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Research Article

On the Cauchy Problem of an Extended Nonlinear Bose-Einstein Equation with Critical Nonlinear Damping in $\mathbb{R}^3$

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Abstract: We study the well-posedness of the initial value problem for an NLS-type equation with harmonic trapping potential and a non-local term under the influence of a nonlinear damping term in $\mathbb{R}^3$ which models the formation of quantum droplets in a trapped dipolar Bose-Einstein condensate (BEC) and states as follows:

$$i\frac{\partial\psi}{\partial t}=(-\Delta+|x|^2)\psi+\alpha\left(|\psi|^2+|\psi|^3\right)\psi-i\gamma|\psi|^4\psi+\beta(K*|\psi|^2)\psi,\ \ x\in\mathbb{R}^3.$$

Only theoretical predictions and experimental results on the criterion of their self-boundedness were established. So far, no rigorous mathematical explanation has been proved. The main objective of this study is to validate these predictions and to prove rigourously that the presence of the nonlinear damping prevents the collapse and ensures the existence of global in-time solutions (the stability).

***Keywords*:** Schrödinger equations, initial-value problems, Bose-Einstein condensates, quantum droplets

MSC: 35E15, 35Q55

1. Introduction

Dipolar quantum droplets are new quantum objects that are self-bound, i.e: instead of collapsing the system formed stable droplets. As mentioned in a seminal paper by Petrov DS [1], ultracold quantum gases can exist in the form of self-bound droplets. This liquid-like behavior which originates from the interplay of attractive mean-field interactions and the repulsive effect of quantum fluctuations (LHY), which was firstly predicted long ago by Lee TD, Huang K, and Yang CN [2], was successfully realized by driving a dipolar Bose-Einstein condensate (BEC) into the strongly dipolar regime (see [3]) and observed in a number of experiments in other dipolar mixtures (see [4-8] and the references therein).

Since their discovery, many properties of dipolar quantum droplets have been made in recent theoretical predictions and experimental works and appear in [6, 7, 9-11] and recently in the good review [12], to which we refer the reader, where the authors have summarized the recent results which predict and establish the existence of quantum droplets. The most important theoretical prediction is their self-bounded state which, to the best of our knowledge and from

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mathematical point of view, has not yet been explained rigorously. Only experimental results have been obtained.

Instead of collapsing, which would take place in the mean field approximation, the self-boundedness criterion of the droplets in dipolar condensates relies upon the effect of corrections due to quantum fluctuations LHY (see [6, 9, 13]) which are effectively represented by local quantic self-repulsive terms in a generalized extended non-local Gross-Pitaevskii equation (eGPE) (see [9]). This equation is given as follows

$$i\hbar\frac{\partial\psi}{\partial t}=\left[-\frac{\hbar^2\nabla^2}{2m}+V_{ext}+g|\psi|^2+\frac{32g\sqrt{a_s^3}}{3\sqrt{\pi}}(1+\frac{3}{2}\epsilon_{dd}^2)|\psi|^3\right]$$

$$+\left[\int_{\mathbb{R}^3}V_{dd}(x-y)|\psi(y)|^2\,dy-i\frac{\hbar L_3}{2}|\psi|^4\right], \tag{1}$$

where $g=4\pi\hbar^2\frac{a_s}{m}$ describes the local interaction between dipoles in the condensate with a_s the s-wave scattering length parameter, the singular potential

$$V_{dd}(x)=\frac{\mu_0\mu^2}{4\pi}\frac{1-3\cos^2(\theta)}{|x|^3},\quad x\in\mathbb{R}^3,$$

is the dipole-dipole interaction potential of polarized particles, with θ being the angle between the polarization direction and the external trapping confinement potential is modeled by V_{ext} and supposed to be quadratic.

For the sake of simplicity, we fix the dipole axis as the vector (0, 0, 1). Moreover, in order to simplify the mathematical analysis of the equation (1), we consider the dimensionless form of (eGPE) that states as follows

$$i\frac{\partial\psi}{\partial t}=(-\Delta+V)\psi+\alpha\left(|\psi|^2+|\psi|^3\right)\psi-i\gamma|\psi|^4\psi+\beta(K*|\psi|^2)\psi, \tag{2}$$

where the unknown $\psi=\psi(t,x)$ maps $\mathbb{R}_+\times\mathbb{R}^3$ into $\mathbb{C}$ is the wave-function of the condensate, the confinement potential is harmonic $V(x)=|x|^2$ and the interaction kernel is given by

$$K(x)=\frac{1-3\cos^2(\theta)}{|x|^3},\ x\in\mathbb{R}^3.$$

The physical parameters α and β, which describe the strength of the nonlinearities belong to $\mathbb{R}$ and the damping parameter $\gamma>0$.

It should be emphasized that in the conservative case ($\gamma=0$), the equation (2) describes the so-called "dipolar Bose-Einstein condensates", namely a condensate made out of particles possessing a permanent electric or magnetic dipole moment. This resultant equation has been studied by many authors among which we cite [14-17]. In [14], the authors established local and global existence results as well as blowup solutions. Either the stability and instability criterions of standing waves or the existence of ground state solutions were discussed in [15, 16]. Moreover, the Scattering theory for the corresponding equation was studied by Bellazzini J and Forcella L in [17]. They have proved that under the energy threshold given by the ground state, all global in time solutions behave as free waves asymptotically in time.

In the absence of the nonlocal terms ($\beta=0$), a nonlinear Schrodinger-type equation augmented by nonlinear damping terms was considered in [18, 19]. In recent work, when considering a dependent anisotropic trapping potential in (2), Alouini B and Hajaiej H have focused on the asymptotic dynamics of the corresponding solutions.

Furthermore, taking into account thhe three-body loss ($\alpha>0$) which describes the tendency of atoms to recombine into molecules and to be lost from the condensate, only experimental results have been achieved that I will quote a few, including [6, 9, 20].

In what follows, we denote $A = -\Delta + V$ the positive self-adjoint operator whose domain in $\mathbb{L}^2(\mathbb{R}^3)$ is

$$D(A) = \left\{ \psi \in \mathbb{L}^2 \mathbb{R}^3 \text{ such that } A\psi \in \mathbb{L}^2(\mathbb{R}^3) \right\}.$$

We supplement (2) with initial data at $t = 0$

$$u(0) = u_0, \tag{3}$$

belonging to the phase space

$$\sum = D(A^{\frac{1}{2}}) = \left\{ \psi \in \mathbb{L}^2(\mathbb{R}^3) \text{ such that } \nabla\psi \text{ and } x\psi \in \mathbb{L}^2(\mathbb{R}^3) \right\}. \tag{4}$$

In this article, we mathematically validate the principal characteristic of these quantum droplets by providing a rigorous mathematical analysis through which we prove that the stability of these droplets, which is reflected by the existence of global in time solutions for (2), is none other than a consequence of the presence of the nonlinear damping term in the extended Gross-Pitaevskii equation under study which acts as a counterbalance against the collapse.

Since not even the existence and uniqueness of solutions to (2) has been established yet, which gives all originality to the current work, the study of the corresponding Cauchy problem will be the main task of this paper. The idea is to combine different techniques presented either in [18, 19] when studying the Cauchy problem for NLS-type equations with nonlinear damping or in [14] where the study of a Bose-Einstein equation with nonlocal nonlinearity was established. The main result of this paper states as follows:

Theorem 1.1 Let $\gamma_0 \in \Sigma$. Then the Cauchy problem (2)-(3) has a unique global in time solution γ belonging to $\mathscr{C}(\mathbb{R}_+, \Sigma)$. Moreover, $\gamma \in \mathbb{L}^\infty(\mathbb{R}_+, \Sigma)$.

This article is organized as follows: in section 2, we introduce some notations and some mathematical tools. Section 3 will be dedicated to proving a local in-time solution and finally, in section 4 we establish the global-in-time criterion of these solutions throughout some a priori estimates. Finally, the conclusion will be the object of the last section.

2. Mathematical background and notations

To begin with, we will be regularly referring to the mixed space-time seminorms

$$\| \psi \|_{\mathbb{L}_T^q \mathbb{L}_x^p} = \left(\int_0^T \|\psi(s)\|_{\mathbb{L}^p(\mathbb{R}^3)}^q ds \right)^{\frac{1}{q}},$$

where $T > 0$ will be made precise in the sequel. For later use, we denote for a given $1 < p < +\infty$

$$\sum\nolimits_p = D_p(A^{\frac{1}{2}}), \tag{5}$$

$$= \left\{ \psi \in \mathbb{L}^p(\mathbb{R}^3), \; A^{\frac{1}{2}}\psi \in \mathbb{L}^p(\mathbb{R}^3) \right\}, \tag{6}$$

$$= \left\{ \psi \in \mathbb{L}^p(\mathbb{R}^3) \text{ such that } \nabla\psi \text{ and } x\psi \in \mathbb{L}^p(\mathbb{R}^3) \right\}. \tag{7}$$

Along with the request, the usual Sobolev spaces denoted by $W^{1,p} = \mathrm{W}^{1,p}(\mathbb{R}^3)$, $1 < p < +\infty$, are defined as follows

$$W^{1,p} = \left\{\psi \in \mathbb{L}^p(\mathbb{R}^3) \text{ such that } \nabla\psi \in \mathbb{L}^p(\mathbb{R}^3)\right\}.$$

The space Σ_p defined above endowed with the norm

$$\|\psi\|_{\Sigma_p} = \left(\|\psi\|^p_{\mathbb{L}^p(\mathbb{R}^3)} + \|A^{\frac{1}{2}}\psi\|^p_{\mathbb{L}^p(\mathbb{R}^3)}\right)^{\frac{1}{p}}, \tag{8}$$

which is equivalent (see [21]) to

$$\|\psi\|_{\Sigma_p} = \left(\|\psi\|^p_{\mathbb{L}^p(\mathbb{R}^3)} + \|\nabla\psi\|^p_{\mathbb{L}^p(\mathbb{R}^3)} + \|x\psi\|^p_{\mathbb{L}^p(\mathbb{R}^3)}\right)^{\frac{1}{p}}, \tag{9}$$

is a complete Banach space. The Hilbert space $\Sigma_2 = \Sigma$ is endowed with the scalar product defined by

$$(\psi,\varphi)_\Sigma = (\psi,\varphi) + \left(\nabla\psi,\nabla\varphi\right) + (x\psi, x\varphi), \;\; \forall\psi,\varphi \in \Sigma,$$

where

$$(\psi,\varphi) = \Re e\int_{\mathbb{R}^3}\psi(x)\overline{\varphi(x)}dx.$$

For the sack of simplicity we extensively use, for a given $p \in [1, +\infty]$ and $s \in (1, +\infty)$, the notation $\mathbb{L}^p_T\Sigma_s$ for the anisotropic space $\mathbb{L}^p([0, T], \Sigma_s)$.

In order to prove the well-posedness of (2) we shall rely on the use of Strichartz estimates. Thanks to the Mehler's formula (see [22]) the Schrödinger propagator e^{-itA}, with $A = -\Delta + |x|^2$, is bounded on $\mathbb{L}^2(\mathbb{R}^3)$. Moreover, it enjoys the following dispersive property (see [19, 22]), for small time, that reads

$$\|e^{-itA}\psi\|_{\mathbb{L}^\infty(\mathbb{R}^3)} \le C\frac{\|\psi\|_{\mathbb{L}^1(\mathbb{R}^3)}}{|t|^{\frac{3}{2}}}. \tag{10}$$

This, by interpolation argument, ensures that for all $p \in [2, +\infty]$ and for all $t > 0$

$$\|e^{-itA}\psi\|_{\mathbb{L}^p(\mathbb{R}^3)} \le C\frac{\|\psi\|_{\mathbb{L}^{p'}(\mathbb{R}^3)}}{|t|^{3(\frac{1}{2}-\frac{1}{p})}}, \tag{11}$$

where p' denotes the conjugate exponent of p.

Definition 2.1 A pair (q, r) is admissible if $2 \le r \le 6$ and

$$\frac{2}{q} = 3\left(\frac{1}{2} - \frac{1}{r}\right).$$

Thanks to the dispersive estimate (11), we have the following Strichartz-type estimates (see [19, 22, 23]).

Lemma 2.2 Let (q, r) be an admissible pair and $0 < T$. Then

$$\| e^{-itA}\psi \|_{\mathbb{L}^q_T \mathbb{R}^r_x} \leq C(r) T^{\frac{1}{q}} \| \psi \|_{\mathbb{L}^2(\mathbb{R}^3)} . \tag{12}$$

Moreover, for all $t \in [0, T]$

$$\left\| \int_0^t e^{-i(t-s)A} F(s) ds \right\|_{\mathbb{L}^q_T \mathbb{L}^r_x} \leq C(r, \rho) T^{\frac{1}{q}} \| F \|_{\mathbb{L}^{\delta'}_T \mathbb{L}^{\rho'}_x} , \tag{13}$$

for all admissible pair (δ, ρ).

Next, we recall the following Sobolev embedding (see [24]) and the well-known Gagliardo-Nirenberg inequality (see [25]).

Lemma 2.3 Let $1 \leq p < +\infty$. Then the following statements hold true

1. If $1 \leq p < 3$, $W^{1,p}(\mathbb{R}^3) \to \mathbb{L}^q(\mathbb{R}^3)$ for all $q \in [p, \frac{3p}{3-p}]$.

Moreover, for all $\psi \in W^{1,p}(\mathbb{R}^3)$

$$\| \psi \|_{\mathbb{L}^q(\mathbb{R}^3)} \leq C \| \psi \|^{1-3(\frac{1}{p}-\frac{1}{q})}_{\mathbb{L}^p(\mathbb{R}^3)} \| \nabla \psi \|^{3(\frac{1}{p}-\frac{1}{q})}_{\mathbb{L}^p(\mathbb{R}^3)} . \tag{14}$$

2. If $p = 3$, $W^{1,p}(\mathbb{R}^3) \to \mathbb{L}^q(\mathbb{R}^3)$ for all $q \in [p, +\infty)$.
3. If $p > 3$, $W^{1,p}(\mathbb{R}^3) \to \mathbb{L}^q(\mathbb{R}^3)$ for all $q \in [p, +\infty)$.

For the nonlocal term, note that even though the interaction kernel K in (2) is indeed highly singular (like $\frac{1}{|x|^3}$), it defines a rather smooth operator (see [14] for more details) thanks to the Calderón-Zygmund theorem (see [26, 27]).

Lemma 2.4 The operator $K : \psi \mapsto K * \psi$ can be extended to a continuous operator on $\mathbb{L}^p(\mathbb{R}^3)$ for all $1 < p < +\infty$.

In the end of this section it should be noted that throughout this article, the constants Cs are numerical positive constants that vary from one line to another and the notation $A \lesssim B$ means the existence of $C > 0$ such that $A \leq CB$.

3. Local in time solutions

Let us start with proving that the initial value problem (2)-(3) is locally well-posed.

Proposition 3.1 Let $\psi_0 \in \Sigma$. Then there exist $T > 0$ depending only on $\| \psi_0 \|_\Sigma$ and a unique solution ψ of the problem (2)-(3) belonging to

$$\mathbb{X}_T = \mathscr{C}([0, T], \Sigma) \cap \mathbb{L}^{\frac{8}{3}}_T \Sigma_4 \cap \mathbb{L}^{\frac{20}{9}}_T \Sigma_5 \cap \mathbb{L}^{10}_T \Sigma_{\frac{30}{13}} .$$

Proof. The Duhamel's formulation for (2) states as follows

$$\psi(t) = e^{-itA}\psi_0 - i\lambda_1 \int_0^t e^{-i(t-s)A} (|\psi(s)|^2 + |\psi(s)|^3)\psi(s) ds$$

$$-i\lambda_2 \int_0^t e^{-i(t-s)A} (K * |\psi(s)|^2)\psi(s) ds - \gamma \int_0^t e^{-i(t-s)A} |\psi(s)|^4 \psi(s) ds. \tag{15}$$

Let (q, r) be an admissible pair. Denoting $L\psi$ either ψ, $x\psi$ or $\nabla\psi$, we deduce from the Hölder inequality that

$$\left\|\int_0^t e^{-i(t-s)A} L(|\psi(s)|^2 \psi(s))ds\right\|_{\mathbb{L}_T^q \mathbb{L}_x^r} \lesssim \left\||\psi|^2 L(\psi)\right\|_{\mathbb{L}_T^{\frac{8}{5}} \mathbb{L}_x^{\frac{4}{3}}},$$

$$\lesssim \|\psi\|^2_{\mathbb{L}_T^8 \mathbb{L}_x^4} \|L(\psi)\|_{\mathbb{L}_T^{\frac{8}{3}} \mathbb{L}_x^4},$$

which, thanks to Lemma 2.3, leads to

$$\left\|\int_0^t e^{-i(t-s)A} L(|\psi(s)|^2 \psi(s))ds\right\|_{\mathbb{L}_T^q \mathbb{L}_x^r} \lesssim T^{\frac{1}{4}} \|\psi\|^2_{\mathbb{L}_T^\infty \Sigma} \|L(\psi)\|_{\mathbb{L}_T^{\frac{8}{3}} \mathbb{L}_x^4}.$$

Hence, in accordance with the norm equivalence (9), we have

$$\left\|\int_0^t e^{-i(t-s)A} |\psi(s)|^2 \psi(s)ds\right\|_{\mathbb{L}_T^q \Sigma_r} \lesssim T^{\frac{1}{4}} \|\psi\|^2_{\mathbb{L}_T^\infty \Sigma} \|\psi\|_{\mathbb{L}_T^{\frac{8}{3}} \Sigma_4}. \tag{16}$$

Proceeding in an identical manner as before, we obtain

$$\left\|\int_0^t e^{-i(t-s)A} L(|\psi(s)|^3 \psi(s))ds\right\|_{\mathbb{L}_T^q \mathbb{L}_x^r} \lesssim \left\||\psi|^3 L(\psi)\right\|_{\mathbb{L}_T^{\frac{20}{11}} \mathbb{L}_x^{\frac{5}{4}}},$$

$$\lesssim \|\psi\|^3_{\mathbb{L}_T^{30} \mathbb{L}_x^5} \|L(\psi)\|_{\mathbb{L}_T^{\frac{20}{9}} \mathbb{L}_x^5},$$

$$\lesssim T^{\frac{1}{10}} \|\psi\|^3_{\mathbb{L}_T^\infty \Sigma} \|L(\psi)\|_{\mathbb{L}_T^{\frac{20}{9}} \mathbb{L}_x^5},$$

and then we have

$$\left\|\int_0^t e^{-i(t-s)A} |\psi(s)|^3 \psi(s)ds\right\|_{\mathbb{L}_T^q \Sigma_r} \lesssim T^{\frac{1}{10}} \|\psi\|^3_{\mathbb{L}_T^\infty \Sigma} \|\psi\|_{\mathbb{L}_T^{\frac{20}{9}} \Sigma_5}. \tag{17}$$

Independently, applying the Hölder inequality leads to

$$\left\|\int_0^t e^{-i(t-s)A} L(|\psi(s)|^4 \psi(s))ds\right\|_{\mathbb{L}_T^q \mathbb{L}_x^r} \lesssim \left\||\psi|^4 L(\psi)\right\|_{\mathbb{L}_T^2 \mathbb{L}_x^{\frac{6}{5}}},$$

$$\lesssim \left\| \|\psi\|^4_{\mathbb{L}_x^{10}} \|L(\psi)\|_{\mathbb{L}_x^{\frac{30}{13}}} \right\|_{\mathbb{L}_T^2}.$$

Thanks to Lemma 2.3, the Sobolev space $W^{1,\frac{30}{13}}(\mathbb{R}^3)$is continuously embedded in $\mathbb{L}^{10}(\mathbb{R}^3)$. Thus

$$\left\|\int_0^t e^{-i(t-s)A} L(|\psi(s)|^4 \psi(s))ds\right\|_{\mathbb{L}_T^q\mathbb{L}_x^r} \lesssim \left\| \|\nabla\psi\|^4_{\mathbb{L}_x^{\frac{30}{13}}} \|L(\psi)\|_{\mathbb{L}_x^{\frac{30}{13}}} \right\|_{\mathbb{L}_T^2},$$

$$\lesssim \|\nabla\psi\|^4_{\mathbb{L}_T^{10}\mathbb{L}_x^{\frac{30}{13}}} \|L(\psi)\|_{\mathbb{L}_T^{10}\mathbb{L}_x^{\frac{30}{13}}}.$$

Consequently,

$$\left\|\int_0^t e^{-i(t-s)A} |\psi(s)|^4 \psi(s)ds\right\|_{\mathbb{L}_T^q\Sigma_r} \lesssim \|\nabla\psi\|^4_{\mathbb{L}_T^{10}\mathbb{L}_x^{\frac{30}{13}}} \|\psi\|_{\mathbb{L}_T^{10}\Sigma_{\frac{30}{13}}}. \tag{18}$$

For the nonlocal term, applying the Hölder inequality leads to

$$\left\|\int_0^t e^{-i(t-s)A} (K*|\psi(s)|^2)\psi(s)ds\right\|_{\mathbb{L}_T^q\mathbb{L}_x^r} \lesssim \|(K*|\psi(s)|^2)\psi\|_{\mathbb{L}_T^{\frac{8}{5}}\mathbb{L}_x^{\frac{4}{3}}},$$

$$\lesssim \|K*|\psi|^2\|_{\mathbb{L}_T^4\mathbb{L}_x^2} \|\psi\|_{\mathbb{L}_T^{\frac{8}{3}}\mathbb{L}_x^4},$$

from which and in accordance with Lemma 2.4 one has

$$\left\|\int_0^t e^{-i(t-s)A} (K*|\psi(s)|^2)\psi(s)ds\right\|_{\mathbb{L}_T^q\mathbb{L}_x^r} \lesssim \|\psi\|^2_{\mathbb{L}_T^8\mathbb{L}_x^4} \|\psi\|_{\mathbb{L}_T^{\frac{8}{3}}\mathbb{L}_x^4},$$

$$\lesssim T^{\frac{1}{4}} \|\psi\|^2_{\mathbb{L}_T^\infty\Sigma} \|\psi\|_{\mathbb{L}_T^{\frac{8}{3}}\mathbb{L}_x^4}. \tag{19}$$

Similarly,

$$\left\|\int_0^t e^{-i(t-s)A} x(K*|\psi(s)|^2)\psi(s)ds\right\|_{\mathbb{L}_T^q\mathbb{L}_x^r} \lesssim T^{\frac{1}{4}} \|\psi\|^2_{\mathbb{L}_T^\infty\Sigma} \|x\psi\|_{\mathbb{L}_T^{\frac{8}{3}}\mathbb{L}_x^4}. \tag{20}$$

Now observe that

$$\left\|\int_0^t e^{-i(t-s)A} \nabla[(K*|\psi(s)|^2)\psi(s)]ds\right\|_{\mathbb{L}_T^q\mathbb{L}_x^r}$$

$$\lesssim \|K*|\psi|^2\|_{\mathbb{L}_T^4\mathbb{L}_x^2} \|\nabla\psi\|_{\mathbb{L}_T^{\frac{8}{3}}\mathbb{L}_x^4} + \left\| \|K*\nabla(|\psi|^2)\|_{\mathbb{L}_x^2} \|\psi\|_{\mathbb{L}_x^4} \right\|_{\mathbb{L}_T^{\frac{8}{5}}}. \tag{21}$$

which, thanks again to the Hölder inequality and the Lemma 2.4, leads to

$$\left\|\int_0^t e^{-i(t-s)A}\nabla[(K*|\psi(s)|^2)\psi(s)]ds\right\|_{\mathbb{L}_T^q\mathbb{L}_x^r}$$

$$\lesssim \|\psi\|^2_{\mathbb{L}_T^8\mathbb{L}_x^4}\|\nabla\psi\|_{\mathbb{L}_T^{\frac{8}{3}}\mathbb{L}_x^4}+\|K*\nabla(|\psi|^2)\|_{\mathbb{L}^2_{T,x}}\|\psi\|_{\mathbb{L}_T^8\mathbb{L}_x^4}, \tag{22}$$

$$\lesssim \|\psi\|^2_{\mathbb{L}_T^8\mathbb{L}_x^4}\|\nabla\psi\|_{\mathbb{L}_T^{\frac{8}{3}}\mathbb{L}_x^4}. \tag{23}$$

Hence, in accordance with (19) and (20) it follows that

$$\left\|\int_0^t e^{-i(t-s)A}(K*|\psi(s)|^2)\psi(s)ds\right\|_{\mathbb{L}_T^q\Sigma_r}\lesssim T^{\frac{1}{4}}\|\psi\|^2_{\mathbb{L}_T^\infty\Sigma}\|\psi\|_{\mathbb{L}_T^{\frac{8}{3}}\Sigma_4}. \tag{24}$$

Denoting $\Phi(\psi)$ the right hand side of (15) and

$$\mathscr{I}=\left\{(+\infty,2),\ (\frac{8}{3},4),\ (\frac{20}{9},5),\ (10,\frac{30}{13})\right\},$$

then gathering the estimates (16), (17), (18) and (24) one has

$$\sup_{(q,r)\in\mathscr{I}}\|\Phi(\psi)\|_{\mathbb{L}_T^q\Sigma_r}$$

$$\le C_1\|\psi_0\|_\Sigma+C_2T^{\frac{1}{4}}\|\psi\|^2_{\mathbb{L}_T^\infty\Sigma}\|\psi\|_{\mathbb{L}_T^{\frac{8}{3}}\Sigma_4}$$

$$+C_3T^{\frac{1}{10}}\|\psi\|^3_{\mathbb{L}_T^\infty\Sigma}\|\psi\|_{\mathbb{L}_T^{\frac{20}{9}}\Sigma_5}+C_4\|\nabla\psi\|^4_{\mathbb{L}_T^{10}\mathbb{L}_x^{\frac{30}{13}}}\|\psi\|_{\mathbb{L}_T^{10}\Sigma_{\frac{30}{13}}}, \tag{25}$$

where $(C_i)_{1\le i\le 4}$ are nonnegative constants that depend only on α, β and γ.

Now Let $\eta>0$ and $R>0$. Then we consider the ball

$$\mathscr{B}=\{\psi,\|\psi\|_{\mathbb{X}_T}\le R\}\cap\{\psi,\|\nabla\psi\|_{\mathbb{L}_T^{10}\mathbb{L}_x^{\frac{30}{13}}}\le\eta\}.$$

According to (25), choosing $R=2C_1\|\psi_0\|_\Sigma$ then $\eta>0$ and $T>0$ small enough such that

$$C_4\eta^4R\le\frac{1}{4}\quad\text{and}\quad C_2T^{\frac{1}{4}}R^3+C_3T^{\frac{1}{10}}R^4\le\frac{1}{4},$$

ensure that Φ maps $\mathscr{B}$ into itself and the result of the current proposition yields by applying a fixed point argument.

4. Global-in-time solutions

For the purpose of proving the self-boundedness (global-in-time) of the the solutions, we may proceed by establishing the uniform (in time) control of the energy throughout some a priori estimates.

4.1 *Some a priori estimates*

Let $[0, T^*)$ be the maximal time interval on which the local solution u of (2)-(3), deduced from Proposition 3.1, is well-defined.

Lemma 4.1 Let ψ be a solution of (2) defined on the maximal interval $[0, T^*)$. Then we have

$$\|\psi(t)\|_{\mathbb{L}^2(\mathbb{R}^3)} \le \|\psi_0\|_{\mathbb{L}^2(\mathbb{R}^3)}, \ \forall\, t \in [0, T^*), \tag{26}$$

$$\int_0^{T^*}\int_{\mathbb{R}^3} |\psi(t,x)|^6\, dx dt \le C(\|u_0\|_{\mathbb{L}^2(\mathbb{R}^3)}). \tag{27}$$

Proof. The scalar product of (2) by $i\psi$ leads to

$$\frac{1}{2}\frac{d}{dt}\|\psi(t)\|^2_{\mathbb{L}^2(\mathbb{R}^3)} + \gamma\|\psi(t)\|^6_{\mathbb{L}^6} = 0.$$

Thus, for $t < T^*$

$$\|\psi(t)\|_{\mathbb{L}^2(\mathbb{R}^3)} \le \|\psi_0\|_{\mathbb{L}^2(\mathbb{R}^3)},$$

and then we deduce that

$$\int_0^{T^*} \|\psi(t)\|^6_{\mathbb{L}^6}\, dt \le \frac{\|\psi_0\|^2_{\mathbb{L}^2(\mathbb{R}^3)}}{2\gamma}.$$

Now in order to derive some a priori estimates on the energy of the solutions of (2)-(3) and due to the nonlinear damping, we follow [19]. For a given $\varrho > 0$, we consider the following modified energy functional:

$$E_\varrho(t) = \|A^{\frac{1}{2}}\psi(t)\|^2_{\mathbb{L}^2(\mathbb{R}^3)} + \frac{\alpha}{2}\|\psi(t)\|^4_{\mathbb{L}^4(\mathbb{R}^3)} + \frac{2\alpha}{5}\|\psi(t)\|^5_{\mathbb{L}^5(\mathbb{R}^3)} \tag{28}$$

$$+\frac{\varrho}{3}\|\psi(t)\|^6_{\mathbb{L}^6(\mathbb{R}^3)} + \frac{\beta}{2}\int_{\mathbb{R}^3}(K * |\psi(t)|^2)|\psi(t,x)|^2\, dx. \tag{29}$$

Lemma 4.2 Let $\alpha, \beta \in \mathbb{R}^*$ and let ψ be a solution of (2) defined on the maximal interval $[0, T^*)$ and $0 < \varrho \le \frac{\gamma}{2}$. Then there exists $C(\|\psi_0\|_\Sigma) > 0$ depending only on α, β, γ and $\|\psi_0\|_\Sigma$ such that

$$E_\varrho(t) \le C(\|\psi_0\|_\Sigma), \ \forall\, t \in [0, T^*). \tag{30}$$

Moreover, the following estimates hold

$$\int_0^{T^*}\int_{\mathbb{R}^3}|\psi(t,x)|^{10}\,dxdt \le C(\|\psi_0\|_\Sigma), \tag{31}$$

$$\int_0^{T^*}\int_{\mathbb{R}^3}(K*|\psi(t)|^2)|\psi(t,x)|^6\,dxdt \le C(\|\psi_0\|_\Sigma), \tag{32}$$

$$\int_0^{T^*}\int_{\mathbb{R}^3}|u(t,x)|^4|\nabla\psi(t,x)|^2\,dxdt \le C(\|\psi_0\|_\Sigma), \tag{33}$$

$$\int_0^{T^*}\int_{\mathbb{R}^3}|x|^2|\psi(t,x)|^6\,dxdt \le C(\|\psi_0\|_\Sigma). \tag{34}$$

Proof. To begin with, the scalar product of (2) by $(\psi_t+\gamma|\psi|^4\psi)$ leads to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{2}}\psi\|^2_{\mathbb{L}^2(\mathbb{R}^3)}+\lambda_1\frac{d}{dt}\left(\frac{1}{4}\|\psi\|^4_{\mathbb{L}^4(\mathbb{R}^3)}+\frac{1}{5}\|\psi\|^5_{\mathbb{L}^5(\mathbb{R}^3)}\right)\\&+\frac{\beta}{4}\frac{d}{dt}\int_{\mathbb{R}^3}(K*|\psi|^2)|\psi|^2\,dx+\gamma\alpha\left(\|\psi\|^8_{\mathbb{L}^8(\mathbb{R}^3)}+\|\psi\|^9_{\mathbb{L}^9(\mathbb{R}^3)}\right)\\&+\gamma(A\psi,\,|\psi|^4\psi)+\gamma\lambda_2\int_{\mathbb{R}^3}(K*|\psi(t)|^2)|\psi|^6\,dx=0.\end{aligned} \tag{35}$$

Independently, we infer from (2) that the scalar product of $\psi_t+\gamma|\psi|^4\psi$ by $|\psi|^4\psi$ gives

$$\frac{1}{2}\frac{d}{dt}\|\psi\|^6_{\mathbb{L}^6(\mathbb{R}^3)}+\gamma\|\psi\|^{10}_{\mathbb{L}^{10}(\mathbb{R}^3)}=4\int_{\mathbb{R}^3}|\psi|^2\,\Im m(\overline{\psi}\nabla\psi)\Re e(\overline{\psi}\nabla\psi)dx. \tag{36}$$

Hence, gathering (35) and (36) it follows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}E_\varrho(t)=&-\gamma\left(A^{\frac{1}{2}}(|\psi|^4\psi),\,A^{\frac{1}{2}}\psi\right)-\gamma\lambda_1\left(\|\psi\|^8_{\mathbb{L}^8(\mathbb{R}^3)}+\|\psi\|^9_{\mathbb{L}^9(\mathbb{R}^3)}\right)\\&-\gamma\beta\int_{\mathbb{R}^3}(K*|\psi|^2)|\psi|^6\,dx+4\varrho\int_{\mathbb{R}^3}|\psi|^2\,\Im m(\overline{\psi}\nabla\psi)\Re e(\overline{\psi}\nabla\psi)dx\\&-\gamma\varrho\|\psi\|^{10}_{\mathbb{L}^{10}(\mathbb{R}^3)}.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}E_\varrho(t)=&-\gamma\int_{\mathbb{R}^3}|\psi|^4|\nabla\psi|^2\,dx-\gamma\int_{\mathbb{R}^3}|x|^2|\psi|^6\,dx-\gamma\varrho\|\psi\|^{10}_{\mathbb{L}^{10}(\mathbb{R}^3)}\\&-\gamma\alpha\left(\|\psi\|^8_{\mathbb{L}^8(\mathbb{R}^3)}+\|\psi\|^9_{\mathbb{L}^9(\mathbb{R}^3)}\right)-\gamma\beta\int_{\mathbb{R}^3}(K*|\psi|^2)|\psi|^6\,dx\\&-4\gamma\|\psi\Re e(\overline{\psi}\nabla\psi)\|^2_{\mathbb{L}^2(\mathbb{R}^3)}+4\varrho\int_{\mathbb{R}^3}|\psi|^2\,\Re e(\overline{\psi}\nabla\psi)\Im m(\overline{\psi}\nabla\psi)dx.\end{aligned} \tag{37}$$

By the use of the following inequality

$$\int_{\mathbb{R}^3} |\psi|^2 |\Re e(\overline{\psi}\nabla\psi)| |\Im m(\overline{\psi}\nabla\psi)| dx \leq \frac{1}{2}\int_{\mathbb{R}^3} |\psi|^4 |\nabla\psi|^2 \, dx,$$

we infer from (37) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}E_\varrho(t) \leq &-(\gamma-2\varrho)\int_{\mathbb{R}^3}|\psi|^4|\nabla\psi|^2\,dx - \gamma\int_{\mathbb{R}^3}|x|^2|\psi|^6\,dx - \gamma\varrho\|\psi\|^{10}_{\mathbb{L}^{10}(\mathbb{R}^3)} \\ &+\gamma|\alpha|\left(\|\psi\|^8_{\mathbb{L}^8(\mathbb{R}^3)} + \|\psi\|^9_{\mathbb{L}^9(\mathbb{R}^3)}\right) + \gamma|\beta|\int_{\mathbb{R}^3}|K*|\psi|^2||\psi|^6\,dx \\ &-4\gamma\|\psi\Re e(\overline{\psi}\nabla\psi)\|^2_{\mathbb{L}^2(\mathbb{R}^3)}. \end{aligned} \tag{38}$$

Now we shall focus on the positive terms in the right hand side of (38):

$$\gamma|\alpha|\left(\|\psi\|^8_{\mathbb{L}^8(\mathbb{R}^3)} + \|\psi\|^9_{\mathbb{L}^9(\mathbb{R}^3)}\right) \text{ and } \gamma|\beta|\int_{\mathbb{R}^3}|K*|\psi|^2||\psi|^6\,dx.$$

For the first one, using interpolation argument we have

$$\|\psi\|^8_{\mathbb{L}^8(\mathbb{R}^3)} \leq \|\psi\|^3_{\mathbb{L}^6(\mathbb{R}^3)}\|\psi\|^5_{\mathbb{L}^{10}(\mathbb{R}^3)},$$

$$\|\psi\|^9_{\mathbb{L}^9(\mathbb{R}^3)} \leq \|\psi\|^{\frac{3}{2}}_{\mathbb{L}^6(\mathbb{R}^3)}\|\psi\|^{\frac{15}{2}}_{\mathbb{L}^{10}(\mathbb{R}^3)}.$$

This, with the use of the Young inequality, ensures that on the one hand

$$\|\psi\|^8_{\mathbb{L}^8(\mathbb{R}^3)} \leq C_0\|\psi\|^6_{\mathbb{L}^6(\mathbb{R}^3)} + \frac{\varrho}{4|\alpha|}\|\psi\|^{10}_{\mathbb{L}^{10}(\mathbb{R}^3)}, \tag{39}$$

and on the other hand

$$\|\psi\|^9_{\mathbb{L}^9(\mathbb{R}^3)} \leq \widetilde{C}_0\|\psi\|^6_{\mathbb{L}^6(\mathbb{R}^3)} + \frac{\varrho}{4(|\alpha|+|\beta|)}\|\psi\|^{10}_{\mathbb{L}^{10}(\mathbb{R}^3)}. \tag{40}$$

For the second one, thanks to the Hölder inequality,

$$\int_{\mathbb{R}^3}|K*|\psi|^2||\psi|^6\,dx \leq \|\psi\|^6_{\mathbb{L}^9(\mathbb{R}^3)}\|K*|\psi|^2\|_{\mathbb{L}^3(\mathbb{R}^3)},$$

which, thanks to Lemma 2.4, leads to

$$\int_{\mathbb{R}^3}|K*|\psi|^2||\psi|^6\,dx \leq C\|\psi\|^2_{\mathbb{L}^6(\mathbb{R}^3)}\|\psi\|^6_{\mathbb{L}^9(\mathbb{R}^3)},$$

and then

$$\int_{\mathbb{R}^3} |K*|\psi|^2\| \psi|^6 \, dx \le \|\psi\|_{\mathbb{L}^9(\mathbb{R}^3)}^9 + C_1 \|\psi\|_{\mathbb{L}^6(\mathbb{R}^3)}^6. \tag{41}$$

Hence, gathering (39), (40) and (41), we deduce from (38) that

$$\frac{1}{2}\frac{d}{dt}E_\varrho(t) \le -(\gamma-2\varrho)\int_{\mathbb{R}^3} |\psi|^4 |\nabla\psi|^2 \, dx - \gamma\int_{\mathbb{R}^3} |x|^2 |\psi|^6 \, dx$$

$$-\frac{\gamma\varrho}{2}\|\psi\|_{\mathbb{L}^{10}(\mathbb{R}^3)}^{10} - 4\gamma\|\psi \Re e(\overline{\psi}\nabla\psi)\|_{\mathbb{L}^2(\mathbb{R}^3)}^2 + C\|\psi\|_{\mathbb{L}^6(\mathbb{R}^3)}^6, \tag{42}$$

where $C = C(\gamma, \alpha, \beta) > 0$.

Choosing $0 < \varrho \le \frac{\gamma}{2}$ and in accordance with Lemma 4.1, we deduce from (42) that

$$\forall\, t \in [0, T^*),\ E_\varrho(t) \le C(\|\psi_0\|_\Sigma).$$

Moreover, the estimates (31), (32), (33) and (34) follows immediately and the proof is therefore achieved.

A straightforward consequence of the Lemma 4.2 states as follows.

Lemma 4.3 Let ψ be the solution of the initial value problem (2)-(3) defined on the maximal interval $[0, T^*)$. Then

$$\|\psi(t)\|_\Sigma \le C(\|\psi_0\|_\Sigma),\ \forall\, t \in [0, T^*).$$

Proof. On the one hand, using interpolation argument, one has

$$\|\psi\|_{\mathbb{L}^4(\mathbb{R}^3)}^4 \le \|\psi\|_{\mathbb{L}^2(\mathbb{R}^3)} \|\psi\|_{\mathbb{L}^6(\mathbb{R}^3)}^3,$$

$$\|\psi\|_{\mathbb{L}^5(\mathbb{R}^3)}^5 \le \|\psi\|_{\mathbb{L}^2(\mathbb{R}^3)}^{\frac{1}{2}} \|\psi\|_{\mathbb{L}^6(\mathbb{R}^3)}^{\frac{9}{2}}.$$

Then, thanks to the Young inequality, we obtain

$$|\alpha|\left(\frac{1}{2}\|\psi\|_{\mathbb{L}^4(\mathbb{R}^3)}^4 + \frac{2}{5}\|\psi\|_{\mathbb{L}^5(\mathbb{R}^3)}^5\right) \le C(\alpha)\|\psi\|_{\mathbb{L}^2(\mathbb{R}^3)}^2 + \frac{\varrho}{12}\|\psi\|_{\mathbb{L}^6(\mathbb{R}^3)}^6. \tag{43}$$

On the other hand, thanks to the Cauchy-Schwarz inequality and the Lemma 2.4, one has

$$\int_{\mathbb{R}^3} |K*|\psi|^2)|\psi|^2 \, dx \le \|\psi\|_{\mathbb{L}^4(\mathbb{R}^3)}^2 \|K*|\psi|^2\|_{\mathbb{L}^2(\mathbb{R}^3)}$$

$$\lesssim \|\psi\|_{\mathbb{L}^4(\mathbb{R}^3)}^4.$$

Hence

$$\frac{|\beta|}{2}\int_{\mathbb{R}^3} |K*|\psi|^2)|\psi|^2 \, dx \le C(\beta)\|\psi\|_{\mathbb{L}^2(\mathbb{R}^3)}^2 + \frac{\varrho}{12}\|\psi\|_{\mathbb{L}^6(\mathbb{R}^3)}^6. \tag{44}$$

Gathering (43) and 44, it follows from (29) that

$$E_\varrho(t) \geq \| A^{\frac{1}{2}}(\psi)(t)\|^2_{\mathbb{L}^2(\mathbb{R}^3)} - C\|\psi\|^2_{\mathbb{L}^2(\mathbb{R}^3)} + \frac{\varrho}{6}\|\psi\|^6_{\mathbb{L}^6(\mathbb{R}^3)},$$

$$\geq \| A^{\frac{1}{2}}(\psi)(t)\|^2_{\mathbb{L}^2(\mathbb{R}^3)} - C\|\psi\|^2_{\mathbb{L}^2(\mathbb{R}^3)},$$

which, thanks to the Lemma 2.3, Lemma 4.1 and the Lemma 4.2, achieves the proof.

4.2 *Proof of Theorem 1.1*

Let $T > 0$. Recalling that $L\psi$ either ψ, $x\psi$ or $\nabla\psi$ and then apply Lemma 2.2 to (15) it follows, thanks to the Hölder inequality, that

$$\|L\psi\|_{\mathbb{L}^q_T\mathbb{L}^r_x} \lesssim T^{\frac{1}{q}}\left(\|L(\psi_0)\|_{\mathbb{L}^2(\mathbb{R}^3)} + \|L(|\psi|^2\psi)\|_{\mathbb{L}^1_T\mathbb{L}^2_x} + \|L(|\psi|^3\psi)\|_{\mathbb{L}^{\frac{20}{11}}_T\mathbb{L}^{\frac{5}{4}}_x}\right)$$

$$+T^{\frac{1}{q}}\left(\|L(|\psi|^4\psi)\|_{\mathbb{L}^2_T\mathbb{L}^{\frac{6}{5}}_x} + \|L[(K*|\psi|^2)\psi]\|_{\mathbb{L}^1_T\mathbb{L}^2_x}\right).$$

On the one hand,

$$\|L(|\psi|^2\psi)\|_{\mathbb{L}^1_T\mathbb{L}^2_x} \lesssim T^{\frac{1}{q}+\frac{1}{2}}\|\psi\|^2_{\mathbb{L}^{10}_{T,x}}\|L(\psi)\|_{\mathbb{L}^{\frac{10}{3}}_{T,x}}, \tag{45}$$

similarly,

$$\|L(|\psi|^3\psi)\|_{\mathbb{L}^{\frac{20}{11}}_T\mathbb{L}^{\frac{5}{4}}_x} \lesssim T^{\frac{1}{q}+\frac{5}{11}}\|\psi\|^2_{\mathbb{L}^{10}_{T,x}}\|L(\psi)\|_{\mathbb{L}^{\frac{10}{3}}_{T,x}}, \tag{46}$$

and

$$\|L(|\psi|^4\psi)\|_{\mathbb{L}^2_T\mathbb{L}^{\frac{6}{5}}_x} \lesssim T^{\frac{1}{q}}\|\psi\|^4_{\mathbb{L}^{10}_{T,x}}\|L(\psi)\|_{\mathbb{L}^\infty_T\mathbb{L}^2_x}. \tag{47}$$

On the other hand,

$$\|L[(K*|\psi|^2)\psi]\|_{\mathbb{L}^1_T\mathbb{L}^2_x} \lesssim T^{\frac{1}{q}+\frac{1}{2}}\|\psi\|^2_{\mathbb{L}^{10}_{T,x}}\|L(\psi)\|_{\mathbb{L}^{\frac{10}{3}}_{T,x}}. \tag{48}$$

Hence, gathering (45), (46), (47) and (48), we obtain that

$$\| L\psi \|_{\mathbb{L}_T^q \mathbb{L}_x^r} \lesssim T^{\frac{1}{q}} \left(\| \psi_0 \|_{\Sigma} + T^{\frac{1}{2}} \| \psi \|_{\mathbb{L}_{T,x}^{10}}^2 \| L(\psi) \|_{\mathbb{L}_{T,x}^{\frac{10}{3}}} \right)$$

$$+ T^{\frac{1}{q}} \left(T^{\frac{5}{11}} \| \psi \|_{\mathbb{L}_{T,x}^{10}}^3 \| L(\psi) \|_{\mathbb{L}_T^{\infty} \mathbb{L}_x^2} + \| \psi \|_{\mathbb{L}_{T,x}^{10}}^4 \| L(\psi) \|_{\mathbb{L}_T^{10} \mathbb{L}_x^{\frac{30}{13}}} \right). \tag{49}$$

Consequently,

$$\sup_{(q,r)\in\mathscr{I}} \| \psi \|_{\mathbb{L}_T^q \Sigma_r} \lesssim \| \psi_0 \|_{\Sigma} + T^{\frac{1}{2}} \| \psi \|_{\mathbb{L}_{T,x}^{10}}^2 \sup_{(q,r)\in\mathscr{I}} \| \psi \|_{\mathbb{L}_T^q \Sigma_r}$$

$$+ T^{\frac{5}{11}} \| \psi \|_{\mathbb{L}_{T,x}^{10}}^3 \sup_{(q,r)\in\mathscr{I}} \| \psi \|_{\mathbb{L}_T^q \Sigma_r}$$

$$+ \| \psi \|_{\mathbb{L}_{T,x}^{10}}^4 \sup_{(q,r)\in\mathscr{I}} \| \psi \|_{\mathbb{L}_T^q \Sigma_r}. \tag{50}$$

Taking into consideration the results of Lemma 4.2, one can split the time interval $[0, T]$ as follows

$$[0, T] = \bigcup_{k=1}^{N} [t_k, t_{k+1}], \; t_1 = 0 \text{ and } t_{N+1} = T,$$

such that

$$\| \psi(t) \|_{\mathbb{L}^{10}([t_k, t_{k+1}], \mathbb{L}_x^{10})} \leq \epsilon,$$

for a given $\epsilon > 0$ small enough depending only on $\|\psi_0\|_{\Sigma}$.

In accordance with (50),

$$\sup_{(q,r)\in\mathscr{I}} \| \psi \|_{\mathbb{L}^q([t_1, t_2], \Sigma_r)} \lesssim \| \psi_0 \|_{\Sigma} + t_2^{\frac{1}{2}} \epsilon^2 \sup_{(q,r)\in\mathscr{I}} \| \psi \|_{\mathbb{L}^q([t_1, t_2], \Sigma_r)}$$

$$+ t_2^{\frac{5}{11}} \epsilon^3 \sup_{(q,r)\in\mathscr{I}} \| \psi \|_{\mathbb{L}^q([t_1, t_2], \Sigma_r)}$$

$$+ \epsilon^4 \sup_{(q,r)\in\mathscr{I}} \| \psi \|_{\mathbb{L}^q([t_1, t_2], \Sigma_r)}, \tag{51}$$

from which we deduce, by a continuity argument, that

$$\sup_{(q,r)\in\mathscr{I}} \| \psi \|_{\mathbb{L}^q([t_1, t_2], \Sigma_r)} \leq C(\| \psi_0 \|_{\Sigma}).$$

Similarly, one can obtain for each $1 \leq k \leq N$, that

$$\sup_{(q,r)\in\mathscr{I}} \|\psi\|_{\mathbb{L}^q([t_k,t_{k+1}],\Sigma_r)} \le C(\|\psi(t_k)\|_\Sigma),$$

independently of the length of the interval. Therefore we deduce that

$$\sup_{(q,r)\in\mathscr{I}} \|\psi\|_{\mathbb{L}^q_T\Sigma_r} \le C(\|\psi_0\|_\Sigma),$$

and the unique solution remains global in time. Since this bound is independent of $T > 0$ the solution is uniformly bounded and the main result of this paper yields.

5. Conclusion

In this article we have mathematically succeeded, through some combined technics, to provide a clear demonstration to highlight the self-bounded state criterion of these quantum droplets under the LHY effect, modeled by the critical nonlinear damping in (2). Despite the difficulty caused by this critical damping, we have proved that, under its influence, all the existent solutions for (2) are global in time and uniformly bounded. Hence, the collapse phenomenon is arrested and blow-up does not occur. This is consistent with either the theoretical predictions or the experimental results in the recent researches under the thematic area "Quantum Droplets".

Our work, with its novelty and its originality, opens intriguing perspectives for the exploration of the different characteristics of these solutions, namely their asymptotic dynamics under the influence of the confinement potential which will be the aim of a future work.

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Conflict of interest

The author declare that there is no conflicting of interests.

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