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Research Article

A Note on Hesitant Fuzzy Approximation Spaces and Hesitant Fuzzy Topological Spaces

Shizhan Lu[1], Yuanyuan Liu[2], Zhu Fu[3], Longsheng Cheng[4*]

[1] School of Management, Jiangsu University, Zhengjiang 212013, China
[2] Department of Public Teaching, Zhengzhou Health Vocational College, Zhengzhou 450000, China
[3] School of Economics and Management, Jiangsu University of Science and Technology, Zhenjiang 212100, China
[4] School of Economics and Management, Nanjing University of Science and Technology, Nanjing 210094, China
E-mail: chenglongshengnj@163.com

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Abstract: A hesitant fuzzy element is defined as a discrete array characterized by complexity and diversity. Consequently, certain inadequate propositions have been identified in the existing literature on hesitant fuzzy sets. The development of a mathematical knowledge system relies on well-established theoretical foundations; even minor deficiencies in these foundations can undermine the integrity of subsequent knowledge frameworks. Topology and approximation space theory serve as essential pillars for a broad range of theoretical constructs. To rectify the flawed propositions in hesitant fuzzy approximation spaces and hesitant fuzzy topological spaces, this study proposes new intersection and union operators for hesitant fuzzy sets and refines the fundamental propositions of hesitant fuzzy sets under these operators. By presenting the inadequate propositions from the literature on these two mathematical structures and replacing the original intersection and union operations with the newly defined ones, all previously identified inadequacies can be rigorously addressed. Consequently, the theories derived from these propositions are also rigorously valid.

***Keywords*:** hesitant fuzzy set, hesitant fuzzy topological space, hesitant fuzzy approximation space

MSC: 03E72, 54A40, 41A65

1. Introduction

The fuzzy set, introduced by Zadeh [1], is a well-established tool for handling uncertainty with various applications, including fuzzy clustering [2, 3], discriminant analysis [4–6], neuro-fuzzy inference [7], classification [8], tracking [9, 10], and clean energy transformation analysis [11]. Following the advent of fuzzy sets, other branches of uncertainty mathematics, such as interval-valued fuzzy sets [12, 13], intuitionistic fuzzy sets [14, 15], and hesitant fuzzy sets [16–18], have also experienced rapid development.

Hesitant fuzzy sets, in which membership degrees are expressed as discrete arrays, demonstrate a superior capability for representing real-world indecisiveness compared with fuzzy sets, interval-valued fuzzy sets, and intuitionistic fuzzy sets. Hesitant fuzzy sets have been extensively applied in a wide range of fields, including decision-making [19–21], attribute reduction [22], classification [23], linguistic perception [24], diagnosing [25], programming [26], image segmentation [27], and forecasting [28], among others.

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A hesitant fuzzy element [29] is defined as a discrete array, exhibiting inherent complexity and diversity. Consequently, several propositions reported in the literature on the mathematical structures of hesitant fuzzy sets have been shown to exhibit certain inadequacies. The construction of a coherent mathematical knowledge system fundamentally depends on sound and well-established theoretical foundations. Even minor deficiencies in these foundational theories may undermine the consistency and reliability of subsequently developed knowledge frameworks. Theoretical research on hesitant fuzzy sets has progressed substantially over recent years. If these theoretical foundations are affected by propositions with inadequacies, a significant portion of existing research may be rendered invalid, thereby diminishing the value of prior scholarly efforts. Accordingly, this study focuses on revising certain operators of hesitant fuzzy sets to validate the inadequate propositions and to promote the stable development of the hesitant fuzzy set framework.

Topology [30] is recognized as a fundamental discipline that forms the core of contemporary mathematical theory. The applications of topology span a wide range of fields, such as precision feeding systems [31], object detection and localization [32], and structural analysis [33]. The theory of approximation spaces has been widely applied in decision-making processes [13, 34], neural network architectures [35], and deep learning methodologies [36]. Collectively, these two mathematical frameworks provide an essential foundation for the development of numerous related theoretical systems.

Given the fundamental importance of topology and approximation space theories, this study investigates certain propositions in hesitant fuzzy topology and hesitant fuzzy approximation spaces that exhibit inadequacies. To this end, we re-examine the fundamental propositions of hesitant fuzzy sets and find that, under the original intersection and union operators, hesitant fuzzy sets fail to satisfy many important basic properties, such as the absorption law and the distributive law. Owing to these deficiencies in the fundamental propositions, the theories derived from them are likewise inadequate.

To address these deficiencies in the fundamental propositions, we propose new intersection and union operators for hesitant fuzzy sets. Under the proposed operators, the hesitant fuzzy sets satisfy important basic properties, such as the absorption law and the distributive law. Furthermore, it is found that once the new intersection and union operators replace the original ones, the previously flawed theorems also become rigorously valid. Consequently, the theorems derived from these propositions are likewise preserved. This approach enables the preservation of existing research achievements in hesitant fuzzy topological space and hesitant fuzzy approximation space. Furthermore, the knowledge systems established on hesitant fuzzy sets can continue to develop in a robust and internally consistent manner.

This article focuses on the following tasks:

(1) Identify the cause of the deficiencies in the current hesitant fuzzy set knowledge system, i.e., the original intersection and union operators fail to satisfy some fundamental properties of hesitant fuzzy sets, such as the absorption law and the distributive law.

(2) Propose new intersection and union operators for hesitant fuzzy sets.

(3) Refine the fundamental propositions of hesitant fuzzy sets under the new operators.

(4) Correct the flawed theorems in hesitant fuzzy approximation spaces and hesitant fuzzy topological spaces.

2. Foundations of hesitant fuzzy sets

This section reviews some foundations of hesitant fuzzy sets.

2.1 *Reviews of fuzzy sets and hesitant fuzzy sets*

Definition 1 [1] A fuzzy set F on U is a mapping $F : U \to [0,1]$.

Definition 2 [1] F_1 and F_2 are two fuzzy sets on U, F_1 is a subset of F_2 if $F_1(x) \leqslant F_2(x)$ for all $x \in U$, denoted $F_1 \subset F_2$.

Definition 3 [29] A hesitant fuzzy element is a non-empty, finite subset of $[0,1]$.

Definition 4 [16] A hesitant fuzzy set on U is defined as a function that, when applied to U, returns a subset of $[0,1]$.

In the following, $HF(U)$ denotes the set of all hesitant fuzzy sets defined over U.

Definition 5 [16] For each $x \in U$, and a hesitant fuzzy set H, the lower bound and upper bound of $H(x)$ are defined lower bound $H^-(x) = inf\ H(x)$, upper bound $H^+(x) = sup\ H(x)$.

Definition 6 [16] Given two hesitant fuzzy sets represented by their membership functions H_1 and H_2, their union and intersection are defined

$$\text{union } (H_1 \cup H_2)(x) = \{h \in H_1(x) \cup H_2(x) : h \geqslant sup(H_1^-(x), H_2^-(x))\},$$

$$\text{intersection } (H_1 \cap H_2)(x) = \{h \in H_1(x) \cup H_2(x) : h \leqslant inf(H_1^+(x), H_2^+(x))\}.$$

Example 1 Let $U = \{x\}$, $H_1(x) = \{0.3, 0.3, 0.6\}$ and $H_2(x) = \{0.5, 0.8\}$.

$$H_1^-(x) = 0.3, H_2^-(x) = 0.5, \max\{H_1^-(x), H_2^-(x)\} = 0.5. (H_1 \cup H_2)(x) = \{0.5, 0.6, 0.8\}.$$

$$H_1^+(x) = 0.6, H_2^+(x) = 0.8, \min\{H_1^+(x), H_2^+(x)\} = 0.6. (H_1 \cap H_2)(x) = \{0.3, 0.3, 0.5, 0.6\}.$$

Definition 7 [16] For $x \in U$ and $H \in HF(U)$, the complement of H is denoted H^c, where

$$H^c(x) = \cup_{\gamma \in H(x)} \{1 - \gamma\}.$$

2.2 *Inclusion definitions of hesitant fuzzy sets*

For two hesitant fuzzy sets H_1 and H_2, it has been defined in the references [37–41] that H_1 is a hesitant fuzzy subset of H_2 if $H_1(x) \subset H_2(x)$ holds for all $x \in U$. Nonetheless, the references [37–41] do not provide a clear elucidation of the condition $H_1(x) \subset H_2(x)$. For instance, given $H_1(x) = \{0.1, 0.9\}$ and $H_2(x) = \{0.6, 0.8\}$, it remains unclear how to ascertain whether $H_1(x) \subset H_2(x)$ or $H_2(x) \subset H_1(x)$ based on the values $\{0.1, 0.9\}$ and $\{0.6, 0.8\}$.

The hesitant fuzzy element encompasses multiple values, rendering a single inclusion relationship inadequate for comprehensively characterizing the diverse interrelations among hesitant fuzzy sets. It is imperative to recognize that multiple inclusion relationships exist among various hesitant fuzzy sets. Lu and Xu et al. [18] introduced several categories of inclusion relationships among different hesitant fuzzy sets and illustrated these relationships through a specific example (as demonstrated in Example 2).

Example 2 [18] Let $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ be a set of decision-making schemes and H be an expert team that consists of three experts. $H(U) = \frac{\{0.9,0.2\}}{x_1} + \frac{\{0.6,0.6,0.5\}}{x_2} + \frac{\{0.7,0.5,0.5\}}{x_3} + \frac{\{0.8,0.6,0.5\}}{x_4} + \frac{\{0.9,0.3,0.1\}}{x_5} + \frac{\{0.9,0.8,0.7\}}{x_6}$ are the estimated values of schemes provided by experts, in which the estimated values for x_1 are 0.9 and 0.2 that are obtained through the evaluations made by experts. One of the three experts fails to evaluate the scheme x_1.

(1) Here, x_1 has an estimated value of 0.9, which is greater than or equal to all the estimated values for x_2, and thus it is possible that the scheme x_1 is better than the scheme x_2, denoted as $H(x_2) \subset_p H(x_1)$.

(2) Here, $0.55 = \text{mean}[H(x_1)] < \text{mean}[H(x_2)] = 0.567$, where mean$[\cdot]$ is the mean value operator. In comparing the mean values of estimated values, we find that the scheme x_2 is better than the scheme x_1, denoted as $H(x_1) \subset_m H(x_2)$.

(3) The best estimated value of x_3 is 0.7, which is greater than or equal to the best estimated value of x_2. On the other hand, the worst estimated value of x_3 is 0.5, which is greater than or equal to the worst estimated value of x_2. To compare the respective best and worst cases of schemes x_2 and x_3, it is acceptable that the scheme x_3 is better than the scheme x_2, denoted as $H(x_2) \subset_a H(x_3)$.

(4) To compare the estimated values for schemes x_3 and x_4 one by one ($0.7 \leqslant 0.8$, $0.5 \leqslant 0.6$, and $0.5 \leqslant 0.5$), it is strongly credible that the scheme x_4 is better than the scheme x_3, denoted as $H(x_3) \subset_s H(x_4)$.

(5) We can obtain $H(x_1) \subset_s H(x_5)$ after truncating the tail estimated value of x_5, i.e., deleting the estimated value 0.1 of x_5. This case is denoted as $H(x_1) \subset_{st} H(x_5)$ and is recorded briefly as $H(x_1) \subset_t H(x_5)$.

(6) The worst estimated value of x_6 is greater than or equal to the best estimated value of x_3. Thus, it is necessary that the scheme x_6 is better than the scheme x_3, which is denoted as $H(x_3) \subset_n H(x_6)$.

Definition 8 [18] Let H_1 and H_2 be two hesitant fuzzy sets on U. Several kinds of inclusion relationships of two hesitant fuzzy sets are defined as follows,

(1) If $H_1^+(x) \leqslant H_2^+(x)$, then $H_1(x) \subset_p H_2(x)$. If $H_1(x) \subset_p H_2(x)$ for all $x \in U$, then $H_1 \subset_p H_2$. If $H_1 \subset_p H_2$ and $H_2 \subset_p H_1$, then $H_1 =_p H_2$.

(2) If $H_1^+(x) \leqslant H_2^+(x)$ and $H_1^-(x) \leqslant H_2^-(x)$, then $H_1(x) \subset_a H_2(x)$. If $H_1(x) \subset_a H_2(x)$ for all $x \in U$, then $H_1 \subset_a H_2$. If $H_1 \subset_a H_2$ and $H_2 \subset_a H_1$, then $H_1 =_a H_2$.

(3) If $\text{mean}[H_1(x)] \leqslant \text{mean}[H_2(x)]$, then $H_1(x) \subset_m H_2(x)$. If $H_1(x) \subset_m H_2(x)$ for all $x \in U$, then $H_1 \subset_m H_2$. If $H_1 \subset_m H_2$ and $H_2 \subset_m H_1$, then $H_1 =_m H_2$.

(4) Let $H_1(x) = V = \{v_1, v_2, \cdots, v_k\}$ and $H_2(x) = W = \{w_1, w_2, \cdots, w_l\}$ be two descending sequences. If $k \geqslant l$ and $w_i \geqslant v_i$ for $1 \leqslant i \leqslant l$, then $H_1(x) \subset_s H_2(x)$. If $H_1(x) \subset_s H_2(x)$ for all $x \in U$, then $H_1 \subset_s H_2$. If $H_1 \subset_s H_2$ and $H_2 \subset_s H_1$, then $H_1 =_s H_2$.

(5) Let $H_1(x) = V = \{v_1, v_2, \cdots, v_k\}$ and $H_2(x) = W = \{w_1, w_2, \cdots, w_l\}$ be two descending sequences. If $k < l$ and $w_i \geqslant v_i$ for $1 \leqslant i \leqslant k$, then $H_1(x) \subset_t H_2(x)$. If $H_1(x) \subset_t H_2(x)$ for all $x \in U$, then $H_1 \subset_t H_2$. It is obvious that $H_1 \subset_t H_2$ and $H_2 \subset_t H_1$ cannot hold simultaneously.

(6) If $H_1^+(x) \leqslant H_2^-(x)$, then $H_1(x) \subset_n H_2(x)$. If $H_1(x) \subset_n H_2(x)$ for all $x \in U$, then $H_1 \subset_n H_2$. If $H_1 \subset_n H_2$ and $H_2 \subset_n H_1$, then $H_1 =_n H_2$.

3. New union and intersection operators of hesitant fuzzy sets

The hesitant fuzzy set does not adhere to the absorption law and the distribution law, as shown in Example 3 and 4. Certain literature has employed the absorption and distribution laws, resulting in inadequacies in existing propositions. The evolution of the mathematical knowledge system is contingent upon the research findings of preceding generations. Should these prior results contain inadequacies, it may precipitate the collapse of the knowledge system. This section seeks to implement minimal adjustments to the operations of hesitant fuzzy sets to rectify the current propositions with inadequacies. This approach aims to facilitate the stable development of hesitant fuzzy sets and prevent the disintegration of the knowledge system.

3.1 *Absorption law and distributive law based on traditional union and intersection operators*

Hesitant fuzzy sets do not adhere to the absorption law or the distributive law. Two examples illustrating this assertion are presented in Example 3 and Example 4.

Example 3 Let $U = \{x\}$, $A = \frac{0.3,0.6}{x}$ and $B = \frac{0.55,0.7}{x}$.

$A \cup B = \frac{0.55,0.6,0.7}{x}$, $(A \cup B) \cap A = \frac{0.3,0.55,0.6,0.6}{x}$. $0.55 \in ((A \cup B) \cap A)(x)$, however, $0.55 \notin A(x)$. Hence, $(A \cup B) \cap A \neq A$.

$A \cap B = \frac{0.3,0.55,0.6}{x}$, $(A \cap B) \cup A = \frac{0.3,0.3,0.55,0.6,0.6}{x}$. $0.55 \in ((A \cap B) \cup A)(x)$, however, $0.55 \notin A(x)$. Hence, $(A \cap B) \cup A \neq A$.

Example 4 Let $U = \{x\}$, $A = \frac{\{0.1,0.35\}}{x}$, $B = \frac{\{0.55,0.7\}}{x}$, $C = \frac{\{0.3,0.6\}}{x}$.

$((A \cap B) \cup C)(x) = \{0.3, 0.35, 0.6\}$, $((A \cup C) \cap (B \cup C))(x) = \{0.3, 0.35, 0.55, 0.6, 0.6\}$.

$0.55 \notin ((A \cap B) \cup C)(x)$, however, $0.55 \in ((A \cup C) \cap (B \cup C))(x)$, then $(A \cap B) \cup C \neq (A \cup C) \cap (B \cup C)$.

$((A \cup B) \cap C)(x) = \{0.3, 0.55, 0.6\}$, $((A \cap C) \cup (B \cap C))(x) = \{0.3, 0.3, 0.35, 0.55, 0.6\}$.

$0.35 \notin ((A \cup B) \cap C)(x)$, however, $0.35 \in ((A \cap C) \cup (B \cap C))(x)$, then $(A \cup B) \cap C \neq (A \cap C) \cup (B \cap C)$.

3.2 *New intersection and union operations, and the propositions derived therefrom*

Deepak [42] considered adjusting the intersection and union operators, and presented the following candidate method.

Definition 9 [42] For two hesitant fuzzy sets H_1 and H_2 on U, the intersection and union operators are defined as follows:

$$(H_1 \sqcup H_2)(x) = H_2(x), \text{if } \text{mean}(H_1(x)) \leqslant \text{mean}(H_2(x)) \text{ for } x \in U,$$

$$(H_1 \sqcap H_2)(x) = H_1(x), \text{if } \text{mean}(H_1(x)) \leqslant \text{mean}(H_2(x)) \text{ for } x \in U,$$

where $\text{mean}(H(x)) = \frac{\sum h \in H(x)}{|H(x)|}$.

Definition 9 only takes the information of one of the two sets, losing the feature of combining the information of the two sets through intersection and union operations. We give new intersection and union operators as follows.

Definition 10 Let H_1 and H_2 be two hesitant fuzzy sets on U. For a given $x \in U$, if $|H_1(x)| < |H_2(x)|$, the value $\text{mean}(H_1(x))$ is utilized to populate the array $H_1(x)$, ensuring that $|H_1(x)| = |H_2(x)|$. Subsequently, the arrays $H_1(x) = \{a_1, a_2, \ldots, a_l\}$ and $H_2(x) = \{b_1, b_2, \ldots, b_l\}$ are arranged in ascending order. The intersection and union operators are defined as follows:

$$(H_1 \barcup H_2)(x) = \{a_1 \vee b_1, a_2 \vee b_2, \cdots, a_l \vee b_l\},$$

$$(H_1 \barcap H_2)(x) = \{a_1 \wedge b_1, a_2 \wedge b_2, \cdots, a_l \wedge b_l\}.$$

Note that, (i) Definition 10 is a candidate method and should not be interpreted as the sole approach; (ii) Definition 10 neither negates nor conflicts with the original intersection and union operations.

Example 5 Let $U = \{x, y\}$, $H_1 = \frac{0.3,0.5}{x} + \frac{0.2,0.3,0.5}{y}$ and $H_2 = \frac{0.3,0.5,0.6}{x} + \frac{0.4,0.6}{y}$.

The mean values are employed to populate $H_1(x)$ and $H_2(y)$, yielding $H_1(x) = \{0.3, 0.4, 0.5\}$ and $H_2(y) = \{0.4, 0.5, 0.6\}$.

$H_1 \barcup H_2 = \frac{0.3\vee0.3,0.4\vee0.5,0.5\vee0.6}{x} + \frac{0.2\vee0.4,0.3\vee0.5,0.5\vee0.6}{y} = \frac{0.3,0.5,0.6}{x} + \frac{0.4,0.5,0.6}{y}$.

$H_1 \barcap H_2 = \frac{0.3,0.4,0.5}{x} + \frac{0.2,0.3,0.5}{y}$.

Definition 11 Let H_1 and H_2 be two hesitant fuzzy sets on U. For a given $x \in U$, if $|H_1(x)| < |H_2(x)|$, the value $\text{mean}(H_1(x))$ is utilized to populate the array $H_1(x)$, ensuring that $|H_1(x)| = |H_2(x)|$. Subsequently, the arrays $H_1(x) = \{a_1, a_2, \ldots, a_l\}$ and $H_2(x) = \{b_1, b_2, \ldots, b_l\}$ are arranged in ascending order. If $a_i \leqslant b_i$ for $1 \leqslant i \leqslant l$, then $H_1(x) \bar{\subset} H_2(x)$. If $H_1(x) \bar{\subset} H_2(x)$ for all $x \in U$, then $H_1 \bar{\subset} H_2$.

If $H_1 \bar{\subset} H_2$ and $H_2 \bar{\subset} H_1$, then $H_1 = H_2$.

In Example 5, the sets H_1 and H_2 satisfy the relation $H_1 \bar{\subset} H_2$.

For the hesitant fuzzy set after the mean value filling, it is considered equivalent to the set prior to the mean value filling. For instance, $H_2 = \frac{0.3,0.5,0.6}{x} + \frac{0.4,0.6}{y} = \frac{0.3,0.5,0.6}{x} + \frac{0.4,\overline{0.5},0.6}{y}$. Then, we will obtain the following propositions.

Proposition 1 For $A, B, C \in HF(U)$, the following statements hold.

(1) $(A^c)^c = A$.

(2) $A \barcup A = A$ and $A \barcap A = A$.

(3) $A \barcup B = B \barcup A$ and $A \barcap B = B \barcap A$.

(4) $A \bar{\subset} A \cup B$ and $B \bar{\subset} A \cup B$.

(5) $A \barcap B \bar{\subset} A$ and $A \barcap B \bar{\subset} B$.

(6) $(A \barcup B) \barcup C = A \barcup (B \barcup C)$ and $(A \barcap B) \barcap C = A \barcap (B \barcap C)$.

(7) $(A \barcup B) \barcap A = A$ and $(A \barcap B) \barcup A = A$.

(8) $(A \barcup B) \barcap C = (A \barcap C) \barcup (B \barcap C)$ and $(A \barcap B) \barcup C = (A \barcup C) \barcap (B \barcup C)$.

(9) $(A \barcup B)^c = A^c \barcap B^c$ and $(A \barcap B)^c = A^c \barcup B^c$.

Proof.

(1)–(6) are obvious.

(7) For $x \in U$, denoted $A(x) = \{a_1, a_2, \cdots, a_l\}$ and $B(x) = \{b_1, b_2, \cdots, b_l\}$. Since $(a_i \vee b_i) \wedge a_i = a_i$ for $1 \leqslant i \leqslant l$, then $((A \barcup B) \barcap A)(x) = A(x)$. Since $(a_i \wedge b_i) \vee a_i = a_i$ for $1 \leqslant i \leqslant l$, then $((A \barcap B) \barcup A)(x) = A(x)$.

(8) For $x \in U$, denoted $A(x) = \{a_1, a_2, \cdots, a_l\}$, $B(x) = \{b_1, b_2, \cdots, b_l\}$ and $C(x) = \{c_1, c_2, \cdots, c_l\}$. Since $(a_i \vee b_i) \wedge c_i = (a_i \wedge c_i) \vee (b_i \wedge c_i)$ for $1 \leqslant i \leqslant l$, then $((A \barcup B) \barcap C)(x) = ((A \barcap C) \barcup (B \barcap C))(x)$. Furthermore, since $(a_i \wedge b_i) \vee c_i = (a_i \vee c_i) \wedge (b_i \vee c_i)$ for $1 \leqslant i \leqslant l$, then $((A \barcap B) \barcup C)(x) = ((A \barcup C) \barcap (B \barcup C))(x)$.

(9) For $x \in U$, denoted $A(x) = \{a_1, a_2, \cdots, a_l\}$ and $B(x) = \{b_1, b_2, \cdots, b_l\}$. Since $1 - (a_i \vee b_i) = (1 - a_i) \wedge (1 - b_i)$ for $1 \leqslant i \leqslant l$, then $((A \barcup B)^c)(x) = (A^c \barcap B^c)(x)$. Since $1 - (a_i \wedge b_i) = (1 - a_i) \vee (1 - b_i)$ for $1 \leqslant i \leqslant l$, then $((A \barcap B)^c)(x) = (A^c \barcup B^c)(x)$. □

Proposition 2 For $A, B, C, D \in HF(U)$, the following statements hold.

(1) If $A \bar{\subset} B$, then $A \barcup B = B$ and $A \barcap B = A$.

(2) If $A \bar{\subset} B$, then $A \barcup C \bar{\subset} B \barcup C$ and $A \barcap C \bar{\subset} B \barcap C$.

(3) If $A \bar{\subset} B$ and $C \bar{\subset} D$, then $A \barcup C \bar{\subset} B \barcup D$ and $A \barcap C \bar{\subset} B \barcap D$.

(4) If $A \bar{\subset} C$ and $B \bar{\subset} C$, then $A \barcup B \bar{\subset} C$.

(5) If $C \bar{\subset} A$ and $C \bar{\subset} B$, then $C \bar{\subset} A \barcap B$.

Proof.

(1) is obvious.

For a random $x \in U$, denoted $A(x) = \{a_1, a_2, \cdots, a_l\}$, $B(x) = \{b_1, b_2, \cdots, b_l\}$, $C(x) = \{c_1, c_2, \cdots, c_l\}$ and $D(x) = \{d_1, d_2, \cdots, d_l\}$.

(2) Since $A \bar{\subset} B$, then $a_i \leqslant b_i$ for $1 \leqslant i \leqslant l$, then $a_i \vee c_i \leqslant b_i \vee c_i$ for $1 \leqslant i \leqslant l$, i.e., $(A \barcup C)(x) \bar{\subset} (B \barcup C)(x)$. On the other hand, $a_i \wedge c_i \leqslant b_i \wedge c_i$ for $1 \leqslant i \leqslant l$, i.e., $(A \barcap C)(x) \bar{\subset} (B \barcap C)(x)$.

(3) Since $A \bar{\subset} B$ and $C \bar{\subset} D$, then $a_i \leqslant b_i$ and $c_i \leqslant d_i$ for $1 \leqslant i \leqslant l$, then $a_i \vee c_i \leqslant b_i \vee d_i$ for $1 \leqslant i \leqslant l$, i.e., $(A \barcup C)(x) \bar{\subset} (B \barcup D)(x)$. On the other hand, $a_i \wedge c_i \leqslant b_i \wedge d_i$ for $1 \leqslant i \leqslant l$, then $(A \barcap C)(x) \bar{\subset} (B \barcap D)(x)$.

(4) Since $A \bar{\subset} C$ and $B \bar{\subset} C$, then $a_i \leqslant c_i$ and $b_i \leqslant c_i$ for $1 \leqslant i \leqslant l$, then $a_i \vee b_i \leqslant c_i$ for $1 \leqslant i \leqslant l$, i.e., $(A \barcup B)(x) \bar{\subset} C(x)$.

(5) Since $C \bar{\subset} A$ and $C \bar{\subset} B$, then $c_i \leqslant a_i$ and $c_i \leqslant b_i$ for $1 \leqslant i \leqslant l$, then $c_i \leqslant a_i \wedge b_i$ for $1 \leqslant i \leqslant l$, i.e., $C(x) \bar{\subset} (A \barcap B)(x)$. □

The symbols used for propositions in sections 4 and 5 are those from the cited references.

4. Hesitant fuzzy approximation spaces

This section discusses several propositions related to hesitant fuzzy approximation spaces, which exhibit inadequacies under the original intersection and union operations. When the new intersection and union operations proposed in this study are used to replace the original ones, these previous propositions with inadequacies become valid, thereby preventing the spread of misunderstandings within the theoretical framework of hesitant fuzzy approximation space.

4.1 *Lower and upper approximations based on original operators of union and intersection*

The references [43–45] present the propositions $\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B)$ and $\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$. Our views on these propositions differ, as illustrated in Examples 6 and 7. By substituting the original intersection and union operators with the new intersection and union operators proposed in this study, all of these propositions are valid, as shown in Proposition 3.

Example 6 Let $U = \{x_1, x_2\}$, $A = \frac{\{0.3,0.6\}}{x_1} + \frac{\{0.3,0.5\}}{x_2}$ and $B = \frac{\{0.4,0.7\}}{x_1} + \frac{\{0.55,0.6\}}{x_2}$. $R(U,U)$ is shown as follows,

$$R(U,U) = \begin{bmatrix} \{1\} & \{0.3,0.4,0.7\} \\ \{0.3,0.4,0.7\} & \{1\} \end{bmatrix}.$$

$A\cap B=\frac{\{0.3,0.4,0.6\}}{x_1}+\frac{\{0.3,0.5\}}{x_2}$.

$\underline{R}(A\cap B)(x_1)=\bigcap\limits_{y\in U}[R^c(x_1,y)\cup(A\cap B)(y)]=[\{0\}\cup\{0.3,0.4,0.6\}]\cap[\{0.3,0.6,0.7\}\cup\{0.3,0.5\}]=\{0.3,0.4,0.6\}\cap\{0.3,0.3,0.5,0.6,0.7\}=\{0.3,0.3,0.3,0.4,0.5,0.6,0.6\}$.

$\underline{R}(A)(x_1)=\bigcap\limits_{y\in U}[R^c(x_1,y)\cup A(y)]=[\{0\}\cup\{0.3,0.6\}]\cap[\{0.3,0.6,0.7\}\cup\{0.3,0.5\}]=\{0.3,0.6\}\cap\{0.3,0.3,0.5,0.6,0.7\}=\{0.3,0.3,0.3,0.5,0.6,0.6\}$.

$\underline{R}(B)(x_1)=\bigcap\limits_{y\in U}[R^c(x_1,y)\cup B(y)]=[\{0\}\cup\{0.4,0.7\}]\cap[\{0.3,0.6,0.7\}\cup\{0.55,0.6\}]=\{0.4,0.7\}\cap\{0.55,0.6,0.6,0.7\}=\{0.4,0.55,0.6,0.6,0.7,0.7\}$.

$(\underline{R}(A)\cap\underline{R}(B))(x_1)=\{0.3,0.3,0.3,0.5,0.6,0.6\}\cap\{0.4,0.55,0.6,0.6,0.7,0.7\}=\{0.3,0.3,0.3,0.4,0.5,0.55,0.6,0.6,0.6,0.6\}$.

$0.55\in(\underline{R}(A)\cap\underline{R}(B))(x_1)$ and $0.55\notin\underline{R}(A\cap B)(x_1)$, then $(\underline{R}(A)\cap\underline{R}(B))(x_1)\neq\underline{R}(A\cap B)(x_1)$. Hence, $\underline{R}(A)\cap\underline{R}(B)\neq\underline{R}(A\cap B)$.

Example 7 Let $U=\{x_1,x_2\}$, $A=\frac{\{0.3,0.6\}}{x_1}+\frac{\{0.3,0.45\}}{x_2}$ and $B=\frac{\{0.4,0.7\}}{x_1}+\frac{\{0.5,0.6\}}{x_2}$. $R(U,U)$ is shown as follows,

$$R(U,U)=\begin{bmatrix}\{1\} & \{0.3,0.4,0.7\}\\ \{0.3,0.4,0.7\} & \{1\}\end{bmatrix}.$$

$A\cup B=\frac{\{0.4,0.6,0.7\}}{x_1}+\frac{\{0.5,0.6\}}{x_2}$.

$\overline{R}(A\cup B)(x_1)=\bigcup\limits_{y\in U}[R(x_1,y)\cap(A\cup B)(y)]=[\{1\}\cap\{0.4,0.6,0.7\}]\cup[\{0.3,0.4,0.7\}\cap\{0.5,0.6\}]=\{0.4,0.6,0.7\}\cup\{0.3,0.4,0.5,0.6\}=\{0.4,0.4,0.5,0.6,0.6,0.7\}$.

$\overline{R}(A)(x_1)=\bigcup\limits_{y\in U}[R(x_1,y)\cap A(y)]=[\{1\}\cap\{0.3,0.6\}]\cup[\{0.3,0.4,0.7\}\cap\{0.3,0.45\}]=\{0.3,0.6\}\cup\{0.3,0.3,0.4,0.45\}=\{0.3,0.3,0.3,0.4,0.45,0.6\}$.

$\overline{R}(B)(x_1)=\bigcup\limits_{y\in U}[R(x_1,y)\cap B(y)]=[\{1\}\cap\{0.4,0.7\}]\cup[\{0.3,0.4,0.7\}\cap\{0.5,0.6\}]=\{0.4,0.7\}\cup\{0.3,0.4,0.5,0.6\}=\{0.4,0.4,0.5,0.6,0.7\}$.

$(\overline{R}(A)\cup\overline{R}(B))(x_1)=\{0.3,0.3,0.3,0.4,0.45,0.6\}\cup\{0.4,0.4,0.5,0.6,0.7\}=\{0.4,0.4,0.4,0.45,0.5,0.6,0.6,0.7\}$.

$0.45\in(\overline{R}(A)\cup\overline{R}(B))(x_1)$ and $0.45\notin\overline{R}(A\cup B)(x_1)$, then $(\overline{R}(A)\cup\overline{R}(B))(x_1)\neq\overline{R}(A\cup B)(x_1)$. Hence, $\overline{R}(A)\cup\overline{R}(B)\neq\overline{R}(A\cup B)$.

4.2 *Lower and upper approximations based on new operators of union and intersection*

Definition 12 Let U be a nonempty and finite universe of discourse. A hesitant fuzzy relation $R:(x,y)\to R(x,y)$ on U is a mapping that transforms the binary relation of (x,y) into a hesitant fuzzy element, where $R(x,y)$ is a hesitant fuzzy element for $x,y\in U$.

Definition 13 Let U be a nonempty and finite universe of discourse and R be a hesitant fuzzy relation. For $A\in HF(U)$, the lower and upper approximations of A, denoted by $\underline{R}(A)$ and $\overline{R}(A)$ respectively, defined as $\underline{R}(A)=\{\frac{\underline{R}(A)(x)}{x}:x\in U\}$ and $\overline{R}(A)=\{\frac{\overline{R}(A)(x)}{x}:x\in U\}$, where

$$\begin{cases}\underline{R}(A)(x)=\bar{\cap}_{y\in U}\{R^c(x,y)\bar{\cup}A(y)\},\\ \overline{R}(A)(x)=\bar{\cup}_{y\in U}\{R(x,y)\bar{\cap}A(y)\}.\end{cases}$$

And (U,R) is called a hesitant fuzzy approximation space.

Example 8 Let $U=\{x_1,x_2,x_3\}$ and $A=\frac{\{0.3,0.8\}}{x_1}+\frac{\{0.5,0.6\}}{x_2}+\frac{\{0.3,0.7\}}{x_3}$. $R(U,U)$ is shown as follows,

$$R(U,U)=\begin{bmatrix}\{1\} & \{0.3,0.5\} & \{0.4,0.7\}\\ \{0.3,0.5\} & \{1\} & \{0.6,0.9\}\\ \{0.4,0.7\} & \{0.6,0.9\} & \{1\}\end{bmatrix}.$$

$\underline{R}(A)(x_1)=\{\{0,0\}\cup\{0.3,0.8\}\}\cap\{\{0.5,0.7\}\cup\{0.5,0.6\}\}\cap\{\{0.3,0.6\}\cup\{0.3,0.7\}\}=\{0.3,0.8\}\cap\{0.5,0.7\}\cap\{0.3,0.7\}=\{0.3,0.7\}$

$\underline{R}(A)(x_2)=\{0.3,0.6\}$ and $\underline{R}(A)(x_3)=\{0.3,0.6\}$.

Then, $\underline{R}(A)=\frac{\{0.3,0.7\}}{x_1}+\frac{\{0.3,0.6\}}{x_2}+\frac{\{0.3,0.6\}}{x_3}$.

$\overline{R}(A)(x_1)=\{\{1,1\}\cap\{0.3,0.8\}\}\cup\{\{0.3,0.5\}\cap\{0.5,0.6\}\}\cup\{\{0.4,0.7\}\cap\{0.3,0.7\}\}=\{0.3,0.8\}\cup\{0.3,0.5\}\cup\{0.3,0.7\}=\{0.3,0.8\}$

$\overline{R}(A)(x_2)=\{0.5,0.7\}$ and $\overline{R}(A)(x_3)=\{0.5,0.7\}$.

Then, $\overline{R}(A)=\frac{\{0.3,0.8\}}{x_1}+\frac{\{0.5,0.7\}}{x_2}+\frac{\{0.5,0.7\}}{x_3}$.

Proposition 3 Let (U,R) be a hesitant fuzzy approximation space, $A,B\in HF(U)$. The following statements hold, where $H^{\emptyset}(x)=\{0\}$ and $H^{U}(x)=\{1\}$ for all $x\in U$.

(1) $\overline{R}(H^{\emptyset})=H^{\emptyset}$ and $\underline{R}(H^{U})=H^{U}$.

(2) If $A\tilde{\subset}B$, then $\underline{R}(A)\tilde{\subset}\underline{R}(B)$ and $\overline{R}(A)\tilde{\subset}\overline{R}(B)$.

(3) $\underline{R}(A^c)=\overline{R}(A)^c$ and $\underline{R}(A)^c=\overline{R}(A^c)$.

(4) $\underline{R}(A)\cup\underline{R}(B)\tilde{\subset}\underline{R}(A\cup B)$.

(5) $\overline{R}(A\cap B)\tilde{\subset}\overline{R}(A)\cap\overline{R}(B)$.

(6) $\underline{R}(A\cap B)=\underline{R}(A)\cap\underline{R}(B)$.

(7) $\overline{R}(A\cup B)=\overline{R}(A)\cup\overline{R}(B)$.

(8) If $R(x,x)=\{1\}$ for all $x\in U$, then $\underline{R}(A)\tilde{\subset}A\tilde{\subset}\overline{R}(A)$.

(9) If $R(x,x)=\{1\}$ for all $x\in U$, then $\underline{R}(H^{\emptyset})=\overline{R}(H^{\emptyset})=H^{\emptyset}$ and $\underline{R}(H^{U})=\overline{R}(H^{U})=H^{U}$.

Proof.

(1) It is obvious.

(2) Since $A\tilde{\subset}B$, by (2) of Proposition 2, we have $R^c(x,y)\cup A(y)\tilde{\subset}R^c(x,y)\cup B(y)$ and $R(x,y)\cap A(y)\tilde{\subset}R(x,y)\cap B(y)$, then $\underline{R}(A)\tilde{\subset}\underline{R}(B)$ and $\overline{R}(A)\tilde{\subset}\overline{R}(B)$ are obvious.

(3) By (9) of Proposition 1, $\cap_{y\in U}\{R^c(x,y)\cup A^c(y)\}=\cap_{y\in U}\{[R(x,y)\cap A(y)]^c\}=[\cup_{y\in U}\{R(x,y)\cap A(y)\}]^c$, then $\underline{R}(A^c)=\overline{R}(A)^c$.

$\underline{R}(A)^c=\overline{R}(A^c)$ can be proved by the similar way above.

(4) $A\tilde{\subset}A\cup B$ and $B\tilde{\subset}A\cup B$, by the result of (2), we have $\underline{R}(A)\tilde{\subset}\underline{R}(A\cup B)$ and $\underline{R}(B)\tilde{\subset}\underline{R}(A\cup B)$. By (4) of Proposition 2, $\underline{R}(A)\cup\underline{R}(B)\tilde{\subset}\underline{R}(A\cup B)$.

(5) By (5) of Proposition 2, it can be proved by the similar way of (4) above.

(6) By (8) of Proposition 1, $\underline{R}(A\cap B)(x)=\cap_{y\in U}\{R^c(x,y)\cup(A\cap B)(y)\}=\cap_{y\in U}\{[R^c(x,y)\cup(A)(y)]\cap[R^c(x,y)\cup(B)(y)]\}=[\cap_{y\in U}\{R^c(x,y)\cup A(y)\}]\cap[\cap_{y\in U}\{R^c(x,y)\cup B(y)\}]=\underline{R}(A)(x)\cap\underline{R}(B)(x)$.

(7) It can be proved by the similar way of (6).

(8) Since $R(x,x)=\{1\}$, then $R^c(x,x)=\{0\}$. $\underline{R}(A)(x)=\cap_{y\in U}\{R^c(x,y)\cup A(y)\}$
$=[R^c(x,x)\cup A(x)]\cap[\cap_{y\in U,y\neq x}\{R^c(x,y)\cup A(y)\}]=A(x)\cap[\cap_{y\in U,y\neq x}\{R^c(x,y)\cup A(y)\}]\tilde{\subset}A(x)$.

$A(x)=[\{R(x,x)\cap A(x)]\tilde{\subset}[\{R(x,x)\cap A(x)]\cup[\cup_{y\in U,y\neq x}\{R(x,y)\cap A(y)\}]=\overline{R}(A)(x)$.

(9) By the result of (8), $\underline{R}(H^{\emptyset})\tilde{\subset}H^{\emptyset}$, then $\underline{R}(H^{\emptyset})=H^{\emptyset}$; $H^{U}\tilde{\subset}\overline{R}(H^{U})$, then $H^{U}=\overline{R}(H^{U})$.

By the result of (1), $\underline{R}(H^{\emptyset})=\overline{R}(H^{\emptyset})=H^{\emptyset}$ and $\underline{R}(H^{U})=\overline{R}(H^{U})=H^{U}$. □

5. Hesitant fuzzy topological spaces

This section randomly selects several propositions concerning hesitant fuzzy topologies, which exhibit inadequacies under the original intersection and union operations. When the new intersection and union operations proposed in this study replace the original ones, these previous propositions with inadequacies are rendered valid, thereby preventing the proliferation of misinterpretations within the theoretical framework of hesitant fuzzy topology. For this purpose, Subsection 5.1 first discusses the empty hesitant fuzzy set and full hesitant fuzzy set, while Subsection 5.2 examines the construction of hesitant fuzzy topologies under the new intersection and union operations.

5.1 *Empty hesitant fuzzy set and full hesitant fuzzy set*

Within the current literature on hesitant fuzzy sets, two distinct definitions for the empty hesitant fuzzy set H^0 and the full hesitant fuzzy set H^1 in hesitant fuzzy topological spaces are identified: (i) $H^0(x) = \{0\}$ and $H^1(x) = \{1\}$ for all $x \in U$ [16, 41]; (ii) $H^0(x) = \emptyset$ and $H^1(x) = [0,1]$ for all $x \in U$ [38–40, 46]. The references [38–40, 46] have cited either [16] or [41] to derive the definitions of the empty hesitant fuzzy set and the full hesitant fuzzy set. However, the two definitions in [38–40, 46] differ from those in [16, 41].

First, we review the constructions of classical sets [47] and fuzzy sets [1, 48–52]. Consider the universe of discourse defined as $U = \{x_1, x_2, x_3\}$, wherein two classical sets, such as $A = \{x_1\}$ and $B = \{x_2, x_3\}$, can be represented in the form of fuzzy sets, specifically $A = \{\frac{1}{x_1}, \frac{0}{x_2}, \frac{0}{x_3}\}$ and $B = \{\frac{0}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}\}$. The representation $A = \{\frac{1}{x_1}, \frac{0}{x_2}, \frac{0}{x_3}\}$ signifies that the probability of x_1 being a member of set A is 100%, while the probabilities for x_2 and x_3 in relation to set A are both 0%. The range of probability values, initially confined to $\{0,1\}$, is extended to the interval $[0,1]$, thereby yielding a fuzzy set, such as $F = \{\frac{0.3}{x_1}, \frac{0.8}{x_2}, \frac{0.5}{x_3}\}$. In contrast to fuzzy sets, hesitant fuzzy sets incorporate a characteristic of hesitation, such as $H = \{\frac{0.3,0.5}{x_1}, \frac{0.8,0.9}{x_2}, \frac{0.5,0.6,0.8}{x_3}\}$. When the elements within $H(x)$ converge to identical values, the hesitant fuzzy sets reduce to traditional fuzzy sets, such as $H = \{\frac{0,0}{x_1}, \frac{1,1,1}{x_2}, \frac{0.8,0.8}{x_3}\} = \{\frac{0}{x_1}, \frac{1}{x_2}, \frac{0.8}{x_3}\}$.

The expression $H^0(x) = \emptyset$ signifies that the probability domain for x being a member of H^0 is indeed empty, rendering any discussion regarding the membership of x in H^0 meaningless. $H^1(x) = [0,1]$ implies that it is complete ignorance for $x \in H^1$ (all probabilities are possible) [16]. The condition $H^1(x) = \{1\}$ for every $x \in U$ indicates that the probability of x belonging to H^1 is unequivocally 100% across the entire set U. Conversely, the representation $H^1(x) = [0,1]$ denotes that the probability of x belonging to H^1 is characterized by a random value within the interval $[0,1]$. The definitions provided in [16, 41] are deemed to be more reasonable compared to the definitions in [38–40, 46].

5.2 *Constructing hesitant fuzzy topology*

Even for two identical hesitant fuzzy sets, applying original intersection or union operations (Definition 6) to them can generate a new hesitant fuzzy set. This implies that any hesitant fuzzy topology closed under such original operations must necessarily contain infinitely many elements, thereby rendering the construction of hesitant fuzzy topologies based on these operations particularly challenging. However, under the new intersection and union operations (Definition 10) proposed in this study, two identical hesitant fuzzy sets do not generate a new hesitant fuzzy set. These newly defined operations thus significantly facilitate the construction of hesitant fuzzy topologies.

Definition 14 Let U be a nonempty set and $\tau \subset HF(U)$. τ is called a hesitant fuzzy topology if it satisfies the following axioms:

(1) $H^0, H^1 \in \tau$.

(2) If $H, G \in \tau$, then $H \barwedge G \in \tau$.

(3) If $H_\alpha \in \tau$ for $\alpha \in \Gamma$, then $\barvee_{\alpha\in\Gamma} H_\alpha \in \tau$.

Example 9 Let $U = \{x, y\}$. Given two hesitant fuzzy sets $H_1 = \frac{0.2,0.5}{x} + \frac{0.6,0.7}{y}$ and $H_2 = \frac{0.3,0.5,0.6}{x} + \frac{0.5,0.8}{y}$, it is easy to construct a hesitant fuzzy topology τ, shown as follows.

(1) $|H_1(x)| = 2 \neq 3 = |H_2(x)|$ and $\text{mean}(H_1(x)) = 0.35$, filling the mean value 0.35 into $H_1(x)$ to obtain $H_1 = \frac{0.2,0.35,0.5}{x} + \frac{0.6,0.7}{y}$.

(2) Let $H_3 = H_1 \widetilde{\cap} H_2 = \frac{0.2\wedge 0.3, 0.35\wedge 0.5, 0.5\wedge 0.6}{x} + \frac{0.6\wedge 0.5, 0.7\wedge 0.8}{y} = \frac{0.2, 0.35, 0.5}{x} + \frac{0.5, 0.7}{y}$. Then, $H_3 = H_1 \widetilde{\cap} H_2$ must be an element of τ.

(3) Let $H_4 = H_1 \widetilde{\cup} H_2 = \frac{0.2\vee 0.3, 0.35\vee 0.5, 0.5\vee 0.6}{x} + \frac{0.6\vee 0.5, 0.7\vee 0.8}{y} = \frac{0.3, 0.5, 0.6}{x} + \frac{0.6, 0.8}{y}$. Then, $H_4 = H_1 \widetilde{\cup} H_2$ must be an element of τ.

(4) $H^0 = \frac{0}{x} + \frac{0}{y}$ and $H^1 = \frac{1}{x} + \frac{1}{y}$ are two elements of τ.

(5) To verify the conditions (2) and (3) in Definition 14 for $\{H^0, H^1, H_1, H_2, H_3, H_4\}$, $H \widetilde{\cap} G \in \tau$ for all $H, G \in \tau$, and $\widetilde{\cup}_{\alpha\in\Gamma} H_\alpha \in \tau$ for all $H_\alpha \in \tau$ and $\alpha \in \Gamma$, then $\tau = \{H^0, H^1, H_1, H_2, H_3, H_4\}$ can be a hesitant fuzzy topology that includes the elements H_1 and H_2.

5.3 *Inferences based on intersection operation of hesitant fuzzy sets*

Sections 5.3–5.5 discuss several propositions with inadequacies. Subsequently, by replacing the original intersection and union operations with the newly proposed ones, these propositions become valid.

Case 1 Definition 2 in [38] does not delineate a specific type of inclusion relationship. Consequently, it is difficult to ascertain whether the proposition satisfies the conditions outlined by the definition. The (HFN3) of Proposition 6 in [38] states that "if $N, M \in HFN(x_\lambda)$, then $N \widetilde{\cap} M \in HFN(x_\lambda)$". This proposition is not valid for the inclusion relationship "$\subset_m$", shown in Example 10. By substituting the original intersection and union operators with the new intersection and union operators, the (HFN3) of Proposition 6 in [38] will be valid, as illustrated in Proposition 4.

Example 10 Let $U = \{x, y\}$, $H_1 = \frac{0.2, 0.8}{x} + \frac{0.2, 0.5}{y}$, $H_2 = \frac{0.3, 0.7}{x} + \frac{0.3, 0.6}{y}$, $x_\lambda = \frac{0.4, 0.6}{x} + \frac{0}{y}$, and $H_1, H_2 \in \tau$.

$\text{mean}(\{0.4, 0.6\}) \leqslant \text{mean}(\{0.2, 0.8\})$ and $\text{mean}(\{0.4, 0.6\}) \leqslant \text{mean}(\{0.3, 0.7\})$, then $x_\lambda \in H_1 \subset_m H_1 \in \tau$ and $x_\lambda \in H_2 \subset_m H_2 \in \tau$, i.e., $H_1, H_2 \in HFN(x_\lambda)$.

$H_1 \widetilde{\cap} H_2 = \frac{0.2, 0.3, 0.7}{x} + \frac{0.2, 0.3, 0.5}{y}$. Since $\text{mean}(\{0.4, 0.6\}) \not\leqslant \text{mean}(\{0.2, 0.3, 0.7\})$, then $x_\lambda \notin H_1 \widetilde{\cap} H_2$. For all $H \in \tau$, if $H \subset_m H_1 \widetilde{\cap} H_2$ (i.e., $\text{mean}(H(x)) \leqslant \text{mean}((H_1 \widetilde{\cap} H_2)(x))$), then $x_\lambda \notin H$, i.e., $H_1 \widetilde{\cap} H_2 \notin HFN(x_\lambda)$.

Proposition 4 If $N, M \in HFN(x_\lambda)$, then $N \widetilde{\cap} M \in HFN(x_\lambda)$.

Proof. If $N, M \in HFN(x_\lambda)$, then there are $U, V \in \tau$ such that $x_\lambda \in U \tilde{\subset} N$ and $x_\lambda \in V \tilde{\subset} M$.

x_λ can be denoted as a hesitant fuzzy set H^*, where $H^*(x) = \lambda$ and $H^*(y) = \{0\}$ for $y \in U - \{x\}$. Then, $H^* \tilde{\subset} U$ and $H^* \tilde{\subset} V$. By the result of (5) in Proposition 2, $H^* \tilde{\subset} V$ and $H^* \tilde{\subset} U$, then $H^* \tilde{\subset} U \widetilde{\cap} V$, i.e., $x_\lambda \in U \widetilde{\cap} V$.

By the result of (3) in Proposition 2, if $U \tilde{\subset} N$ and $V \tilde{\subset} M$, then $U \widetilde{\cap} V \tilde{\subset} N \widetilde{\cap} M$. Then, $x_\lambda \in U \widetilde{\cap} V \tilde{\subset} N \widetilde{\cap} M$ and $U \widetilde{\cap} V \in \tau$. Therefore, $N \widetilde{\cap} M \in HFN(x_\lambda)$. □

Case 2 The proof of Proposition 3.3 in [53] states that if $h \subset int_H(h^*)$ then $h \cap k \subset int_H(h^*) \cap k$. However, the reference [53] does not specify any particular type of inclusion relationship. The proposition that if $A \subset_m B$ then $A \cap C \subset_m B \cap C$ is not valid for the inclusion relationship "$\subset_m$", shown in Example 11. By substituting the original intersection and union operators with the new intersection and union operators, the Proposition 3.3 in [53] will be valid, as shown in Proposition 5.

Example 11 Let $A(x) = \{0.5, 0.6\}$, $B(x) = \{0.2, 0.95\}$ and $C(x) = \{0.6\}$.

$A(x) \subset_m B(x)$, however, $\{0.5, 0.6, 0.6\} = A(x) \cap C(x) \not\subset_m B(x) \cap C(x) = \{0.2, 0.6\}$.

Proposition 5 Let (X, τ, L) be a hesitant fuzzy ideal topological space and $h, k \in HFS(X)$. If $h \in IO(X, \tau)$, then $h \widetilde{\cap} k \in IO(X, \tau)$.

Proof. Since $h \in IO(X, \tau)$, then $h \tilde{\subset} \text{int}_H(h^*)$. By (2) of Proposition 2, $h \widetilde{\cap} k \tilde{\subset} \text{int}_H(h^*) \cap k$.

$k \in \tau$ implies $k = \text{int}_H(k)$. Then, $h \widetilde{\cap} k \tilde{\subset} \text{int}_H(h^*) \cap int_H(k) = \text{int}_H(h^* \widetilde{\cap} k)$. Since $A^* \widetilde{\cap} B \tilde{\subset} (A \widetilde{\cap} B)^*$, then $h^* \widetilde{\cap} k \tilde{\subset} (h \widetilde{\cap} k)^*$, then $int_H(h^* \widetilde{\cap} k) \tilde{\subset} \text{int}_H(h \widetilde{\cap} k)^*$.

$h \widetilde{\cap} k \tilde{\subset} \text{int}_H(h^* \cap k) \tilde{\subset} \text{int}_H(h \widetilde{\cap} k)^*$, then $h \widetilde{\cap} k \in IO(X, \tau)$. □

5.4 *Inferences based on union operation of hesitant fuzzy sets*

Definition 2 in [40] does not delineate a specific type of inclusion relationship. Consequently, it is difficult to ascertain whether the proposition satisfies the conditions outlined by the definition.

The proof of Theorem 3 in [40] uses a proposition that if $\widetilde{\cup}_{i\in N} \text{int}_H(cl_H(h_i)) \subset \text{int}_H(\widetilde{\cup}_{i\in N} cl_H(h_i))$ and $\widetilde{\cup}_{i\in N} cl_H(\text{int}_H(h_i)) \subset cl_H(\widetilde{\cup}_{i\in N} \text{int}_H(h_i))$ then $[\widetilde{\cup}_{i\in N} \text{int}_H(cl_H(h_i))] \widetilde{\cup} [\widetilde{\cup}_{i\in N} cl_H(\text{int}_H(h_i))] \subset [\text{int}_H(\widetilde{\cup}_{i\in N} cl_H(h_i))] \widetilde{\cup} [cl_H(\widetilde{\cup}_{i\in N} \text{int}_H(h_i))]$. This

proposition is not valid for the inclusion relationship "$\subset_m$", shown in Example 12. The original intersection and union operators are substituted by the new ones proposed in this study. Theorem 3 in [40] will be valid, as shown in Proposition 6.

Example 12 Let $\widetilde{\cup}_{i\in N}\,\mathrm{int}_H(cl_H(h_i))(x)=\{0.4,0.5\}$, $\mathrm{int}_H(\widetilde{\cup}_{i\in N}cl_H(h_i))(x)=\{0.1,0.9\}$, $\widetilde{\cup}_{i\in N}cl_H(\mathrm{int}_H(h_i))(x)=\{0.1,0.7\}$ and $cl_H(\widetilde{\cup}_{i\in N}\,\mathrm{int}_H(h_i))(x)=\{0.1,0.8\}$.

$\widetilde{\cup}_{i\in N}\,\mathrm{int}_H(cl_H(h_i))(x)\subset_m \mathrm{int}_H(\widetilde{\cup}_{i\in N}cl_H(h_i))(x)$ and $\widetilde{\cup}_{i\in N}cl_H(\mathrm{int}_H(h_i))(x)\subset_m cl_H(\widetilde{\cup}_{i\in N}\,\mathrm{int}_H(h_i))(x)$. However, $[\widetilde{\cup}_{i\in N}\,\mathrm{int}_H(cl_H(h_i))]\widetilde{\cup}[\widetilde{\cup}_{i\in N}cl_H(\mathrm{int}_H(h_i))](x)\not\subset_m[\mathrm{int}_H(\widetilde{\cup}_{i\in N}cl_H(h_i))]\widetilde{\cup}[cl_H(\widetilde{\cup}_{i\in N}\,\mathrm{int}_H(h_i))](x)$.

Proposition 6 If $\{h_i : i\in N\}$ is a collection of hesitant fuzzy b-open (resp. hesitant fuzzy α-open, hesitant fuzzy preopen, hesitant fuzzy semiopen and hesitant fuzzy β-open) sets of a hesitant fuzzy topological space (X,τ), then $\bar{\cup}_{i\in N}h_i$ is a hesitant fuzzy b-open (resp. hesitant fuzzy α-open, hesitant fuzzy preopen, hesitant fuzzy semiopen and hesitant fuzzy β-open) set.

Proof. According to statement (3) of Proposition 2, if $A\bar{\subset}B$ and $C\bar{\subset}D$, then $A\bar{\cup}C\bar{\subset}B\bar{\cup}D$.

Since $h_i\bar{\subset}\mathrm{int}_H(cl_H(h_i))\bar{\cup}cl_H(\mathrm{int}_H(h_i))$ for $i\in N$, then

$$\bar{\cup}_{i\in N}h_i\bar{\subset}\bar{\cup}_{i\in N}[\mathrm{int}_H(\mathrm{cl}_H(h_i))\bar{\cup}\mathrm{cl}_H(\mathrm{int}_H(h_i))]\bar{\subset}[\bar{\cup}_{i\in N}\,\mathrm{int}_H(\mathrm{cl}_H(h_i))]\bar{\cup}[\bar{\cup}_{i\in N}\,\mathrm{cl}_H(\mathrm{int}_H(h_i))]$$

$$\bar{\subset}[\mathrm{int}_H(\bar{\cup}_{i\in N}\,\mathrm{cl}_H(h_i))]\bar{\cup}[\mathrm{cl}_H(\bar{\cup}_{i\in N}\,\mathrm{int}_H(h_i))]\bar{\subset}[\mathrm{int}_H(\mathrm{cl}_H(\bar{\cup}_{i\in N}h_i))]\bar{\cup}[\mathrm{cl}_H(\mathrm{int}_H(\bar{\cup}_{i\in N}h_i))].$$

Then, $\bar{\cup}_{i\in N}h_i$ is hesitant fuzzy b-open.

The proofs of other cases are similar to the proof of the case of hesitant fuzzy b-open. □

5.5 *Inferences based on distributive law of hesitant fuzzy sets*

The proof of Theorem 6 in [40] uses the distributive law of hesitant fuzzy sets. Example 4 shows that hesitant fuzzy sets do not satisfy the distributive law based on the original intersection and union operators. Theorem 6 in [40] will be valid after substituting the original intersection and union operators with the new intersection and union operators proposed in this study, as illustrated in Proposition 7.

Proposition 7 Let (X,τ) be a hesitant fuzzy topological space, $h_U\in\tau$ and $h_A\in HS(X)$. If h_A is hesitant fuzzy b-open, then $h_U\bar{\cap}h_A$ is hesitant fuzzy b-open.

Proof. By the result of (8) in Proposition 1, the hesitant fuzzy sets satisfy the distributive law based on the new intersection and union operators ($\bar{\cap}$ and $\bar{\cup}$).

Since h_A is a is hesitant fuzzy b-open, then

$$\begin{aligned}h_U\bar{\cap}h_A\bar{\subset}h_U\bar{\cap}[\mathrm{int}_H(\mathrm{cl}_H(h_A))\bar{\cup}\mathrm{cl}_H(\mathrm{int}_H(h_A))]&=[h_U\bar{\cap}\mathrm{int}_H(\mathrm{cl}_H(h_A))]\bar{\cup}[h_U\bar{\cap}\mathrm{cl}_H(\mathrm{int}_H(h_A))]\\&=[\mathrm{int}_H(h_U)\bar{\cap}\mathrm{int}_H(\mathrm{cl}_H(h_A))]\bar{\cup}[h_U\bar{\cap}\mathrm{cl}_H(\mathrm{int}_H(h_A))]\\&\bar{\subset}[\mathrm{int}_H(h_U\bar{\cap}\mathrm{cl}_H(h_A))]\bar{\cup}[\mathrm{cl}_H(h_U\bar{\cap}\mathrm{int}_H(h_A))]\\&\bar{\subset}[\mathrm{int}_H(\mathrm{cl}_H(h_U\bar{\cap}h_A))]\bar{\cup}[\mathrm{cl}_H(\mathrm{int}_H(h_U\bar{\cap}h_A))].\end{aligned}$$

Then, $h_U\bar{\cap}h_A$ is hesitant fuzzy b-open. □

6. Conclusion

Topology serves as the foundational pillar of contemporary mathematics. Numerous other mathematical theoretical systems have emerged as extensions of topology. Furthermore, approximation space theory constitutes a significant foundational knowledge system within the field of computer science.

Although significant advancements have been made in the theoretical frameworks of hesitant fuzzy approximation space and hesitant fuzzy topology, some propositions within these frameworks reveal inadequacies. The future development of mathematical knowledge systems is contingent upon prior research outcomes, and any inadequacies in existing propositions may jeopardize the integrity and reliability of subsequent advancements. In order to protect established theoretical achievements from being compromised by such imperfections and to ensure the continued robust evolution of hesitant fuzzy approximation spaces and hesitant fuzzy topology, this study tries to make minor revisions to the operators of hesitant fuzzy sets. These revisions are designed to ensure that all presently inadequate propositions are rigorously validated. It is shown that by substituting the original intersection and union operations with the modified ones proposed in this study, the presently identified propositions with inadequacies can be rigorously validated. As a result, all prior research findings are preserved, allowing these knowledge systems to continue their development in a stable and sustainable manner.

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Conflict of interest

The authors declare no competing financial interest.

References

[1] Zadeh LA. Fuzzy sets. *Information and control*. 1965; 8: 338–353. Available from: https://doi.org/10.1016/S0019-9958(65)90241-X.

[2] Hu C, Xu H, Fu Z, Wu B, Zhang R, Zhi C. Non-destructive Identification of the geographical origin of red jujube by near-infrared spectroscopy and fuzzy clustering methods. *International Journal of Food Properties*. 2023; 26(2): 3275–3290. Available from: https://doi.org/10.1080/10942912.2023.2281883.

[3] Wu X, Zhou J, Wu B, Sun J, Dai C. Identification of tea varieties by mid-infrared diffuse reflectance spectroscopy coupled with a possibilistic fuzzy c-means clustering with a fuzzy covariance matrix. *Journal of Food Process Engineering*. 2019; 42(8): e13298. Available from: https://doi.org/10.1111/jfpe.13298.

[4] Qi Z, Wu X, Wu X, Yang Y, Wu B, Fu H. Discrimination of the red jujube varieties using a portable NIR spectrometer and fuzzy improved linear discriminant analysis. *Foods*. 2022; 11(5): 763. Available from: https://doi.org/10.3390/foods11050763.

[5] Wu X, Fang Y, Wu B, Liu M. Application of Near-Infrared Spectroscopy and Fuzzy Improved Null Linear Discriminant Analysis for Rapid Discrimination of Milk Brands. *Foods*. 2023; 12(21): 3929. Available from: https://doi.org/10.3390/foods12213929.

[6] Wu B, Tang W, Zhou J, Jia H, Shen H, Qi Z. Near-infrared spectroscopy combined with fuzzy fast pseudoinverse linear discriminant analysis to discriminate mee tea grades. *Heliyon*; 10(5): e27732.

[7] Jin M, Zhao Z, Chen S, Chen J. Improved piezoelectric grain cleaning loss sensor based on adaptive neuro-fuzzy inference system. *Precision Agriculture*. 2022; 23(4): 1174–1188. Available from: https://doi.org/10.1007/s11119-022-09879-2.

[8] Wu X, Wang Y, Wu B, Sun J. Classification of Fritillaria using a portable near-infrared spectrometer and fuzzy generalized singular value decomposition. *Industrial Crops and Products*. 2024; 218: 119032. Available from: https://doi.org/10.1016/j.indcrop.2024.119032.

[9] Liu W, Zhou J, Liu Y, Zhang T, Yan M, Chen J, et al. An ultrasonic ridge-tracking method based on limiter sliding window filter and fuzzy pure pursuit control for ridge transplanter. *Agriculture*. 2024; 14(10): 1713. Available from: https://doi.org/10.3390/agriculture14101713.

[10] Song Q, Shi J, Li X, Du D, Wang A, Cui X, et al. Path Tracking Control of Rice Transplanter Based on Fuzzy Sliding Mode and Extended Line-of-Sight Guidance Method. *Agronomy*. 2026; 16(2): 215. Available from: https://doi.org/10.3390/agronomy16020215.

[11] Yin S, Sun M, Xu T. Carbon emission reduction pathways in China's rural production and household sectors: A TOE and fsQCA-Based analysis. *Energy Strategy Reviews*. 2026; 64: 102095. Available from: https://doi.org/10.1016/j.esr.2026.102095.

[12] Gorzalczany MB. A method of inference in approximate reasoning based on interval-valued fuzzy sets. *Fuzzy Sets and Systems*. 1987; 21: 1–17. Available from: https://doi.org/10.1016/0165-0114(87)90148-5.

[13] Lu S, Cheng L. Pure rational and non-pure rational decision methods on interval-valued fuzzy soft β-covering approximation spaces. *Expert Systems with Applications*. 2025; 288: 128262. Available from: https://doi.org/10.1016/j.eswa.2025.128262.

[14] Atanassov KT. Intuitionistic fuzzy sets. *Fuzzy Sets And Systems*. 1986; 32: 87–96. Available from: https://doi.org/10.1016/S0165-0114(86)80034-3.

[15] Khan MJ, Jiang S, Ping W, Akram M. Construction methods for entropy measures of circular intuitionistic fuzzy sets and their application. *Engineering Applications of Artificial Intelligence*. 2025; 160: 111809. Available from: https://doi.org/10.1016/j.engappai.2025.111809.

[16] Torra V. Hesitant fuzzy sets. *International Journal of Intelligent Systems*. 2010; 25: 529–539. Available from: https://doi.org/10.1002/int.20418.

[17] Alcantud JCR, Torra V. Decomposition theorems and extension principles for hesitant fuzzy sets. *Information Fusion*. 2018; 41: 48–56. Available from: https://doi.org/10.1016/j.inffus.2017.08.005.

[18] Lu S, Xu Z, Fu Z, Cheng L, Yang T. Foundational theories of hesitant fuzzy sets and hesitant fuzzy information systems and their applications for multi-strength intelligent classifiers. *Information Sciences*. 2025; 714: 122212. Available from: https://doi.org/10.1016/j.ins.2025.122212.

[19] Li L, Xu Y. An extended hesitant fuzzy set for modeling multi-source uncertainty and its applications in multiple-attribute decision-making. *Expert Systems With Applications*. 2024; 238: 121834. Available from: https://doi.org/10.1016/j.eswa.2023.121834.

[20] Xin G, Ying L. Multi-attribute decision-making based on comprehensive hesitant fuzzy entropy. *Expert Systems With Applications*. 2024; 237: 121459. Available from: https://doi.org/10.1016/j.eswa.2023.121459.

[21] Mahmood T, Hayat RZ, Ahmmad J, ur Rehman U, Pamucar D. Energy storage system selection for AI-controlled microgrids using complex hesitant fuzzy MCDM approach based on Dombi operators. *Contemporary Mathematics*. 2025; 6(3): 3269–3300. Available from: https://doi.org/10.37256/cm.6320256576.

[22] Zhang J, Hu Q, Mi J, Fu C. Hesitant fuzzy three-way concept lattice and its attribute reduction. *Applied Intelligence*. 2024; 54(3): 2445–2457. Available from: https://doi.org/10.1007/s10489-024-05317-0.

[23] Peng Z, Zhang X. A New Similarity Measure of Hesitant Fuzzy Sets and Its Application. *International Journal of Fuzzy Systems*. 2024; 27: 1585–1599. Available from: https://doi.org/10.1007/s40815-024-01857-2.

[24] Abuasaker W, Nguyen J, Ruiz FJ, Sánchez M, Agell N. Perceptual maps to aggregate assessments from different rating profiles: A hesitant fuzzy linguistic approach. *Applied Soft Computing*. 2023; 147: 110803. Available from: https://doi.org/10.1016/j.asoc.2023.110803.

[25] l Bandar A. Analyzing the impact of data visualization applications for diagnosing the health conditions through hesitant fuzzy-based hybrid medical expert system. *Ain Shams Engineering Journal*. 2024; 15(5): 102705. Available from: https://doi.org/10.1016/j.asej.2024.102705.

[26] Saghi S, Nazemi A, Effati S, Ranjbar M. Simplex algorithm for hesitant fuzzy linear programming problem with hesitant cost coefficient. *Iranian Journal of Fuzzy Systems*. 2023; 20(1): 137–152. Available from: https://doi.org/10.22111/IJFS.2023.7351.

[27] Zeng W, Ma R, Li D, Yin Q, Xu Z. Distance measure of hesitant fuzzy sets and its application in image segmentation. *International journal of fuzzy systems*. 2022; 24(7): 3134–3143. Available from: https://doi.org/10.1007/s40815-022-01328-6.

[28] Pant M, Mehra N. Strong (α, k)-cut and computational-based segmentation based novel hesitant fuzzy time series forecasting model. *Applied Soft Computing*. 2024; 153: 111251. Available from: https://doi.org/10.1016/j.asoc.2024.111251.

[29] Xia M, Xu Z. Hesitant fuzzy information aggregation in decision making. *International Journal of Approximate Reasoning*. 2011; 52: 395–407. Available from: https://doi.org/10.1016/j.ijar.2010.09.002.

[30] Manetti M. *Topology*. Springer Nature; 2023. Available from: https://doi.org/10.1002/9781119563549.ch5.

[31] Chen C, Liu X, Liu C, Pan Q. Development of the precision feeding system for sows via a rule-based expert system. *International Journal of Agricultural and Biological Engineering*. 2023; 16(2): 187–198. Available from: https://doi.org/10.25165/j.ijabe.20231602.7427.

[32] Hu T, Wang W, Gu J, Xia Z, Zhang J, Wang B. Research on apple object detection and localization method based on improved YOLOX and RGB-D images. *Agronomy*. 2023; 13(7): 1816. Available from: https://doi.org/10.3390/agronomy13071816.

[33] Cheng M, Hou X, Gu J, Xia Z, Zhang J, Wang B. Structural Insights into the Catalytic Cycle of Inulin Fructotransferase: From Substrate Anchoring to Product Releasing. *Journal of Agricultural and Food Chemistry*. 2024; 72(30): 17030–17040. Available from: https://doi.org/10.1021/acs.jafc.4c03615.s001.

[34] Shakiba A. Distributed decision making with S-approximation spaces. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*. 2021; 29(02): 281–311. Available from: https://doi.org/10.1142/S021848852150013X.

[35] Gribonval R, Kutyniok G, Nielsen M, Voigtlaender F. Approximation spaces of deep neural networks. *Constructive approximation*. 2022; 55(1): 259–367. Available from: https://doi.org/10.1007/s00365-021-09543-4.

[36] Manna T, Anitha A. Precipitation prediction by integrating rough set on fuzzy approximation space with deep learning techniques. *Applied Soft Computing*. 2023; 139: 110253. Available from: https://doi.org/10.1016/j.asoc.2023.110253.

[37] Babitha KV, John SJ. Hesitant fuzzy soft sets. *Journal of New Results in Science*. 2013; 3: 98–107. Available from: https://api.semanticscholar.org/CorpusID:59931500.

[38] Lee JG, Hur K. Hesitant fuzzy topological spaces. *Mathematics*. 2020; 8(2): 188. Available from: https://doi.org/10.3390/math8020188.

[39] Swaminathan A, Sivaraja S. Hesitant fuzzy paraopen and hesitant fuzzy mean open sets. *Journal of Applied and Pure Mathematics*. 2022; 4: 141–150. Available from: https://doi.org/10.23091/japm.2022.141.

[40] Ibrahim HZ. On some weaker hesitant fuzzy open sets. *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*. 2021; 70(2): 888–899. Available from: https://doi.org/10.31801/cfsuasmas.867250.

[41] Kim J, Jun YB, Lim PK, Lee JG, Hur K. The category of hesitant H-fuzzy sets. *Annals of Fuzzy Mathematics and Informatics*. 2022; 18(1): 57–74. Available from: https://doi.org/10.30948/afmi.2019.18.1.57.

[42] Deepak D, Mathew B, John SJ, Garg H. A topological structure involving hesitant fuzzy sets. *Journal of Intelligent & Fuzzy Systems*. 2019; 36(6): 6401–6412. Available from: https://doi.org/10.3233/jifs-182673.

[43] Yang X, Song X, Qi Y, Yang J. Constructive and axiomatic approaches to hesitant fuzzy rough set. *Soft Computing*. 2014; 18(6): 1067–1077. Available from: https://doi.org/10.1007/s00500-013-1127-2.

[44] Zhang H, Shu L, Xiong L. On novel hesitant fuzzy rough sets. *Soft Computing*. 2019; 23(22): 11357–11371. Available from: https://doi.org/10.1007/s00500-019-04037-9.

[45] Zhang H, He Y, Ma W. An approximation reduction approach in multi-granulation hesitant fuzzy decision information system. *Journal of Intelligent & Fuzzy Systems*. 2019; 37(1): 1555–1567. Available from: https://doi.org/10.3233/jifs-18586.

[46] Swaminathan A, Ozel C, AlShammari IE. Hesitant Fuzzy Compactness and Hesitant Fuzzy Regularity in Hesitant Fuzzy Topological Spaces. *European Journal of Pure and Applied Mathematics*. 2023; 16(3): 1381–1388. Available from: https://doi.org/10.29020/nybg.ejpam.v16i3.4768.

[47] Patras F. *The Essence of Numbers, Lecture Notes in Mathematics book series*. Springer Press; 2020. Available from: https://doi.org/10.1007/978-3-030-56700-2.

[48] Nisar KS, Farman M. A review on fuzzy fractional order modeling in health systems with application to cardiovascular disease. *International Journal of Mathematics and Computer in Engineering*. 2026; 4: 235–260. Available from: https://doi.org/10.2478/ijmce-2026-0014.

[49] Çitil HG. On a fuzzy initial value problem. *International Journal of Mathematics and Computer in Engineering*. 2024; 3(1): 13–20. Available from: https://doi.org/10.2478/ijmce-2025-0002.

[50] Cui B, Cui X, Wei X, Zhu Y, Ma Z, Zhao Y, et al. Design and Testing of a Tractor Automatic Navigation System Based on Dynamic Path Search and a Fuzzy Stanley Model. *Agriculture*. 2024; 14(12): 2136. Available from: https://doi.org/10.3390/agriculture14122136.

[51] Li Y, Qi B, Bao E, Tong Z, Lian Y, Sun M. Design and analysis of a sowing depth detection and control device for a wheat row planter based on fuzzy PID and multi-sensor fusion. *Agronomy*. 2025; 15(6): 1490. Available from: https://doi.org/10.3390/agronomy15061490.

[52] Shen Y, Wu X, Tan Y, Liu J. Qualitative analysis of lambda-cyhalothrin on Chinese cabbage using mid-infrared spectroscopy combined with fuzzy feature extraction algorithms. *Agriculture*. 2021; 11(3): 275. Available from: https://doi.org/10.3390/agriculture11030275.

[53] Thgeel RM, Hassen HA, Mohammed MG. I-Open (Closec) set and I-Continuous, I-Open (Closed) map in Hesitant Fuzzy Ideal Topological Space. *Turkish Online Journal of Qualitative Inquiry*. 2021; 12(7): 11219–11226.