathcal{Q}'(s_k)]^{-1}\mathcal{Q}(u_k),\ \ k = 0, 1, 2, \ldots \tag{3}$$

They proved the convergence analysis of this scheme by applying the Lipschitz continuity of $\mathcal{Q}'$, that is,

$$\| \mathcal{Q}'(s^*)^{-1}(\mathcal{Q}(s) - \mathcal{Q}(s^*)) \| \leq l_0 \| s - s^* \|,\ \forall s \in \Omega$$

and

$$\| \mathcal{Q}'(s^*)^{-1}(\mathcal{Q}(s) - \mathcal{Q}(t)) \| \leq l \| s - t \|,\ \forall s, t \in \Omega.$$

However, there are many instances where the technique based on Lipschitz conditions fails. As an illustration, we take the following integral equation given in [27].

$$\mathcal{Q}(s)(w) = s(w) - 3\int_0^1 G_1(w, y)\, s(y)^{\frac{5}{4}}\, dy, \tag{4}$$

where $s(w) \in C[0, 1]$ and $G_1(x, y)$ is Green's function defined on [0, 1] × [0, 1] by

$$G_1(w, y) = \begin{cases} (1-w)y, & \text{if } y \leq w \\ w(1-y), & \text{if } w \leq y \end{cases}.$$

Then,

$$\| \mathcal{Q}'(s) - \mathcal{Q}'(t) \| \leq \frac{15}{32} \| s - t \|^{\frac{1}{4}}.$$

It is noteworthy that $\mathcal{Q}'$ is not Lipschitz continuous. Hence, the existing work [22] cannot be implemented to address the above problem. However, $\mathcal{Q}'$ belongs to the Hölder continuity class. So, we establish the local convergence of the iterative scheme (3) by employing Hölder continuity of $\mathcal{Q}'$.

In addition, the benefits of our analysis are: this study generalizes the local analysis based on Lipschitz constants given in [22] and expands the applicability of the formula (3). We avoid the use of the extra condition based on the boundedness of $\mathcal{Q}'$. Also, we produce larger radii of convergence balls in comparison to the earlier study [22]. The most important point is that the proposed technique provides these advantages without using any extra conditions.

This manuscript is outlined as follows: Section 1 contains the introduction part. In Section 2, the analysis of local convergence of the iterative scheme (3) is described. Section 3 offers numerical applications of the proposed analytical results. The conclusion is provided in the final section.

2. Local convergence analysis

In order to expand the applicability and the convergence domain of the scheme (3), we establish the local analysis with the help of some real parameters and functions. Let $B(c, \gamma)$ and $\overline{B}(c, \gamma)$ denote open and closed balls, respectively, in $\mathcal{X}$ for $c \in \mathcal{X}$ and $\gamma > 0$. $BL(\mathcal{X}, \mathcal{Y})$ stands for the set of all linear and bounded operators from $\mathcal{X}$ to $\mathcal{Y}$. Let $q \in (0, 1]$. Considering two positive parameters w_0 and w_1 with $w_0 \leq w_1$, we define $\mathcal{H}_1$ on $\left[0, \left(\frac{1}{w_0}\right)^{\frac{1}{q}}\right)$ by

$$\mathcal{H}_1(x) = \frac{w_1 x^q}{(q+1)(1-w_0 x^q)} \tag{5}$$

and the parameter

$$\varDelta_1 = \left(\frac{(q+1)}{(q+1)w_0 + w_1}\right)^{\frac{1}{q}} < \left(\frac{1}{w_0}\right)^{\frac{1}{q}}.$$

Observe that $\mathcal{H}_1(\varDelta_1) = 1$. Again, we define functions $\mathcal{H}_2$ and $\mathcal{A}_2$ on $\left[0, \left(\frac{1}{w_0}\right)^{\frac{1}{q}}\right)$ by

$$\mathcal{H}_2(x) = \frac{w_0}{2}(\mathcal{H}_1(x)^q + 1)x^q \tag{6}$$

and

$$\mathcal{A}_2(x) = \mathcal{H}_2(x) - 1.$$

Now, $\mathcal{A}_2(0) = -1 < 0$ and $\lim_{x \to \left(\left(\frac{1}{w_0}\right)^{\frac{1}{q}}\right)^-} \mathcal{A}_2(x) = +\infty$. We apply intermediate value theorem (IVT) to show the existence of the zeros of the function $\mathcal{A}_2(x)$ in the interval $\left(0, \left(\frac{1}{w_0}\right)^{\frac{1}{q}}\right)$. Let us denote the smallest zero of $\mathcal{A}_2(x)$ in $\left(0, \left(\frac{1}{w_0}\right)^{\frac{1}{q}}\right)$ by $\varDelta_2$. Again, we define $\mathcal{H}_3$ and $\mathcal{A}_3$ on $[0, \varDelta_2)$ by

$$\mathcal{H}_3(x) = \mathcal{H}_1(x) + \frac{w_0(\mathcal{H}_1(x)^q x^q + x^q)(1+x^q)}{(1-w_0 x^q)(2(1-\mathcal{H}_2(x)))} \tag{7}$$

and

$$\mathcal{A}_3(x) = \mathcal{H}_3(x) - 1.$$

Now, $\mathcal{A}_3(0) = -1 < 0$ and $\lim_{x \to \varDelta_2^-} \mathcal{A}_3(x) = +\infty$. So, application of IVT gives us the smallest zero $\varDelta_3$ of $\mathcal{A}_3(x)$ in the interval $(0, \Delta_2)$. Lastly, we introduce $\mathcal{H}_4$ and $\mathcal{A}_4$ on $[0, \Delta_2)$ by

$$\mathcal{H}_4(x) = \mathcal{H}_3(x) + \frac{[w_0(\mathcal{H}_1(x)^q x^q + 3x^q) + 2](1 + x^q)\mathcal{H}_3(x)}{(1 - w_0 x^q)(2(1 - \mathcal{H}_2(x)))} \tag{8}$$

and

$$\mathcal{A}_4(x) = \mathcal{H}_4(x) - 1.$$

Now, $\mathcal{A}_4(0) = -1 < 0$ and $\lim_{x \to \Delta_2^-} \mathcal{A}_4(x) = +\infty$. Then the minimal zero Δ_4 of $\mathcal{A}_4(x)$ exists in $(0, \Delta_2)$ due to IVT. We choose

$$R = min\{\Delta_1, \Delta_3, \Delta_4\} \tag{9}$$

to confirm the followings.

$$0 \le \mathcal{H}_1(x) < 1, \tag{10}$$

$$0 \le \mathcal{H}_2(x) < 1, \tag{11}$$

$$0 \le \mathcal{H}_3(x) < 1 \tag{12}$$

and

$$0 \le \mathcal{H}_4(x) < 1, \tag{13}$$

for each $x \in [0, R)$. Also, we consider the following assumptions to prove the main convergence result. Let $\mathcal{Q} : \Omega \subseteq \mathcal{X} \to \mathcal{Y}$ be a Fréchet derivable operator such that

$$\mathcal{Q}(s^*) = 0,\ \mathcal{Q}'(s^*)^{-1} \in BL(\mathcal{Y}, \mathcal{X}), \tag{14}$$

$$\| \mathcal{Q}'(s^*)^{-1}(\mathcal{Q}'(s) - \mathcal{Q}'(s^*)) \| \le w_0 \| s - s^* \|^q,\ \forall s \in \Omega \tag{15}$$

and

$$\| \mathcal{Q}'(s^*)^{-1}(\mathcal{Q}'(s) - \mathcal{Q}'(t)) \| \le w_1 \| s - t \|^q,\ \forall s, t \in \Omega_0, \tag{16}$$

where $\Omega_0 = B\left(s^*,\ (\frac{1}{w_0})^{\frac{1}{q}}\right) \cap \Omega$.

Many authors [22, 28] use an extra assumption

$$\| \mathcal{Q}'(s^*)^{-1}\mathcal{Q}'(s) \| \le M. \tag{17}$$

We remove this additional condition using the following results.

Lemma 1 If $\mathcal{Q}$ obeys (15) and $\bar{B}(s^*, R) \subseteq \Omega$, then $\forall s \in B(s^*, R)$, we get

$$\| \mathcal{Q}'(s^*)^{-1}\mathcal{Q}'(s) \| \leq 1 + w_0 \| s - s^* \|^q \tag{18}$$

and

$$\| \mathcal{Q}'(s^*)^{-1}\mathcal{Q}(s) \| \leq (1 + w_0 \| s - s^* \|^q) \| s - s^* \|. \tag{19}$$

Proof. Applying (15), we get

$$\| \mathcal{Q}'(s^*)^{-1}\mathcal{Q}'(s) \| \leq 1 + \| \mathcal{Q}'(s^*)^{-1}(\mathcal{Q}'(s) - \mathcal{Q}'(s^*)) \| \leq 1 + w_0 \| s - s^* \|^q .$$

For $\xi \in [0, 1]$,

$$\| s^* + \xi(s - s^*) - s^* \| = \xi \| s - s^* \| < R.$$

So, $s^* + \xi(s - s^*) \in B(s^*, R)$ and

$$\| \mathcal{Q}'(s^*)^{-1}\mathcal{Q}'(s^* + \xi(s - s^*)) \| \leq 1 + w_0\xi \| s - s^* \|^q \leq 1 + w_0 \| s - s^* \|^q . \tag{20}$$

We apply the mean value theorem and (20) to find

$$\begin{aligned} \| \mathcal{Q}'(s^*)^{-1}\mathcal{Q}(s) \| &= \| \mathcal{Q}'(s^*)^{-1}(\mathcal{Q}(s) - \mathcal{Q}(s^*)) \| \\ &\leq \| \mathcal{Q}'(s^*)^{-1}\mathcal{Q}'(s^* + \xi(s - s^*))(s - s^*) \| \\ &\leq (1 + w_0 \| s - s^* \|^q) \| s - s^* \|. \end{aligned}$$

Theorem 1 Let $s^* \in \Omega$ and the conditions (14)-(16) hold true and

$$\bar{B}(s^*, R) \subseteq \Omega, \tag{21}$$

where R is given in (9). Then, for any starting point $s_0 \in B(s^*, R)$ the iterative formula (3) produces the well defined sequence $\{s_k\}$ such that $\{s_k\}_{k \geq 0} \in B(s^*, R)$ and $\lim_{k \to \infty} s_k = s^*$. Also, the following items hold true for all $k \geq 0$

$$\| t_k - s^* \| \leq \mathcal{H}_1(\| s_k - s^* \|) \| s_k - s^* \| < \| s_k - s^* \| < R, \tag{22}$$

$$\| u_k - s^* \| \leq \mathcal{H}_3(\| s_k - s^* \|) \| s_k - s^* \| < \| s_k - s^* \| < R \tag{23}$$

and

$$\| s_{k+1} - s^* \| \leq \mathcal{H}_4(\| s_k - s^* \|) \| s_k - s^* \| < \| s_k - s^* \| < R, \tag{24}$$

where the real functions $\mathcal{H}_1$, $\mathcal{H}_3$ and $\mathcal{H}_4$ are introduced in (5), (7) and (8) respectively. For $\Delta \in \left[R, (\frac{q+1}{w_0})^{\frac{1}{q}}\right)$, s^* uniquely solves the equation $\mathcal{Q}(s) = 0$ in $\bar{B}(s^*, \Delta) \cap \Omega$.

Proof. Since $s_0 \in B(s^*, R)$, we obtain

$$\| \mathcal{Q}'(s^*)^{-1}(\mathcal{Q}'(s_0) - \mathcal{Q}'(s^*)) \| \leq w_0 \| s_0 - s^* \|^q < w_0 R^q < 1 \tag{25}$$

using (9) and (15). Now, Banach lemma [1, 5, 6] guarantees that $\mathcal{Q}'(s_0)^{-1} \in BL(\mathcal{Y}, \mathcal{X})$ and

$$\| \mathcal{Q}'(s_0)^{-1}\mathcal{Q}'(s^*) \| \leq \frac{1}{1 - w_0 \| s_0 - s^* \|^q} < \frac{1}{1 - w_0 R^q}. \tag{26}$$

Therefore, t_0 is well defined. Again, we get from (3) that

$$\begin{aligned} t_0 - s^* &= s_0 - s^* - \mathcal{Q}'(s_0)^{-1}\mathcal{Q}(s_0) \\ &= -\left[\mathcal{Q}'(s_0)^{-1}\mathcal{Q}'(s^*)\right]\left[\int_0^1 \mathcal{Q}'(s^*)^{-1}(\mathcal{Q}'(s^* + \xi(s_0 - s^*)) - \mathcal{Q}'(s_0))(s_0 - s^*)\, d\xi\right]. \end{aligned} \tag{27}$$

Using (5), (9), (10), (26) and (27), we get

$$\begin{aligned} \| t_0 - s^* \| &\leq \left[\| \mathcal{Q}'(s_0)^{-1}\mathcal{Q}'(s^*) \|\right]\left[\left\| \int_0^1 \mathcal{Q}'(s^*)^{-1}(\mathcal{Q}'(s^* + \xi(s_0 - s^*)) - \mathcal{Q}'(s_0))(s_0 - s^*)\, d\xi \right\|\right] \\ &\leq \frac{w_1 \| s_0 - s^* \|^q}{(q+1)(1 - w_0 \| s_0 - s^* \|^q)} \| s_0 - s^* \| \\ &= \mathcal{H}_1(\| s_0 - s^* \|) \| s_0 - s^* \| \leq \| s_0 - s^* \| < R. \end{aligned} \tag{28}$$

So, (22) is true for $k = 0$. Now, we claim that $[\mathcal{Q}'(s_0) + \mathcal{Q}'(t_0)]^{-1} \in BL(\mathcal{Y}, \mathcal{X})$. The equations (6), (9), (11), (15) and (28) are used to obtain

$$\begin{aligned} &\left\| (2\mathcal{Q}'(s^*))^{-1}\left[\mathcal{Q}'(s_0) + \mathcal{Q}'(t_0) - 2\mathcal{Q}'(s^*)\right] \right\| \\ &\leq \frac{1}{2}\left[\| \mathcal{Q}'(s^*)^{-1}(\mathcal{Q}'(s_0) - \mathcal{Q}'(s^*)) \| + \| \mathcal{Q}'(s^*)^{-1}(\mathcal{Q}'(t_0) - \mathcal{Q}'(s^*)) \|\right] \\ &\leq \frac{w_0}{2}\left[\| s_0 - s^* \|^q + \| t_0 - s^* \|^q\right] \\ &\leq \frac{w_0}{2}[\| s_0 - s^* \|^q + \mathcal{H}_1(\| s_0 - s^* \|)^q \| s_0 - s^* \|^q] \\ &= \frac{w_0}{2}[\mathcal{H}_1(\| s_0 - s^* \|)^q + 1] \| s_0 - s^* \|^q \\ &= \mathcal{H}_2(\| s_0 - s^* \|) < \mathcal{H}_2(R) < 1. \end{aligned}$$

Now, we obtain $[\mathcal{Q}'(s_0) + \mathcal{Q}'(t_0)]^{-1} \in BL(\mathcal{Y}, \mathcal{X})$ and

$$\left\|\left[\mathcal{Q}'(s_0)+\mathcal{Q}'(t_0)\right]^{-1}\mathcal{Q}'(s^*)\right\| \leq \frac{1}{2(1-\mathcal{H}_2(\|s_0-s^*\|))}. \tag{29}$$

using Banach lemma on invertible operators. We obtain from (3), (7), (9), (12), (19), (26), (28) and (29) that

$$\begin{aligned}
\|u_0-s^*\| &\leq \|t_0-s^*\| + \|\mathcal{Q}'(s_0)^{-1}\mathcal{Q}'(s^*)\| \\
&\times \left(\|\mathcal{Q}'(s^*)^{-1}(\mathcal{Q}'(t_0)-\mathcal{Q}'(s^*))\| + \|\mathcal{Q}'(s^*)^{-1}(\mathcal{Q}'(s_0)-\mathcal{Q}'(s^*))\|\right) \\
&\times \|(\mathcal{Q}'(s_0)+\mathcal{Q}'(t_0))^{-1}\mathcal{Q}'(s^*)\|\,\|\mathcal{Q}'(s^*)^{-1}\mathcal{Q}(s_0)\| \\
&\leq \mathcal{H}_1(\|s_0-s^*\|)\|s_0-s^*\| \\
&+ \frac{w_0(\|t_0-s^*\|^q + \|s_0-s^*\|^q)(1+w_0\|s_0-s^*\|^q)\|s_0-s^*\|}{(1-w_0\|s_0-s^*\|^q)(2(1-\mathcal{H}_2(\|s_0-s^*\|)))} \\
&\leq \mathcal{H}_1(\|s_0-s^*\|)\|s_0-s^*\| \\
&+ \frac{w_0(\mathcal{H}_1(\|s_0-s^*\|)^q\|s_0-s^*\|^q + \|s_0-s^*\|^q)(1+w_0\|s_0-s^*\|^q)\|s_0-s^*\|}{(1-w_0\|s_0-s^*\|^q)(2(1-\mathcal{H}_2(\|s_0-s^*\|)))} \\
&= \left(\mathcal{H}_1(\|s_0-s^*\|) + \frac{w_0(\mathcal{H}_1(\|s_0-s^*\|)^q\|s_0-s^*\|^q + \|s_0-s^*\|^q)(1+w_0\|s_0-s^*\|^q)}{(1-w_0\|s_0-s^*\|^q)(2(1-\mathcal{H}_2(\|s_0-s^*\|)))}\right)\|s_0-s^*\| \\
&= \mathcal{H}_3(\|s_0-s^*\|)\|s_0-s^*\| < \|s_0-s^*\| < R. \qquad (30)
\end{aligned}$$

Finally, we use (3), (8), (9), (13), (19), (26), (28) (29) and (30) to deduce

$$\begin{aligned}
\|s_1-s^*\| &\leq \|u_0-s^*\| + \|(\mathcal{Q}'(s_0)+\mathcal{Q}'(t_0))^{-1}\mathcal{Q}'(s^*)\| \\
&\times \left(\|\mathcal{Q}'(s^*)^{-1}(\mathcal{Q}'(t_0)-\mathcal{Q}'(s^*))\| + \|\mathcal{Q}'(s^*)^{-1}(\mathcal{Q}'(s_0)-\mathcal{Q}'(s^*))\|\right. \\
&\left. + 2\|\mathcal{Q}'(s^*)^{-1}(\mathcal{Q}'(s_0)-\mathcal{Q}'(s^*))\| + 2\|\mathcal{Q}'(s^*)^{-1}\mathcal{Q}'(s^*)\|\right) \\
&\times \|\mathcal{Q}'(s_0)^{-1}\mathcal{Q}'(s^*)\| \left\|\int_0^1 \mathcal{Q}'(s^*)^{-1}\mathcal{Q}'(s^*+\xi(u_0-s^*))(u_0-s^*)\,d\xi\right\| \\
&\leq \mathcal{H}_3(\|s_0-s^*\|)\|s_0-s^*\| \\
&+ \frac{[w_0(\|t_0-s^*\|^q + 3\|s_0-s^*\|^q)+2](1+w_0\|s_0-s^*\|^q)\|u_0-s^*\|}{(1-w_0\|s_0-s^*\|^q)(2(1-\mathcal{H}_2(\|s_0-s^*\|)))}
\end{aligned}$$

$$\leq \mathcal{H}_3(\| s_0 - s^* \|) \| s_0 - s^* \|$$

$$+ \frac{[w_0(\mathcal{H}_1(\| s_0 - s^* \|)^q \| s_0 - s^* \|^q + 3\| s_0 - s^* \|^q) + 2](1 + w_0 \| s_0 - s^* \|^q)\mathcal{H}_3(\| s_0 - s^* \|) \| s_0 - s^* \|}{(1 - w_0 \| s_0 - s^* \|^q)(2(1 - \mathcal{H}_2(\| s_0 - s^* \|)))}$$

$$= \Bigg(\mathcal{H}_3(\| s_0 - s^* \|)$$

$$+ \frac{[w_0(\mathcal{H}_1(\| s_0 - s^* \|)^q \| s_0 - s^* \|^q + 3\| s_0 - s^* \|^q) + 2](1 + w_0 \| s_0 - s^* \|^q)\mathcal{H}_3(\| s_0 - s^* \|)}{(1 - w_0 \| s_0 - s^* \|^q)(2(1 - \mathcal{H}_2(\| s_0 - s^* \|)))} \Bigg) \| s_0 - s^* \|$$

$$= \mathcal{H}_4(\| s_0 - s^* \|) \| s_0 - s^* \| < \| s_0 - s^* \| < R. \tag{31}$$

Thus, the estimate (24) is true for $k = 0$. We obtain the estimates (22)-(24) by using s_k, t_k and s_{k+1} instead of s_0, t_0 and s_1 respectively in the earlier estimations. Since $\|s_{k+1} - s^*\| \leq \mathcal{H}_4(R)\|s_k - s^*\| < R$, we confirm that $s_{k+1} \in B(s^*, R)$ and $\lim_{k\to\infty} s_k = s^*$. The uniqueness part is proved by considering another solution $t^*(\neq s^*) \in B(s^*, \Delta) \cap \Omega$ of $\mathcal{Q}(s) = 0$. Let $\mathcal{G} = \int_0^1 \mathcal{Q}'(t^* + \xi(s^* - t^*))\, d\xi$. From equation (15), we get

$$\| \mathcal{Q}'(s^*)^{-1}(\mathcal{G} - \mathcal{Q}'(s^*)) \| \leq \int_0^1 w_0 \| t^* + \xi(s^* - t^*) - s^* \|^q \, d\xi$$

$$\leq \frac{w_0}{q+1} \| s^* - t^* \|^q$$

$$\leq \frac{w_0 \Delta^q}{q+1} < 1.$$

The invertability of $\mathcal{G}$ and the identity $0 = \mathcal{Q}(s^*) - \mathcal{Q}(t^*) = \mathcal{G}(s^* - t^*)$ imply that $s^* = t^*$.

3. Numerical examples

We validate our theoretical results in this section. We compare the convergence radii of the method (3), obtained by our proposed analysis, with the radii obtained in the earlier study [22]. It is concluded that the proposed technique produces bigger convergence radii for the method (3) than the previous work [22].

Example 1 [22] Let $\mathcal{Q}$ is defined on $\bar{B}(0, 1)$ for $(s_1, s_2, s_3)^t$ by

$$\mathcal{Q}(s) = (e^{s_1} - 1, \frac{e-1}{2} s_2^2 + s_2, s_3)^t$$

We have $s^* = (0, 0, 0)^t$, $q = 1$, $w_0 = e - 1$ and $w_1 = 1.789572397$. The radius R is obtained using the proposed analytical result (see Table 1).

Table 1. Comparison of convergence radius for Example 1

Proposed Analysis				Earlier study [22]
Δ_1	Δ_3	Δ_4	R	r
0.382692	0.215469	0.114489	0.114489	0.075870

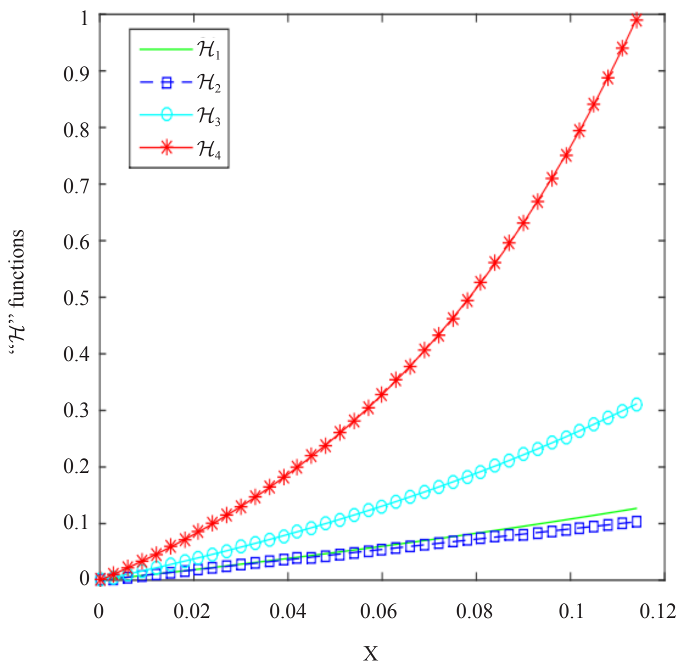

Figure 1. $\mathcal{H}$ functions for Example 1

Example 2 [22] Define $\mathcal{Q}$ on $\Omega = \left[-\frac{1}{2}, \frac{5}{2}\right]$ by

$$\mathcal{Q}(s) = \begin{cases} s^3 \ln(s^2) + s^5 - s^4, & \text{if } s \neq 0 \\ 0, & \text{if } s = 0 \end{cases}.$$

We have $s^* = 1$, $q = 1$ and $w_0 = w_1 = 96.6628$. The value of R is given in Table 2.

Table 2. Comparison of convergence radius for Example 2

Proposed Analysis				Earlier study [22]
Δ_1	Δ_3	Δ_4	R	r
0.006896	0.003865	0.002055	0.002055	0.001362

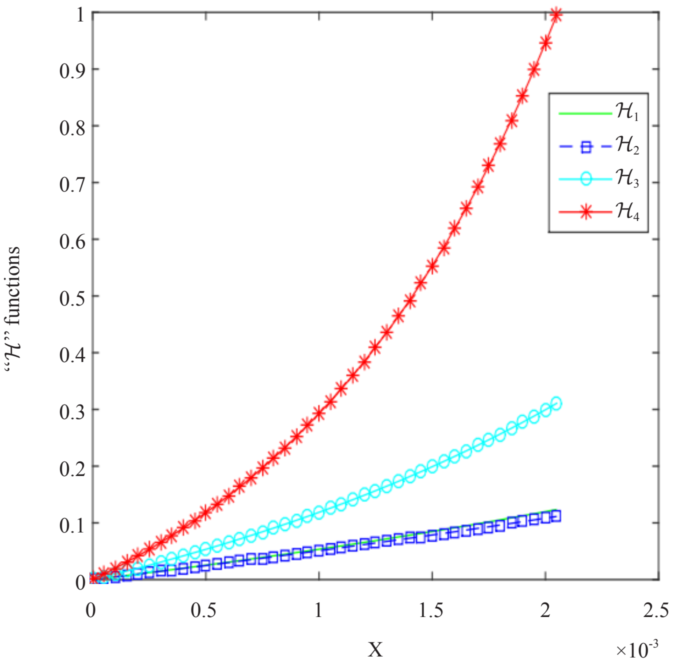

Figure 2. $\mathcal{H}$ functions for Example 2

Example 3 [22] Consider the mixed Hammerstein type integral equation given by

$$\mathcal{Q}(s)(w) = s(w) - \int_0^1 G_1(w, y)\left(s(y)^{\frac{5}{2}} + \frac{s(y)^2}{2} \right) dy,$$

where $s(w) \in C[0, 1]$. We have $s^* = 0$. Also, $q = 1$ and $w_0 = w_1 = \frac{1}{8}\left(\frac{5}{2}\sqrt{2}+1\right)$. In Table 3, the radius R is provided.

Table 3. Comparison of convergence radius for Example 3

Proposed Analysis				Earlier study [22]
Δ_1	Δ_3	Δ_4	R	r
1.175899	0.658908	0.350378	0.350378	0.232141

Example 4 [28] Consider the nonlinear integral equation given by

$$\mathcal{Q}(s)(w) = s(w) - 3\int_0^1 G_1(w, y)\, s(y)^{\frac{5}{4}}\, dy,$$

where $s(w) \in C[0, 1]$ and $G_1(w, y)$ is Green's function. We have $s^* = 0$. Also, $q = 0.25$ and $w_0 = w_1 = \frac{15}{32}$. The value of R is presented in Table 4.

Table 4. Comparison of convergence radius for Example 4

Proposed Analysis				Earlier study [22]
Δ_1	Δ_3	Δ_4	R	r
1.973081	0.060989	0.005851	0.005851	Cannot be computed

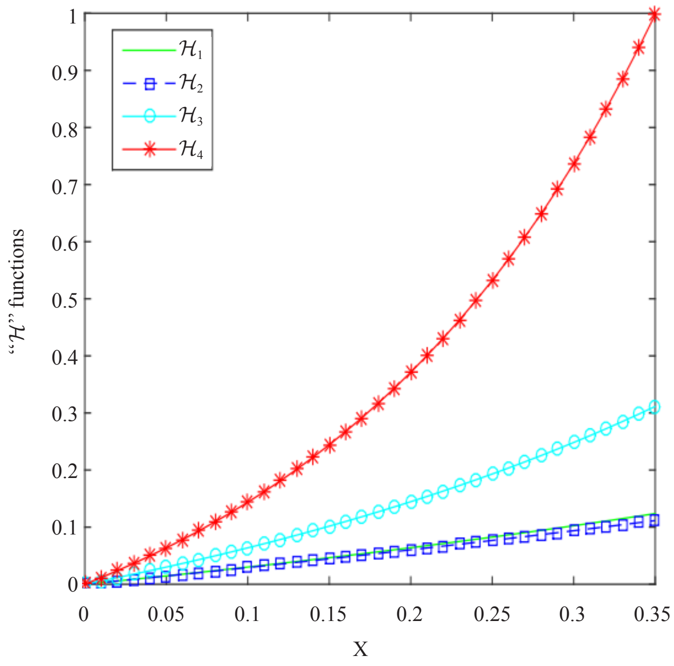

Figure 3. $\mathcal{H}$ functions for Example 3

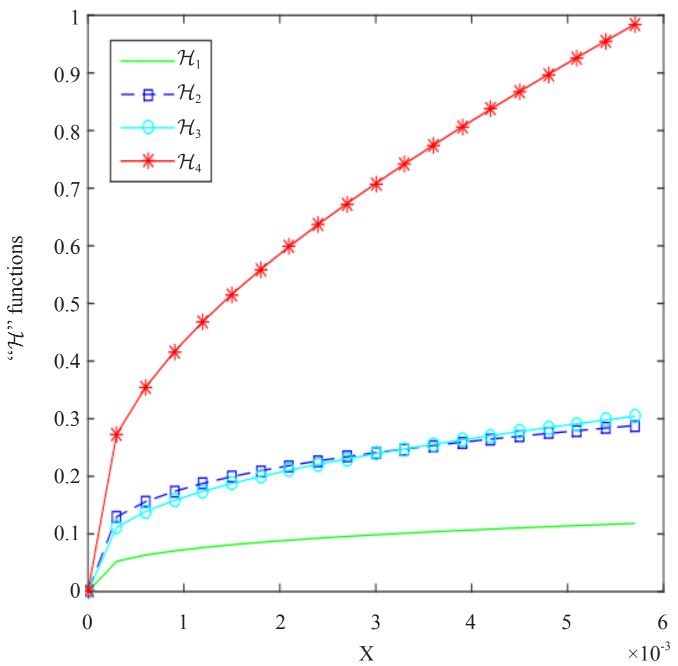

Figure 4. $\mathcal{H}$ functions for Example 4

Thus, we show the convergence of the scheme (3) for Example 4 with radius $R = 0.005851$.

4. Conclusions

Local convergence analysis of a fifth-order nonlinear equation solver is presented to extend its applicability and convergence domain. This study generalized the local analysis based on Lipschitz condition. It is concluded from the numerical tests that our analysis provides a larger domain of convergence, in comparison with the previous work, for the scheme (3). Also, our theoretical conclusions worked well in the situation where the earlier analysis based on Lipschitz constants cannot be used.

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