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Research Article

New Explicit Solutions of the Zoomeron Equation Using Optimum Indexing Approach

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Abstract: This paper presents the nonlinear physical model called the (2+1)-dimensional Zoomeron Equation that is one of the equations describing incognito evolution. We apply the Power Index Method (PIM) to find analytic solutions of the equation by using function transformation. The PIM is a straight arithmetical technique in which the indexes of independent variables are adjusted in our chosen similarity transformation so that partial differential equations may be efficiently reduced to ordinary differential equations. Several exact solutions of the nonlinear Zoomeron equation are constructed, which are graphically represented for certain values of constant parameters. The suggested approach has been successfully used to identify new precise solutions for the nonlinear Partial Differential Equation (PDE).

***Keywords*:** nonlinear Zoomeron equation, exact analytic solutions, Kummer hypergeometric function, Power Index Method (PIM)

MSC: 35C05, 35C08

1. Introduction

The discovery of exact solutions to Nonlinear Partial Differential Equations (NLPDEs) can help to elucidate physical processes. As a result, some important analytic methods are developed and applied to the various types of NLPDEs such as Burger's Equation, Korteweg-De-Vries (KdV) equation, Nonlinear Schrodinger (NLS), the Euler equation of gesticulation, Benjamin-Ono (BO), Benjamin-Bona-Mahony (BBM), Camassa-Holm (CH), and Degasperis-Procesi (DP) equalization, Sine-Gordon (SG), and Landau-Ginzburg-Higgs Equation, etc. in [1, 2]. The Ablowitz-Kaup-Newell-Segur Water Wave Equation and Burger's equations were found to have new exact solutions by Khalil and Khudija are using the Power Index Method. Lie Symmetry Transformations are one type of variable transformation that can occasionally be found that converts a nonlinear PDE into a nonlinear Ordinary Differential Equation (ODE) [3, 4]. The fractional Schrödinger equation plays an important role in modeling quantum system exhibiting anomalous diffusion and nonlocal interactions. It has applications in quantum mechanics, fractional quantum field theory, optical wave propagation, Bose-Einstein condensates with memory effects. The fractional formulation provides a more general framework thank the classical Schrödinger equation by incorporating long-range interactions and Levy-type stochastic processes, making it useful for describing various physical phenomena that cannot be accurately modeled by integer-order differential equations [5–7].

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Also, multi layered transport in fractal porous media is governed by a system of Partial Differential Equations (PDEs) derived from the theory of elasticity. Therefore, it makes sense to create a method that converts the PDEs to Ordinary Differential Euations (ODEs) in order to acquire the analytical fundamental solution for computational convenience. In [8–10] introduces advanced approximation techniques that could be used to verify the accuracy of the obtained PIM solutions. The spectral analysis tools developed here may help in studying the stability and convergence of the numerical/analytical solutions of the Zoomeron equation.

Russell proposed the idea of a soliton with a constant shape and speed in 1834, which he named the Wave of Translation [11]. In 1976, Calogero and Degasperis [12] planned solitons that move with variable speed and detected a relation between their speed and polarization effects when generalizing the KdV equation [13–16]. Boomeron and Trappon solitons are Zoomeron equation solutions that cause the direction of propagation to change only once in Boomeron and several times in Trappon. Many authors have investigated this equation and got its exact solutions. Qawasmeh [17] has applied the sine-cosine function method to construct the traveling wave solutions. In [18], Khan and Akbar investigated solutions using Modified Simple Equation (MSE) method. In [19], applied the extended direct algebraic technique to find new analytical solutions. In [20], novel exact solutions of Zoomeron equation were found by using the extended direct algebraic method. Several more approaches have been adopted to find the exact solution of the equation, see for details [21–23].

In this paper, we will use polynomial (or function) transformation to reduce PDEs to nonlinear ODEs by using a newly introduced optimum indexing approach that is called Power Index Method (PIM). By choosing particular values of indexes of variables t, x, y and find analytic solutions of ODEs and exact solutions of PDEs. Our exact solutions are in explicit general form and contains independent variables with non-restricted parameters.

$$(\frac{w_{xy}}{w})_{tt} - (\frac{w_{xy}}{w})_{xx} + 2(w^2)_{xt} = 0, \tag{1}$$

equivalently

$$\frac{w_{xtty}}{w} - \frac{w_t w_{xty}}{w^2} + 2\frac{w_t^2 w_{xy}}{w^3} - \frac{w_{tt} w_{xy}}{w^2} - \frac{w_{xxxy}}{w} + 2\frac{w_x w_{xxy}}{w^2}$$

$$- \frac{w_{xxxy}}{w} + 2\frac{w_x w_{xxy}}{w^2} - 2\frac{w_x^2 w_{xy}}{w^3} - \frac{w_{xx} w_{xy}}{w^2} + 4w_t w_x + 4w w_{tx} = 0. \tag{2}$$

where $w(x, y, t)$ is the amplitude of the relevant wave model.

2. Description of Power Index Method (PIM)

Planning the Power Index Method (PIM) procedure for locating exact solutions to nonlinear Partial Differential Equations (PDEs) is the aim of this section. Ahmad and Bibi [1, 2] were the ones who first used this technique. The following are PIM's primary steps.

Step 1 Suppose we have a nonlinear PDE for $w(x, y, t)$ in the form

$$Q(w, w_x, w_y, w_t, w_{xx}, w_{yy}, ...), \tag{3}$$

where Q is a polynomial in its arguments which includes nonlinear terms and highest-order derivatives.

Step 2 We have introduce new variable ξ as

$$\xi = \xi(x,y,t) := x^m y^n t^r \tag{4}$$

and transformation

$$w(x,y,t) = x^l y^k t^s f(\xi) \tag{5}$$

By using the chain rule of differentiation, each term of PDE can be expressed in term of $f(\xi)$ and its derivatives with coefficient functions of independent variables x,y,t with their indexes p_x, p_y, p_t. The indexes of x,y,t in each term can be described in Table 1 in the following general form

$$p_x = a_1 m + a_2 l$$

$$p_y = b_1 n + b_2 k,$$

$$p_t = c_1 r + c_2 k,$$

where $a_i, b_i, c_i, i = 1,2$ are constants.

$$w_x = \frac{\partial}{\partial x}(x^l y^k t^s f(\xi)) = (l x^{l-1} y^k t^s) f(\xi) + m x^{l+m-1} y^{n+k} t^{r+s} f'(\xi) \tag{6}$$

It is required that each p_x must be expressed in terms of the index of x in ξ for some values of m and l. Similarly each p_y and p_t should perform as a multiple of indexes of y and t in ξ respectively. The coefficient indexes of x,y,t in (6) of terms $f(\xi)$ and $f'(\xi)$ have the form

Table 1. Table of indexes of x, y and t in (6)

Indexes	f	f'
p_x	$l-1$	$l+m-1$
p_y	k	$n+k$
p_t	s	$r+s$

After some calculation, we can find following relations

$$l = \alpha_1 m + 1$$

$$k = -(\alpha_1 + 1)n$$

$$r = \frac{-1}{\alpha_1 + 1} s.$$

It shows that the coefficients $p_x(f), p_y(f), p_t(f)$ can be reduced in term of ξ. If we choose $\alpha_2 = 1 + \alpha_1$ then the coefficients indexes of $p_x(f'), p_y(f'), p_t(f')$ can also be expressed in term of ξ.

Step 3 We write all indexes of one independent variable like x and find its relations between defined indexes like m and n of x. We pick some relations of indexes of x, y, t so that each coefficient of function $f(\xi)$ and its derivatives can be expressed in term of the new variable ξ. Following ODE is our objective in this method

$$P(\xi, f, f', f'', ...) = 0, \tag{7}$$

where P is a polynomial of $f(\xi)$ and its derivatives.

Step 4 The ODE (7) can be solved by using a computerized symbolic package like Maple or any other method that exists in the literature for nonlinear ODEs. The exact solutions of PDE (1) can be obtained by using (4) and relations $l = l(m), k = k(n)$ and $s = s(r)$

$$w(x, y, t) = x^l y^k t^s f(x^m y^n t^r) \tag{8}$$

Figure 1 is the flowchart that gives a complete idea of the solution steps of PDE (1).

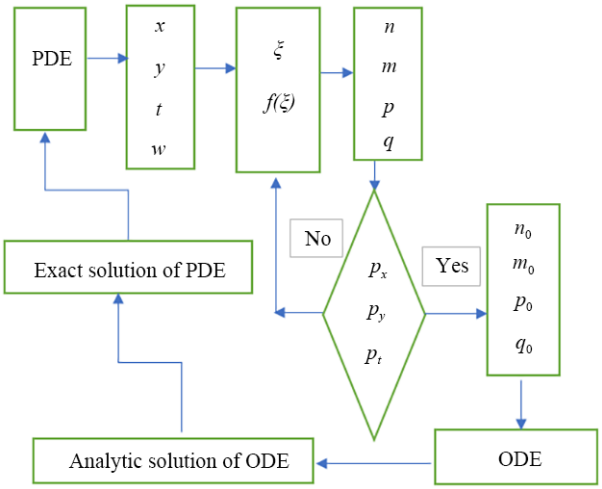


Figure 1. Flow chart of solution (8) of the equation (1)

3. Exact solutions of Zoomeron by using function transformations

Case 1 Let us start to choose a new variable that is a function of x and t with power only in t. We intentionally considered a special class of similarity solutions that are independent of the spatial variable y. The purpose of this reduction was to simplify the original (2+1)-dimensional Zoomeron equation into a tractable ordinary differential equation while preserving the essential nonlinear wave behavior. Therefore, the transformation

$$\xi(t,x) = xt^m, w(t,x,y) = t^n h(\xi), m,n \in R. \tag{9}$$

Using (9), the equation (1) can be expressed into like ODE with indexes of t of the form

$$mxt^{2n+2m-1}[h(\xi)h^{''}(\xi) + h^{'2}(\xi)] + (m+2n)t^{2n+m-1}h(\xi)h^{'}(\xi) = 0. \tag{10}$$

To see the indexes of variable t, we notice that the term with $t-$index $2n+2m-1$ may be transformed in terms of the new variable ξ and the second term with $t-$index $2n+m-1$ must vanish due to the absence of the x variable. It is possible only if m and n have the following relation

$$n = \frac{1}{2}(1-m), m \neq 1. \tag{11}$$

Using the relation (11), the transformation (9) and the ODE (10) reduce to the following form:

$$\xi(t,x) = xt^m, w(t,x,y) = t^{\frac{1}{2}(1-m)}h(\xi). \tag{12}$$

and

$$m\xi[h(\xi)h^{''}(\xi) + h^{'2}(\xi)] + h(\xi)h^{'}(\xi) = 0. \tag{13}$$

It can be written in simplified form as

$$m\xi[h(\xi)h^{'}(\xi)]^{'} + h(\xi)h^{'}(\xi) = 0. \tag{14}$$

If we choose $g(\xi) = h(\xi)h^{'}(\xi)$ then (14) has the form

$$m\xi g^{'}(\xi) + g(\xi) = 0. \tag{15}$$

The analytic solutions of ODE (15) are given by

$$g(\xi) = C_1 \xi^{\frac{-1}{m}} \tag{16}$$

By using relation $g(\xi) = h(\xi)h^{'}(\xi)$, the analytic solution of (14) will be

$$h_1(\xi) = \sqrt{\frac{2C_2(m-1)\xi^{\frac{-1}{m}} + 2C_1 m\xi}{(m-1)\xi^{\frac{-1}{m}}}} \tag{17}$$

and

$$h_2(\xi) = -\sqrt{\frac{2C_2(m-1)\xi^{\frac{-1}{m}} + 2C_1 m\xi}{(m-1)\xi^{\frac{-1}{m}}}} \tag{18}$$

Since $\xi(t,x) = xt^m$, the exact solution of (1) are shown in Figures 2 and 3.

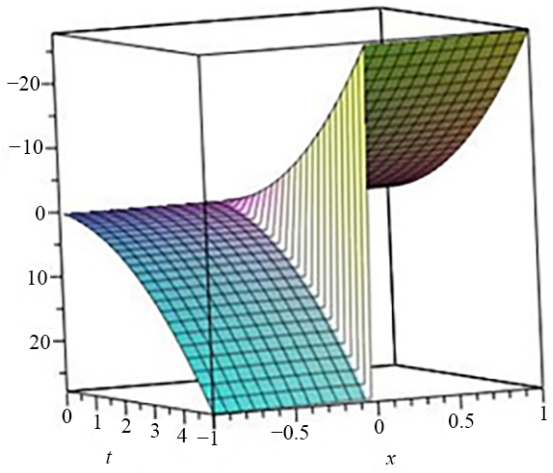


Figure 2. 3D plot of solution (19) with $m = -1$ illustrating localized nonlinear wave propagation

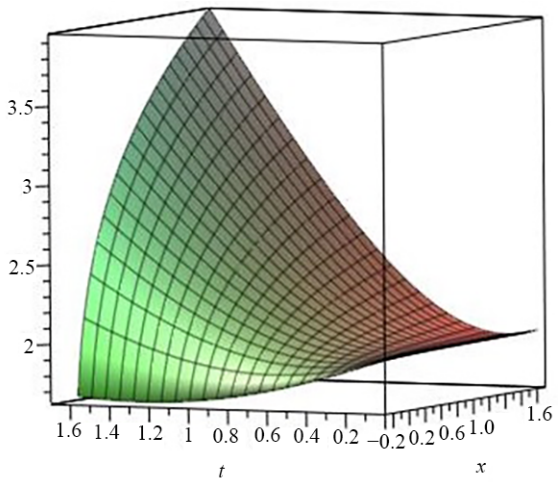


Figure 3. 3D plot of solution (19) with $m = 2$ demonstrating the influence of the parameter m on the wave amplitude and propagation pattern

$$w_1(x,t) = \sqrt{\frac{2C_2(m-1)tx^{\frac{-1}{m}} + 2C_1 mxt^m}{(m-1)t^m x^{\frac{-1}{m}}}}, \tag{19}$$

and

$$w_2(x,t) = -\sqrt{\frac{2C_2(m-1)tx^{\frac{-1}{m}} + 2C_1 mxt^m}{(m-1)t^m x^{\frac{-1}{m}}}}. \tag{20}$$

Again, we choose a relation $n = \frac{-m}{2}$ in (10), then attain a subsequent transformation and ODE

$$\xi(t,x) = xt^m, w(t,x,y) = t^{\frac{-m}{2}} h(\xi), \tag{21}$$

and

$$h(\xi)h''(\xi)) + h'^2(\xi) = 0. \tag{22}$$

The analytic solutions of ODE (22) are

$$h_3(\xi) = \sqrt{2C_1\xi + 2C_2} \tag{23}$$

$$h_4(\xi) = -\sqrt{2C_1\xi + 2C_2}. \tag{24}$$

The exact solution of (1) are

$$w_3(x,t) = \sqrt{\frac{2C_1 xt^m + 2C_2}{t^m}} \tag{25}$$

$$w_4(x,t) = -\sqrt{\frac{2C_1 xt^m + 2C_2}{t^m}}. \tag{26}$$

Case 2 Next, we present another transformation

$$\xi(t,x) = xy^m, w(t,x,y) = y^n h(\xi), m,n \in R. \tag{27}$$

$$w_{xy} = (m+n)y^{n+m-1}h'(\xi) + mxy^{n+2m-1}h''(\xi) \tag{28}$$

$$\frac{w_{xy}}{w} = (m+n)y^{m-1}\frac{h^{'}(\xi)}{h(\xi)} + mxy^{2m-1}\frac{h^{''}(\xi)}{h(\xi)}. \tag{29}$$

Using (27), (28) and (29), the equation (1) can be described into like ODE with coefficients of $h(\xi)$ and its derivatives are powers of x and y.

$$mxy^{4m-1}[h^2(\xi)h^{''''}(\xi) - 2h^{'''}(\xi)h^{'}(\xi)h(\xi) - (h^{''})^2(\xi)h(\xi) + 2(h^{''}(\xi))(h^{'}(\xi))^2]$$

$$+y^{3m-1}[(3m+n)h^2(\xi)h^{'''}(\xi) - (5m+3n)h^{''}(\xi)h^{'}(\xi)h(\xi) + 2(m+n)(h^{'}(\xi))^3] = 0. \tag{30}$$

Since the indexces of y are $4m-1$ and $3m-1$ and the terms xy^{4m-1} and y^{3m-1} must be expressed in term of new variable ξ but the term y^{3m-1} must vanish because it might not be stated in terms of ξ. Therefore, we must make a decision $m = \frac{1}{3}$. Lastly, we have the ODE and transformation below

$$\xi(t,x) = xy^{\frac{1}{3}}, w(t,x,y) = y^n h(\xi) \tag{31}$$

$$\xi[h^2(\xi)h^{''''}(\xi) - 2h^{'''}(\xi)h^{'}(\xi)h(\xi) - (h^{''})^2(\xi)h(\xi) + 2h^{''}(\xi)(h^{'}(\xi))^2]$$

$$+3(m+n)h^2(\xi)h^{'''}(\xi) - (5+9n)h^{''}(\xi)h^{'}(\xi)h(\xi) + 2(1+3n)(h^{'}(\xi))^3 = 0. \tag{32}$$

The analytic solution of (32) is

$$h_5(\xi) = C_3\exp(\frac{-i\xi}{C_1})\ \text{Kummer M}\ (\frac{1}{2}(iC_2 + \frac{3n}{C_1} + \frac{1}{C_1})C_1, 3n+1, \frac{2i\xi}{C_1})$$

$$+C_4\exp(\frac{-i\xi}{C_1})\ \text{Kummer U}\ (\frac{1}{2}(iC_2 + \frac{3n}{C_1} + \frac{1}{C_1})C_1, 3n+1, \frac{2i\xi}{C_1}), \tag{33}$$

where Kummer M and Kummer U are defined in Appendix.

$$w_5(x,y) = C_3y^n\exp(\frac{-ixy^{\frac{1}{3}}}{C_1})\ \text{Kummer M}\ (\frac{1}{2}(iC_2 + \frac{3n}{C_1} + \frac{1}{C_1})C_1, 3n+1, \frac{2ixy^{\frac{1}{3}}}{C_1})$$

$$+C_4y^n\exp(\frac{-ixy^{\frac{1}{3}}}{C_1})\ \text{Kummer U}\ (\frac{1}{2}(iC_2 + \frac{3n}{C_1} + \frac{1}{C_1})C_1, 3n+1, \frac{2ixy^{\frac{1}{3}}}{C_1}) \tag{34}$$

3.1 *Remark 1*

(a) For $m = \frac{1}{3}$ and $n = -1$, then the ODE (32) reduces to the form

$$\xi[h^2(\xi)h^{''''}(\xi) - 2h^{'''}(\xi)h^{'}(\xi)h(\xi) - (h^{''})^2(\xi)h(\xi)]$$

$$+2\xi h''(\xi)(h'(\xi))^2+4h''(\xi)h'(\xi)h(\xi)-4(h'(\xi))^3=0. \tag{35}$$

The analytic solution of (35) is

$$h_6(\xi)=C_3\xi^3\exp(\frac{-i\xi}{C_2})\ \text{Kummer M}\ (\frac{1}{2}(iC_2+\frac{4}{C_1})C_1,4,\frac{2i\xi}{C_1})$$

$$+C_4\xi^3\exp(\frac{-i\xi}{C_2})\ \text{Kummer U}\ (\frac{1}{2}(iC_2+\frac{4}{C_1})C_1,4,\frac{2i\xi}{C_1}). \tag{36}$$

Since $\xi(t,x)=xy^{\frac{1}{3}}$ and $w(t,x,y)=y^{-1}h(\xi)$, the exact solution of (1) can become shown in Figures 4 and 5 as

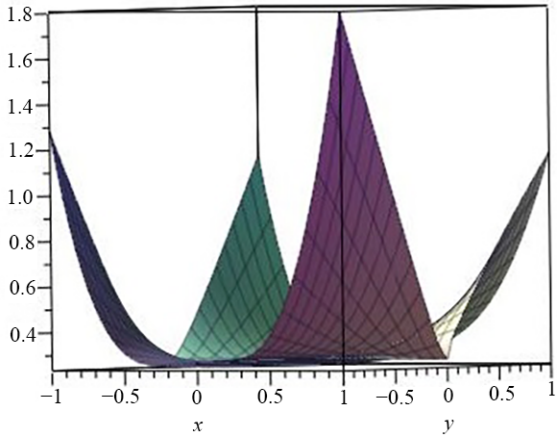


Figure 4. 3D plot of abs(w) of solution (37) with $m=1/3$ and $n=-1/3$ showing the magnitude distribution of the Kummer-type solution

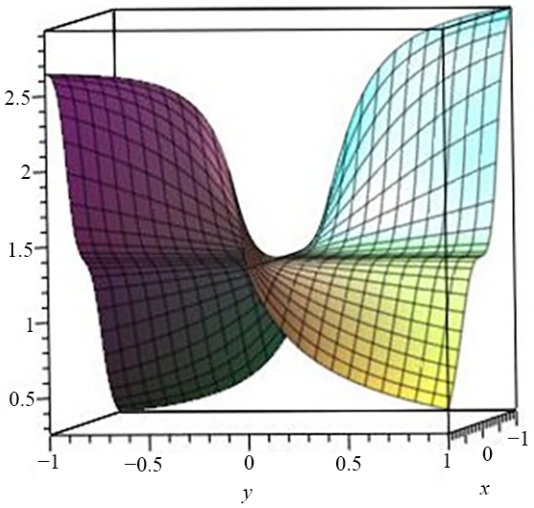


Figure 5. 3D plot of arg(w) of solution (37) with $m=1/3$ and $n=-1/3$ illustrating the phase structure of the complex Kummer-type wave solution

$$w_6(x,y) = C_3 x^3 \exp\left(\frac{-ixy^{\frac{1}{3}}}{C_2}\right) \text{ Kummer M } \left(\frac{1}{2}\left(iC_2 + \frac{4}{C_2}\right)C_2, 4, \frac{2ixy^{\frac{1}{3}}}{C_2}\right)$$

$$+ C_4 x^3 \exp\left(\frac{-ixy^{\frac{1}{3}}}{C_2}\right) \text{ Kummer U } \left(\frac{1}{2}\left(iC_2 + \frac{4}{C_1}\right)C_1, 4, \frac{2ixy^{\frac{1}{3}}}{C_1}\right). \tag{37}$$

(b) For $m = \frac{1}{3}$ and $n = -\frac{1}{3}$, then the ODE (32) becomes to the form

$$\xi[h^2(\xi)h''''(\xi) - 2h'''(\xi)h'(\xi)h(\xi) - (h'')^2(\xi)h(\xi)]$$

$$+ 2\xi h''(\xi)(h'(\xi))^2 - 2h''(\xi)h'(\xi)h(\xi) - 2h'''(\xi)h^2(\xi) = 0. \tag{38}$$

The analytic solution of (38) is

$$h_7(\xi) = C_3 \text{ WhittakerM } \left(\frac{-i}{2}C_2C_1, \frac{1}{2}, \frac{2i\xi}{C_1}\right) + C_4 \text{ WhittakerU } \left(\frac{-i}{2}C_2C_1, \frac{1}{2}, \frac{2i\xi}{C_1}\right)$$

Since $\xi(t,x) = xy^{\frac{1}{3}}$ and $w(t,x,y) = y^{-\frac{1}{3}}h(\xi)$, the exact solution of (1) can be obtained as Figures 6 and 7

$$w_7(x,y) = C_3 y^{\frac{-1}{3}} \text{ WhittakerM } \left(\frac{-i}{2}C_2C_1, \frac{1}{2}, \frac{2ixy^{\frac{1}{3}}}{C_2}\right)$$

$$+ C_4 y^{\frac{-1}{3}} \text{ WhittakerU } \left(\frac{-i}{2}C_2C_1, \frac{1}{2}, \frac{2ixy^{\frac{1}{3}}}{C_1}\right). \tag{39}$$

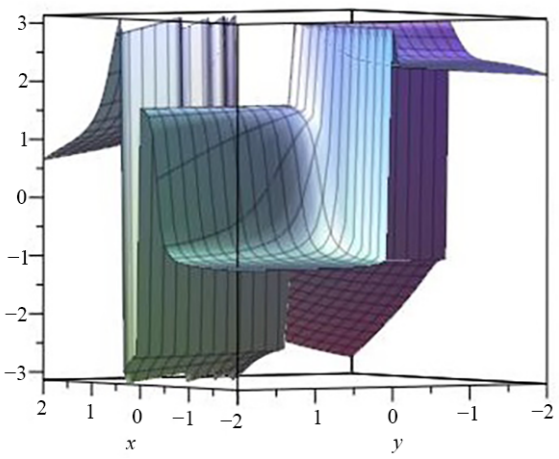


Figure 6. 3D plot of abs(w) of (39) with $m = 1/3$ and $n = -1/3$ illustrating the amplitude behavior of the Whittaker-function solution

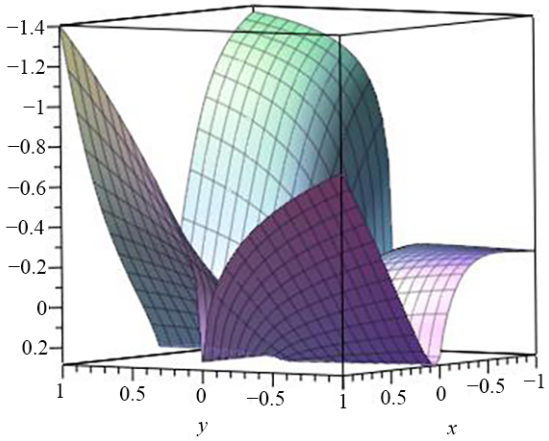


Figure 7. 3D plot of solution (39) with $m = 1/3$ and $n = -1/3$ showing nonlinear dispersive wave propagation

(c) For $m = \frac{1}{3}$ and $n = -\frac{5}{9}$, then the ODE (32) reduces to the form

$$3\xi[h^2(\xi)h''''(\xi) - 2h'''(\xi)h'(\xi)h(\xi) - (h'')^2(\xi)h(\xi)]$$

$$+6\xi h''(\xi)(h'(\xi))^2 + 4h'''(\xi)h^2(\xi) - 4(h'(\xi))^3 = 0. \tag{40}$$

The analytic solution of (35) shown in Figures 8 and 9

$$h_8(\xi) = C_3\xi^{\frac{5}{3}}\exp(\frac{-i\xi}{C_2})\ \text{Kummer M}\ (\frac{1}{2}(iC_1 + \frac{8}{3C_2})C_2, 4, \frac{2i\xi}{C_2})$$

$$+C_4\xi^{\frac{5}{3}}\exp(\frac{-i\xi}{C_2})\ \text{Kummer U}\ (\frac{1}{6}(3iC_1 + \frac{8}{C_2})C_2, \frac{8}{3}, \frac{2i\xi}{C_2}). \tag{41}$$

Since $\xi(t,x) = xy^{\frac{1}{3}}$ and $w(t,x,y) = y^{-\frac{5}{9}}h(\xi)$, the exact solution of (1) can be expressed as

$$w_8(x,y) = C_3x^{\frac{5}{3}}\exp(\frac{-ixy^{\frac{1}{3}}}{C_2})\ \text{Kummer M}\ (\frac{1}{2}(iC_2 + \frac{4}{C_2})C_1, 4, \frac{2ixy^{\frac{1}{3}}}{C_2})$$

$$+C_4x^{\frac{5}{3}}\exp(\frac{-ixy^{\frac{1}{3}}}{C_2})\ \text{Kummer U}\ (\frac{1}{6}(3iC_1 + \frac{8}{C_2})C_2, \frac{8}{3}, \frac{2ixy^{\frac{1}{3}}}{C_2}). \tag{42}$$

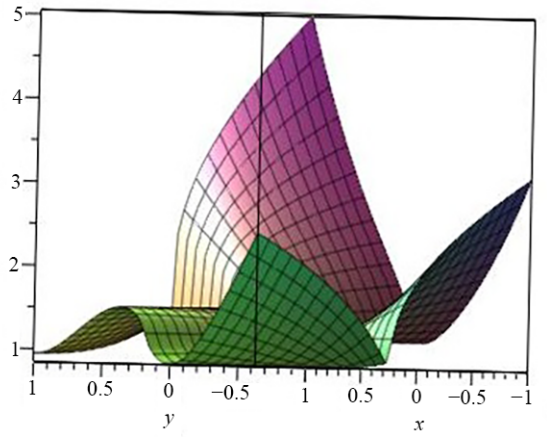


Figure 8. 3D plot of abs(w) of (41) with $m = 1/3$ and $n = -5/9$ demonstrating localized wave amplitude structures

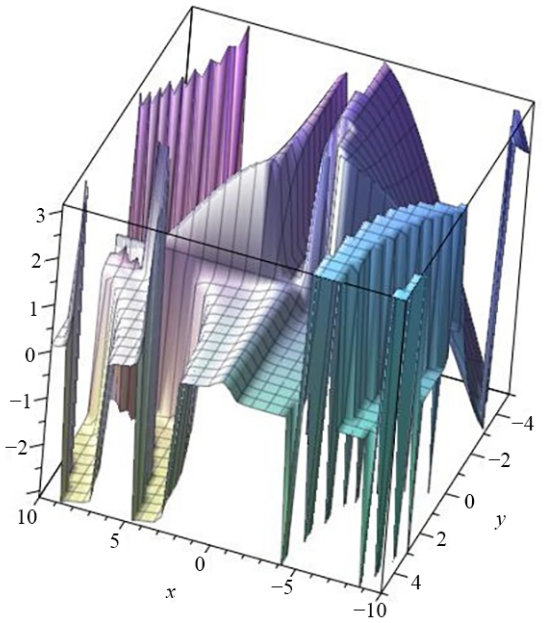


Figure 9. 3D plot of solution (41) with $m = 1/3$ and $n = -5/9$ illustrating oscillatory nonlinear wave behavior

(d) To find the value of m where bothe indexces of y are equal so $4m-1$ must be equal to $3m-1$, it is only possible when $m = 0$. The ODE (30) reduces to

$$h^2(\xi)h'''(\xi) - 3h''(\xi)h'(\xi)h(\xi) + 2(h'(\xi))^3 = 0. \tag{43}$$

The analytic solution of ODE (43) is

$$h_9(\xi) = \exp(\frac{1}{2}\xi(C_2\xi + C_1)). \tag{44}$$

The exact solution of PDE (1) becomes

$$w_9(x,y) = y^n \exp(\frac{1}{2}x(C_2x+C_1)). \tag{45}$$

Case 3 Next, we present another transformation by choosing $m = \frac{1}{4}$ (As Exponent matches the selected parameter value $m = \frac{1}{4}$):

$$\xi(t,x) = xy^{\frac{1}{4}}, w(t,x,y) = y^n h(\xi), m,n \in R. \tag{46}$$

Using (46), the equation (1) can be described into ODE of the form

$$\xi[h^2(\xi)h''''(\xi) - 2h'''(\xi)h'(\xi)h(\xi) - (h'')^2(\xi)h(\xi) + 2(h''(\xi))(h'(\xi))^2]$$

$$+ [(3+4n)h^2(\xi)h'''(\xi) - (12n+5)h''(\xi)h'(\xi)h(\xi) + 2(4n+1)(h'(\xi))^3] = 0. \tag{47}$$

The analytic solution of (47) is

$$h_10(\xi) = C_3 \exp(\frac{-i\xi}{C_2}) \text{ Kummer M } (\frac{1}{2}(iC_2 + \frac{4n}{C_1} + \frac{1}{C_1})C_1, 4n+1, \frac{2i\xi}{C_1})$$

$$+ C_4 \exp(\frac{-i\xi}{C_2}) \text{ Kummer U } (\frac{1}{2}(iC_2 + \frac{4n}{C_1} + \frac{1}{C_1})C_1, 4n+1, \frac{2i\xi}{C_1}) \tag{48}$$

$$w_9(x,t) = C_3 y^n \exp(\frac{-ixy^{\frac{1}{4}}}{C_1}) \text{ Kummer M } (\frac{1}{2}(iC_2 + \frac{4n}{C_1} + \frac{1}{C_1})C_1, 4n+1, \frac{2ixy^{\frac{1}{4}}}{C_1})$$

$$+ C_4 y^n \exp(\frac{-ixy^{\frac{1}{4}}}{C_1}) \text{ Kummer U } (\frac{1}{2}(iC_2 + \frac{4n}{C_1} + \frac{1}{C_1})C_1, 4n+1, \frac{2ixy^{\frac{1}{4}}}{C_1}). \tag{49}$$

3.2 *Remark 2*

(a) For $m = \frac{1}{4}$ and $n = -\frac{1}{4}$, then the ODE (47) reduces to the form

$$\xi[h^2(\xi)h''''(\xi) - 2h'''(\xi)h'(\xi)h(\xi) - (h'')^2(\xi)h(\xi) + 2h'''(\xi)(h(\xi))^2]$$

$$- 2h''(\xi)h'(\xi)h(\xi) + 2h''(\xi)(h(\xi))^2 = 0. \tag{50}$$

The analytic solution of (50) is

$$h_{11}(\xi) = C_3 \text{ WhittakerM } (\frac{-1}{2} iC_1C_2, \frac{2i\xi}{C_1}) + C_4 \text{ WhittakerU } (\frac{1}{2} iC_1C_2, \frac{1}{2}, \frac{2i\xi}{C_1}) \tag{51}$$

Figure of equation (52) shown in Figures 10 and 11

$$w_{11}(x,t) = C_3 y^{\frac{-1}{4}} \text{ WhittakerM } (\frac{-1}{2} iC_1C_2,, \frac{2ixy^{\frac{1}{4}}}{C_1}) + C_4 y^{\frac{-1}{4}} \text{ WhittakerU } (\frac{-1}{2} iC_1C_2,, \frac{2ixy^{\frac{1}{4}}}{C_1}). \tag{52}$$

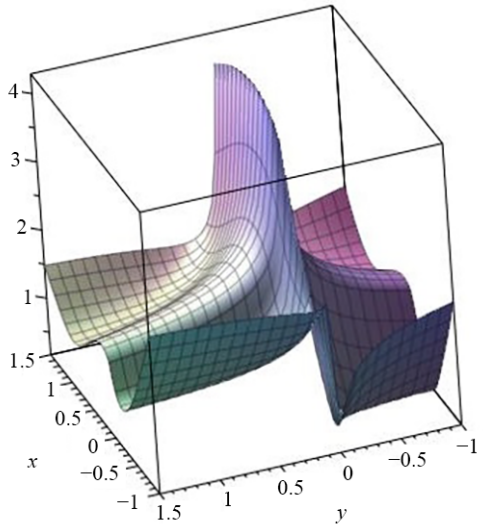


Figure 10. 3D plot of abs(w) of (52) with $m = 1/3$ and $n = -1/4$ illustrating the magnitude behavior of the Whittaker-function solution

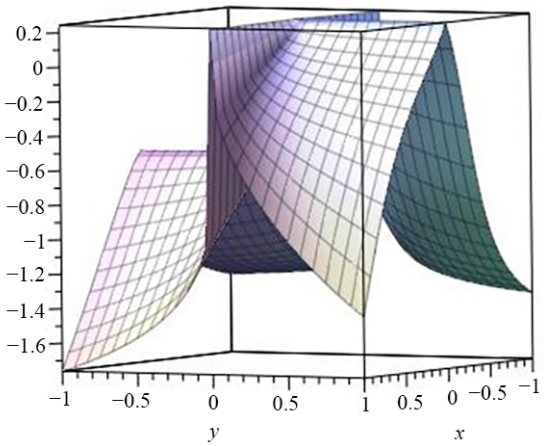


Figure 11. 3D plot of solution (52) with $m = 1/3$ and $n = -1/4$ demonstrating nonlinear wave propagation and phase variation

(b) For $m = \frac{1}{3}$ and $n = -\frac{3}{4}$, then the ODE (47) becomes to the form

$$\xi[h^2(\xi)h''''(\xi) - 2h'''(\xi)h'(\xi)h(\xi) - (h'')^2(\xi)h(\xi) + 2h''(\xi)(h'(\xi))^2]$$

$$-4h''(\xi)h'(\xi)h(\xi) + 4(h'(\xi))^3 = 0. \tag{53}$$

The analytic solution of (53) is

$$h_{12}(\xi) = C_3\xi^3 \exp(\frac{-\xi i}{C_2}) \text{ Kummer M } (\frac{1}{2}(iC_2 + \frac{4}{C_1})C_1, 4, \frac{2i\xi}{C_1})$$

$$+C_4\xi^3 \exp(\frac{-\xi i}{C_2}) \text{ Kummer U } (\frac{1}{2}(iC_2 + \frac{4}{C_1})C_1, 4, \frac{2i\xi}{C_1}) \tag{54}$$

$$w_{12}(x,t) = C_3x^3 \exp(\frac{-xy^{\frac{1}{3}}i}{C_2}) \text{ Kummer M } (\frac{1}{2}(iC_2 + \frac{-4}{C_1})C_1, 4, \frac{2ixy^{\frac{1}{4}}}{C_2}) \tag{55}$$

Figure shown in Figures 12 and 13

$$+C_4x^3 \exp(\frac{-ixy^{\frac{1}{3}}}{C_2}) \text{ Kummer U } (\frac{1}{2}(iC_2 + \frac{-4}{C_1})C_1, 4, \frac{2ixy^{\frac{1}{4}}}{C_1}). \tag{56}$$

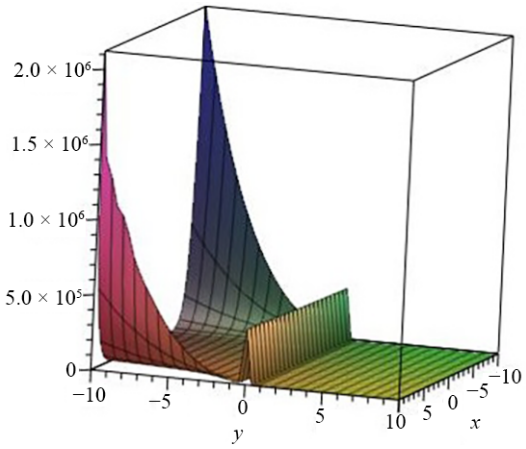


Figure 12. 3D plot of abs(w) of (56) with $m = 1/3$ and $n = -3/4$ showing the amplitude distribution of the Kummer-type analytical solution

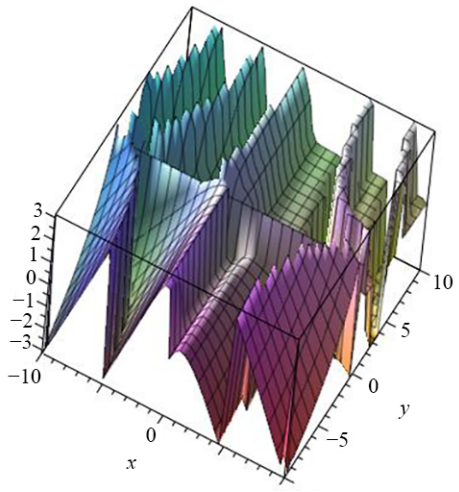


Figure 13. 3D plot of solution (56) with $m = 1/3$ and $n = -3/4$ illustrating the effect of parameter selection on nonlinear wave propagation

(c) For $m = \frac{1}{4}$ and $n = -\frac{5}{12}$, then the ODE (32) reduces to the form

$$3\xi[h^2(\xi)h''''(\xi) - 2h'''(\xi)h'(\xi)h(\xi) - (h'')^2(\xi)h(\xi)]$$

$$-2h'''(\xi)h^2(\xi) - 4(h'(\xi))^3 = 0. \tag{57}$$

The analytic solution of (35) is

$$h_{13}(\xi) = C_3\xi^{\frac{5}{3}}\exp(\frac{-i\xi}{C_2})\ \text{Kummer M}\ (\frac{1}{6}(i3C_2 + \frac{8}{C_1})C_1, \frac{8}{3}, \frac{2i\xi}{C_1})$$

$$+ C_4\xi^{\frac{5}{3}}\exp(\frac{-\xi}{C_2})\ \text{Kummer U}\ (\frac{1}{6}(i3C_2 + \frac{8}{C_1})C_1, \frac{8}{3}, \frac{2i\xi}{C_1}). \tag{58}$$

Figure of equation (59) shown in Figures 14 and 15

$$w_{13}(x,t) = C_3x^{\frac{5}{3}}\exp(\frac{-ixy^{\frac{1}{4}}}{C_2})\ \text{Kummer M}\ (\frac{1}{6}(3iC_2 + \frac{8}{C_1})C_1, \frac{8}{3}, \frac{2ixy^{\frac{1}{3}}}{C_1})$$

$$+ C_4x^{\frac{5}{3}}\exp(\frac{-ixy^{\frac{1}{4}}}{C_2})\ \text{Kummer U}\ (\frac{1}{6}(3iC_2 + \frac{8}{C_1})C_1, \frac{8}{3}, \frac{2ixy^{\frac{1}{4}}}{C_1}). \tag{59}$$

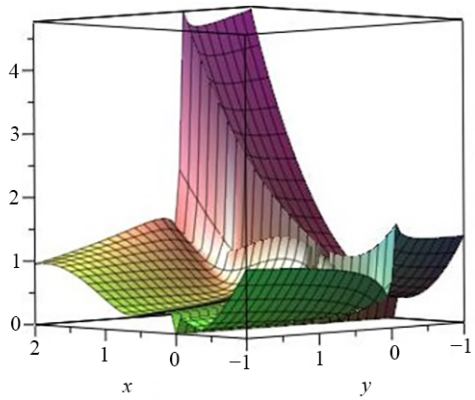


Figure 14. 3D plot of abs(w) of (59) with $m = 1/3$ and $n = -5/12$ demonstrating amplitude localization of the obtained analytical solution

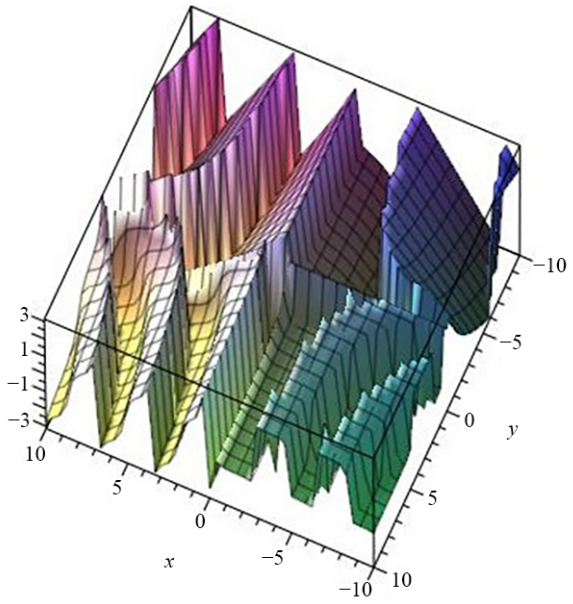


Figure 15. 3D plot of solution (59) with $m = 1/3$ and $n = -5/12$ showing nonlinear oscillatory wave structures

4. Conclusion

The Zoomeron equation has been successfully solved using the Power Index Method (PIM) in this study, yielding completely new close-form solutions. We have found that the suggested approach is quite effective and completely compatible with the problem's complexity. Additionally, it was claimed that this approach can be used to solve problems of a nonlinear physical nature. The novelty of the present work lies in the application of the Power Index Method (PIM) to derive new explicit analytical solutions of the nonlinear (2+1)-dimensional Zoomeron equation. Unlike many existing studies that mainly focus on numerical approximations or standard traveling-wave reductions, the present approach systematically constructs similarity transformations that reduce the governing PDE into solvable ordinary differential equations. The obtained solutions include generalized forms expressed through Kummer hyper geometric and Whittaker functions, which have not been previously reported for the considered Zoomeron model. Moreover, the work provides analytical and graphical investigations of different wave structures generated by the equation. The combination of similarity reduction, exact special-function solutions, and graphical interpretation constitutes the main contribution of this study.

5. Results and discussion

The obtained analytical solutions demonstrate that the Power Index Method (PIM) is an effective reduction technique for constructing explicit solutions of the nonlinear Zoomeron equation. By introducing suitable similarity transformations, the original (2+1)-dimensional nonlinear partial differential equation was successfully reduced into ordinary differential equations, which admit exact solutions in terms of elementary functions, Kummer hyper geometric functions, and Whittaker functions. The graphical behavior of the solutions indicates the existence of localized wave structures, solitary wave patterns, and dispersive wave propagation depending on the selected parameter values. The solutions remain bounded for particular parameter ranges, while oscillatory and decaying behaviors are observed in other cases, illustrating the rich nonlinear dynamics of the Zoomeron model. Furthermore, the obtained solutions provide insight into the propagation of nonlinear waves in higher-dimensional media. The figures reveal that the amplitude, shape, and propagation speed of the waves are strongly influenced by the transformation parameters and integration constants. In particular, the Kummer and Whittaker-type solutions describe more generalized wave structures compared with elementary traveling wave solutions. A comparison among different cases shows that the proposed transformations generate multiple classes of exact solutions with distinct qualitative behavior. These results confirm that the Power Index Method can serve as a useful analytical framework for studying nonlinear evolution equations arising in mathematical physics.

In addition, the present work improves the understanding of exact solution structures of the Zoomeron equation by combining similarity reductions with special-function representations. Unlike purely numerical approaches, the derived closed-form solutions provide direct analytical expressions that can be used for further stability analysis, asymptotic investigation, and verification of computational schemes. The presented graphical illustrations support the analytical findings and demonstrate the physical relevance of the obtained solutions.

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Conflict of interest

The authors declare no competing financial interest.

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Appendix. Kummer M and Kummer U functions

In order to confirm the correctness of the obtained analytical solutions, direct substitution of the derived Kummer and Whittaker-type solutions into the reduced ordinary differential equations and subsequently into the original Zoomeron equation was performed using symbolic computation. The residual expressions vanished identically for selected parameter values, confirming the validity of the obtained exact solutions. The Kummer M was defined by Kummer in 1837 and the Kummer U was introduced by France Sco Tricomi in 1947. The Kummer M is defined as

$$\text{Kummer M}\,(\mu,\nu,\xi)=\sum_{n=0}^{\infty}\frac{(\mu)_n}{(\nu)_n}\xi^n, \tag{A1}$$

where $(\mu)_n=\mu(\mu+1)(\mu+2)...(\mu+n-1)$ and $(\mu)_0=1$. The Kummer U is defined as

$$\text{Kummer U}\,(\mu,\nu,\xi)=\frac{\Gamma(\nu-1)}{\Gamma(\mu+1-\nu)}\,\text{Kummer M}\,(\mu,\nu,\xi)$$

$$+\xi^{1-\nu}\frac{\Gamma(1-\nu)}{\Gamma(\mu)}\,\text{Kummer M}\,(\mu+1-\nu,2-\nu,\xi). \tag{A2}$$