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Research Article

Extending the Convergence of Two Similar Sixth Order Schemes for Solving Equations under Generalized Conditions

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Abstract: The applicability of two competing efficient sixth convergence order schemes is extended for solving Banach space valued equations. In previous works, the seventh derivative was used, even though it did not explicitly appear in the schemes. But we use only the first derivative that appears on the scheme. Moreover, bounds on the error distances and results on the uniqueness of the solution are provided not given in the earlier works based on ω-continuity conditions. Our technique extends other schemes analogously, since it is so general. Numerical examples complete this work.

***Keywords*:** seventh convergence order, ω-continuity, local convergence, Banach space

AMS: 65H10, 65G99, 49M15

1. Introduction

In [1], a multistep class of iterative methods is considered for approximating a solution x_* of the equation

$$\mathcal{G}(x) = 0, \tag{1}$$

where $\mathcal{G} : \Omega \subseteq B_1 \to B_2$ is nonlinear operator between the Banach spaces B_1, B_2 and Ω is a nonempty and open set. The solution x_* is sought in closed form. But this is achieved only on special occasions. That is why iterative schemes are developed generating sequences converging to x_* under suitable convergence criteria [2-6]. By $\mathcal{G}'$ we denote the Frećhet derivative of operator $\mathcal{G}$ (see Definition 2.1 that follows).

We are concerned with the following three-step schemes developed for $x_0 \in \Omega$ and all $n = 0, 1, 2, \ldots$ by

$$y_n = x_n - \frac{2}{3}\mathcal{G}'(x_n)^{-1}\mathcal{G}(x_n)$$

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$$z_n = x_n - \frac{1}{2}(3\mathcal{G}'(y_n) - \mathcal{G}'(x_n))^{-1} \times (3\mathcal{G}'(y_n) + \mathcal{G}'(x_n))\mathcal{G}'(x_n)^{-1}\mathcal{G}(x_n)$$

$$x_{n+1} = z_n - (\frac{1}{2}(3\mathcal{G}'(y_n) - \mathcal{G}'(x_n))^{-1} \times (3\mathcal{G}'(y_n) + \mathcal{G}'(x_n)))^2\mathcal{G}'(x_n)^{-1}\mathcal{G}(z_n) \quad (2)$$

and

$$y_n = x_n - \frac{2}{3}\mathcal{G}'(x_n)^{-1}\mathcal{G}(x_n)$$

$$z_n = x_n - \frac{1}{2}(3\mathcal{G}'(y_n) - \mathcal{G}'(x_n))^{-1} \times (3\mathcal{G}'(y_n) + \mathcal{G}'(x_n))\mathcal{G}'(x_n)^{-1}\mathcal{G}(x_n)$$

$$x_{n+1} = z_n - 2(3\mathcal{G}'(y_n) - \mathcal{G}'(x_n))^{-1}\mathcal{G}(z_n). \quad (3)$$

The sixth convergence order of scheme (2) and scheme (3) is shown in [1, 7], respectively using Taylor series expansions and hypotheses up to the seventh derivative of $\mathcal{G}$ not appearing on these schemes. These hypotheses limit the applicability of scheme (2). As a motivational example, consider function. Let $f: [-\frac{1}{2}, \frac{3}{2}] \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} t^3 \log t^2 + t^5 - t^4 & if\ t \neq 0 \\ 0 & if\ t = 0. \end{cases}$$

Then, we obtain $f'''(t) = 6\log t^2 + 60t^2 - 24t + 22$. So, function f''' is not bounded on $[-\frac{1}{2}, \frac{3}{2}]$. Hence, the results in [7] cannot be used to solve equation (1) using scheme (2) or scheme (3). Moreover, no upper bounds on $\|x_n - x_*\|$ or results on the uniqueness of x_* were presented either. In this article, we develop a technique using only the derivatives of order one (that appears on the schemes) and provide upper bounds on $\|x_n - x_*\|$ and uniqueness results. This way, we extend the applicability of these schemes. We have used, the Computational Order of Convergence (COC) and Approximate Computational Order of Convergence (ACOC) to determine the convergence order which does not require the usage of higher-order derivatives or divided differences (see Remark 2.2) [8-12]. This is done in Section 2 and Section 3. Numerical examples appear in Section 4. This technique can be used on other methods [13-16].

Similar work has been done in [6]. But the relations do not imply each other and are hard to determine the ones in this paper based on [6].

2. Convergence

Some standard definitions and results are restated in order to make the paper as self-contained as possible. More on this topic can be found in [4, 9-11, 17]. The set $L(X, Y)$ denotes the space of bounded linear operators from X into Y.

Definition 2.1 The operator $F: \Omega \subset X \to Y$ is Fréchet-differentiable at $x \in \Omega$ if there exists an $A \in L(X, Y)$ such that

$$\lim_{h \to 0} \frac{1}{\|h\|} \|\mathcal{G}(x+h) - \mathcal{G}(x) - A(x)\| = 0.$$

The linear operator A is denoted by $\mathcal{G}'(x)$ and is called the Fréchet derivative of F at x.

Next, we present the celebrated Banach Lemma on invertible operators.

Lemma 2.2 Let $A, B \in L(X, Y)$ with $A^{-1} \in L(Y, X)$ and $\|A^{-1}\| \|A - B\| < 1$. Then $B^{-1} \in L(Y, X)$ and

$$\| B^{-1}\| \leq \frac{\| A^{-1}\|}{1-\| A^{-1}\| \ \| A-B\|}.$$

Next, we develop parameters and scalar functions. Let $M = [0, \infty)$.

Assume equation:

(i)

$$\varsigma_0(t) - 1 = 0, \tag{4}$$

has a least positive solution ρ_0 for some function $\varsigma_0 : M \to M$ nondecreasing and continuous. Let $M_0 = [0, \rho_0)$.

(ii)

$$h_1(t) - 1 = 0 \tag{5}$$

has a least solution $R_1 \in (0, \rho_0)$ for functions $\varsigma : M_0 \to M$, $\varsigma_1 : M_0 \to M$ nondecreasing and continuous such that for

$$h_1(t) = \frac{\int_0^1 \varsigma((1-\tau)t)d\tau + \frac{1}{3}\int_0^1 \varsigma_1(\theta t)d\theta}{1-\varsigma_0(t)}.$$

(iii)

$$1 - p(t) = 0 \tag{6}$$

has a least solution $\rho_p \in (0, \rho_0)$ for $p(t) = \frac{1}{2}(3\varsigma_0(h_1(t)t) + \varsigma_0(t))$. Set $M_p = [0, \rho_p)$.

(iv)

$$h_2(t) = 0 \text{ and } \varsigma_0(h_2(t)t) - 1 = 0 \tag{7}$$

has least solutions $R_2, \rho_1 \in (0, \rho_p)$, respectively for functions $h_2 : M_p \to M$ defined by

$$h_2(t) = \frac{\int_0^1 \varsigma((1-\theta)t)d\theta}{1-\varsigma_0(t)} + \frac{3(\varsigma_0(h_1(t)t) + \varsigma_0(t))\int_0^1 \varsigma_1(\theta t)d\theta}{4(1-\varsigma_0(t))(1-p(t))}.$$

Let $\rho = \min\{\rho_0, \rho_1\}$ and $M_1 = [0, \rho)$.

(v)

$$h_3(t) - 1 = 0,$$

has a least solution $R_3 \in (0, \rho)$, where

$$h_3(t) = \left[\frac{\int_0^1 \varsigma((1-\theta)h_2(t)t)d\theta}{1-\varsigma_0(h_2(t)t)} + \frac{(\varsigma_0(h_2(t)t)+\varsigma_0(t))\int_0^1 \varsigma_1(\theta h_2(t)t)d\theta}{(1-\varsigma_0(h_2(t)t))(1-\varsigma_0(t))} \right.$$

$$\left. + \frac{3}{4}\left(\frac{3\varsigma_0(h_1(t)t)+\varsigma_0(t)}{(1-p(t)}\right) \times \left(1+\frac{3(3\varsigma_1(h_1(t)t)+\varsigma_1(t))}{4(1-p(t))}\right)\right] \frac{\int_0^1 \varsigma_1(\theta h_2(t)t)d\theta}{1-\varsigma_0(t)} \tag{8}$$

Notice that the preceding multiplications are well defined as products between real functions.
We shall show that

$$R = \min\{R_j\}, j = 1,\ 2,\ 3 \tag{9}$$

is a radius of convergence for scheme (2). By these definitions, we have that for each $t \in [0, R)$

$$0 \leq \varsigma_0(t) < 1 \tag{10}$$

$$0 \leq p(t) < 1,\ 0 \leq \xi_0(h_2(t)t) < 0 \tag{11}$$

and

$$0 \leq h_j(t) < 1. \tag{12}$$

Moreover, define $B(x_*, \varepsilon) = \{x \in X : \|x - x_*\| < \varepsilon\}$ and $\overline{B}(x_*, \varepsilon) = \{x \in X : \|x - x_*\| \leq \varepsilon\}$, $\varepsilon > 0$.
The conditions $(\mathcal{A})$ shall be used in the local convergence analysis of scheme (2) that follows.
$(\mathcal{A}_1)$ There exists a simple solution x_* of equation (4)
$(\mathcal{A}_2)$ There exists function $\varsigma_0 : M \to M$ nondecreasing and continuous such that for all $x \in \Omega$

$$\| \mathcal{G}'(x_*)^{-1}(\mathcal{G}'(x) - \mathcal{G}'(x_*))\| \leq \varsigma_0(\| x - x_*\|).$$

Set $\Omega_0 = \Omega \cap B(x_*, \rho_0)$.
$(\mathcal{A}_3)$ There exists function $\varsigma : M_0 \to M$, $\varsigma_1 : M_0 \to M$ nondecreasing and continuous such that for each $x, y \in \Omega_0$

$$\| \mathcal{G}'(x_*)^{-1}(\mathcal{G}'(x) - \mathcal{G}'(y))\| \leq \varsigma(\| x - y\|)$$

and

$$\| \mathcal{G}'(x_*)^{-1}\mathcal{G}'(x)\| \leq \varsigma_1(\| x - x_*\|).$$

$(\mathcal{A}_4)$ $\overline{B}(x_*, \tilde{R}) \subseteq \Omega$, where $\tilde{R}$ to be determined later, and
$(\mathcal{A}_5)$ There exists $\rho_* \geq R$ such that

$$\int_0^1 \varsigma_0(\rho_* \theta)d\theta < 1.$$

Set $\Omega_1 = \Omega \cap \overline{B}(x_*, \rho)$.

Next, based on the conditions ($\mathcal{A}$) and the preceding notation, we show the main local convergence analysis for scheme (2).

Theorem 2.3 Assume conditions ($\mathcal{A}$) hold with $\tilde{R} = R$, and starter $x_0 \in B(x_*, R) - \{x_*\}$. Then, sequence $\{x_n\}$ developed by scheme (2) is well defined in $B(x_*, R)$, remains in $B(x_*, R)$ and converges to x_*. Moreover, x_* is the only solution of equation (4) in the set Ω_1 given in ($\mathcal{A}_5$)

Proof. Let $e_n = \|x_n - x_*\|$. Choose $u \in B(x_*, R) - \{x_*\}$. By using ($\mathcal{A}_2$), (9) and (10)

$$\| \mathcal{G}'(x_*)^{-1}(\mathcal{G}'(u) - \mathcal{G}'(x_*))\| \leq \varsigma_0(\| u - x_*\|) \leq \varsigma_0(R) < 1,$$

which together with the Banach lemma on invertible operators given in Lemma 2.2 and scheme (2) give

$$\| \mathcal{G}'(u)^{-1}\mathcal{G}'(x_*)\| \leq \frac{1}{1 - \varsigma_0(\| u - x_*\|)} \tag{13}$$

and that y_0 exists. So, we can write by the first substep of scheme (2) for $n = 0$ and ($\mathcal{A}_1$) that

$$y_0 - x_* = x_0 - x_* - \mathcal{G}'(x_0)^{-1}\mathcal{G}(x_0) + \frac{1}{2}\mathcal{G}'(x_0)^{-1}\mathcal{G}(x_0). \tag{14}$$

Then, in view of (9), (12) (for $j = 1$), ($\mathcal{A}_3$), (13) (for $u = x_0$), (14), and the triangle inequality, we have in turn

$$\begin{aligned}
\|y_0 - x_*\| &\leq \left\|x_0 - x_* - \mathcal{G}'(x_0)^{-1}\mathcal{G}(x_0)\right\| + \frac{1}{3}\left\|\mathcal{G}'(x_0)^{-1}\mathcal{G}'(x_*)\right\|\left\|\mathcal{G}'(x_*)^{-1}\mathcal{G}(x_0)\right\| \\
&\leq \left\|\mathcal{G}'(x_0)^{-1}\mathcal{G}'(x_*)\right\| \times \int_0^1 \left\|\mathcal{G}'(x_*)^{-1}(\mathcal{G}'(x_* + \theta(x_0 - x_*)) - \mathcal{G}'(x_0))\right\| d\theta e_0 \\
&\leq \frac{\int_0^1 \varsigma((1-\theta)e_0)d\theta + \frac{1}{3}\int_0^1 \varsigma_1(\theta e_0)d\theta}{1 - \varsigma_0(e_0)} e_0 \\
&\leq h_1(e_0)e_0 \\
&\leq e_0 < R,
\end{aligned} \tag{15}$$

so $y_0 \in B(x_*, R)$.

Next, we show $(3\mathcal{G}'(y_0) - \mathcal{G}'(x_0))$ is invertible. Using ($\mathcal{A}_2$), (9), (11) and (15), we get in turn that

$$\begin{aligned}
&\left\|(2\mathcal{G}'(x_*))^{-1}(3\mathcal{G}'(y_0) - \mathcal{G}'(x_0) - 2\mathcal{G}'(x_*))\right\| \\
&\leq \frac{1}{2}\left[3\left\|\mathcal{G}'(x_*)^{-1}(\mathcal{G}'(y_0) - \mathcal{G}'(x_*))\right\| + \left\|\mathcal{G}'(x_*)^{-1}(\mathcal{G}'(x_0) - \mathcal{G}'(x_*))\right\|\right] \\
&\leq \frac{1}{2}\left(3\varsigma_0(\|y_0 - x_*\|) + \varsigma_0(e_0)\right) \leq p(e_0) \leq p(R) < 1,
\end{aligned}$$

so, again by Lemma 2.2

$$\left\|(3\mathcal{G}'(y_0)-\mathcal{G}'(x_0))^{-1}\mathcal{G}'(x_*)\right\| \le \frac{1}{2(1-p(e_0))}, \tag{16}$$

and z_0, x_1 exist. Then, by (9), (12) (for $j = 2, 3$), (13) (for $u = x_0$), (15), and (16), we obtain in turn

$$\|z_0 - x_*\| = \left\|x_0 - x_* - \mathcal{G}'(x_0)^{-1}\mathcal{G}(x_0)\left[I - \frac{1}{2}(3\mathcal{G}'(y_0)-\mathcal{G}'(x_0))^{-1}(3\mathcal{G}'(y_0)+\mathcal{G}'(x_0))\right]\mathcal{G}'(x_0)^{-1}\mathcal{G}(x_0)\right\|$$

$$\le \left[\frac{\int_0^1 \varsigma((1-\theta)e_0)d\theta}{1-\varsigma_0(e_0)} + \frac{3}{4}\frac{(\varsigma_0(\|y_0-x_*\|)+\varsigma_0(e_0))\int_0^1\varsigma_1(\theta e_0)d\theta}{(1-\varsigma_0(e_0))(1-p(e_0))}\right]e_0$$

$$\le h_2(e_0)e_0 \le e_0, \tag{17}$$

and

$$\|x_1 - x_*\| \le \left\|z_0 - x_* - \mathcal{G}'(z_0)^{-1}\mathcal{G}(z_0) + \mathcal{G}'(z_0)^{-1}(\mathcal{G}'(x_0)-\mathcal{G}'(z_0))\mathcal{G}'(x_0)^{-1}\mathcal{G}(z_0)\right.$$

$$\left. + (I - (\frac{1}{2}(3\mathcal{G}'(y_0)-\mathcal{G}'(x_0))^{-1}(3\mathcal{G}'(y_0)+\mathcal{G}'(x_0)))^2)\mathcal{G}'(x_0)^{-1}\mathcal{G}(z_0)\right\|$$

$$\le \left[\frac{\int_0^1\varsigma((1-\theta)\|z_0-x_*\|)d\theta}{1-\varsigma_0(\|z_0-x_*\|)} + \frac{(\varsigma_0(\|z_0-x_*\|)+\varsigma_0(e_0))\int_0^1\varsigma_1(\theta\|z_0-x_*\|)d\theta}{(1-\varsigma_0(\|z_0-x_*\|))(1-\varsigma_0(e_0))}\right.$$

$$+\left(\frac{3(3\varsigma_0(\|y_0-x_*\|)+\varsigma_0(e_0))}{4(1-p(e_0))}\right)\times\left(1+\frac{3(3\varsigma_0(\|y_0-x_*\|)+\varsigma_1(e_0))}{4(1-p(e_0))}\right)$$

$$\left.+\frac{\int_0^1\varsigma_1(\theta\|z_0-x_*\|)d\theta}{1-\varsigma_0(e_0)}\right]\|z_0-x_*\|$$

$$\le h_3(e_0)e_0 \le e_0, \tag{18}$$

so, $z_0, x_1 \in B(x_*, R)$, where we also used that the large parenthesis can be written as

$$\left(I - (\frac{1}{2}(3\mathcal{G}'(y_0)-\mathcal{G}'(x_0))^{-1}(3\mathcal{G}'(y_0)+\mathcal{G}'(x_0)))\right)\times\left(I + (\frac{1}{2}(3\mathcal{G}'(y_0)-\mathcal{G}'(x_0))^{-1}(3\mathcal{G}'(y_0)+\mathcal{G}'(x_0)))\right)$$

and

$$\left[I-\frac{1}{2}(3\mathcal{G}'(y_0)-\mathcal{G}'(x_0))^{-1}(3\mathcal{G}'(y_0)+\mathcal{G}'(x_0))\right]$$

$$=(3\mathcal{G}'(y_0)-\mathcal{G}'(x_0))^{-1}[(3\mathcal{G}'(y_0)-\mathcal{G}'(x_0))-\frac{1}{2}(3\mathcal{G}'(y_0)+\mathcal{G}'(x_0))]$$

$$=\frac{3}{2}(3\mathcal{G}'(y_0)-\mathcal{G}'(x_0))^{-1}(\mathcal{G}'(y_0)-\mathcal{G}'(x_0)). \tag{19}$$

$$\left(I-(\frac{1}{2}(3\mathcal{G}'(y_0)-\mathcal{G}'(x_0))^{-1}(3\mathcal{G}'(y_0)+\mathcal{G}'(x_0)))\right)$$

$$=\frac{1}{2}(3\mathcal{G}'(y_0)-\mathcal{G}'(x_0))^{-1}(6\mathcal{G}'(y_0)-2\mathcal{G}'(x_0)-3\mathcal{G}'(y_0)-\mathcal{G}'(x_0)))$$

$$=\frac{3}{2}(3\mathcal{G}'(y_0)-\mathcal{G}'(x_0))^{-1}(\mathcal{G}'(y_0)-\mathcal{G}'(x_0)),$$

$$\left\|(I-(\frac{1}{2}(3\mathcal{G}'(y_0)-\mathcal{G}'(x_0))^{-1}(3\mathcal{G}'(y_0)+\mathcal{G}'(x_0))))\right\|\le\frac{3}{4}\frac{3\varsigma_0(\|y_0-x_*\|)+\varsigma_0(e_0)}{1-p(e_0)},$$

$$\left\|(I+(\frac{1}{2}(3\mathcal{G}'(y_0)-\mathcal{G}'(x_0))^{-1}(3\mathcal{G}'(y_0)+\mathcal{G}'(x_0))))\right\|\le 1+\frac{3}{4}\frac{3\varsigma_0(\|y_0-x_*\|)+\varsigma_0(e_0)}{1-p(e_0)}.$$

Hence, so far we have shown estimates (15), (17) and (18), for $n = 0$ and $y_0, z_0, x_1 \in B(x_*, R)$.
Next, we suppose estimates

$$\|y_m-x_*\|\le h_1(e_m)e_m,$$

$$\|z_m-x_*\|\le h_2(e_m)e_m$$

and

$$\|x_{m+1}-x_*\|\le h_3(e_m)e_m$$

hold for $m = 0, 1, 2, \ldots, n$. We shall show that they hold for $m = n + 1$. Then, by repeating these calculations with x_0, y_0, z_0, x_1 replaced by x_m, y_m, z_m, x_{m+1} respectively, we complete the induction and get

$$\|x_{m+1}-x_*\|\le\gamma\|x_m-x_*\|<R \tag{20}$$

where $\gamma = h_3(e_0) \in [0, 1)$, so $\lim_{m\to\infty} x_m = x_*$ and $x_{m+1} \in B(x_*, R)$.

Finally, let $q \in \Omega_1$ with $\mathcal{G}(q) = 0$. Consider $S = \int_0^1 \mathcal{G}'(x_* + \theta(q - x_*))d\theta$. It follows by $(\mathcal{A}_2)$ that

$$\left\| \mathcal{G}'(x_*)^{-1}(S - \mathcal{G}'(x_*)) \right\| \le \int_0^1 \varsigma_0(\theta \| x_* - q \|) d\theta \le \int_0^1 \varsigma_0(\theta \rho) < 1,$$

so $x_* = q$, since S is invertibile and $0 = \mathcal{G}(q) - \mathcal{G}(x_*) = S(q - x_*)$.

Remark 2.4 1. In view of $(\mathcal{A}_2)$ and the estimate

$$\left\| F'(x^*)^{-1} F'(x) \right\| = \left\| F'(x^*)^{-1}(F'(x) - F'(x^*)) + I \right\|$$

$$\le 1 + \left\| F'(x^*)^{-1}(F'(x) - F'(x^*)) \right\| \le 1 + \varsigma_0(\| x - x^* \|)$$

the second condition in $(\mathcal{A}_3)$ can be dropped and ς_1 can be replaced by

$$\varsigma_1(t) = 1 + \varsigma_0(t)$$

or

$$\varsigma_1(t) = 1 + \varsigma_0(R), \text{ or } \varsigma_1(t) = 2,$$

since $t \in [0, \rho_0)$.

2. The results obtained here can be used for operators F satisfying autonomous differential equations [3] of the form

$$F'(x) = P(F(x))$$

where P is a continuous operator. Then, since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then, we can choose: $P(x) = x + 1$.

3. Let $\varsigma_0(t) = L_0 t$, and $\varsigma(t) = Lt$. In [5], we showed that $r_A = \dfrac{2}{2L_0 + L}$ is the convergence radius of Newton's method:

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n) \text{ for each } n = 0, 1, 2, \cdots \tag{21}$$

under the conditions $(\mathcal{A}_1)$-$(\mathcal{A}_3)$. It follows from the definition of r_A, that the convergence radius R of the method (2) cannot be larger than the convergence radius r_A of the second order Newton's method (21). As already noted in [5] r_A is at least as large as the convergence radius given by Rheinboldt [15]

$$r_R = \frac{2}{3L}, \tag{22}$$

where L_1 is the Lipschitz constant on D. The same value for r_R was given by Traub [16]. In particular, for $L_0 < L_1$ we have that

$$r_R < r_A$$

and

$$\frac{r_R}{r_A} \to \frac{1}{3} \text{ as } \frac{L_0}{L_1} \to 0.$$

That is the radius of convergence r_A is at most three times larger than Rheinboldt.

4. We can compute the computational order of convergence (COC) defined by

$$\lambda = \ln\left(\frac{d_{n+1}}{d_n}\right) / \ln\left(\frac{d_n}{d_{n-1}}\right)$$

or the approximate computational order of convergence

$$\lambda_1 = \ln\left(\frac{e_{n+1}}{e_n}\right) / \ln\left(\frac{e_n}{e_{n-1}}\right).$$

Next, we develop analogously the local convergence analysis of scheme (3).

But this time

$$\bar{h}_3(t) = \left[\frac{\int_0^1 \varsigma((1-\theta)h_2(t)t)d\theta}{1-\varsigma_0(h_2(t)t)} + \frac{(\varsigma_0(t)+3\varsigma_0(h_1(t)t)+2\varsigma_0(h_2(t)t))\int_0^1 \varsigma_1(\theta h_2(t)t)d\theta}{2(1-\varsigma_0(t)t)(1-p(t))}\right] h_2(t).$$

Let $\bar{R} = \min\{R_1, R_2, \bar{R}_3\}$, where $\bar{R}_3$ is the least zero in $D_2 - \{0\}$ of equation $\bar{h}_3(t) - 1$ (assuming it exists). Denote by (A)' conditions (A) but $\tilde{R} = \bar{R}$. The definition of $\bar{h}_3$ is motivated by the estimate

$$\begin{aligned} \| x_1 - x_* \| &\le \| z_0 - x_* - \mathcal{G}'(z_0)^{-1}\mathcal{G}(z_0) + \mathcal{G}'(z_0)^{-1}(3\mathcal{G}'(y_0) \\ &\quad -2\mathcal{G}'(z_0) - \mathcal{G}'(x_0)) \times (3\mathcal{G}'(y_0) - \mathcal{G}'(x_0))^{-1}\mathcal{G}(z_0)\| \\ &\le \bar{h}_3(e_0)e_0 \le e_0 < \bar{R}. \end{aligned} \tag{23}$$

Hence, we arrived at the corresponding local convergence result for scheme (3).

Theorem 2.5 Suppose conditions (A)' for $\tilde{R} = \bar{R}$ hold. Then, the conclusions of Theorem 2.3 hold for scheme (3) with $\bar{R}$, $\bar{h}_3$ replacing R, h_3, respectively.

Table 1. Radius for example 3.1

Radius	$\varsigma_1(t) = 1$	$\varsigma_1(t) = 1 + \varsigma_0(t)$
R_1	0.44444	0.4
R_2	0.277466	0.259201
R_3	0.0938063	0.0896749
$\bar{R}_3$	0.250116	0.234502

3. Numerical examples

We use examples to find the radius of convergence for the schemes. Conditions ($\mathcal{A}$) are used and Definition 2.1 to determine the scalar functions. The radii are found by solving the "h" equations.

Example 3.1 Let $B_1 = B_2 = \Omega = \mathbb{R}$. Define $\mathcal{G}(x) = \sin x$. Using conditions ($\mathcal{A}$) and Definition 2.1 we get $\mathcal{G}'(x) = \cos x$. Moreover, notice that $x_* = 0$. By plugging these on conditions ($\mathcal{A}_1$)-($\mathcal{A}_3$), we see that they are satisfied provided we choose $\varsigma_0(t) = \varsigma(t) = t$ and $\varsigma_1(t) = 1$. Then, by (9) we have the following radii (Table 1).

Example 3.2 Let $B_1 = B_2 = C[0, 1]$, the space of continuous functions defined on [0, 1] with the max norm. Let $\Omega = \bar{U}(0, 1)$. Define function F on Ω by

$$\mathcal{G}(\psi)(x) = \psi(x) - 5\int_0^1 x\theta\psi(\theta)^3 d\theta. \tag{24}$$

We have by Definition 2.1 that

$$\mathcal{G}'(\psi(\mu))(x) = \mu(x) - 15\int_0^1 x\theta\psi(\theta)^2 \mu(\theta)d\theta, \text{ for each } \mu \in D.$$

Moreover, notice that $x_* = 0$. By plugging these on conditions ($\mathcal{A}_1$)-($\mathcal{A}_3$), we see that they are satisfied provided that $\varsigma_0(t) = \varsigma_1(t) = \frac{15}{2}t,\ \varsigma_1(t) = 2$. Then, by (9) we have the following radii (Table 2).

Table 2. Radius for example 3.2

Radius	$\varsigma_1(t) = 2$	$\varsigma_1(t) = 1 + \varsigma_0(t)$
R_1	0.0296296	0.0727273
R_2	0.0231907	0.0533333
R_3	0.00330118	0.0119567
$\bar{R}_3$	0.0199532	0.0312669

Example 3.3 Let $B_1 = B_2 = \mathbb{R}^3$, $\Omega = U(0, 1)$, $x_* = (0, 0, 0)^T$, and define F on Ω by

$$\mathcal{G}(x) = \mathcal{G}(u_1, u_2, u_3) = (e^{u_1} - 1, \frac{e-1}{2}u_2^2 + u_2, u_3)^T. \tag{25}$$

For the points $u = (u_1, u_2, u_3)^T$, the Fréchet derivative is given because of Definition 2.1 by

$$\mathcal{G}'(u) = \begin{pmatrix} e^{u_1} & 0 & 0 \\ 0 & (e-1)u_2 + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using the norm of the maximum of the rows and since $\mathcal{G}'(x_*) = diag(1, 1, 1)$, we get that conditions ($\mathcal{A}$) are verified if we choose $\varsigma_0(t) = (e-1)t,\ \varsigma(t) = e^{\frac{1}{e-1}}t$, and $\varsigma_1(t) = e^{\frac{1}{e-1}}$. Then, by (9) we have the following radii (Table 3).

Table 3. Radius for example 3.3

Radius	$\varsigma_1(t) = e^{\frac{1}{e-1}}$	$\varsigma_1(t) = 1 + \varsigma_0(t)$
R_1	0.0154407	0.229929
R_2	0.110108	0.149959
R_3	0.0183481	0.0520623
$\bar{R}_3$	0.0955553	0.206634

Example 3.4 Returning back to the motivational example at the introduction of this study, we have for $x_* = 1$ that condition ($\mathcal{A}$) are satisfied provided that we choose $\varsigma_0(t) = \varsigma(t) = 96.662907t$, $\varsigma_1(t) = 1.0631$. Then, by (9) we have the following radii (Table 4).

Table 4. Radius for example 3.4

Radius	$\varsigma_1(t) = 1.0631$	$\varsigma_1(t) = 1 + \varsigma_0(t)$
R_1	0.00445282	0.00413809
R_2	0.0104863	0.00268149
R_3	0.000879059	0.0009277
$\bar{R}_3$	0.00249313	0.0024257

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