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Research Article

# Stability and Well-Posedness Analysis of the Fractional Shallow Water Wave Models with Applications

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**Abstract:** In this work, we present a comprehensive analytical investigation of the fractional Whitham-Broer-Kaup Equations (WBKEs), which model nonlinear oscillatory behavior in shallow water wave phenomena. The equations are formulated using the Atangana-Baleanu-Caputo (ABC) fractional derivative to capture nonlinear oscillations and the dynamics of tsunami-induced shallow water waves. Within the framework of Banach spaces endowed with the compact-open topology, the existence, uniqueness, and Hyers-Ulam stability of the fractional WBKEs are rigorously established via fixed point theorems. Furthermore, a hybrid analytical technique combining the double Laplace transform and the Adomian decomposition method, referred to as the Double Laplace Transform Adomian Decomposition Method (DLADM), is developed to obtain approximate series solutions. This method effectively handles nonlinearities and fractional-order effects, yielding rapidly convergent solutions that agree well with exact analytical results. The findings also demonstrate the robustness of the fractional model under small perturbations and its effectiveness in describing nonlinear shallow water wave dynamics.

***Keywords***: fractional Whitham-Broer-Kaup Equations (WBKEs), double Laplace transform, Atangana-Baleanu-Caputo (ABC) operator, stability

**MSC:** 44A10, 35B35, 26A33

## 1. Introduction

Fractional calculus (FC) is an extension of classical calculus that permits derivatives and integrals of arbitrary, non-integer orders. The origin of this concept can be traced back to the $17^{\text{th}}$ century, when mathematicians such as Leibniz and L'Hôpital discussed the possibility of fractional-order differentiation. Its systematic development, however, took shape in the $19^{\text{th}}$ century through the contributions of Liouville, Riemann, and Grunwald, who independently formulated definitions for fractional integrals and derivatives [1, 2]. Since then, the theoretical framework of FC has


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evolved considerably, unifying various forms of differential and integral operators and providing a rigorous mathematical foundation for analyzing systems that exhibit nonlocal and hereditary behavior. In recent decades, FC has garnered increasing attention, with researchers exploring new applications and properties of fractional integrals and derivatives. The advancement of numerical techniques in fractional calculus has significantly enhanced the ability to solve fractional differential equations (FDEs), thereby broadening its applicability across engineering and various scientific disciplines [3, 4]. Fractional operators, such as those defined in the sense of Caputo, Riemann-Liouville, Atangana-Baleanu, and Caputo-Fabrizio, have been proposed to model various physical phenomena with nonlocal kernels. These operators have been instrumental in bridging the gap between integer order models and experimental observations. The rise of modern computational algorithms has also accelerated the practical use of FC in complex simulations, supporting developments in modeling viscoelastic materials, electromagnetic systems, biological population dynamics, and diffusion processes with memory effects [5]. partial differential equations (PDEs) are fundamental tools for describing a wide range of physical and scientific phenomena, such as plasma physics, fluid flow, quantum mechanics, and electromagnetic processes. A significant extension of this field is the study of fractional systems of PDEs, which generalize classical integer-order derivatives to derivatives of non-integer order. In recent years, this fractional framework has drawn considerable attention due to its superior ability to represent complex processes that cannot be sufficiently modeled using standard integer order equations [6, 7]. The incorporation of fractional derivatives allows for the accurate representation of systems with memory effects and spatial heterogeneity. As a result, fractional partial differential equations (FPDEs) provide a more comprehensive modeling approach for nonlocal interactions, anomalous diffusion, viscoelastic behavior, and chaotic dynamics in nonlinear systems [8]. Fractional PDE systems are widely applied across diverse disciplines, including biology, geophysics, finance, engineering, physics. Some notable applications in applied sciences involve electrodynamics, economic and accounting systems, chaos modeling, biological population dynamics, and fluid mechanics [9]. Mursaleen, [10] introduced blending-type $q$-Baskakov operators based on wavelet transformations and studies their approximation and convergence properties. In [11], proposed quadratic function preserving wavelet-type Baskakov operators to improve accuracy in function approximation. FDEs are increasingly recognized for their applicability and versatility in various scientific domains. They are particularly relevant when describing systems governed by hereditary laws or memory effects, such as charge transport in disordered media, groundwater flow in porous rocks, and heat conduction in heterogeneous materials. The rapid growth in this field has also stimulated extensive research on the mathematical properties of FPDEs which ensure that models maintain both physical relevance [12]. In recent years, the investigation of nonlinear PDEs has emerged as a significant focus of research. Numerous authors have developed various mathematical methods to analyze approximate solutions to these equations. For instance, Kumar [13] developed a powerful method for solving nonlinear problems. Yang [6] introduced a new integral method to address the steady heat transfer problem, Mohyud Din et al. [8] utilized perturbation methods. Xie et al. [14] explored solutions using hyperbolic techniques. Biazar and Aminikhah [9] worked on coupled Burgers equations, employing perturbation techniques. Ahmad et al. [15] employed both the Adomian decomposition method and He's polynomial to solve the Whitham-Broer-Kaup Equations (WBKEs). Mohyud Din et al. [16] explored several integer-order PDEs using homotopy perturbation techniques. Yuzbasi and Sahin [17] presented solutions for one-dimensional parabolic convection-diffusion model problems. Ahmad et al. [18] proposed a perturbation method for the Caputo-Fabrizio PDE. Additionally, these diverse analytical methods highlight the continuous effort to obtain efficient, stable, and convergent solutions to nonlinear and fractional PDEs, which often arise in practical scientific modeling. Studies have shown that the nonlinear evolution of physical waves phenomena can be effectively captured by the coupled WBKEs [19], which are expressed in their classical form:

$$\begin{cases} \omega_\xi(\kappa, \xi) + \omega(\kappa, \xi)\omega_\kappa(\kappa, \xi) + \psi_\kappa(\kappa, \xi) + \beta\, \omega_{\kappa\kappa}(\kappa, \xi) = 0, \\ \\ \psi_\xi(\kappa, \xi) + \left[\omega(\kappa, \xi)\psi(\kappa, \xi)\right]_\kappa - \beta\, \psi_{\kappa\kappa}(\kappa, \xi) + \beta'\, \omega_{\kappa\kappa\kappa}(\kappa, \xi) = 0, \end{cases} \tag{1}$$

where $\psi$ represents the fluid's height velocity, and $\omega$ denotes its horizontal velocity, both of which exhibit significant deviations from their equilibrium states. The parameters $\beta$ and $\beta'$ correspond to different diffusion powers. These equations describe nonlinear wave propagation, incorporating both dispersive and convective effects, and have been successfully applied to model shallow water dynamics, plasma oscillations, and nonlinear acoustic waves. A number of works have also concentrated on the nonlinear WBKEs (1). Within the broader framework of fractional differential equations, various fractional integral and derivative operators have been developed to model hereditary characteristics of physical processes. Consequently, many efficient analytical and numerical approaches have been designed to obtain approximate solutions for physical and biological fractional PDEs. Notable examples include the Laplace transform method [20], the fractional Newton method [21], the extended direct algebraic mapping approach [22], the homotopy perturbation method [23], the sine-Gordon technique [24], the Laplace residual power series method (RPSM) [25], the reproducing kernel Hilbert space method applied to the Fornberg-Whitham type equation [26], nonlocal fractional equations [27] which are applied to the variational iteration method [28], the telegraph equation [29], the monotone iterative method for reaction-diffusion equations [30], the expansion method [31], and a modified Adams-Bashforth method [32]. More recently, Nadeem et al. [33] combined the Yang transform with the residual power series method to establish the Yang residual power series scheme, which was successfully applied to a two-dimensional fuzzy fractional-order heat problem.

Consider the Atangana-Baleanu-Caputo (ABC) fractional WBKEs [34] in shallow water wave phenomena:

$$\begin{cases} {}^{ABC}\mathscr{D}^{\alpha}_{0,\,\xi}\,\omega(\kappa,\,\xi)+\omega(\kappa,\,\xi)\omega_{\kappa}(\kappa,\,\xi)+\psi_{\kappa}(\kappa,\,\xi)+\beta\,\omega_{\kappa\kappa}(\kappa,\,\xi)=0, \\ {}^{ABC}\mathscr{D}^{\alpha}_{0,\,\xi}\,\psi(\kappa,\,\xi)+[\omega(\kappa,\,\xi)\psi(\kappa,\,\xi)]_{\kappa}-\beta\,\psi_{\kappa\kappa}(\kappa,\,\xi)+\beta'\,\omega_{\kappa\kappa\kappa}(\kappa,\,\xi)=0, \\ \omega(\kappa,\,0)=h_0(\kappa), \quad \psi(\kappa,\,0)=\hat{h}_0(\kappa), \end{cases} \tag{2}$$

where ${}^{ABC}\mathscr{D}^{\alpha}_{0,\,\xi}$ is the ABC fractional derivative of order $0<\alpha\leq 1$, $\kappa$ is spatial dimension of wave, $\xi$ denotes the time, $\omega$ represents the depth averaged horizontal velocity of wave, $\psi$ denotes free surface elevation, and the parameters $\beta$, $\beta'$ are coefficients controlling dissipation and dispersion. This fractional model captures the influence of memory and hereditary properties on nonlinear wave propagation and extends the applicability of the WBKEs to systems where dissipative and dispersive effects coexist with long-term temporal dependence. The behavior of the fractional WBKEs (2) is influenced by time and model parameters. The time variable $\xi$ describes the evolution of the wave profile, while the fractional order $\alpha$ reflects memory effects in the system. The parameters $\beta$ and $\beta_0$ control the dispersive properties, where $\beta$ governs wave smoothing and spreading, and $\beta_0$ affects higher-order oscillatory structures. Thus, variations in these parameters lead to different wave dynamics and enhance the physical relevance of the model. A wide range of analytical and numerical methods has been introduced for obtaining solutions to the ABC WBKEs (2). Among these are the residual power series method [35], the redefined quintic B-spline basis with the Von Neumann technique [36], the Adomian decomposition method [37], the Yang transformation combined with the Adomian technique [38], the q-Homotopy Analysis Transform Method (q-HATM) [39], the Yang residual power series method (YRPST) [40], the Sumudu decomposition method [41], the Laplace transform coupled with the Adomian method [42], the natural decomposition method [39], and the scaling transform method [43]. These studies collectively emphasize the flexibility and accuracy of hybrid analytical numerical schemes for fractional nonlinear PDEs.

This study is motivated by the need for more accurate models of nonlinear shallow water waves that exhibit memory and nonlocal effects. The fractional WBKEs (2) provide a better framework for such dynamics. Using Banach spaces with compact-open topology and a hybrid double Laplace transform Adomian decomposition method (DLADM), the work ensures well-posedness and stability while yielding efficient analytical solutions. This paper begins with an introduction to fractional calculus and its applications in modeling systems with memory and nonlocal effects, presenting the fractional WBKEs (2) for shallow water wave dynamics. The second section introduces the ABC fractional operator and double Laplace transform, forming the analytical foundation. In the third section, existence, uniqueness, and Hyers-Ulam stability of the fractional WBKEs (2) are rigorously established using fixed point theorems in Banach spaces endowed with the compact-open topology. The fourth section focuses on solving fractional WBKEs (2) by applying a hybrid DLADM to derive rapidly convergent approximate analytical solutions. The fifth section presents numerical and graphical comparisons to validate the accuracy and convergence of the proposed method, while the final section concludes with remarks on the efficiency and stability of the developed fractional framework. The proposed DLADM differs from existing methods by combining the double Laplace transform with the Adomian decomposition method, which enhances its capability to handle nonlinear fractional systems. Unlike standard Adomian decomposition method (ADM) or single Laplace approaches, the DLADM provides faster convergence and reduces computational complexity. Moreover, it effectively treats both linear and nonlinear terms without requiring discretization or perturbation, making it more efficient and accurate for solving fractional WBKEs.

## 2. On double Laplace with ABC operator

**Definition 1** [44] Let $\omega$ be any map on $\Omega \times [0,\, \xi_0]$, where $\xi_0 > 0$. The ABC integral operator of $\omega(\kappa,\, \xi)$ for order $\alpha > 0$ is given by

$$ {}^{ABC}\mathscr{I}^{\alpha}_{0,\,\xi}\, \omega(\kappa,\, \xi) = \frac{1-\alpha}{A^{\alpha}}\, \omega(\kappa,\, \xi) + \frac{\alpha}{\Gamma(\alpha)A^{\alpha}} \int_0^{\xi} (\xi-\sigma)^{\alpha-1}\omega(\kappa,\, \sigma)d\sigma, \quad 0<\alpha<1 \tag{3} $$

where $A^{\cdot}$ is a condition referred to as the normalization function with $A^0 = A^1 = 1$.

The normalization condition $A^0 = A^1 = 1$ ensures that the fractional operator is consistent with the classical cases. In particular, for $\alpha = 0$ the operator reduces to the identity, while for $\alpha = 1$ it coincides with the standard integral operator. This property preserves the physical and mathematical interpretation at integer orders.

**Definition 2** [44] Let $\omega$ be any map on $\Omega \times [0,\, \xi_0]$. The ABC derivative operator of $\omega(\kappa,\, \xi)$ for order $\alpha > 0$ is given by

$$ {}^{ABC}\mathscr{D}^{\alpha}_{0,\,\xi}\, \omega(\kappa,\, \xi) = \begin{cases} \dfrac{A^{\alpha}}{1-\alpha} \displaystyle\int_0^{\xi} E_{\alpha}\left(\dfrac{-\alpha(\xi-\mu)^{\alpha}}{1-\alpha}\right) \omega^{(n)}_{\mu^n}(\kappa,\, \mu)d\mu, & n-1<\alpha<n, \\ \\ \omega^{(n)}_{\xi^n}(\kappa,\, \xi), \quad \alpha := n \in \mathbb{N}, & \end{cases} \tag{4} $$

where $E_{\alpha}(\cdot)$ is the Mittag-Leffler function

$$ E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \qquad \alpha > 0. $$

Recall [45], for $\xi \geq 0$ and for $n-1 < \alpha < n$ we have ${}^{ABC}\mathscr{D}^{\alpha}_{0,\,\xi}\, {}^{ABC}\mathscr{I}^{\alpha}_{0,\,\xi}\, \omega(\kappa,\, \xi) = \omega(\kappa,\, \xi)$.

**Theorem 1** [46] Let $\omega$ be any map on $\Omega \times [0,\, \xi_0]$. Then

$$ {}^{ABC}\mathscr{I}^{\alpha}_{0,\,\xi}\, {}^{ABC}\mathscr{D}^{\alpha}_{0,\,\xi}\, \omega(\kappa,\, \xi) = \omega(\kappa,\, \xi) - \sum_{k=0}^{n-1} \frac{\xi^k}{k!} \omega^{(k)}_{\xi}(\kappa,\, \xi)\Big|_{\xi=0}. \tag{5} $$

Let $\omega$ be a map on $\Omega \times [0,\, \xi_0]$. The Laplace transform $\mathbb{L}_\kappa$ [47] of $\omega(\kappa,\, \xi)$ with respect to $\kappa$ is defined by

$$\mathbb{L}_\kappa[\omega(\kappa,\, \xi)] = \int_0^\infty \omega(\kappa,\, \xi)e^{-\rho\kappa}d\kappa \;\; Re(\rho) > 0. \tag{6}$$

The inverse $\mathbb{L}_\kappa^{-1}$ of $\mathbb{L}_\kappa[\omega(\kappa,\, \xi)]$ is given by $\omega(\kappa,\, \xi) = \mathbb{L}_\kappa^{-1}[\mathbb{L}_\kappa[\omega(\kappa,\, \xi)]]$. The double Laplace transform $\mathbb{L}$ [47] of $\omega(\kappa,\, \xi)$ is defined by

$$\omega_{\mathbb{L}}(\rho,\, \rho) := \mathbb{L}[\omega(\kappa,\, \xi)] = \int_0^\infty \omega(\kappa,\, \xi)e^{-(\rho\kappa+\rho\xi)}d\kappa d\xi \;\; Re(\rho),\, Re(\rho) > 0. \tag{7}$$

The inverse $\omega_{\mathbb{L}}^{-1}$ of $\omega_{\mathbb{L}}(\rho,\, \rho)$ is given by

$$\omega(\kappa,\, \xi) = \omega_{\mathbb{L}}^{-1}[\omega_{\mathbb{L}}(\rho,\, \rho)] = \frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} e^{\rho\kappa}\frac{1}{2\pi \mathrm{i}}\int_{d-\mathrm{i}\infty}^{d+\mathrm{i}\infty} \omega_{\mathbb{L}}(\rho,\, \rho)e^{\rho\xi}d\rho d\rho. \tag{8}$$

Recall [47] that the double Laplace transform and its inverse are linear maps and $\mathbb{L}[\kappa^n\xi^m] = \frac{m!n!}{\rho^n\rho^m}$.

**Theorem 2** [44] The transform $\mathbb{L}$ of $\omega_{\kappa^m}^{(m)}(\kappa,\, \xi)$ is given by

$$\mathbb{L}\left[\omega_{\kappa^m}^{(m)}(\kappa,\, \xi)\right] = \rho^m\omega_{\mathbb{L}}(\rho,\, \rho) - \sum_{j=0}^{m-1}\rho^{m-j-1}\mathbb{L}_\xi\left[\omega_{\kappa^j}^{(j)}(\kappa,\, \xi)\Big|_{\kappa=0}\right]. \tag{9}$$

**Theorem 3** [44] The transform $\mathbb{L}$ of the ABC operator is given by

1. $\mathbb{L}\left[{}^{ABC}\mathscr{D}_{0,\,\xi}^{\alpha}\omega(\kappa,\, \xi)\right] = \dfrac{A^\alpha}{(1-\alpha)\rho^\alpha+\alpha}\left[\rho^\alpha\omega_{\mathbb{L}}(\rho,\, \rho) - \sum_{j=0}^{n-1}\rho^{\alpha-j-1}\mathbb{L}_\kappa\left[\omega_{\xi^j}^{(j)}(\kappa,\, \xi)\Big|_{\xi=0}\right]\right]$;
2. $\mathbb{L}\left[{}^{ABC}\mathscr{D}_{0,\,\kappa}^{\beta}\omega(\kappa,\, \xi)\right] = \dfrac{A^\alpha}{(1-\alpha)\rho^\alpha+\alpha}\left[\rho^\beta\omega_{\mathbb{L}}(\rho,\, \rho) - \sum_{j=0}^{m-1}\rho^{\beta-j-1}\mathbb{L}_\xi\left[\omega_{\kappa^j}^{(j)}(\kappa,\, \xi)\Big|_{\kappa=0}\right]\right]$,

where $n-1 < \alpha < n$ and $m-1 < \beta < m$.

## 3. Fixed point framework for fractional WBKEs (2)

In this section, we develop a mathematical framework to analyze the existence, uniqueness, and stability of solutions for the fractional WBKEs (2). Using Banach spaces equipped with the compact-open topology, we employ fixed point theorems to establish the well posedness of the system. The approach ensures that the fractional WBKEs possesses continuous, unique, and Hyers-Ulam stable solutions under suitable conditions.

### 3.1 *On locally compact with continuity property*

A subset $\Omega$ of a topological space $C$ is said to be locally compact if every point $x \in \Omega$ possesses a neighborhood $N$ whose closure $\overline{N}$ is compact in $C$. Moreover, $\Omega$ is said to satisfy the Hausdorff property if, for every pair of distinct points $x,\, y \in \Omega$, there exist disjoint open sets $G$ and $E$ in the subspace $\Omega$ such that $x \in G$ and $y \in E$. Define the operators $\overline{\omega},\, \overline{\psi} : [\Omega \times [0,\, \xi_0]] \times \mathbb{R}^{\Omega\times[0,\,\xi_0]} \times \mathbb{R}^{\Omega\times[0,\,\xi_0]} \to \mathbb{R}$ by

$$\overline{\omega}(\kappa,\, \xi,\, \omega,\, \psi) = -\omega(\kappa,\, \xi)\omega_\kappa(\kappa,\, \xi) - \psi_\kappa(\kappa,\, \xi) - \beta\, \omega_{\kappa\kappa}(\kappa,\, \xi) \tag{10}$$

and

$$\overline{\psi}(\kappa,\,\xi,\,\omega,\,\psi)=-[\omega(\kappa,\,\xi)\psi(\kappa,\,\xi)]_{\kappa}+\beta\,\psi_{\kappa\kappa}(\kappa,\,\xi)-\beta'\,\omega_{\kappa\kappa\kappa}(\kappa,\,\xi) \tag{11}$$

where $\mathbb{R}^{\Omega\times[0,\,\xi_0]}$ denotes the family of all functions from $\Omega\times[0,\,\xi_0]$ into $\mathbb{R}$. Apply the ABC integral operator of the system (2) and use (5) to get that

$$\omega(\kappa,\,\xi)=h_0(\kappa)+\frac{1-\alpha}{A^{\alpha}}\,\overline{\omega}(\kappa,\,\xi,\,\omega,\,\psi)+\frac{\alpha}{\Gamma(\alpha)A^{\alpha}}\int_0^{\xi}(\xi-\sigma)^{\alpha-1}\,\overline{\omega}(\kappa,\,\sigma,\,\omega,\,\psi)d\sigma \tag{12}$$

and

$$\psi(\kappa,\,\xi)=\hat{h}_0(\kappa)+\frac{1-\alpha}{A^{\alpha}}\,\overline{\psi}(\kappa,\,\xi,\,\omega,\,\psi)+\frac{\alpha}{\Gamma(\alpha)A^{\alpha}}\int_0^{\xi}(\xi-\sigma)^{\alpha-1}\,\overline{\psi}(\kappa,\,\sigma,\,\omega,\,\psi)d\sigma. \tag{13}$$

Define the operators $\mathscr{R}_{\omega},\,\mathscr{R}_{\psi}:\mathbb{R}^{\Omega\times[0,\,\xi_0]}\times\mathbb{R}^{\Omega\times[0,\,\xi_0]}\to\mathbb{R}^{\Omega\times[0,\,\xi_0]}$ by

$$\mathscr{R}_{\omega}(\omega,\,\psi)(\kappa,\,\xi)=h_0(\kappa)+\frac{1-\alpha}{A^{\alpha}}\,\overline{\psi}(\kappa,\,\xi,\,\omega,\,\psi)+\frac{\alpha}{\Gamma(\alpha)A^{\alpha}}\int_0^{\xi}(\xi-\sigma)^{\alpha-1}\,\overline{\omega}(\kappa,\,\sigma,\,\omega,\,\psi)d\sigma \tag{14}$$

and

$$\mathscr{R}_{\psi}(\omega,\,\psi)(\kappa,\,\xi)=\hat{h}_0(\kappa)+\frac{1-\alpha}{A^{\alpha}}\,\overline{\psi}(\kappa,\,\xi,\,\omega,\,\psi)+\frac{\alpha}{\Gamma(\alpha)A^{\alpha}}\int_0^{\xi}(\xi-\sigma)^{\alpha-1}\,\overline{\psi}(\kappa,\,\sigma,\,\omega,\,\psi)d\sigma. \tag{15}$$

Consider the Banach space $\mathscr{B}:=C_{\Omega_0}\times C_{\Omega_0}$ with the norm

$$\|(\omega,\,\psi)\|_{\mathscr{B}}=\max_{(\kappa,\,\xi)\in\Omega\times[0,\,\xi_0]}\left\{|\omega(\kappa,\,\xi)|\,,\,|\psi(\kappa,\,\xi)|\right\}.$$

We say $\overline{\omega}$ has Lipschitz property if there is constant $\delta_{\omega}>0$ such that

$$|\overline{\omega}(\kappa,\,\xi,\,\omega,\,\psi)-\overline{\omega}(\kappa,\,\xi,\,\omega_1,\,\psi_1)|\leq\delta_{\omega}\,\|(\omega,\,\psi)-(\omega_1,\,\psi_1)\|_{\mathscr{B}}$$

for all $(\kappa,\,\xi)\in\Omega\times[0,\,\xi_0]$.

**Theorem 4** If $\overline{\omega}$ and $\overline{\psi}$ satisfies Lipschitz conditions, then $\mathscr{R}_{\omega}$ and $\mathscr{R}_{\psi}$ satisfies Lipschitz conditions.

**Proof.** By the Lipschitz property of $\overline{\omega}$ and $\overline{\psi}$ there are $\delta_{\omega},\,\delta_{\psi}>0$ such that

$$|\overline{\omega}(\kappa,\,\xi,\,\omega,\,\psi)-\overline{\omega}(\kappa,\,\xi,\,\omega_1,\,\psi_1)|\leq\delta_{\omega}\,\|(\omega,\,\psi)-(\omega_1,\,\psi_1)\|_{\mathscr{B}}$$

and

$$|\overline{\psi}(\kappa,\,\xi,\,\omega,\,\psi)-\overline{\psi}(\kappa,\,\xi,\,\omega_1,\,\psi_1)|\leq\delta_\psi\,\|(\omega,\,\psi)-(\omega_1,\,\psi_1)\|_{\mathscr{B}}.$$

Hence

$$\begin{aligned}
&|\mathscr{R}_\omega(\omega,\,\psi)(\kappa,\,\xi)-\mathscr{R}_\omega(\omega_1,\,\psi_1)(\kappa,\,\xi)|\\
&\leq\frac{1-\alpha}{A^\alpha}\,|\overline{\omega}(\kappa,\,\xi,\,\omega,\,\psi)-\overline{\omega}(\kappa,\,\xi,\,\omega_1,\,\psi_1)|\\
&\quad+\frac{\alpha}{\Gamma(\alpha)A^\alpha}\left|\int_0^\xi(\xi-\sigma)^{\alpha-1}\,\overline{\omega}(\kappa,\,\sigma,\,\omega,\,\psi)d\sigma-\int_0^\xi(\xi-\sigma)^{\alpha-1}\,\overline{\omega}(\kappa,\,\sigma,\,\omega_1,\,\psi_1)d\sigma\right|\\
&\leq\frac{1-\alpha}{A^\alpha}\,|\overline{\omega}(\kappa,\,\xi,\,\omega,\,\psi)-\overline{\omega}(\kappa,\,\xi,\,\omega_1,\,\psi_1)|\\
&\quad+\frac{\alpha}{\Gamma(\alpha)A^\alpha}\int_0^\xi|\xi-\sigma|^{\alpha-1}\,|\overline{\omega}(\kappa,\,\xi,\,\omega,\,\psi)-\overline{\omega}(\kappa,\,\xi,\,\omega_1,\,\psi_1)|\,d\sigma\\
&\leq\frac{(1-\alpha)\delta_\omega}{A^\alpha}\,\|(\omega,\,\psi)-(\omega_1,\,\psi_1)\|_{\mathscr{B}}+\frac{\xi_0^\alpha\delta_\omega}{\Gamma(\alpha)A^\alpha}\,\|(\omega,\,\psi)-(\omega_1,\,\psi_1)\|_{\mathscr{B}}.
\end{aligned}$$

Then we have

$$\|\mathscr{R}_\omega(\omega,\,\psi)-\mathscr{R}_\omega(\omega_1,\,\psi_1)\|\leq\left[\frac{(1-\alpha)\delta_\omega}{A^\alpha}+\frac{\xi_0^\alpha\delta_\omega}{\Gamma(\alpha)A^\alpha}\right]\|(\omega,\,\psi)-(\omega_1,\,\psi_1)\|_{\mathscr{B}}. \tag{16}$$

Similarly, we have

$$\left\|\mathscr{R}_\psi(\omega,\,\psi)-\mathscr{R}_\psi(\omega_1,\,\psi_1)\right\|\leq\left[\frac{(1-\alpha)\delta_\psi}{A^\alpha}+\frac{\xi_0^\alpha\delta_\psi}{\Gamma(\alpha)A^\alpha}\right]\|(\omega,\,\psi)-(\omega_1,\,\psi_1)\|_{\mathscr{B}}. \tag{17}$$

□

Define the operators $\mathscr{H}_\omega,\,\mathscr{H}_\psi:\mathscr{B}\to\mathbb{R}^{\Omega\times[0,\,\xi_0]}$ as

$$\mathscr{H}_\omega(\omega,\,\psi)=-\omega\,\omega_\kappa-\psi_\kappa-\beta\;\omega_{\kappa\kappa} \tag{18}$$

and

$$\mathscr{H}_\psi(\omega,\,\psi)=-[\omega\psi]_\kappa+\beta\;\psi_{\kappa\kappa}-\beta'\;\omega_{\kappa\kappa\kappa} \tag{19}$$

respectively, where $C_{\Omega_0} := C(\Omega \times [0, \xi_0]) \subseteq \mathbb{R}^{\Omega \times [0, \xi_0]}$ denotes the family of all continuous functions from $\Omega \times [0, \xi_0]$ into $\mathbb{R}$. For the Lipschitz continuity of the operators $\mathscr{H}_\omega$ and $\mathscr{H}_\psi$, note that if $\overline{\varpi}$ and $\overline{\psi}$ satisfies Lipschitz conditions, then we have

$$|\langle \mathscr{H}_\omega(\omega, \psi) - \mathscr{H}_\omega(\omega_1, \psi_1), (\omega, \psi) - (\omega_1, \psi_1)\rangle| \leq \delta_\omega \|(\omega, \psi) - (\omega_1, \psi_1)\|^2_{\mathscr{B}}$$

and

$$|\langle \mathscr{H}_\psi(\omega, \psi) - \mathscr{H}_\psi(\omega_1, \psi_1), (\omega, \psi) - (\omega_1, \psi_1)\rangle| \leq \delta_\psi \|(\omega, \psi) - (\omega_1, \psi_1)\|^2_{\mathscr{B}}.$$

It is known that in any metrizable topological space, the family of all open balls constitutes a subbasis for its topology. Hence, in the metric space $\mathbb{R}$ equipped with the standard metric $d(\kappa, \xi) = |\kappa - \xi|$, the collection of open balls $B(\kappa_0, r)$ serves as a subbasis for the usual topology on $\mathbb{R}$. Since $\Omega \times [0, \xi_0]$ is a Hausdorff subspace of the Euclidean space $\mathbb{R}^2$, the family

$$CO = \{\langle Q, B(\kappa_0, r)\rangle : Q \text{ is a compact subset of } \Omega \times [0, \xi_0],\ \kappa_0 \in \mathbb{R},\ r > 0\}$$

forms a subbasis for the compact-open topology on $C_{\Omega_0}$, where

$$\langle Q, B(\kappa_0, r)\rangle = \{\omega \in C_{\Omega_0} : \omega[Q] \subset B(\kappa_0, r)\}.$$

In the subsequent analysis, the function space $\mathbb{R}^{\Omega \times [0, \xi_0]}$ is considered with this compact-open topology.

**Theorem 5** If $\overline{\varpi}$ is continuous, then the operator $\mathscr{H}_\omega$ is continuous.

**Proof.** We will show that $\mathscr{H}_\omega^{-1}(\langle Q, B(\kappa_0, r)\rangle)$ is open in $\mathscr{B}$ for every subbasic open $\langle Q, B(\kappa_0, r)\rangle$ of $\mathbb{R}^{\Omega \times [0, \xi_0]}$ for any compact set $Q \subset \Omega \times [0, \xi_0]$, $\kappa_0 \in \mathbb{R}$ and $r > 0$. Note that

$$\begin{aligned} \mathscr{H}_\omega^{-1}(\langle Q, B(\kappa_0, r)\rangle) &= \left\{ (\rho, \rho') \in \mathscr{B} :\ \mathscr{H}_\omega(\rho, \rho')[Q] \subset B(\kappa_0, r) \right\} \\ &= \left\{ (\rho, \rho') \in \mathscr{B} :\ \overline{\varpi}(\{(\rho, \rho')\} \times Q) \subset B(\kappa_0, r) \right\}. \end{aligned}$$

Fix $(\rho_0, \rho_0') \in \mathscr{H}_\omega^{-1}(\langle Q, B(\kappa_0, r)\rangle)$. Then $\overline{\varpi}(\rho_0, \rho_0', h) \in B(\kappa_0, r)$ for all $h \in Q$, i.e. $\{(\rho_0, \rho_0')\} \times Q \subset \overline{\varpi}^{-1}(B(\kappa_0, r))$. Since

$$\overline{\varpi}^{-1}(B(\kappa_0, r)) \subset [\Omega \times [0, \xi_0]] \times C_{\Omega_0} \times C_{\Omega_0}$$

is open and $\overline{\varpi}$ is continuous, then for each $h \in Q$ there exist open neighborhoods $O_h \subset C_{\Omega_0} \times C_{\Omega_0}$ of $(\rho_0, \rho_0')$ and $V_h \subset \mathbb{R}^2$ of $h$ such that

$$O_h \times V_h\ \subset\ \overline{\varpi}^{-1}(B(\kappa_0, r)).$$

The family $\{V_h\}_{h\in Q}$ is an open cover of the compact set $Q$, so choose $h_1, \ldots, h_n \in Q$ with $Q \subset \bigcup_{i=1}^{n} V_{h_i}$, and set

$$O := \bigcap_{i=1}^{n} O_{h_i}.$$

Then $O$ is an open neighborhood of $(\rho_0, \rho_0')$ in $\mathscr{B}$ and hence for any $(\rho, \rho') \in O$ and any $k \in Q$, there exists $i$ with $k \in V_{h_i}$ such that

$$(h, k) \in O_{h_i} \times V_{h_i} \subset \overline{\varpi}^{-1}(B(\kappa_0, r)).$$

Then $\overline{\varpi}(\rho, \rho', Q) \subset B(\kappa_0, r)$; equivalently, $\mathscr{H}_\omega(\rho, \rho') \in \langle Q, B(\kappa_0, r)\rangle$. Hence $O \subset \mathscr{H}_\omega^{-1}(\langle Q, B(\kappa_0, r)\rangle)$, that is, $\mathscr{H}_\omega^{-1}(\langle Q, B(\kappa_0, r)\rangle)$ is open. Therefore $\mathscr{H}_\omega$ is continuous. □

**Theorem 6** If the operator $\mathscr{H}_\omega$ is continuous and $\Omega$ is locally compact set in $\mathbb{R}$ then the restriction $\overline{\varpi}$ on $\Omega \times [0, \xi_0] \times \mathscr{B}$ is continuous.

**Proof.** We prove directly that $\overline{\varpi}^{-1}(B(\kappa, r))$ is open in $[\Omega \times [0, \xi_0]] \times \mathscr{B}$ for each open ball $B(\kappa, r) \subset \mathbb{R}$. Fix an arbitrary open ball $B(\kappa_0, r) \subset \mathbb{R}$ and take a point $(\xi_0', \xi_0'', \rho_0, \rho_0') \in \overline{\varpi}^{-1}(B(\kappa_0, r))$; that is, $\mathscr{H}_\omega(\rho_0, \rho_0')(\xi_0', \xi_0'') \in B(\kappa_0, r)$. Since $\Omega$ is locally compact in $\mathbb{R}$ and so its is Hausdorff, then $\Omega \times [0, \xi_0]$ is locally compact Hausdorff. Hence there exists an open neighborhood $V$ of $(\xi_0', \xi_0'')$ such that $\overline{V}$ is compact containing $(\xi_0', \xi_0'')$, where $\overline{V}$ is the clouser set of $V$. Set $Q := \overline{V}$; then $Q$ is compact and $(\xi_0', \xi_0'') \in V \subset Q$. Since $\mathscr{H}_\omega(\rho_0, \rho_0')(\xi_0', \xi_0'') \in B(\kappa_0, r)$ and $(\xi_0', \xi_0'') \in Q$, it follows that $\mathscr{H}_\omega(\rho_0, \rho_0')[Q] \subset \mathbb{R}$ intersects $B(\kappa_0, r)$ at least at $(\xi_0', \xi_0'')$. Define the subbasic open set

$$\langle Q, B(\kappa_0, r)\rangle = \{(\rho, \rho') \in \mathbb{R}^{\Omega\times[0,\xi_0]} \times \mathbb{R}^{\Omega\times[0,\xi_0]} : \mathscr{H}_\omega(\rho, \rho')[Q] \subset B(\kappa_0, r)\}.$$

Since $\mathscr{H}_\omega$ is continuous and $\langle Q, B(\kappa_0, r)\rangle$ is open set in $\mathbb{R}^{\Omega\times[0,\xi_0]}$, then there exists an open neighborhood $O \subset \mathscr{B}$ of $(\rho_0, \rho_0')$ such that $\mathscr{H}_\omega(O) \subset \langle Q, B(\kappa_0, r)\rangle$.

Equivalently, for every $x \in O$ we have $\mathscr{H}_\omega(x)[Q] \subset B(\kappa_0, r)$. Now consider the product open set

$$V \times O \subset [\Omega \times [0, \xi_0]] \times \mathscr{B}. \tag{20}$$

If $(\xi, \xi', \rho, \rho') \in V \times O$ and since $(\xi, \xi') \in V \subset Q$ and $\mathscr{H}_\omega(\rho, \rho')[Q] \subset B(\kappa_0, r)$, then we get $\overline{\varpi}(\xi, \xi', \rho, \rho') = \mathscr{H}_\omega(\rho, \rho')(\xi, \xi') \in B(\kappa_0, r)$. Therefore $O \times V \subset \overline{\varpi}^{-1}(B(\kappa_0, r))$. We have found, for the arbitrary $(\xi_0', \xi_0'', \rho_0, \rho_0') \in \overline{\varpi}^{-1}(B(\kappa_0, r))$, an open neighborhood $V \times O$ of $(\xi_0', \xi_0'', \rho_0, \rho_0')$ with $O \times V \subset \overline{\varpi}^{-1}(B(\kappa_0, r))$; hence $\overline{\varpi}^{-1}(B(\kappa_0, r))$ is open. Since $B(\kappa_0, r) \subset \mathbb{R}$ is arbitrary, then $\overline{\varpi}$ is continuous. □

In theorems above (5, 6), similarly for the continuity of the operator $\mathscr{H}_\psi$ with $\overline{\psi}$.

## 3.2 *Well-posedness of fractional WBKEs (2)*

Here, we examine the well-posedness of the fractional WBKEs (2) by establishing the existence and uniqueness of its solutions in a Banach space with the compact-open topology. Using fixed point theorems, the required conditions ensuring that the system admits a unique and stable solution are derived. To achieve this, define the operator $\mathscr{R} : \mathscr{B} \to \mathscr{B}$ as

$$\mathscr{R}(\omega, \psi)(\kappa, \xi) = (\mathscr{R}_\omega(\omega, \psi)(\kappa, \xi), \mathscr{R}_\psi(\omega, \psi)(\kappa, \xi)) \tag{21}$$

for all $(\kappa, \xi) \in \Omega \times [0, \xi_0]$. To guarantee the existence and uniqueness of solutions for the system (2), we consider the following set of assumptions:

C1: The domain of the system (2) is a space $C^3[\Omega \times [0, \xi_0]]$ with respect to compact-open topology, where $\Omega$ is locally compact bounded set in $\mathbb{R}$.

C2: $\exists$ constants $\Delta_{\overline{\omega}}, \Delta_{\overline{\psi}} > 0$ such that $|\overline{\omega}(\kappa, \xi, \omega)| \leq \Delta_{\overline{\omega}}$ and $|\overline{\psi}(\kappa, \xi, \omega, \psi)| \leq \Delta_{\overline{\psi}}$.

C3: $\overline{\omega}$ and $\overline{\psi}$ satisfies Lipschitz conditions with Lipschitz constants $\delta_\omega, \delta_\psi > 0$, respectively.

**Lemma 1** If $C1$ is satisfied then the restrictions of the operators $\mathscr{H}_\omega$ and $\mathscr{H}_\psi$ are continuous.

**Proof.** Since the operators $\partial_\kappa, \partial_\kappa^2, \partial_\kappa^3 : C^3[\Omega \times [0, \xi_0]] \to C[\Omega \times [0, \xi_0]]$ are continuous. Then easily to see that

$$\mathscr{H}_\omega(\omega, \psi) = -\omega\, \omega_\kappa - \psi_\kappa - \beta\, \omega_{\kappa\kappa} \text{ and } \mathscr{H}_\psi(\omega, \psi) = -[\omega\psi]_\kappa + \beta\, \psi_{\kappa\kappa} - \beta'\, \omega_{\kappa\kappa\kappa}$$

are continuous. □

**Theorem 7** If $C1$ and $C2$ hold then the system (2) has solutions.

**Proof.** To get this solution we will use Schaefer's fixed point theorem. For any $\varepsilon > 0$, let

$$D_\varepsilon = \{(\omega, \psi) \in \mathscr{B} : \|(\omega, \psi)\|_{\mathscr{B}} \leq \varepsilon\}.$$

Firstly, for the continuity property of $\mathscr{R}$, we will prove that $\|\mathscr{R}(\omega_n, \psi_n) - \mathscr{R}(\omega, \psi)\|_{\mathscr{B}} \to 0$ for any convergent sequence $(\omega_n, \psi_n) \to (\omega, \psi)$ in $\mathscr{B}$. Let $(\omega_n, \psi_n) \to (\omega, \psi)$ in $\mathscr{B}$. For any $(\kappa, \xi) \in \Omega \times [0, \xi_0]$ we have

$$\begin{aligned}
&|\mathscr{R}_\omega(\omega_n, \psi_n)(\kappa, \xi) - \mathscr{R}_\omega(\omega, \psi)(\kappa, \xi)| \\
&\leq \frac{1-\alpha}{A^\alpha} |\overline{\omega}(\kappa, \xi, \omega_n, \psi_n) - \overline{\omega}(\kappa, \xi, \omega, \psi)| \\
&\quad + \frac{\alpha}{\Gamma(\alpha) A^\alpha} \left| \int_0^\xi (\xi - \sigma)^{\alpha-1} \left[\overline{\omega}(\kappa, \sigma, \omega_n, \psi_n) - \overline{\omega}(\kappa, \sigma, \omega, \psi)\right] d\sigma \right| \\
&\leq \frac{1-\alpha}{A^\alpha} |\overline{\omega}(\kappa, \xi, \omega_n, \psi_n) - \overline{\omega}(\kappa, \xi, \omega, \psi)| \\
&\quad + \frac{\alpha}{\Gamma(\alpha) A^\alpha} \int_0^\xi (\xi - \sigma)^{\alpha-1} |\overline{\omega}(\kappa, \sigma, \omega_n, \psi_n) - \overline{\omega}(\kappa, \sigma, \omega, \psi)| d\sigma.
\end{aligned} \tag{22}$$

By Lemma (1), $\mathscr{H}_\omega$ is continuous. Since $\Omega$ is a locally compact Hausdorff, then by Theorem (6), $\overline{\omega}$ is continuous. Hence for any $(\kappa, \xi) \in \Omega \times [0, \xi_0]$ we have

$$|\overline{\omega}(\kappa, \xi, \omega_n, \psi_n) - \overline{\omega}(\kappa, \xi, \omega, \psi)| \to 0.$$

By the boundedness of $\Omega$, we have $\omega_n \to \omega$ uniformly in $D_\varepsilon$. Hence for all $n \in \mathbb{N}$ and for any $(\kappa, \xi) \in \Omega \times [0, \xi_0]$, $|\omega_n| < \mathscr{C}_\omega$ and $|\omega| < \mathscr{C}_\omega$ for some constants $\mathscr{C}_\omega$. So by using the dominated convergence theorem, we get

$$\sup_{(\kappa,\,\xi)\in\Omega\times[0,\,\xi_0]}\|\mathscr{R}_\omega(\omega_n,\,\psi_n)(\kappa,\,\xi)-\mathscr{R}_\omega(\omega,\,\psi)(\kappa,\,\xi)\|\to 0.$$

Similarly,

$$\sup_{(\kappa,\,\xi)\in\Omega\times[0,\,\xi_0]}\left\|\mathscr{R}_\psi(\omega_n,\,\psi_n)(\kappa,\,\xi)-\mathscr{R}_\psi(\omega,\,\psi)(\kappa,\,\xi)\right\|\to 0.$$

That is, $\|\mathscr{R}(\omega_n,\,\psi_n)-\mathscr{R}(\omega,\,\psi)\|_{\mathscr{B}}\to 0$ and hence $\mathscr{R}$ is continuous.

Secondly, we prove that the boundedness of sets is topological property under the continuous map $\mathscr{R}$. Let $(\omega,\,\psi)\in D_\varepsilon$. Then

$$\begin{aligned}
&|\mathscr{R}_\omega(\omega,\,\psi)(\kappa,\,\xi)|\\
&\le |h_0(\kappa)|+\frac{1-\alpha}{A^\alpha}\,|\overline{\omega}(\kappa,\,\xi,\,\omega,\,\psi)|+\frac{\alpha}{\Gamma(\alpha)A^\alpha}\int_0^\xi(\xi-\sigma)^{\alpha-1}\,|\overline{\omega}(\kappa,\,\sigma,\,\omega,\,\psi)|\,d\sigma\\
&\le |h_0(\kappa)|+\frac{(1-\alpha)\Delta_{\overline{\omega}}}{A^\alpha}+\frac{\xi_0^\alpha\Delta_{\overline{\omega}}}{\Gamma(\alpha)A^\alpha}
\end{aligned}\tag{23}$$

and

$$\begin{aligned}
&\left|\mathscr{R}_\psi(\omega,\,\psi)(\kappa,\,\xi)\right|\\
&\le \left|\hat{h}_0(\kappa)\right|+\frac{1-\alpha}{A^\alpha}\,|\overline{\psi}(\kappa,\,\xi,\,\omega,\,\psi)|+\frac{\alpha}{\Gamma(\alpha)A^\alpha}\int_0^\xi(\xi-\sigma)^{\alpha-1}\,|\overline{\psi}(\kappa,\,\sigma,\,\omega,\,\psi)|\,d\sigma\\
&\le \left|\hat{h}_0(\kappa)\right|+\frac{(1-\alpha)\Delta_{\overline{\psi}}}{A^\alpha}+\frac{\xi_0^\alpha\Delta_{\overline{\psi}}}{\Gamma(\alpha)A^\alpha}.
\end{aligned}\tag{24}$$

Hence $\mathscr{R}(D_\varepsilon)\subseteq D_{\varepsilon'}$ where

$$\varepsilon'=\max\left\{|h_0(\kappa)|+\frac{(1-\alpha)\Delta_{\overline{\omega}}}{A^\alpha}+\frac{\xi_0^\alpha\Delta_{\overline{\omega}}}{\Gamma(\alpha)A^\alpha},\,\left|\hat{h}_0(\kappa)\right|+\frac{(1-\alpha)\Delta_{\overline{\psi}}}{A^\alpha}+\frac{\xi_0^\alpha\Delta_{\overline{\psi}}}{\Gamma(\alpha)A^\alpha}\right\}.$$

Since $\varepsilon'$ is independent on $(\omega,\,\psi)$ then $\mathscr{R}(D_\varepsilon)$ is uniformly bounded set.

Thirdly, for the equicontinuity property of $\mathscr{R}$, let $(\kappa_1,\,\xi_1),\,(\kappa,\,\xi)\in\Omega\times[0,\,\xi_0]$ be arbitrary points. Note that

$$
\begin{aligned}
&|\overline{\omega}(\omega,\, \psi)(\kappa,\, \xi) - \overline{\omega}(\omega,\, \psi)(\kappa_1,\, \xi_1)| \\
&\leq \frac{1-\alpha}{A^\alpha}\, |\overline{\omega}(\kappa,\, \xi,\, \omega,\, \psi) - \overline{\omega}(\kappa_1,\, \xi_1,\, \omega,\, \psi)| \\
&\quad + \frac{\alpha}{\Gamma(\alpha)A^\alpha} \int_0^\xi \left[(\xi-\sigma)^{\alpha-1} - (\xi_1-\sigma)^{\alpha-1}\right] |\overline{\omega}(\kappa,\, \sigma,\, \omega,\, \psi) - \overline{\omega}(\kappa_1,\, \sigma,\, \omega,\, \psi)|\, d\sigma \\
&\leq \frac{1-\alpha}{A^\alpha}\, |\overline{\omega}(\kappa,\, \xi,\, \omega,\, \psi) - \overline{\omega}(\kappa_1,\, \xi_1,\, \omega,\, \psi)| \\
&\quad + \frac{\alpha}{\Gamma(\alpha)A^\alpha} \int_0^\xi \left[(\xi-\sigma)^{\alpha-1} - (\xi_1-\sigma)^{\alpha-1}\right] \left[|\overline{\omega}(\kappa,\, \sigma,\, \omega,\, \psi)| + |\overline{\omega}(\kappa_1,\, \sigma,\, \omega,\, \psi)|\right] d\sigma \\
&\leq \frac{1-\alpha}{A^\alpha}\, |\overline{\omega}(\kappa,\, \xi,\, \omega,\, \psi) - \overline{\omega}(\kappa_1,\, \xi_1,\, \omega,\, \psi)| \\
&\quad + \frac{2\varepsilon'}{\Gamma(\alpha)A^\alpha} \int_0^\xi \left|(\xi-\sigma)^{\alpha-1} - (\xi_1-\sigma)^{\alpha-1}\right| d\sigma.
\end{aligned}
\tag{25}
$$

Since the kernel functions are integrable, it follows that $|\mathscr{R}_\omega(\omega,\, \psi)(\kappa,\, \xi) - \mathscr{R}_\omega(\omega,\, \psi)(\kappa_1,\, \xi_1)| \to 0$ as $(\kappa,\, \xi) \to (\kappa_1,\, \xi_1)$. Similarly, $\left|\mathscr{R}_\psi(\omega,\, \psi)(\kappa,\, \xi) - \mathscr{R}_\psi(\omega,\, \psi)(\kappa_1,\, \xi_1)\right| \to 0$ as $(\kappa,\, \xi) \to (\kappa_1,\, \xi_1)$. Therefore, $\|\mathscr{R}(\kappa,\, \xi) - \mathscr{R}(\kappa_1,\, \xi_1)\|_{\mathscr{B}} \to 0$ as $(\kappa,\, \xi) \to (\kappa_1,\, \xi_1)$ which implies that $\mathscr{R}$ possesses the equicontinuity property. Since boundedness and uniform boundedness are preserved under continuous mappings, the Arzelà-Ascoli theorem guarantees that $\mathscr{R}$ is a compact operator.

Finally, we prove that the set $P = \{(\omega,\, \psi) \in \mathscr{B} : (\omega,\, \psi) = \varsigma\mathscr{R}(\omega,\, \psi),\ \varsigma \in [0,\, 1]\}$ is bounded in $\mathscr{B}$. let $(\omega,\, \psi) = \varsigma\mathscr{R}(\omega,\, \psi)$ for some $\varsigma \in [0,\, 1]$. Then we have

$$
\begin{aligned}
|\omega(\kappa,\, \xi)| &= |\varsigma\mathscr{R}_\omega(\omega,\, \psi)(\kappa,\, \xi)| = \varsigma\, |\mathscr{R}_\omega(\omega,\, \psi)(\kappa,\, \xi)| \\
&\leq \varsigma \left[|h_0(\kappa)| + \frac{(1-\alpha)\Delta_{\overline{\omega}}}{A^\alpha} + \frac{\xi_0^\alpha \Delta_{\overline{\omega}}}{\Gamma(\alpha)A^\alpha}\right]
\end{aligned}
\tag{26}
$$

and

$$
\begin{aligned}
|\psi(\kappa,\, \xi)| &= \left|\varsigma\mathscr{R}_\psi(\omega,\, \psi)(\kappa,\, \xi)\right| = \varsigma\left|\mathscr{R}_\psi(\omega,\, \psi)(\kappa,\, \xi)\right| \\
&\leq \varsigma \left[\left|\hat{h}_0(\kappa)\right| + \frac{(1-\alpha)\Delta_{\overline{\psi}}}{A^\alpha} + \frac{\xi_0^\alpha \Delta_{\overline{\psi}}}{\Gamma(\alpha)A^\alpha}\right].
\end{aligned}
\tag{27}
$$

That is, $P$ is bounded in $\mathscr{B}$. Therefore, by applying Schaefer's fixed point theorem, it follows that $\mathscr{R}$ possesses at least one solution in $D_\varepsilon$, which represents a solution of the system (2). □

**Theorem 8** If $C1$-$C3$ hold and

$$(1-\alpha)+\frac{\xi_0^\alpha}{\Gamma(\alpha)}<\frac{A^\alpha}{\delta_\omega+\delta_\psi}$$

then the system (2) has unique solution.

**Proof.** Form (C3) and Theorem (4) we get that for $(\omega,\,\psi),\,(\omega_1,\,\psi_1)\in\mathscr{B}$,

$$\|\overline{\omega}(\omega,\,\psi)-\overline{\omega}(\omega_1,\,\psi_1)\|\leq\left[\frac{(1-\alpha)\delta_\omega}{A^\alpha}+\frac{\xi_0^\alpha\delta_\omega}{\Gamma(\alpha)A^\alpha}\right]\|(\omega,\,\psi)-(\omega_1,\,\psi_1)\|_{\mathscr{B}}$$

and

$$\|\overline{\psi}(\omega,\,\psi)-\overline{\psi}(\omega_1,\,\psi_1)\|\leq\left[\frac{(1-\alpha)\delta_\psi}{A^\alpha}+\frac{\xi_0^\alpha\delta_\psi}{\Gamma(\alpha)A^\alpha}\right]\|(\omega,\,\psi)-(\omega_1,\,\psi_1)\|_{\mathscr{B}}.$$

Hence

$$\begin{aligned}\|\mathscr{R}(\omega,\,\psi)-\mathscr{R}(\omega_1,\,\psi_1)\|&\leq\left[\frac{(1-\alpha)\delta_\omega}{A^\alpha}+\frac{\xi_0^\alpha\delta_\omega}{\Gamma(\alpha)A^\alpha}\right]\|(\omega,\,\psi)-(\omega_1,\,\psi_1)\|_{\mathscr{B}}\\&\quad+\left[\frac{(1-\alpha)\delta_\psi}{A^\alpha}+\frac{\xi_0^\alpha\delta_\psi}{\Gamma(\alpha)A^\alpha}\right]\|(\omega,\,\psi)-(\omega_1,\,\psi_1)\|_{\mathscr{B}}\\&=\left[(1-\alpha)+\frac{\xi_0^\alpha}{\Gamma(\alpha)}\right]\frac{\delta_\omega+\delta_\psi}{A^\alpha}\|(\omega,\,\psi)-(\omega_1,\,\psi_1)\|_{\mathscr{B}}.\end{aligned}$$

Therefore, when the inequality

$$\left[(1-\alpha)+\frac{\xi_0^\alpha}{\Gamma(\alpha)}\right]\frac{\delta_\omega+\delta_\psi}{A^\alpha}<1\ \left(\text{i.e., }\ (1-\alpha)+\frac{\xi_0^\alpha}{\Gamma(\alpha)}<\frac{A^\alpha}{\delta_\omega+\delta_\psi}\right)$$

is satisfied, the operator $\mathscr{R}$ qualifies as a contraction mapping. Accordingly, by invoking the Banach fixed point theorem, we conclude that the system has unique solution. □

## 3.3 *The stability property*

In this subsection, we examine the Hyers-Ulam stability of the fractional WBKEs (2) to evaluate the robustness of its solutions under small perturbations in initial data or model parameters. This type of stability ensures that every approximate solution remains close to an exact one within a bounded tolerance, thereby confirming the reliability of the obtained analytical results. By employing integral inequalities together with the previously established existence and uniqueness results, sufficient conditions are derived to guarantee the Hyers-Ulam stability of the fractional WBKEs (2). Let $(\hat{\omega},\,\hat{\psi})$ denote an approximate solution of (2) satisfying

$$\begin{cases} \left|{}^{ABC}\mathscr{D}_{0,\,\xi}^{\alpha}\,\hat{\omega}(\kappa,\,\xi)-\overline{\omega}(\kappa,\,\xi,\,\hat{\omega},\,\hat{\psi})\right|\leq\mu_1, \\ \\ \left|{}^{ABC}\mathscr{D}_{0,\,\xi}^{\alpha}\,\hat{\psi}(\kappa,\,\xi)-\overline{\psi}(\kappa,\,\xi,\,\hat{\omega},\,\hat{\psi})\right|\leq\mu_2, \end{cases} \tag{28}$$

for all $(\kappa,\,\xi)\in\Omega\times[0,\,\xi_0]$, where $\mu_1,\,\mu_2>0$. Following [48], we say the nonlinear system (2) is Hyers-Ulam stable if we have a unique solution $(\omega,\,\psi)$ of (2) such that

$$\|\hat{\omega}-\omega\|\leq\mu_1'\mu_1 \quad \text{and} \quad \|\hat{\psi}-\psi\|\leq\mu_2'\mu_2,$$

where $\mu_1',\,\mu_2'>0$ are constants independent of $\mu_1,\,\mu_2$, and $\|\cdot\|$ represents the supremum norm on $C[\Omega\times[0,\,\xi_0]]$.

**Theorem 9** If $C1$-$C3$ hold and

$$(1-\alpha)+\frac{\xi_0^{\alpha}}{\Gamma(\alpha)}<\frac{A^{\alpha}}{\delta_{\omega}+\delta_{\psi}}$$

then the system (2) is Hyers-Ulam stable.

**Proof.** The proof is based on the interplay between the existence-uniqueness framework and integral inequalities. Under the contraction condition, the operator satisfies a stability bound that guarantees the deviation between approximate and exact solutions remains controlled, thereby ensuring Hyers-Ulam stability. Let $(\hat{\omega},\,\hat{\psi})$ be approximate solution and $(\omega,\,\psi)$ be unique solution of (2). For the approximate solution $(\hat{\omega},\,\hat{\psi})$ with satisfying the inequalities (28), there are two continuous functions $g$ and $g'$ on $\Omega\times[0,\,\xi_0]$ such that

$$|g(\kappa,\,\xi)|\leq\varepsilon_1,\ \left|g'(\kappa,\,\xi)\right|\leq\varepsilon_2,$$

$${}^{ABC}\mathscr{D}_{0,\,\xi}^{\alpha}\,\hat{\omega}(\kappa,\,\xi)=\overline{\omega}(\kappa,\,\xi,\,\hat{\omega},\,\hat{\psi})+g(\kappa,\,\xi),$$

and

$${}^{ABC}\mathscr{D}_{0,\,\xi}^{\alpha}\,\hat{\psi}(\kappa,\,\xi)=\overline{\psi}(\kappa,\,\xi,\,\hat{\omega},\,\hat{\psi})+g'(\kappa,\,\xi),$$

for all $(\kappa,\,\xi)\in\Omega\times[0,\,\xi_0]$. Here the approximate solution $(\hat{\omega},\,\hat{\psi})$ will be as

$$\hat{\omega}(\kappa,\,\xi)=g_0(\kappa)+\frac{1-\alpha}{A^{\alpha}}\left[\overline{\omega}(\kappa,\,\xi,\,\hat{\omega},\,\hat{\psi})+g(\kappa,\,\xi)\right]$$

$$+\frac{\alpha}{\Gamma(\alpha)A^{\alpha}}\int_0^{\xi}(\xi-\sigma)^{\alpha-1}\,\overline{\omega}(\kappa,\,\sigma,\,\hat{\omega},\,\hat{\psi})d\sigma+\frac{\alpha}{\Gamma(\alpha)A^{\alpha}}\int_0^{\xi}(\xi-\sigma)^{\alpha-1}\,g(\kappa,\,\sigma)d\sigma$$

and

$$\hat{\psi}(\kappa, \xi) = \hat{g}_0(\kappa) + \frac{1-\alpha}{A^{\alpha}}\left[\overline{\psi}(\kappa, \xi, \hat{\omega}, \hat{\psi}) + g(\kappa, \xi)\right]$$

$$+ \frac{\alpha}{\Gamma(\alpha)A^{\alpha}}\int_0^{\xi}(\xi-\sigma)^{\alpha-1}\,\overline{\psi}(\kappa, \sigma, \hat{\omega}, \hat{\psi})d\sigma + \frac{\alpha}{\Gamma(\alpha)A^{\alpha}}\int_0^{\xi}(\xi-\sigma)^{\alpha-1}\,g'(\kappa, \sigma)d\sigma$$

for all $(\kappa, \xi) \in \Omega \times [0, \xi_0]$. Then by Theorem (8) for all $(\kappa, \xi) \in \Omega \times [0, \xi_0]$ we have

$$|\hat{\omega}(\kappa, \xi) - \omega(\kappa, \xi)| \le \frac{1-\alpha}{A^{\alpha}}\left[|\overline{\omega}(\kappa, \sigma, \hat{\omega}, \hat{\psi}) - \overline{\omega}(\kappa, \sigma, \omega, \psi)| + |g(\kappa, \sigma)|\right]$$

$$+ \frac{\alpha}{\Gamma(\alpha)A^{\alpha}}\int_0^{\xi}(\xi-\sigma)^{\alpha-1}\,\left[|\overline{\omega}(\kappa, \sigma, \hat{\omega}, \hat{\psi}) - \overline{\omega}(\kappa, \sigma, \omega, \psi)| + |g(\kappa, \sigma)|\right]d\sigma$$

$$\le \frac{1-\alpha}{A^{\alpha}}\left[\delta_{\omega}\,\|(\hat{\omega}, \hat{\psi}) - (\omega, \psi)\|_{\mathscr{B}} + \varepsilon_1\right]$$

$$+ \frac{\alpha}{\Gamma(\alpha)A^{\alpha}}\int_0^{\xi}(\xi-\sigma)^{\alpha-1}\,\left[\delta_{\omega}\,\|(\hat{\omega}, \hat{\psi}) - (\omega, \psi)\|_{\mathscr{B}} + \varepsilon_1\right]d\sigma$$

$$\le \left[\frac{1-\alpha}{A^{\alpha}} + \frac{\xi_0^{\alpha}}{\Gamma(\alpha)A^{\alpha}}\right]\left[\delta_{\omega}\,\|(\hat{\omega}, \hat{\psi}) - (\omega, \psi)\|_{\mathscr{B}} + \varepsilon_1\right].$$

That is,

$$\|\hat{\omega} - \omega\| \le \mathbb{C}_{\alpha}^{A}\left[\delta_{\omega}\,\|(\hat{\omega}, \hat{\psi}) - (\omega, \psi)\|_{\mathscr{B}} + \varepsilon_1\right]$$

where $\mathbb{C}_{\alpha}^{A} := \dfrac{1-\alpha}{A^{\alpha}} + \dfrac{\xi_0^{\alpha}}{\Gamma(\alpha)A^{\alpha}}$. Similarly,

$$\|\hat{\psi} - \psi\| \le \mathbb{C}_{\alpha}^{A}\left[\delta_{\psi}\,\|(\hat{\omega}, \hat{\psi}) - (\omega, \psi)\|_{\mathscr{B}} + \varepsilon_2\right].$$

Hence

$$\|\hat{\omega} - \omega\| \le \mathbb{C}_{\alpha}^{A}\left[\delta_{\omega}\,\|(\hat{\omega}, \hat{\psi}) - (\omega, \psi)\|_{\mathscr{B}} + \varepsilon_1\right]$$

$$\le \mathbb{C}_{\alpha}^{A}\left[\frac{\mathbb{C}_{\alpha}^{A}\delta_{\omega}}{1-\left(\mathbb{C}_{\alpha}^{A}(\delta_{\omega}+\delta_{\psi})\right)}\max\{\varepsilon_1, \varepsilon_2\}\right] + \mathbb{C}_{\alpha}^{A}\varepsilon_1$$

and

$$\|\hat{\psi}-\psi\| \le \mathbb{C}_{\alpha}^{A}\left[\delta_{\psi}\,\|(\hat{\omega},\,\hat{\psi})-(\omega,\,\psi)\|_{\mathscr{B}}+\varepsilon_2\right]$$

$$\le \mathbb{C}_{\alpha}^{A}\left[\frac{\mathbb{C}_{\alpha}^{A}\delta_{\psi}}{1-\left(\mathbb{C}_{\alpha}^{A}(\delta_{\omega}+\delta_{\psi})\right)}\max\{\varepsilon_1,\,\varepsilon_2\}\right]+\mathbb{C}_{\alpha}^{A}\varepsilon_2.$$

In particular, if we set $\varepsilon := \max\{\varepsilon_1,\,\varepsilon_2\}$, then

$$\|\hat{\omega}-\omega\| \le \varepsilon\,\mathbb{C}_{\alpha}^{A}\left[\frac{\mathbb{C}_{\alpha}^{A}\delta_{\omega}}{1-\left(\mathbb{C}_{\alpha}^{A}(\delta_{\omega}+\delta_{\psi})\right)}+1\right]$$

and

$$\|\hat{\psi}-\psi\| \le \varepsilon\,\mathbb{C}_{\alpha}^{A}\left[\frac{\mathbb{C}_{\alpha}^{A}\delta_{\psi}}{1-\left(\mathbb{C}_{\alpha}^{A}(\delta_{\omega}+\delta_{\psi})\right)}+1\right].$$

Take

$$\varepsilon_1' := \mathbb{C}_{\alpha}^{A}\left[\frac{\mathbb{C}_{\alpha}^{A}\delta_{\omega}}{1-\left(\mathbb{C}_{\alpha}^{A}(\delta_{\omega}+\delta_{\psi})\right)}+1\right] \text{ and } \varepsilon_2' := \mathbb{C}_{\alpha}^{A}\left[\frac{\mathbb{C}_{\alpha}^{A}\delta_{\psi}}{1-\left(\mathbb{C}_{\alpha}^{A}(\delta_{\omega}+\delta_{\psi})\right)}+1\right].$$

The Hyers-Ulam stability constants $\varepsilon_1'$ and $\varepsilon_2'$ measure how closely the approximate solution remains to the exact one. These constants are inversely related to the denominator

$$1-\mathbb{C}_{\alpha}^{A}(\delta_{\omega}+\delta_{\psi}),$$

which characterizes the contraction gap of the operator. A smaller value of $\mathbb{C}_{\alpha}^{A}$, $\delta_{\omega}$, or $\delta_{\psi}$ increases this denominator, thereby reducing the values of $\varepsilon_1'$ and $\varepsilon_2'$ and yielding a stronger form of stability. To reinforce the stability of the system, it is therefore desirable to ensure that

$$\mathbb{C}_{\alpha}^{A}(\delta_{\omega}+\delta_{\psi}) \le 1, \quad \text{i.e.,} \quad (1-\alpha)+\frac{\xi_0^{\alpha}}{\Gamma(\alpha)} < \frac{A^{\alpha}}{\delta_{\omega}+\delta_{\psi}}$$

This condition may be fulfilled by choosing a smaller domain length $\xi_0$, reducing the Lipschitz constants, or increasing the fractional order $\alpha$ so that $\Gamma(\alpha)$ becomes larger. Any of these adjustments increases the denominator, thereby strengthening the Hyers-Ulam stability of the fractional nonlinear system. Consequently, the system (2) is Hyers-Ulam stable. □

## 4. Some applications

This section employs the DLADM to derive approximate analytical solutions for the fractional WBKEs (2). This hybrid approach combines the decomposition of nonlinear terms with the transform properties of the Laplace operator, resulting in an efficient and robust framework for solving fractional problems. The method exhibits rapid convergence, high accuracy, and computational simplicity. To illustrate its effectiveness, several examples and applications are presented, demonstrating the accuracy and reliability of the proposed approach in modeling nonlinear shallow water wave dynamics.

For clarity, the main steps of the proposed DLADM are summarized as follows:

1. Apply the double Laplace transform to the fractional system.
2. Express the solution as an infinite series expansion.
3. Decompose the nonlinear terms using Adomian polynomials.
4. Compute the recursive components of the solution.
5. Apply the inverse double Laplace transform to obtain approximate solutions.
6. Construct the final solution as a convergent series.

By applying the operator $\mathbb{L}$ to both sides of (2) and using Theorem (3), we obtain:

$$\mathbb{L}\left[{}^{ABC}\mathscr{D}_{0,\,\xi}^{\alpha}\,\omega\right] = -\mathbb{L}\left[\omega\omega_{\kappa}\right] - \mathbb{L}\left[\psi_{\kappa}\right] - \beta\,\mathbb{L}\left[\omega_{\kappa\kappa}\right] \tag{29}$$

and

$$\mathbb{L}\left[{}^{ABC}\mathscr{D}_{0,\,\xi}^{\alpha}\,\psi\right] = -\mathbb{L}\left[\psi\omega_{\kappa}\right] - \mathbb{L}\left[\omega\psi_{\kappa}\right] + \beta\,\mathbb{L}\left[\psi_{\kappa\kappa}\right] - \beta'\,\mathbb{L}\left[\omega_{\kappa\kappa\kappa}\right]. \tag{30}$$

Since $0 < \alpha < 1$, then by Theorems (2, 3) we get

$$\omega_{\mathbb{L}} = \frac{1}{\rho}\mathbb{L}_{\kappa}(h_0) - \frac{\Gamma_{\rho,\,\alpha}(A^{\alpha})}{\rho^{\alpha}}\left[\mathbb{L}\left[\omega\omega_{\kappa}\right] + \mathbb{L}\left[\psi_{\kappa}\right] + \beta\mathbb{L}\left[\omega_{\kappa\kappa}\right]\right] \tag{31}$$

and

$$\psi_{\mathbb{L}} = \frac{1}{\rho}\mathbb{L}_{\kappa}(\hat{h}_0) - \frac{\Gamma_{\rho,\,\alpha}(A^{\alpha})}{\rho^{\alpha}}\left[\,\mathbb{L}\left[\psi\omega_{\kappa}\right] + \mathbb{L}\left[\omega\psi_{\kappa}\right] - \beta\mathbb{L}\left[\psi_{\kappa\kappa}\right] + \beta'\mathbb{L}\left[\omega_{\kappa\kappa\kappa}\right]\right] \tag{32}$$

where $\Gamma_{\rho,\,\alpha}(A^{\alpha}) := \dfrac{(1-\alpha)\rho^{\alpha}+\alpha}{A^{\alpha}}$. For the linear terms of the system (2), assume that $\omega$ and $\psi$ are as series of the following forms:

$$\omega = \sum_{n=0}^{\infty}\omega_n \text{ and } \psi = \sum_{n=0}^{\infty}\psi_n. \tag{33}$$

For the non-linear terms of the system (2), then Adomian polynomials with the standard coupled-polynomial forms given as:

$$\omega\omega_\kappa = \sum_{n=0}^{\infty} A_n,\ \omega\psi_\kappa = \sum_{n=0}^{\infty} B_n,\ \psi\omega_\kappa = \sum_{n=0}^{\infty} C_n,\ A_n = \sum_{k=0}^{n} \omega_k \omega_{(n-k)\kappa},$$

$$B_n = \sum_{k=0}^{n} \omega_k \psi_{(n-k)\kappa} \text{ and } C_n = \sum_{k=0}^{n} \psi_k \omega_{(n-k)\kappa}. \tag{34}$$

For three terms we have

$$A_0 = \omega_0\omega_{0\kappa},\ A_1 = 2\,\omega_0\omega_{1\kappa},\ A_2 = 2\,\omega_2\omega_{0\kappa} + \omega_1\omega_{1\kappa},\ \cdots$$

$$B_0 = \omega_0\psi_{0\kappa},\ B_1 = \omega_0\psi_{1\kappa} + \omega_1\psi_{0\kappa},\ B_2 = \omega_2\psi_{0\kappa} + \omega_1\psi_{1\kappa} + \omega_0\psi_{2\kappa},\ \cdots \tag{35}$$

$$C_0 = \psi_0\omega_{0\kappa},\ C_1 = \psi_0\omega_{1\kappa} + \psi_1\omega_{0\kappa},\ C_2 = \psi_2\omega_{0\kappa} + \psi_1\omega_{1\kappa} + \psi_0\omega_{2\kappa},\ \cdots$$

Take the inverse double Laplace of (31) and (32) to get

$$\omega = h_0 - \mathbb{L}^{-1}\left[\frac{\Gamma_{\rho,\,\alpha}(A^\alpha)}{\rho^\alpha}\left[\mathbb{L}\left[\omega\omega_\kappa\right] + \mathbb{L}\left[\psi_\kappa\right] + \beta\mathbb{L}\left[\omega_{\kappa\kappa}\right]\right]\right] \tag{36}$$

and

$$\psi = \hat{h}_0 - \mathbb{L}^{-1}\left[\frac{\Gamma_{\rho,\,\alpha}(A^\alpha)}{\rho^\alpha}\left[\,\mathbb{L}\left[\psi\omega_\kappa\right] + \mathbb{L}\left[\omega\psi_\kappa\right] - \beta\mathbb{L}\left[\psi_{\kappa\kappa}\right] + \beta'\mathbb{L}\left[\omega_{\kappa\kappa\kappa}\right]\right]\right]. \tag{37}$$

From (33) and (34) we have

$$\sum_{n=0}^{\infty}\omega_n = h_0 - \mathbb{L}^{-1}\left[\frac{\Gamma_{\rho,\,\alpha}(A^\alpha)}{\rho^\alpha}\left\{\mathbb{L}\left[\sum_{n=0}^{\infty} A_n\right] + \mathbb{L}\left[\sum_{n=0}^{\infty}\psi_{n\kappa}\right] + \beta\mathbb{L}\left[\sum_{n=0}^{\infty}\omega_{n\kappa\kappa}\right]\right\}\right] \tag{38}$$

and

$$\sum_{n=0}^{\infty}\psi_n = \hat{h}_0 - \mathbb{L}^{-1}\left[\frac{\Gamma_{\rho,\,\alpha}(A^\alpha)}{\rho^\alpha}\left\{\mathbb{L}\left[\sum_{n=0}^{\infty} C_n\right] + \mathbb{L}\left[\sum_{n=0}^{\infty} B_n\right] - \beta\mathbb{L}\left[\sum_{n=0}^{\infty}\psi_{n\kappa\kappa}\right] + \beta'\mathbb{L}\left[\sum_{n=0}^{\infty}\omega_{n\kappa\kappa\kappa}\right]\right\}\right] \tag{39}$$

Matching coefficients of identical powers in (38) and (39) yields the following series solution:

$$
\begin{aligned}
&\omega_0 = h_0\\
&\omega_1 = -\mathbb{L}^{-1}\left[\frac{\Gamma_{\rho,\,\alpha}(A^{\alpha})}{\rho^{\alpha}}\left\{\mathbb{L}\left[A_0\right] + \mathbb{L}\left[\psi_{0\kappa}\right] + \beta\mathbb{L}\left[\omega_{0\kappa\kappa}\right]\right\}\right]\\
&\omega_2 = -\mathbb{L}^{-1}\left[\frac{\Gamma_{\rho,\,\alpha}(A^{\alpha})}{\rho^{\alpha}}\left\{\mathbb{L}\left[A_1\right] + \mathbb{L}\left[\psi_{1\kappa}\right] + \beta\mathbb{L}\left[\omega_{1\kappa\kappa}\right]\right\}\right]\\
&\vdots\\
&\omega_{n+1} = -\mathbb{L}^{-1}\left[\frac{\Gamma_{\rho,\,\alpha}(A^{\alpha})}{\rho^{\alpha}}\left\{\mathbb{L}\left[A_n\right] + \mathbb{L}\left[\psi_{n\kappa}\right] + \beta\mathbb{L}\left[\omega_{n\kappa\kappa}\right]\right\}\right]\\
&\vdots
\end{aligned}
\tag{40}
$$

and

$$
\begin{aligned}
&\psi_0 = \hat{h}_0\\
&\psi_1 = -\mathbb{L}^{-1}\left[\frac{\Gamma_{\rho,\,\alpha}(A^{\alpha})}{\rho^{\alpha}}\left\{\mathbb{L}\left[C_0\right] + \mathbb{L}\left[B_0\right] - \beta\mathbb{L}\left[\psi_{0\kappa\kappa}\right] + \beta'\mathbb{L}\left[\omega_{0\kappa\kappa\kappa}\right]\right\}\right]\\
&\psi_2 = -\mathbb{L}^{-1}\left[\frac{\Gamma_{\rho,\,\alpha}(A^{\alpha})}{\rho^{\alpha}}\left\{\mathbb{L}\left[C_1\right] + \mathbb{L}\left[B_1\right] - \beta\mathbb{L}\left[\psi_{1\kappa\kappa}\right] + \beta'\mathbb{L}\left[\omega_{1\kappa\kappa\kappa}\right]\right\}\right]\\
&\vdots\\
&\psi_{n+1} = -\mathbb{L}^{-1}\left[\frac{\Gamma_{\rho,\,\alpha}(A^{\alpha})}{\rho^{\alpha}}\left\{\mathbb{L}\left[C_n\right] + \mathbb{L}\left[B_n\right] - \beta\mathbb{L}\left[\psi_{n\kappa\kappa}\right] + \beta'\mathbb{L}\left[\omega_{n\kappa\kappa\kappa}\right]\right\}\right]\\
&\vdots
\end{aligned}
\tag{41}
$$

Hence, the series solution can be expressed as follows:

$$
\omega(\kappa,\,\xi) = \sum_{n=0}^{\infty}\omega_n(\kappa,\,\xi) \text{ and } \psi(\kappa,\,\xi) = \sum_{n=0}^{\infty}\psi_n(\kappa,\,\xi). \tag{42}
$$

The coefficient $\beta$ controls the spreading and smoothing of the wave profile as it propagates horizontally. Physically, $\beta$ represents an effective viscosity or dispersion that causes the wave crests to broaden and the amplitude to decrease over time. If $\beta > 0$: the wave becomes smoother, and its peaks spread apart; the energy disperses spatially. If $\beta < 0$: the opposite occurs the wave becomes sharper and can develop shock like features. In the schematic illustration, the broadening of wave crests is indicated by the horizontal spreading of successive peaks, a direct manifestation of the dispersion parameter $\beta$. The parameter $\beta'$ represents high order dispersion, governing the fine structure of the wave, such as small ripples or secondary oscillations between major crests. Mathematically, it appears in the term $\beta' \omega_{\kappa\kappa\kappa}$ and contributes to the formation of oscillatory tails behind the main wave. The small high frequency ripples superimposed on the main surface are associated with the effect of $\beta'$. Increasing $\beta'$ enhances these fine oscillations. This helps explain the later numerical example, we will consider that wave maintains dispersive character and doesn't have dissipative smoothing, that is, $\beta = 0$ and $\beta' = 1$. Consider the following fractional WBKEs in shallow water wave phenomena with $A^{\alpha} = 1$:

$$\begin{cases} {}^{ABC}\mathscr{D}^{\alpha}_{0,\,\xi}\,\omega(\kappa,\,\xi) + \omega(\kappa,\,\xi)\omega_{\kappa}(\kappa,\,\xi) + \psi_{\kappa}(\kappa,\,\xi) = 0, \\ {}^{ABC}\mathscr{D}^{\alpha}_{0,\,\xi}\,\psi(\kappa,\,\xi) + [\omega(\kappa,\,\xi)\psi(\kappa,\,\xi)]_{\kappa} - \omega_{\kappa\kappa\kappa}(\kappa,\,\xi) = 0, \\ \omega(\kappa,\,0) = 5 - 0.2\coth(0.1\kappa + 1), \\ \psi(\kappa,\,0) = -0.02\,\mathrm{csch}(0.1\kappa + 1). \end{cases} \tag{43}$$

If $\alpha = 1$ then the exact solution of (43) is

$$\begin{aligned} \omega(\kappa,\,\xi) &= 5 - 0.2\coth(0.1\kappa - 5\xi + 1) \\ \psi(\kappa,\,\xi) &= -0.02\,\mathrm{csch}(0.1\kappa - 5\xi + 1). \end{aligned} \tag{44}$$

From (40) and (41) we that

$$\begin{aligned} \omega_0(\kappa,\,\xi) &= 5 - 0.2\coth(0.1\kappa + 1) \\ \omega_1(\kappa,\,\xi) &= -\frac{0.05\,\xi^{\alpha}}{\Gamma(\alpha+1)}\mathrm{csch}^2(0.1\kappa + 1) \\ \omega_2(\kappa,\,\xi) &= -\frac{0.01\,\xi^{2\alpha}}{\Gamma(2\alpha+1)}\mathrm{csch}^2(0.1\kappa + 1)\coth(0.1\kappa + 1) \\ &\vdots \end{aligned} \tag{45}$$

and

$$\begin{aligned}
&\psi_0(\kappa,\,\xi)=-0.02\,\operatorname{csch}(0.1\kappa+1)\\
&\psi_1(\kappa,\,\xi)=-\frac{0.05\,\xi^{\alpha}}{\Gamma(\alpha+1)}\operatorname{csch}^2(0.1\kappa+1)\coth(0.1\kappa+1)\\
&\psi_2(\kappa,\,\xi)=-\frac{0.005\,\xi^{2\alpha}}{\Gamma(2\alpha+1)}\operatorname{csch}^4(0.1\kappa+1)(2+\cosh(0.2\kappa+2))\\
&\vdots
\end{aligned}\tag{46}$$

Therefore the solution of (43) is given by

$$\begin{aligned}
\omega(\kappa,\,\xi)=5-0.2\coth(0.1\kappa+1)-\frac{0.05\,\xi^{\alpha}}{\Gamma(\alpha+1)}\operatorname{csch}^2(0.1\kappa+1)\\
-\frac{0.01\,\xi^{2\alpha}}{\Gamma(2\alpha+1)}\operatorname{csch}^2(0.1\kappa+1)\coth(0.1\kappa+1)+\cdots
\end{aligned}\tag{47}$$

and

$$\begin{aligned}
\psi(\kappa,\,\xi)=\;-0.02\,\operatorname{csch}(0.1\kappa+1)-\frac{0.05\,\xi^{\alpha}}{\Gamma(\alpha+1)}\operatorname{csch}^2(0.1\kappa+1)\coth(0.1\kappa+1)\\
-\frac{0.005\,\xi^{2\alpha}}{\Gamma(2\alpha+1)}\operatorname{csch}^4(0.1\kappa+1)(2+\cosh(0.2\kappa+2))+\cdots
\end{aligned}\tag{48}$$

## 5. Numerical discussion

This section presents a detailed numerical analysis of the fractional WBKEs (43) using the DLADM. The goal is to compare the numerical approximations with the exact analytical solutions and to interpret the wave behavior illustrated in the tables and figures. Table 1 provides a pointwise comparison between the exact and DLADM solutions for the horizontal velocity $\omega(\kappa,\,\xi)$ and the free surface elevation $\psi(\kappa,\,\xi)$ at various space time coordinates. The absolute error in $\omega$ is consistently of order $10^{-4}$, while the error in $\psi$ is even smaller, typically between $10^{-6}$ and $10^{-5}$. These uniformly small errors across all tested values of $\kappa$ and $\xi$ demonstrate that the DLADM method provides a highly accurate approximation of the exact solutions, without showing any instability or error growth over time. Furthermore, the symmetry in the table entries reflects the internal structure of the analytic traveling wave solution, which depends on the expression $(0.1\kappa-5\xi+1)$. Figures 1 and **??** show the DLADM curves for $\omega(\kappa,\,\xi)$ and $\psi(\kappa,\,\xi)$ at different times $\xi=1,\,2,\,3$. The analytical form of $\omega$ is based on a hyperbolic coth-profile, which produces a kink like traveling wave. The DLADM curves accurately reproduce this shape, with the wave shifting in the $\kappa$ direction as time increases while maintaining its form. Similarly, the analytical solution for $\psi$ is a hyperbolic profile, corresponding to a localized peak that moves through space. The DLADM curves capture this localized motion cleanly and match the analytical profile extremely well, consistent with the very small errors in Table 1. Figures 3 and 4 present full space time surfaces for $\omega(\kappa,\,\xi)$ and

$\psi(\kappa, \xi)$ obtained using DLADM. The surface for $\omega$ clearly reveals a traveling transition layer: for fixed $\kappa$ the solution evolves smoothly in time, while for fixed $\xi$ one sees the characteristic kink structure. The surface for $\psi$ exhibits a localized ridge moving steadily across the $(\kappa, \xi)$-plane. Both surfaces are smooth and free from numerical oscillations or artificial dispersion, which indicates that the DLADM method is stable and accurately represents the dynamics of the fractional WBKEs. Finally, Figure 5 displays the exact surfaces for $\omega$ and $\psi$ at $\alpha = 1$, serving as visual benchmarks for the numerical results. When comparing Figures 3 and 4 with Figure 5, the agreement is immediate: the shape, propagation speed, peak location, and overall wave structure align almost perfectly. This close correspondence confirms that DLADM not only approximates the pointwise values correctly but also captures the qualitative and global behavior of the exact solutions. In summary, the numerical results lead to several key conclusions. First, DLADM is a reliable and highly accurate method for solving the fractional WBKEs. Second, the fractional model successfully captures the essential features of shallow-water wave dynamics, including dispersion, steepening effects, and localized structures. Third, the near perfect agreement between analytical and numerical results validates both the numerical method and the theoretical formulation of the model, indicating that this approach can be confidently extended to more complex cases where analytical solutions are unavailable.

**Table 1.** Comparing absolute error (AE) of exact solution and DLADM solution of (43) at $\alpha = 1$

| $\kappa$ | $\xi$ | $\omega(\kappa, \xi)_{\text{Exact}}$ | $\omega(\kappa, \xi)_{\text{DLADM}}$ | $\text{AE}_\omega$ | $\psi(\kappa, \xi)_{\text{Exact}}$ | $\psi(\kappa, \xi)_{\text{DLADM}}$ | $\text{AE}_\psi$ |
|---|---|---|---|---|---|---|---|
| 2.00 | 0.10 | 4.66907567 | 4.66860877 | 0.00046691 | −0.02636492 | −0.02636229 | 0.00000264 |
| | 0.20 | 3.98670209 | 3.98630342 | 0.00039867 | −0.09933643 | −0.09932650 | 0.00000993 |
| | 0.30 | 5.68654769 | 5.68597903 | 0.00056865 | 0.06567707 | 0.06567050 | 0.00000657 |
| | 0.40 | 5.30118814 | 5.30065802 | 0.00053012 | 0.02251983 | 0.02251758 | 0.00000225 |
| | 0.50 | 5.23209310 | 5.23156989 | 0.00052321 | 0.01177591 | 0.01177473 | 0.00000118 |
| 4.00 | 0.10 | 4.72078655 | 4.72031447 | 0.00047208 | −0.01948336 | −0.01948142 | 0.00000195 |
| | 0.20 | 4.47361351 | 4.47316615 | 0.00044736 | −0.04869114 | −0.04868627 | 0.00000487 |
| | 0.30 | 7.00666223 | 7.00596156 | 0.00070067 | 0.19966706 | 0.19964709 | 0.00001997 |
| | 0.40 | 5.37240510 | 5.37186786 | 0.00053724 | 0.03141426 | 0.03141112 | 0.00000314 |
| | 0.50 | 5.24984415 | 5.24931917 | 0.00052498 | 0.01497401 | 0.01497251 | 0.00000150 |
| 6.00 | 0.10 | 4.75015585 | 4.74968083 | 0.00047502 | −0.01497401 | −0.01497251 | 0.00000150 |
| | 0.20 | 4.62759490 | 4.62713214 | 0.00046276 | −0.03141426 | −0.03141112 | 0.00000314 |
| | 0.30 | 2.99333777 | 2.99303844 | 0.00029933 | −0.19966706 | −0.19964709 | 0.00001997 |
| | 0.40 | 5.52638649 | 5.52583385 | 0.00055264 | 0.04869114 | 0.04868627 | 0.00000487 |
| | 0.50 | 5.27921345 | 5.27868553 | 0.00052792 | 0.01948336 | 0.01948142 | 0.00000195 |
| 8.00 | 0.10 | 4.76790690 | 4.76743011 | 0.00047679 | −0.01177591 | −0.01177473 | 0.00000118 |
| | 0.20 | 4.69881186 | 4.69834198 | 0.00046988 | −0.02251983 | −0.02251758 | 0.00000225 |
| | 0.30 | 4.31345231 | 4.31302097 | 0.00043135 | −0.06567707 | −0.06567050 | 0.00000657 |
| | 0.40 | 6.01329791 | 6.01269658 | 0.00060133 | 0.09933643 | 0.09932650 | 0.00000993 |
| | 0.50 | 5.33092433 | 5.33039123 | 0.00053309 | 0.02636492 | 0.02636229 | 0.00000264 |

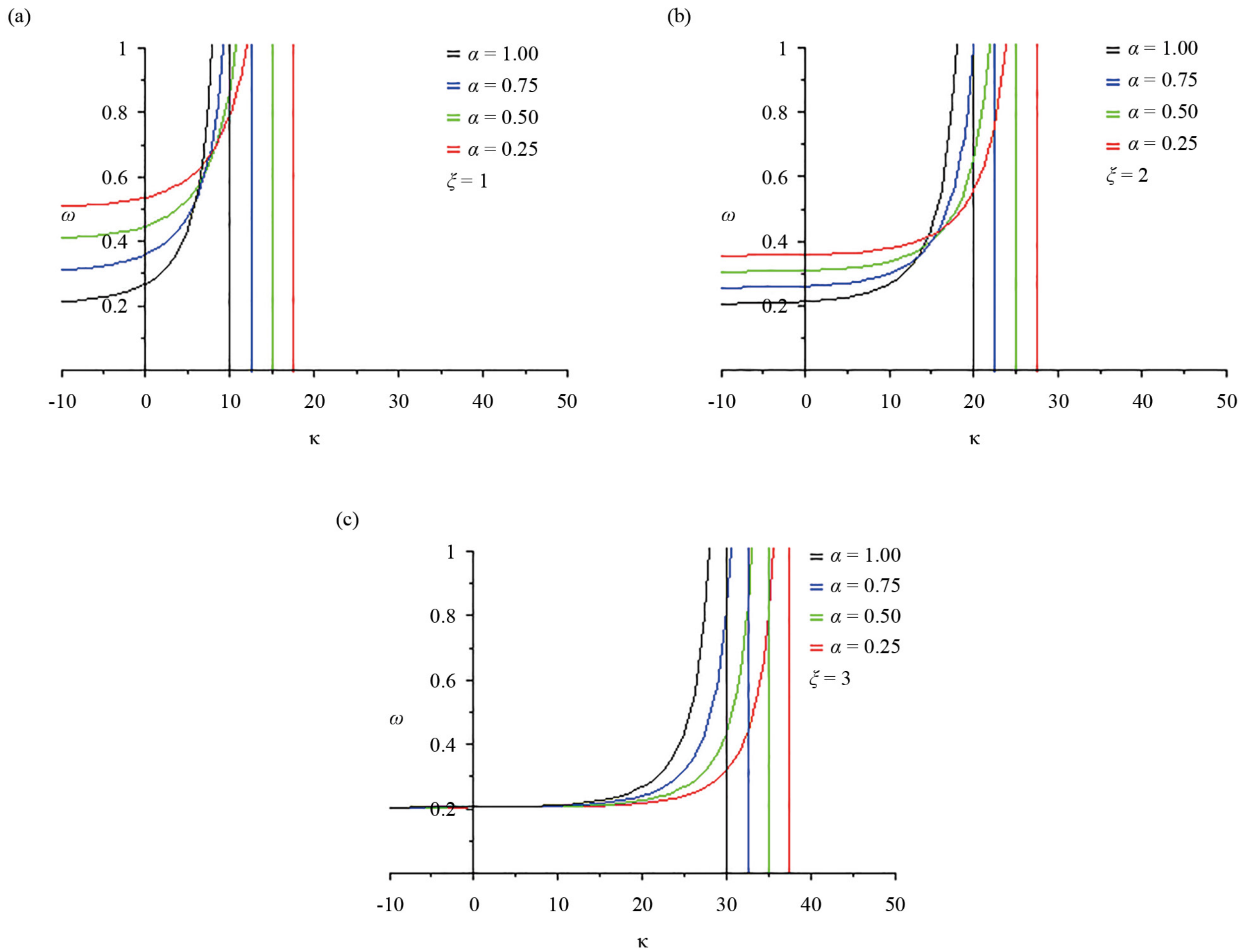


**Figure 1.** DLADM curves for the solutions $\omega(\kappa, \xi)$ of (43)

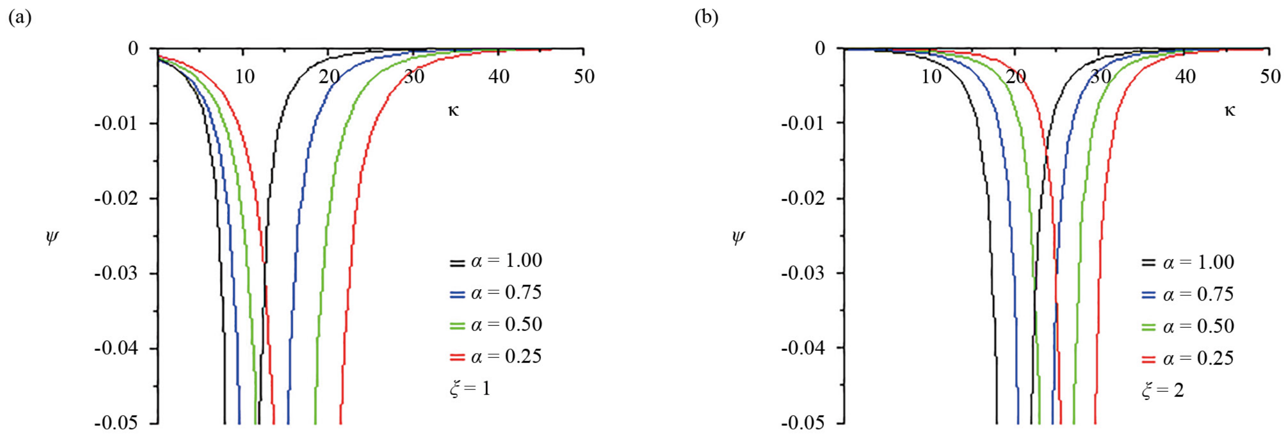


**Figure 2.** DLADM curves for the solutions $\psi(\kappa, \xi)$ of (43)

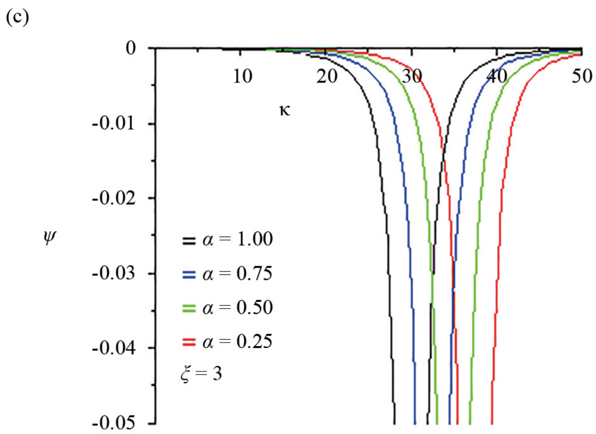


**Figure 2.** (cont.)

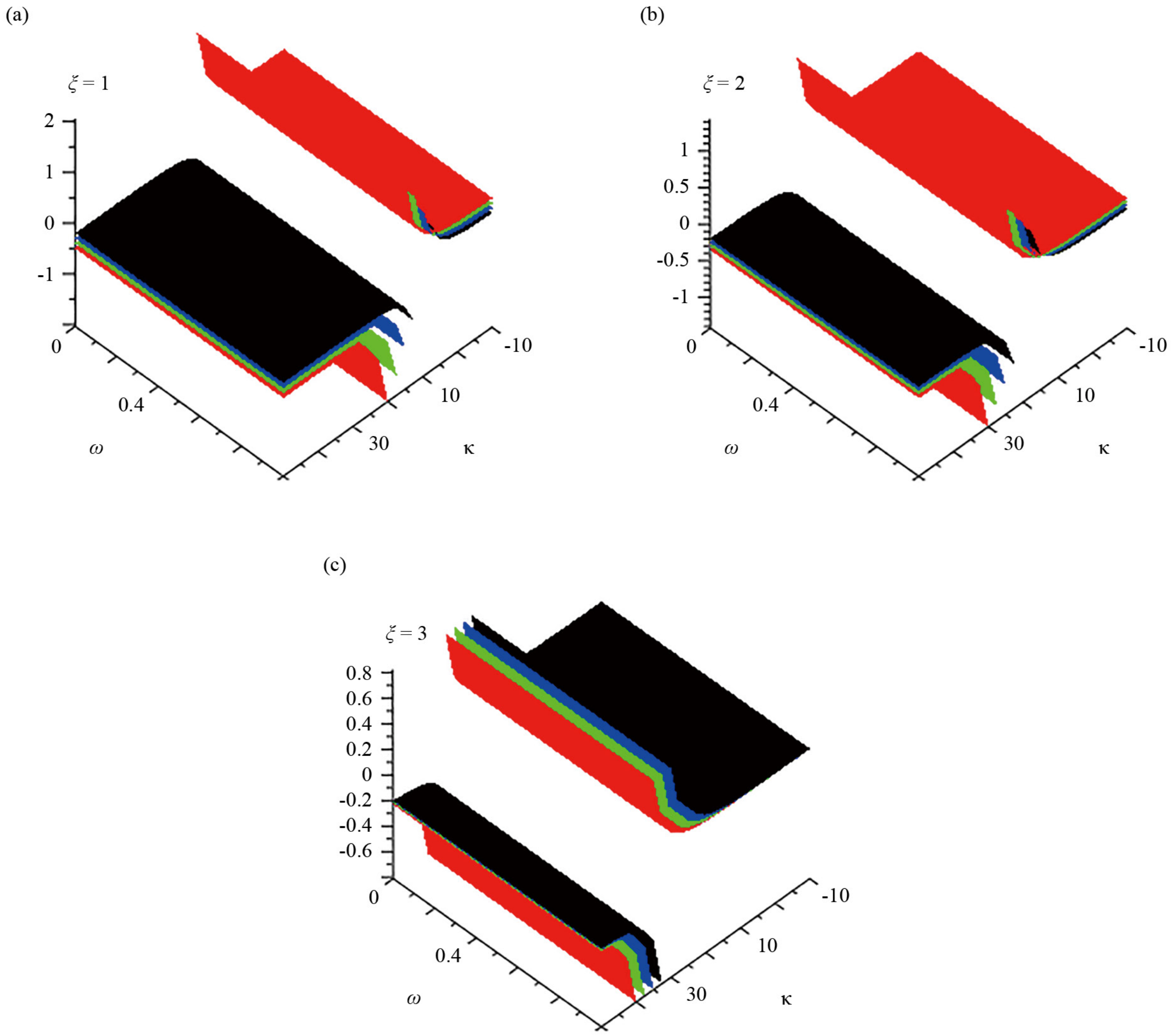


**Figure 3.** DLADM plots for the solutions $\omega(\kappa, \xi)$ of (43)

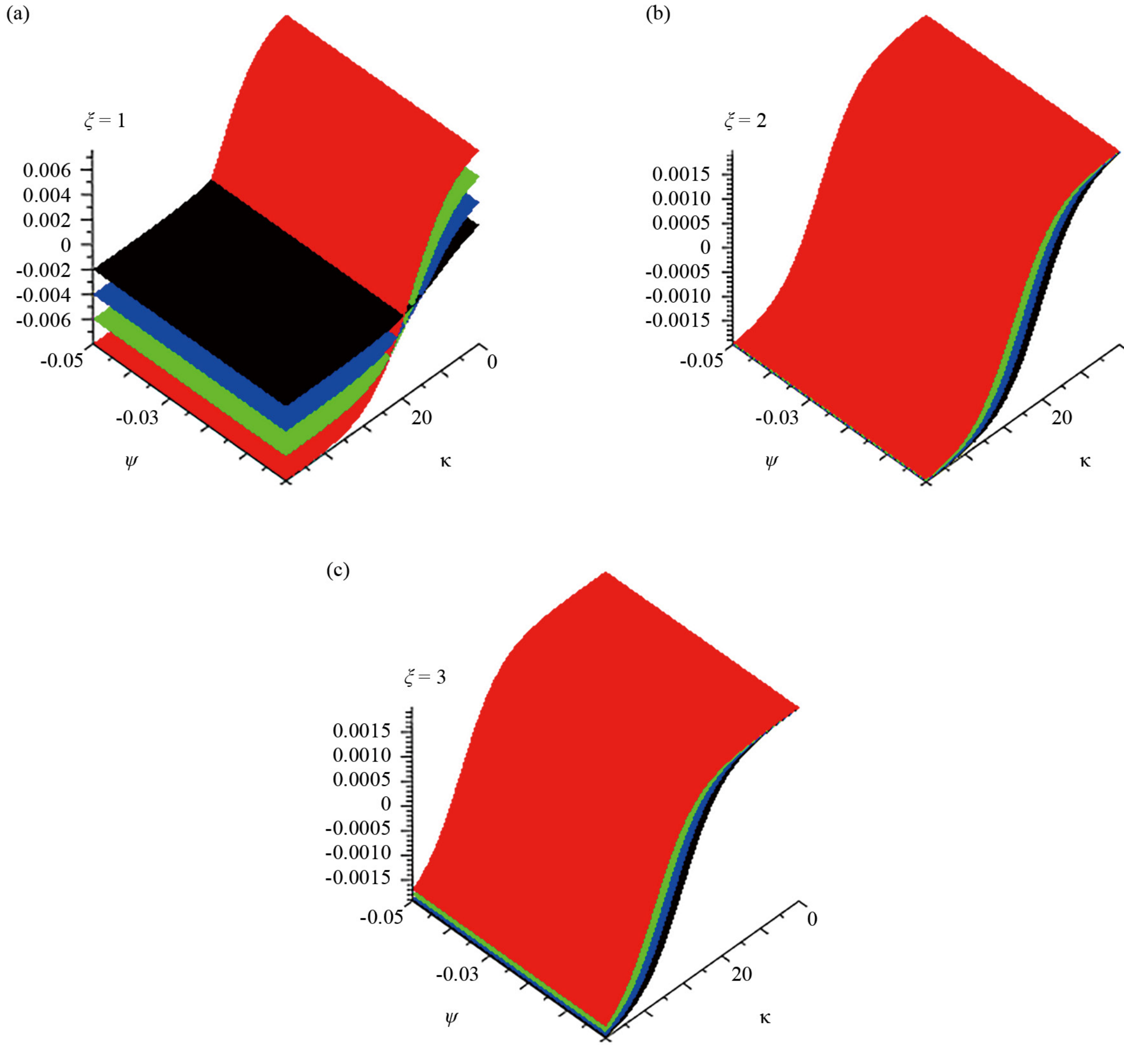


**Figure 4.** DLADM plots for the solutions $\psi(\kappa, \xi)$ of (43)

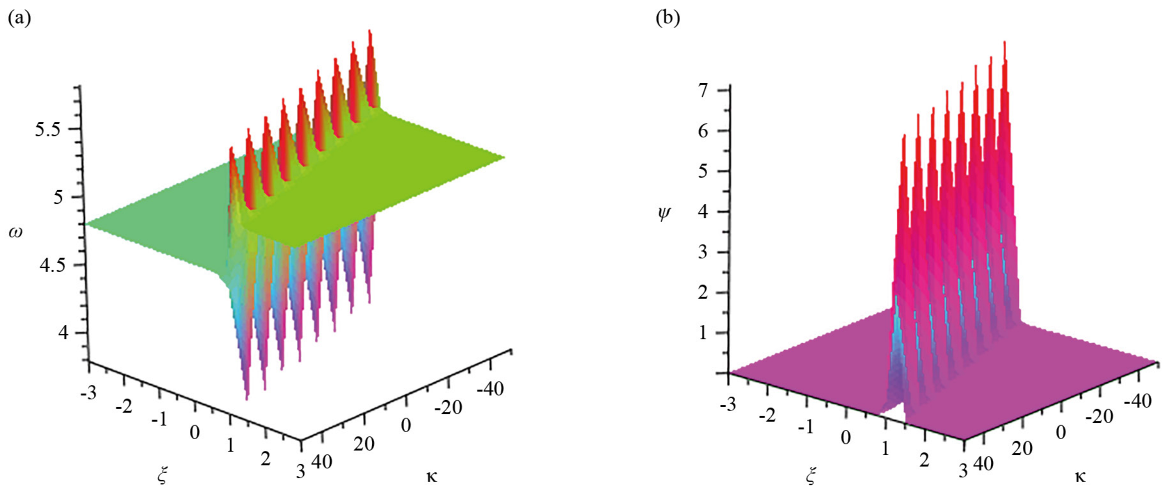


**Figure 5.** The exact solutions $\omega(\kappa, \xi)$ and $\psi(\kappa, \xi)$ of (43) at $\alpha = 1.00$

To further assess the performance of the proposed DLADM, we compare its accuracy with other analytical and numerical methods such as the homotopy perturbation method, residual power series method, and spectral techniques reported in the literature. The obtained results demonstrate that the DLADM provides highly accurate approximations with rapid convergence and smaller absolute errors. Moreover, unlike some existing methods, the proposed approach does not require discretization or linearization, which reduces computational complexity and improves efficiency. In addition, the proposed DLADM is compared with other numerical approaches such as finite difference methods, spectral techniques, and neural network-based models. While finite difference and spectral methods often require discretization and may suffer from stability constraints, and neural network approaches involve high training cost, the DLADM provides a direct analytical approximation with reduced computational effort. These features make the proposed method a competitive and efficient tool for solving fractional nonlinear WBKEs.

## 6. Physical interpretation of the results

The analytical solution obtained for the fractional WBK equations provides significant insight into the physical behavior of nonlinear wave propagation in dispersive media, particularly in shallow water environments. The WBK system describes the interaction between nonlinear convection and dispersive effects in fluid flow. By incorporating the fractional derivative in the sense of the ABC operator, the model extends the classical formulation to include memory and nonlocal effects. Physically, this implies that the present state of the system depends not only on the instantaneous conditions but also on its past history. The fractional order parameter $\alpha \in (0, 1]$ plays a key role in the system dynamics. When $\alpha = 1$, the model reduces to the classical WBK equations, representing standard wave propagation. For $\alpha < 1$, the system exhibits memory-dependent damping and anomalous diffusion, which are commonly observed in complex fluids. The use of the double Laplace transform provides an efficient analytical framework by converting the fractional differential equations into algebraic equations in the transform domain. This approach simplifies the treatment of initial conditions and allows a clearer interpretation of the solution behavior. The stability analysis indicates that the system remains stable under appropriate parameter conditions. Moreover, the fractional effects contribute to enhanced stability by smoothing steep gradients and reducing oscillatory behavior. The ABC operator, due to its non-singular kernel, ensures physically realistic modeling by avoiding singularities associated with classical fractional derivatives. Overall, the fractional WBK model offers a more accurate description of real-world wave phenomena, especially in media where memory effects and energy dissipation are significant. This framework is applicable to various physical systems, including shallow water waves, plasma waves, and other dispersive environments.

## 7. Conclusions

This work presents a solid analytical and numerical framework for studying the fractional WKBEs, which describe nonlinear shallow water waves with memory effects. Using fixed point theory in Banach spaces with the compact-open topology, we established the existence, uniqueness, and Hyers-Ulam stability of the fractional system, ensuring that its solutions are mathematically well posed. A hybrid DLADM was developed to obtain rapidly convergent approximate solutions. The numerical results show excellent agreement with the exact analytical solutions, with very small absolute errors throughout the computational domain. The graphical simulations confirm that DLADM accurately reproduces key wave behaviors, including traveling wave profiles, localized structures, and dispersive effects. Overall, the study demonstrates that the fractional WBK model, together with the DLADM approach, provides an effective and reliable framework for analyzing nonlinear wave dynamics. This methodology can be confidently extended to more complex fractional systems where analytical solutions are difficult or impossible to obtain.

## Author contributions

Conceptualization: F.H.D., F.A., K.M.S.; Methodology: F.H.D., K.M.S., A.K., M.F.A.; Formal analysis: F.H.D., A.K., N.A.; Investigation: F.A., M.F.A., N.A.; Resources: F.A., M.F.A., A.S.; Writing–original draft: F.H.D., A.S.; Writing–review & editing: K.M.S., A.K., F.A., M.F.A., N.A.; Visualization: A.S., K.M.S.; Supervision: K.M.S., A.K.; Project administration: F.H.D.; Funding acquisition: F.H.D.

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## Conflict of interest

The authors declare no competing financial interest.

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