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Research Article

Sharp Inequalities on the Logarithmic Coefficients for Complex-Valued Functions Linked to the Domain of a Four-Leaf Type Function

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Abstract: A primary objective of examining coefficient inequalities for several families of complex-valued functions is to express their coefficients with respect to those of Carathéodory functions. Utilizing the inequalities related to Carathéodory functions enables effective analysis of the properties of coefficient inequalities. This article focuses on deriving sharp inequalities in the logarithmic coefficients for complex-valued functions related to a four-leaf-type function. The coefficient inequalities cover the four logarithmic coefficients, the Zalcman, Fekete-Szegö, and Krushkal inequalities for the specified family. Additionally, we present sharp estimates of second Hankel determinants, where the entries involve logarithmic coefficients. All the derived estimations are sharp.

***Keywords*:** sharp inequalities, logarithmic coefficient inequalities, Carathéodory functions, complex-valued functions, Hankel determinants, four-leaf type function

MSC: 30C45, 30C50, 30C80

1. Introduction and definitions

Complex analysis is a crucial area of modern mathematics with extensive applications in both scientific mathematical fields. Geometric Function Theory is often considered one of the most intriguing disciplines of mathematics. Geometric Function Theory (GFT) is a field within complex analysis focused on examining the geometric behavior of holomorphic functions. This field serves as a cornerstone of applied mathematics, playing a vital role in electronics, fluid dynamics, engineering, partial differential equations, and several other disciplines. This field developed rapidly as a result of various efforts aimed at solving the Bieberbach conjecture [1]. It focuses on estimating the coefficient inequalities of functions belonging to the univalent function class $\mathscr{S}$. This conjecture asserts that for $g \in \mathscr{S}$, expressed through the Taylor-Maclaurin series expansion ς

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$$g(\varsigma) = \varsigma + \sum_{e=2}^{\infty} d_e \varsigma^e, \quad \varsigma \in \mathbb{E}, \tag{1}$$

where $\mathbb{E} := \{\varsigma \in \mathbb{C} : |\varsigma| < 1\}$, then $|d_e| \leq e$ for all $e \geq 2$. Equation (1) gives the series form for a set of holomorphic functions $\mathscr{A}$. The family $\mathscr{S}$ is a subfamily of $\mathscr{A}$ that Koebe initially investigated in 1907. Zalcman posited a conjecture in 1960 that the coefficients of any function in the family $\mathscr{S}$ follow a precise functional

$$\left|d_e^2 - d_{2e-1}\right| \leq (e-1)^2.$$

This results in the publications of various articles [2–4] on the Zalcman inequality and its generalized version

$$\left|\tau d_e^2 - d_{2e-1}\right| \leq \tau e^2 - 2e + 1, \text{ with } \tau \geq 0$$

for specific subfamilies of the family $\mathscr{S}$. Krushkal used analytic homotopy from an unpublished article [5] to prove the Zalcman inequality for $e \leq 6$ in [6], considering $e \geq 2$. If $g \in \mathscr{S}$, it was further shown that:

$$\left|d_e^i - d_2^{i(e-1)}\right| \leq 2^{i(e-1)} - e^i, \text{ with } e, i \geq 2.$$

Ma [7] presented an extended version of the Zalcman inequality for $g \in \mathscr{S}$ in 1999, which is expressed as

$$|d_i d_e - d_{i+e-1}| \leq (i-1)(e-1), \quad i, e \geq 2.$$

He demonstrated it only for a subfamily of $\mathscr{S}$, but the problem remains unresolved for the entire $\mathscr{S}$ family.

Subordination is an essential idea in geometric function theory that has important applications for complex analysis and leads to significant outcomes because it establishes connections between functions and their features. Let us look at the idea of subordination, which illustrates the relationship among analytic functions. When written as $g_1 \prec g_2$, this indicates that g_1 is subordinate to g_2. The definition is that given two functions $g_1, g_2 \in \mathscr{A}$, there exists a Schwarz function ω such that $g_1(\varsigma) = g_2(\omega(\varsigma))$ for $\varsigma \in \mathbb{E}$. If g_2 is univalent in $\mathbb{E}$, we can express this connection as follows.

$$g_1(\varsigma) \prec g_2(\varsigma), \quad \varsigma \in \mathbb{E}$$

if and only if the below conditions hold:

$$g_1(0) = g_2(0) \quad \text{and} \quad g_1(\mathbb{E}) \subset g_2(\mathbb{E}).$$

The Hankel determinant for class $\mathscr{S}$ was first analyzed by Pommerenke [8, 9]. The determinant $H_{2,1}(g)$ corresponds to a specific form of the Fekete-Szegö functional $\left|d_3 - \sigma d_2^2\right|$ for some complex σ. The Fekete-Szegö [10] inequality stands as one of the earliest challenges in the study of coefficients for univalent analytic functions established in 1933. Numerous

researchers have actively focused on exploring the estimate of the Hankel determinant $H_{2,2}(g)$. Table 1 provides significant articles on sharp second Hankel determinants for various subclasses.

Table 1. Sharp estimates on $|H_{2,2}(g)|$ for some subclasses of $\mathscr{S}$

Authors	Family	Sharp estimate	Year	Reference
Sim et al.	$\mathscr{C}(\alpha)$	$\frac{(1-\alpha)^2(6+5\alpha)}{48(1+\alpha)}$	2021	[11]
Tayyah et al.	$\mathscr{S}^*_{LE}$	$\frac{1}{36}$	2025	[12]
Zaprawa	$\mathscr{S}^*_s(e^{\varsigma})$	$\frac{1}{4}$	2022	[13]
Abbas et al.	$\mathscr{M}_\alpha(F(\varsigma))$	$\frac{1}{16(1+2\alpha)^2}$	2024	[14]
Hadi et al.	$\mathscr{S}^*_C$	$\frac{1}{16}$	2026	[15]
El-Ityan et al.	$\mathbb{M}_{\tan}$	$\frac{1}{4}$	2026	[16]
Faisal et al.	$\mathscr{SS}^*_{SG}$	$\frac{1}{16}$	2023	[17]

The logrithmic coefficients χ_s of $g \in \mathscr{S}$ are provided by the below formula

$$G_g(\varsigma) := \log\left(\frac{g(\varsigma)}{\varsigma}\right) = 2\sum_{q=1}^{\infty}\chi_q\varsigma^q, \quad \text{for } \varsigma \in \mathbb{E}.$$

These coefficients are essential in deriving various estimations within the framework of univalent function theory de Branges [18] established in 1985 that

$$\sum_{j=1}^{q} j(q-j+1)\left|\chi_q\right|^2 \leq \sum_{j=1}^{q}\frac{q-j+1}{j}, \quad \text{for } q \geq 1,$$

and equality holds if and only g is expressed as $\varsigma/\left(1-e^{\Phi i}\varsigma\right)^2$ for certain $\Phi \in \mathbb{R}$. Obviously, this inequality encompasses the renowned Bieberbach-Robertson-Milin conjectures concerning the Taylor coefficients of g presented in their most general expression. For a deeper exploration on the proof of de Branges result, we direct readers to [19–21]. In 2005, Kayumov [22] provided a solution to Brennan's conjecture for conformal mappings by focusing on the analysis of logarithmic coefficients.

If $g \in \mathscr{S}$ is given by (1), then its logarithmic coefficients are given by

$$\chi_1 = \frac{1}{2}d_2, \tag{2}$$

$$\chi_2 = \frac{1}{2}\left(d_3 - \frac{1}{2}d_2^2\right), \tag{3}$$

$$\chi_3 = \frac{1}{2}\left(d_4 - d_2d_3 + \frac{1}{3}d_2^3\right), \tag{4}$$

$$\chi_4 = \frac{1}{2}\left(d_5 - d_2 d_4 + d_2^2 d_3 - \frac{1}{2}d_3^2 - \frac{1}{4}d_2^4\right). \tag{5}$$

It can be noted that $H_{2,1}(G_g/2) = \chi_1\chi_3 - \chi_2^2$ is equivalent to the the renowned inequality $H_{2,1}(g) = d_3 - d_2^2$ within the family $\mathscr{S}$ or its subfamilies. Faisal et al. provided a precise estimation for $g \in \mathscr{S}\mathscr{S}^*_{SG}$ in [17] by claiming that $|H_{2,1}(G_g/2)| \leq 1/64$. Moreover, Abbas et al. demonstrated in [14] that $|H_{2,1}(G_g/2)| \leq 1/64(1+2\alpha)^2$ for $\mathscr{M}_\alpha(F(\varsigma))$. Several articles discuss precise bounds of $H_{2,1}(G_g/2)$ for specific subfamilies of univalent functions, as presented in Table 2.

Table 2. Sharp estimates on $|H_{2,2}(G_g/2)|$ for some subclasses of $\mathscr{S}$

Authors	Family	Sharp estimate	Year	Reference
Shi et al.	$\mathscr{R}_{3L}$	$\frac{1}{100}$	2022	[23]
Sunthrayuth et al.	$\mathscr{R}_{\sin}$	$\frac{1}{64}$	2022	[24]
Peng et al.	$\mathscr{R}_{\cosh}$	$\frac{1}{256}$	2024	[25]
Allu and Shaji	$\mathscr{S}^*$	$\frac{1}{8}$	2024	[26]
Ali et al.	$\mathscr{S}^*_{\cos}$	$-\frac{10577}{1028376} + \frac{1549}{4113504}\sqrt{1549}$	2024	[27]
Tayyah et al.	$\mathscr{S}^*_k$	$\frac{1}{36}$	2025	[28]
Attiya et al.	$\mathscr{S}^*_p$	$\frac{1}{64}$	2026	[29]

Recently in 2022, the function $\varphi_{n,\ell} = 1 + n\varsigma/(n+1) + \varsigma^n/(n+1)$ with $n \geq 2$ was introduced by Gandhi et al. in [30] which maps the disc $\mathbb{E}$ onto a region bounded by an epicycloid with $(n-1)$ cusp. A family $\mathscr{S}^*_{n-1,\ell}$ of starlike functions connected with $\varphi_{n,\ell}$ was established in the disc $\mathbb{E}$ and examined some geometrical properties of it. The symbols $\mathscr{S}^*_{3,\ell}$ and $\mathscr{S}^*_{4,\ell}$ indicate the families of starike functions associated with three-leaf-shaped and four-leaf-shaped domains, respectively. Using the function $\varphi_{4,\ell}$, Alshehry and his collaborators [31] introduced a subfamily of starlike functions subordinated to the four-leaf function, defined as follows:

$$\mathscr{S}^*_{4,\ell} := \left\{ g \in \mathscr{S} : \frac{\varsigma g'(\varsigma)}{g(\varsigma)} \prec 1 + \frac{5}{6}\varsigma + \frac{1}{6}\varsigma^5, \quad \varsigma \in \mathbb{E} \right\}.$$

Similar to the definition of $\mathscr{S}^*_{4,\ell}$, Sunthrayuth et al. [32] established a novel subfamily of bounded turning functions defined by

$$\mathscr{R}_{4,\ell} := \left\{ g \in \mathscr{S} : g'(\varsigma) \prec 1 + \frac{5}{6}\varsigma + \frac{1}{6}\varsigma^5, \quad \varsigma \in \mathbb{E} \right\}.$$

The image of the unit disk under $1 + \frac{5}{6}\varsigma + \frac{1}{6}\varsigma^5$ is shown in Figure 1.

Example 1 The family $\mathscr{R}_{4,\ell}$ is non-empty. To illustrate this, we present the following examples of functions $g_1(\varsigma), g_2(\varsigma), g_3(\varsigma)$ together with their corresponding derivatives. We observe that each derivative is subordinate to the function $1 + \frac{5}{6}\varsigma + \frac{1}{6}\varsigma^5$. Indeed, consider:

$$g_1'(\varsigma) = \sqrt{1+\varsigma} = 1 + \frac{1}{2}\varsigma - \frac{1}{8}\varsigma^2 + \cdots, \prec 1 + \frac{5}{6}\varsigma + \frac{1}{6}\varsigma^5,$$

$$g_2'(\varsigma)=\frac{2}{1+e^{-\varsigma}}=1+\frac{1}{2}\varsigma-\frac{1}{24}\varsigma^3+\cdots,\prec 1+\frac{5}{6}\varsigma+\frac{1}{6}\varsigma^5,$$

$$g_3'(\varsigma)=1+\frac{\varsigma^2}{2}\prec 1+\frac{5}{6}\varsigma+\frac{1}{6}\varsigma^5.$$

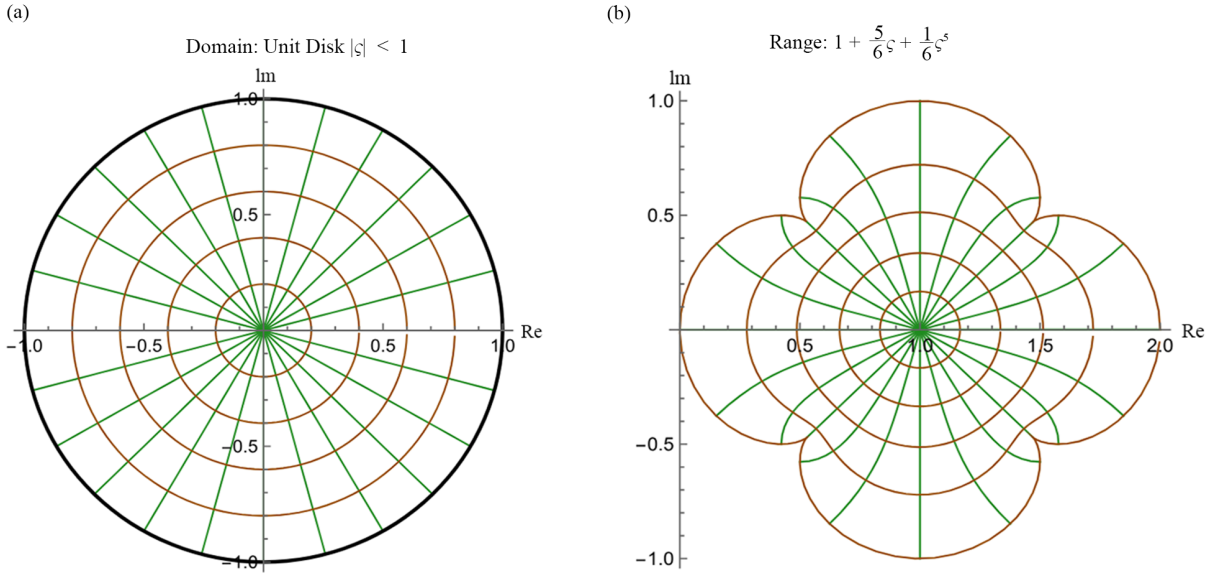


Figure 1. Visual representation of the domain and range for the four-leaf starlike function

Thus, all these functions belong to the class $\mathscr{R}_{4,\ell}$. The following Figure 2 illustrates the images of the unit disk under $g_1'(\varsigma), g_2'(\varsigma), g_3'(\varsigma)$:

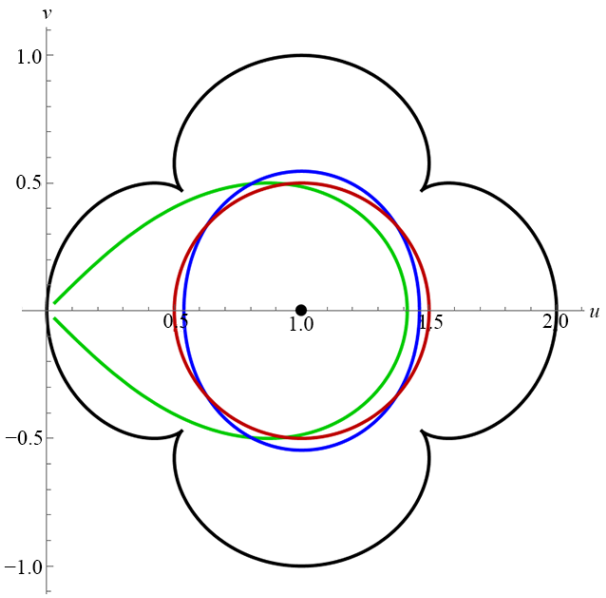


Figure 2. Images of the unit disk under $g_1'(\varsigma), g_2'(\varsigma), g_3'(\varsigma)$ corresponding respectively to the colors green, blue, and red

For the class $\mathscr{S}^*_{4,\ell}$, Alshehry et al. [31] determined the precise estimates for second Hankel determinants with logarithmic coefficients. In 2022, Sunthrayuth et al. [32] investigated the second Hankel determinant, the Krushkal inequality, and the coefficient functionals for the family $\mathscr{R}_{4,\ell}$. In 2024, Guo et al. [33] derived the precise estimate for the third Hankel determinant and demonstrated that $|H_{3,1}(g)| \leq 25/576$ within the class $\mathscr{R}_{4,\ell}$.
In the current research, we established the sharp bounds of logarithmic coefficients, the Krushkal, Fekete-Szegö, and Zalcman inequalities for $\mathscr{R}_{4,\ell}$. Moreover, we achieved the sharp estimates of $H_{2,1}(G_g/2)$ for the defined family.

Motivated by the results presented in Tables 1 and 2, which summarize several recent developments on coefficient problems and Hankel determinants for specific subfamilies of holomorphic functions, we observe that most existing studies focus on specific geometric domains or particular families of starlike and bounded turning functions.

However, the present work extends and unifies these directions by considering a new family of functions associated with a four-leaf-type domain. In this context, we derive sharp inequalities for logarithmic coefficients, including the Zalcman, and Krushkal inequalities, in addition to establishing sharp estimates for the second Hankel determinant involving logarithmic coefficients. It is worth emphasizing that the main novelty of this paper lies in combining the theory of Carathéodory functions and subordination techniques within the framework of the four-leaf-type domain, leading to new sharp results that generalize and improve several previously known works.

2. A set of lemmas

Denote with $\mathscr{P}$ the Carathéodory class consisting of simple functions q in $\mathbb{E}$ such that $\mathfrak{Re}(q(\varsigma)) > 0$ and are normalized:

$$q(\varsigma) = 1 + \sum_{n=1}^{\infty} \mu_n \varsigma^n, \quad \varsigma \in \mathbb{E}. \tag{6}$$

Lemma 1 [34] Let $q \in \mathscr{P}$, then for δ, κ, $\eta \in \overline{\mathbb{E}} := \{\varsigma \in \mathbb{C} : |\varsigma| < 1\}$.

$$2\mu_2 = \kappa\left(4 - \mu_1^2\right) + \mu_1^2, \tag{7}$$

$$4\mu_3 = 2\left(4 - \mu_1^2\right)\mu_1\kappa - \mu_1\kappa^2\left(4 - \mu_1^2\right) + 2\delta\left(1 - |\kappa|^2\right)\left(4 - \mu_1^2\right) + \mu_1^3, \tag{8}$$

$$\begin{aligned} 8\mu_4 = {} & \kappa\left(4 - \mu_1^2\right)\left[\left(\kappa^2 + 3 - 3\kappa\right)\mu_1^2 + 4\kappa\right] - 4\left(4 - \mu_1^2\right)\left(1 - |\kappa|^2\right) \\ & \times \left[-\left(1 - |\delta|^2\right)\eta + \overline{\kappa}\delta^2 + \mu\left(\kappa - 1\right)\delta\right] + \mu_1^4. \end{aligned} \tag{9}$$

Lemma 2 [35] Let ϖ, ρ, υ and ρ satifies $\rho \in (0,1)$ and $\upsilon \in (0,1)$ with

$$8\rho(1-\rho)\left((\upsilon\rho - 2\varpi)^2 + (\upsilon(\rho+\upsilon) - \rho)^2\right) + \upsilon(1-\upsilon)(\rho - 2\rho\upsilon)^2 \leq 4\rho\upsilon^2(1-\upsilon)^2(1-\rho). \tag{10}$$

Let $q \in \mathscr{P}$ has the form (6). Then

$$\left|\varpi\mu_1^4 + \rho\mu_2^2 + 2\upsilon\mu_1\mu_3 - \frac{3}{2}\rho\mu_1^2\mu_2 - \mu_4\right| \leq 2.$$

Lemma 3 [36] If $q \in \mathscr{P}$ and of the form (6), we get

$$|\mu_{n+k} - \sigma\mu_n\mu_k| \leq 2\max\{1, |2\sigma - 1|\} = \begin{cases} 2 & \text{for} \quad 0 \leq \sigma \leq 1; \\ 2|2\sigma - 1| & \text{otherwise.} \end{cases} \tag{11}$$

Lemma 4 [36] If $q \in \mathscr{P}$ has the form (6). Then

$$|\mu_e| \leq 2, \qquad e \geq 1. \tag{12}$$

Lemma 5 [37] If $q \in \mathscr{P}$ is provided by (6) and if $L \in [0,1]$ with $L(2L-1) \leq M \leq L$, then we have

$$\left|\mu_3 - 2L\mu_1\mu_2 + M\mu_1^3\right| \leq 2. \tag{13}$$

3. Logarithmic coefficient inequalities

This section starts by determining the first four logarithmic coefficient inequalities for $g \in \mathscr{R}_{4,\ell}$.

Theorem 1 Let $g \in \mathscr{R}_{4,\ell}$ and be of the form (1). Then

$$|\chi_1| \leq \frac{5}{24}, \tag{14}$$

$$|\chi_2| \leq \frac{5}{36}, \tag{15}$$

$$|\chi_3| \leq \frac{5}{48}, \tag{16}$$

$$|\chi_4| \leq \frac{1}{12}. \tag{17}$$

These inequalities are sharp.

Proof. If $g \in \mathscr{R}_{4,\ell}$, then from the use of subordination relationship we have

$$g'(\varsigma) = 1 + \frac{5}{6}\omega(\varsigma) + \frac{1}{6}\omega^5(\varsigma), \quad \varsigma \in \mathbb{E}.$$

and we define a function

$$q(\varsigma) = \frac{1+\omega(\varsigma)}{1-\omega(\varsigma)} = 1 + \mu_1\varsigma + \mu_2\varsigma^2 + \mu_3\varsigma^3 + \cdots.$$

It is easy to see that $q \in \mathscr{P}$ and

$$\omega(\varsigma)=\frac{q(\varsigma)-1}{q(\varsigma)+1}=\frac{\mu_1\varsigma+\mu_2\varsigma^2+\mu_3\varsigma^3+\mu_4\varsigma^4+\cdots}{2+\mu_1\varsigma+\mu_2\varsigma^2+\mu_3\varsigma^3+\mu_4\varsigma^4+\cdots}. \tag{18}$$

By utilizing the series expansion of (18), we achieve

$$1+\frac{5}{6}\omega(\varsigma)+\frac{1}{6}(\omega(\varsigma))^5=1+\left(\frac{5}{12}\mu_1\right)\varsigma+\left(\frac{5}{12}\mu_2-\frac{5}{24}\mu_1^2\right)\varsigma^2+\left(\frac{5}{12}\mu_3-\frac{5}{12}\mu_1\mu_2+\frac{5}{48}\mu_1^3\right)\varsigma^3$$

$$+\left(\frac{5}{16}\mu_1^2\mu_2-\frac{5}{96}\mu_1^4-\frac{5}{24}\mu_2^2-\frac{5}{12}\mu_1\mu_3+\frac{5}{12}\mu_4\right)\varsigma^4+\cdots, \tag{19}$$

and

$$g'(\varsigma)=\varsigma+\sum_{n=2}^{\infty}nd_n\varsigma^{n-1}=1+2d_2\varsigma+3d_3\varsigma^2+\cdots. \tag{20}$$

When the coefficients of ς, ς^2, ς^3, ς^4 are compared between the equations (19) and (20), we acquire

$$d_2=\frac{5}{24}\mu_1, \tag{21}$$

$$d_3=\frac{5}{36}\mu_2-\frac{5}{72}\mu_1^2, \tag{22}$$

$$d_4=-\frac{5}{48}\mu_1\mu_2+\frac{5}{48}\mu_3+\frac{5}{192}\mu_1^3, \tag{23}$$

$$d_5=-\frac{1}{24}\mu_2^2+\frac{1}{16}\mu_1^2\mu_2-\frac{1}{12}\mu_1\mu_3+\frac{1}{12}\mu_4-\frac{1}{96}\mu_1^4. \tag{24}$$

Plugging (21)–24 into (2)–(5), we acquire

$$\chi_1=\frac{5}{48}\mu_1, \tag{25}$$

$$\chi_2=\frac{5}{72}\mu_2-\frac{35}{768}\mu_1^2, \tag{26}$$

$$\chi_3=\frac{5}{96}\mu_3-\frac{115}{1728}\mu_1\mu_2+\frac{1805}{82944}\mu_1^3, \tag{27}$$

$$\chi_4=-\frac{28849}{2654208}\mu_1^4-\frac{133}{5184}\mu_2^2-\frac{121}{2304}\mu_1\mu_3+\frac{2071}{41472}\mu_1^2\mu_2+\frac{1}{24}\mu_4. \tag{28}$$

Implementing (12), in (25), we obtain

$$|\chi_1| \leq \frac{5}{24}.$$

In order to demonstrate the bound of $|\chi_2|$, we observe that the equality of (26) might be expressed in the following manner

$$\chi_2 = \frac{5}{72}\left(\mu_2 - \frac{21}{32}\mu_1^2\right).$$

Using (11), we have

$$|\chi_2| \leq \frac{5}{36}.$$

From (27), we can write

$$|\chi_3| = \frac{5}{96}\left|\left(\mu_3 - 2\left(\frac{23}{36}\right)\mu_1\mu_2 + \frac{361}{864}\mu_1^3\right)\right|.$$

From (13), let

$$L = \frac{23}{36} \text{ and } M = \frac{361}{864}.$$

It is clear that $0 \leq L \leq 1$, and $L \geq M$ with

$$L(2L-1) = \frac{115}{648} \leq M.$$

Thus all the conditions of (13) are satisfied. Hence we have

$$|\chi_3| \leq \frac{5}{48}.$$

We deduce from (28) that

$$\chi_4 = -\frac{1}{24}\left(\frac{28849}{110592}\mu_1^4 + \left(\frac{133}{216}\right)\mu_2^2 + 2\left(\frac{121}{192}\right)\mu_1\mu_3 - \frac{3}{2}\left(\frac{2071}{2592}\right)\mu_1^2\mu_2 - \mu_4\right).$$

$$= -\frac{1}{24}\left|\varpi\mu_1^4 + \rho\mu_2^2 + 2\upsilon\mu_1\mu_3 - \frac{3}{2}\rho\mu_1^2\mu_2 - \mu_4\right|, \tag{29}$$

where

$$\varpi = \frac{28849}{110592}, \quad \rho = \frac{133}{216}, \quad \upsilon = \frac{121}{192}, \quad \rho = \frac{2071}{2592}.$$

These constants satisfy $\rho \in (0,1)$ and $\upsilon \in (0,1)$ with

$$8\rho(1-\rho)\left((\upsilon\rho - 2\varpi)^2 + (\upsilon(\rho+\upsilon)-\rho)^2\right) + \upsilon(1-\upsilon)(\rho - 2\rho\upsilon)^2 \leq 4\rho\upsilon^2(1-\upsilon)^2(1-\rho).$$

Hence by (29) and (10), we deduce that

$$|\chi_4| \leq \frac{1}{12}.$$

Regarding sharpness, it is seen that the functions (2)–(5) can provide equalities with

$$g_u'(\varsigma) = 1 + \frac{5}{6}(\varsigma^u) + \frac{1}{6}\left(\varsigma^{5u}\right),$$

where $u = 1,2,3,4$. Then

$$g_1(\varsigma) = \int_0^{\varsigma}\left(1 + \frac{5}{6}(t) + \frac{1}{6}\left(t^5\right)\right)dt = \varsigma + \frac{5}{12}\varsigma^2 + \frac{1}{36}\varsigma^6 + \cdots, \tag{30}$$

$$g_2(\varsigma) = \int_0^{\varsigma}\left(1 + \frac{5}{6}\left(t^2\right) + \frac{1}{6}\left(t^{10}\right)\right)dt = \varsigma + \frac{5}{18}\varsigma^3 + \frac{1}{66}\varsigma^{11} + \cdots, \tag{31}$$

$$g_3(\varsigma) = \int_0^{\varsigma}\left(1 + \frac{5}{6}\left(t^3\right) + \frac{1}{6}\left(t^{15}\right)\right)dt = \varsigma + \frac{5}{24}\varsigma^4 + \frac{1}{96}\varsigma^{16} + \cdots, \tag{32}$$

$$g_4(\varsigma) = \int_0^{\varsigma}\left(1 + \frac{5}{6}\left(t^4\right) + \frac{1}{6}\left(t^{20}\right)\right)dt = \varsigma + \frac{1}{6}\varsigma^5 + \frac{1}{126}\varsigma^{21} + \cdots. \tag{33}$$

□

We now aim to compute the Fekete-Szegö inequality for $g \in \mathscr{R}_{4,\ell}$.

Theorem 2 If $g \in \mathscr{R}_{4,\ell}$, and be of the form (1), then

$$\left|\chi_2 - \sigma\chi_1^2\right| \leq \max\left\{\frac{5}{36}, \left|\frac{25|\sigma|+25}{576}\right|\right\}, \text{ for } \sigma \in \mathbb{C}.$$

This functional is best possible.

Proof. Utilizing (25), and (26), we derive

$$\left|\chi_2-\sigma\chi_1^2\right|=\left|\frac{5}{72}\mu_2-\frac{35}{768}\mu_1^2-\frac{25\sigma}{2304}\mu_1^2\right|.$$

Application of (11), leads us to

$$\left|\chi_2-\sigma\chi_1^2\right|\leq\max\left\{\frac{5}{36},\left|\frac{25\left|\sigma\right|+25}{576}\right|\right\}.$$

This required result is sharp and is determined by using (2), (3) and (31). □

We examine the Zalcman inequalities for $g\in\mathscr{R}_{4,\ell}$ in the next results.

Theorem 3 If $g\in\mathscr{R}_{4,\ell}$, and be of the form (1). Then

$$\left|\chi_4-\chi_2^2\right|\leq\frac{1}{12}.$$

The above functional is best possible.

Proof. Utilizing (26) and (28), we gain

$$\left|\chi_4-\chi_2^2\right|=-\frac{1}{24}\left|\frac{68723}{221184}\mu_1^4+\frac{79}{108}\mu_2^2+2\left(\frac{121}{192}\right)\mu_1\mu_3-\frac{3}{2}\left(\frac{4667}{5184}\right)\mu_1^2\mu_2-\mu_4\right|.$$

$$=-\frac{1}{24}\left|\varpi\mu_1^4+\rho\mu_2^2+2\upsilon\mu_1\mu_3-\frac{3}{2}\rho\mu_1^2\mu_2-\mu_4\right|, \tag{34}$$

where

$$\varpi=\frac{68723}{221184},\qquad\rho=\frac{79}{108},\qquad\upsilon=\frac{121}{192},\qquad\rho=\frac{4667}{5184}.$$

These constants satisfy $\rho\in(0,1)$ and $\upsilon\in(0,1)$ with

$$8\rho\left(1-\rho\right)\left(\left(\upsilon\rho-2\varpi\right)^2+\left(\upsilon\left(\rho+\upsilon\right)-\rho\right)^2\right)+\upsilon\left(1-\upsilon\right)\left(\rho-2\rho\upsilon\right)^2\leq4\rho\upsilon^2\left(1-\upsilon\right)^2\left(1-\rho\right).$$

Hence by (34) and (10), we deduce that

$$\left|\chi_4-\chi_2^2\right|\leq\frac{1}{12}.$$

This outcome is sharp and is determined by utilizing (3), (5) and (33). □

Theorem 4 Let $g\in\mathscr{R}_{4,\ell}$ has the series expansion (1). Then

$$|\chi_4 - \chi_1\chi_3| \leq \frac{1}{12}.$$

The inequality is sharp.

Proof. From (25), (27) and (28), we have

$$|\chi_4 - \chi_1\chi_3| = \frac{1}{40}\left|\frac{104597}{331776}\mu_1^4 + \frac{133}{216}\mu_2^2 + 2\left(\frac{89}{128}\right)\mu_1\mu_3 - \frac{3}{2}\left(\frac{4717}{5184}\right)\mu_1^2\mu_2 - \mu_4\right|.$$

$$= \frac{1}{40}\left|\varpi\mu_1^4 + \rho\mu_2^2 + 2\upsilon\mu_1\mu_3 - \frac{3}{2}\rho\mu_1^2\mu_2 - \mu_4\right|, \tag{35}$$

where

$$\varpi = \frac{104597}{331776}, \quad \rho = \frac{133}{216}, \quad \upsilon = \frac{89}{128}, \quad \rho = \frac{4717}{5184}.$$

These constants satisfy $\rho \in (0,1)$ and $\upsilon \in (0,1)$ with

$$8\rho(1-\rho)\left((\upsilon\rho - 2\varpi)^2 + (\upsilon(\rho+\upsilon) - \rho)^2\right) + \upsilon(1-\upsilon)(\rho - 2\rho\upsilon)^2 \leq 4\rho\upsilon^2(1-\upsilon)^2(1-\rho).$$

Hence by (35) and (10), we deduce that

$$|\chi_4 - \chi_1\chi_3| \leq \frac{1}{12}.$$

This outcome is sharp and is determined by utilizing (2), (4), (5) and (33). □

Theorem 5 If $g \in \mathscr{R}_{4,\ell}$. Then

$$|\chi_1\chi_2 - \chi_3| \leq \frac{5}{48}.$$

The outcome is sharp.

Proof. Using (25)–(27), we have

$$|\chi_1\chi_2 - \chi_3| = -\frac{5}{96}\left|\mu_3 - 2\left(\frac{17}{24}\right)\mu_1\mu_2 + \frac{1759}{3456}\mu_1^3\right|.$$

From (13), let

$$L = \frac{17}{24} \text{ and } M = \frac{1759}{3456}.$$

It is clear that $0 \leq L \leq 1$, and $L \geq M$ with

$$L(2L-1) = \frac{85}{288} \leq M.$$

Thus all the conditions of (13) are satisfied. Hence we have

$$|\chi_1\chi_2 - \chi_3| \leq \frac{5}{48}.$$

This result is best possible and is obtained by using (2)–(4) and (32). □

Further, we intend to investigate the Krushkal functionals for $g \in \mathscr{R}_{4,\ell}$.

Theorem 6 If $g \in \mathscr{R}_{4,\ell}$, and be of the form (1). Then

$$\left|\chi_4 - \chi_1^4\right| \leq \frac{1}{12}.$$

The above functional is best possible.

Proof. Incorporating (25) and (28), we arrive at

$$\left|\chi_4 - \chi_1^4\right| = -\frac{1}{24}\left|\frac{19441}{73728}\mu_1^4 + \frac{133}{216}\mu_2^2 + 2\left(\frac{121}{192}\right)\mu_1\mu_3 - \frac{3}{2}\left(\frac{2071}{2592}\right)\mu_1^2\mu_2 - \mu_4\right|.$$

$$= -\frac{1}{24}\left|\varpi\mu_1^4 + \rho\mu_2^2 + 2\upsilon\mu_1\mu_3 - \frac{3}{2}\rho\mu_1^2\mu_2 - \mu_4\right|, \tag{36}$$

where

$$\varpi = \frac{19441}{73728}, \quad \rho = \frac{133}{216}, \quad \upsilon = \frac{121}{192}, \quad \rho = \frac{2071}{2592}.$$

These constants satisfy $\rho \in (0,1)$ and $\upsilon \in (0,1)$ with

$$8\rho(1-\rho)\left((\upsilon\rho - 2\varpi)^2 + (\upsilon(\rho+\upsilon) - \rho)^2\right) + \upsilon(1-\upsilon)(\rho - 2\rho\upsilon)^2 \leq 4\rho\upsilon^2(1-\upsilon)^2(1-\rho).$$

Hence by (36) and (10), we deduce that

$$\left|\chi_4 - \chi_1^4\right| \leq \frac{1}{12}.$$

This outcome is sharp and is determined by using (2), (5) and (33). □

Theorem 7 If $g \in \mathscr{R}_{4,\ell}$, has the form (1) , then

$$\left|\chi_3 - \chi_1^3\right| \leq \frac{5}{48}.$$

This inequality is sharp.

Proof. Using (25) and (27), we derive

$$\left|\chi_3 - \chi_1^3\right| = \frac{5}{96}\left|\mu_3 - 2\left(\frac{23}{36}\right)\mu_1\mu_2 + \frac{1369}{3456}\mu_1^3\right|.$$

From (13), let

$$L = \frac{23}{36} \quad \text{and} \quad M = \frac{1369}{3456}.$$

It is clear that $0 \leq L \leq 1$, and $L \geq M$ with

$$L(2L-1) = -\frac{2}{25} \leq M.$$

Thus all the conditions of (13) are satisfied. Hence we have

$$\left|\chi_3 - \chi_1^3\right| \leq \frac{5}{48}.$$

This result is best possible and is obtained by using (2), (4) and (32). □

4. Second Hankel determinants involving logarithmic coefficients

Lastly, we will explore the second Hankel determinants including logarithmic coefficients for $g \in \mathscr{R}_{4,\ell}$.

Theorem 8 Let $g \in \mathscr{R}_{4,\ell}$, then

$$\left|H_{2,1}\left(G_g/2\right)\right| = \left|\chi_1\chi_3 - \chi_2^2\right| \leq \frac{25}{1296}.$$

This outcome is sharp.

Proof. Plugging (25)–(27), we achieve

$$H_{2,1}\left(G_g/2\right) = \frac{25}{4608}\mu_1\mu_3 - \frac{25}{41472}\mu_1^2\mu_2 + \frac{3025}{15925248}\mu_1^4 - \frac{25}{5184}\mu_2^2.$$

Incorporating Lemma 1 to transform μ_2 and μ_3 into μ_1, and recognizing that μ_1 can be further expressed as μ. Then

$$|H_{2,1}(G_g/2)| = \left| \frac{625}{15925248}\mu^4 - \frac{25\left(4-\mu^2\right)}{18432}\mu^2\kappa^2 + \frac{25\left(4-\mu^2\right)\left(1-|\kappa|^2\right)}{9216}\mu\delta - \frac{25\left(4-\mu^2\right)^2}{20736}\kappa^2 \right|,$$

Implementing the triangle inequality and substituting $|\delta| \leq 1$, $|\kappa| = \xi$, with $\xi \leq 1$. Thus

$$|H_{2,1}(G_g/2)| \leq \frac{625}{15925248}\mu^4 + \frac{25}{18432}\mu^2\left(4-\mu^2\right)\xi^2 + \frac{25}{9216}\mu\left(4-\mu^2\right)\left(1-\xi^2\right) + \frac{25}{20736}\left(4-\mu^2\right)^2\xi^2 := \phi(\mu,\xi).$$

Establishing that $\phi'(\mu,\xi) \geq 0$ is now uncomplicated, resulting in the conclusion that $\phi(\mu,\xi) \leq \phi(\mu,1)$. Assuming $\xi = 1$ produces

$$|H_{2,1}(G_g/2)| \leq \frac{625}{15925248}\mu^4 + \frac{25\left(4-\mu^2\right)}{18432}\mu^2 + \frac{25\left(4-\mu^2\right)^2}{20736} := \phi(\mu,1).$$

By performing a basic calculation, it confirms that $\phi(\mu,1)$. achieves its optimum value at 0. Then

$$|H_{2,1}(G_g/2)| \leq \frac{25}{1296}.$$

The $H_{2,1}(G_g/2)$ is sharp, with equality occurring at (2)–(4) and (31). □

Theorem 9 Suppose that $g \in \mathscr{R}_{4,\ell}$, then

$$|H_{2,2}(G_g/2)| = \left|\chi_2\chi_4 - \chi_3^2\right| \leq \frac{25}{2304}.$$

This is the sharp result.

Proof. Utilizing (26)–(28) along with $\mu_1 = \mu$, it yields

$$\begin{aligned} H_{2,2}(G_g/2) = \frac{1}{55037657088}\Big(&1198105\mu^6 - 7379040\mu^4\mu_2 + 6963840\mu^3\mu_3 + 11450880\mu^2\mu_2^2 \\ &- 104509440\mu^2\mu_4 + 180817920\mu\mu_2\mu_3 - 98058240\mu_2^3 \\ &+ 159252480\mu_2\mu_4 - 149299200\mu_3^2\Big). \end{aligned}$$

Let $p = 4-\mu^2$ in (7)–(9) and using Lemma 1, we acquire

$$H_{2,2}(G_g/2) = \frac{1}{55037657088}\Big\{-12257280p^3\kappa^3 + 39813120p^2\kappa^3 + 12441600\mu^3 p\left(1-|\kappa|^2\right)\kappa\delta$$

$$+12441600\mu^2 p\left(1-|\kappa|^2\right)\overline{\kappa}\delta^2-12441600\mu^2 p\left(1-|\kappa|^2\right)\left(1-|\delta|^2\right)\eta$$

$$+10368000\mu\kappa p^2\left(1-|\kappa|^2\right)\delta-2488320\mu p^2\kappa^2\left(1-|\kappa|^2\right)\delta-39813120p^2\kappa\overline{\kappa}\left(1-|\kappa|^2\right)\delta^2$$

$$+39813120p^2\kappa\left(1-|\kappa|^2\right)\left(1-|\delta|^2\right)\eta-150000\mu^4 p\kappa+3650400\mu^4 p\kappa^2$$

$$+3830400\mu^2 p^2\kappa^2-12441600\mu^2 p\kappa^2-3110400\mu^4 p\kappa^3-15137280\mu^2 p^2\kappa^3$$

$$+622080\mu^2 p^2\kappa^4-37324800p^2\left(1-|\kappa|^2\right)^2\delta^2-1080000\mu^3 p\delta\left(1-|\kappa|^2\right)+15625\mu^6\Big\}.$$

Since $p=4-\mu^2$, then

$$H_{2,2}\left(G_g/2\right)=\frac{1}{55037657088}\left(b_1\left(\mu,\kappa\right)+b_2\left(\mu,\kappa\right)\delta+b_3\left(\mu,\kappa\right)\delta^2+\nabla\left(\mu,\kappa,\delta\right)\eta\right),$$

where

$$b_1\left(\mu,\kappa\right)=15625\mu^6+\left(4-\mu^2\right)\left[\left(4-\mu^2\right)\left(622080\mu^2\kappa^4-2880000\mu^2\kappa^3-9216000\kappa^3\right.\right.$$

$$\left.\left.+3830400\mu^2\kappa^2\right)-150000\mu^4\kappa+3650400\mu^4\kappa^2-12441600\mu^2\kappa^2-3110400\mu^4\kappa^3\right],$$

$$b_2\left(\mu,\kappa\right)=\left(1-|\kappa|^2\right)\left(4-\mu^2\right)\left[\left(4-\mu^2\right)\left(-2488320\mu\kappa^2+10368000\mu\kappa\right)-1080000\mu^3+12441600\mu^3\kappa\right],$$

$$b_3\left(\mu,\kappa\right)=\left(1-|\kappa|^2\right)\left(4-\mu^2\right)\left[\left(4-\mu^2\right)\left(-2488320\left|\kappa\right|^2-37324800\right)+12441600\mu^2\overline{\kappa}\right],$$

$$\nabla\left(\mu,\kappa,\delta\right)=\left(4-\mu^2\right)\left(1-|\delta|^2\right)\left(1-|\kappa|^2\right)\left[-12441600\mu^2+39813120\kappa\left(4-\mu^2\right)\right].$$

Utilizing $|\kappa|=\kappa$, $|\delta|=\zeta$ with $|\eta|\leq 1$, we achieve

$$\left|H_{2,2}\left(G_g/2\right)\right|\leq\frac{1}{55037657088}\left(\left|b_1\left(\mu,\kappa\right)\right|+\left|b_2\left(\mu,\kappa\right)\right|\zeta+\left|b_3\left(\mu,\kappa\right)\right|\zeta^2+\left|\nabla\left(\mu,\kappa,\delta\right)\right|\right)$$

$$\leq\frac{1}{55037657088}\left(Y\left(\mu,\kappa,\zeta\right)\right),\tag{37}$$

where

$$Y(\mu,\kappa,\zeta)=a_1(\mu,\kappa)+a_2(\mu,\kappa)\zeta+a_3(\mu,\kappa)\zeta^2+a_4(\mu,\kappa)\left(1-\zeta^2\right),$$

with

$$a_1(\mu,\kappa)=15625\mu^6+\left(4-\mu^2\right)\left[\left(4-\mu^2\right)\left(622080\mu^2\kappa^4+2880000\mu^2\kappa^3+9216000\kappa^3\right.\right.$$

$$\left.\left.+3830400\mu^2\kappa^2\right)+150000\mu^4\kappa+3650400\mu^4\kappa^2+12441600\mu^2\kappa^2+3110400\mu^4\kappa^3\right],$$

$$a_2(\mu,\kappa)=\left(1-\kappa^2\right)\left(4-\mu^2\right)\left[\left(4-\mu^2\right)\left(2488320\mu\kappa^2+10368000\mu\kappa\right)+1080000\mu^3+12441600\mu^3\kappa\right],$$

$$a_3(\mu,\kappa)=\left(1-\kappa^2\right)\left(4-\mu^2\right)\left[\left(4-\mu^2\right)\left(2488320\kappa^2+37324800\right)+12441600\mu^2\kappa\right],$$

$$a_4(\mu,\kappa)=\left(1-\kappa^2\right)\left(4-\mu^2\right)\left[12441600\mu^2+39813120\kappa\left(4-\mu^2\right)\right].$$

We needs to optimize $Y(\mu,\kappa,\zeta)$ within the cuboid $\Xi:[0,2]\times[0,1]\times[0,1]$. To accomplish this goal, we need to identify the optimal value of $Y(\mu,\kappa,\zeta)$ inside Ξ, on its faces, and along its edges.

1. Interior points of cuboid Ξ :

Suppose that $(\mu,\kappa,\zeta)\in(0,2)\times(0,1)\times(0,1)$, then

$$\frac{\partial Y}{\partial\zeta}=8640\left(4-\mu^2\right)\left(1-\kappa^2\right)\left[(\kappa-1)\zeta\left(576(\kappa-15)\left(4-\mu^2\right)+2880\mu^2\right)\right.$$

$$\left.+\mu\left(288\kappa\left(4-\mu^2\right)(\kappa+300)+5\mu^2(288\kappa+25)\right)\right].$$

Plugging $\frac{\partial Y}{\partial\zeta}=0$ yields

$$\zeta=\frac{\mu\left(288\kappa\left(4-\mu^2\right)(\kappa+300)+5\mu^2(288\kappa+25)\right)}{(\kappa-1)\left(576(15-\kappa)\left(4-\mu^2\right)-2880\mu^2\right)}=\zeta_0.$$

When ζ_0 should belong to $(0,1)$, then it is possible only if

$$1440\mu^3(288\kappa+25)+288\mu\kappa\left(4-\mu^2\right)(\kappa+300)+576(1-\kappa)\left(4-\mu^2\right)(15-\kappa)<288(1-\kappa)\mu^2. \tag{38}$$

and

$$\mu^2>\frac{2304(15-\kappa)}{11520-576\kappa}. \tag{39}$$

A solution is considered a optimal point only when it fulfills both functionals (38) and (39).

Let $m(\kappa) = \frac{2304(15-\kappa)}{11520-576\kappa}$. This function decreases over $(0,1)$. Therefore $\mu^2 > \frac{56}{19}$, and a fundamental computation confirms that (38) fails to hold for $\kappa \in (0,1)$. We confirm that no optimal point for Y is present in $(0,2)\times(0,1)\times(0,1)$.

2. Interior of the faces of cuboid Ξ :

Considering $\mu = 0$, we get

$$I_1(\kappa,\zeta) = 1474560\left[(27(\kappa-1)\zeta^2(\kappa-15)+432\kappa)(1-\kappa^2)+100\kappa^3\right].$$

But $\frac{\partial I_1}{\partial \zeta} \neq 0$ for $\kappa,\zeta \in (0,1)$. So, there is no maxima for I_1 in $(0,1)\times(0,1)$.

Choosing $\mu = 2$, we get

$$Y(2,\kappa,\zeta) = 1000000. \tag{40}$$

If $\kappa = 0$, it gives

$$I_2(\mu,\zeta) = Y(\mu,0,\zeta) = 15625\mu^6 + (4-\mu^2)\left(1080000\mu^3\zeta + (149299200 - 49766400\mu^2)\zeta^2 + 12441600\mu^2\right).$$

Solving $\frac{\partial I_2}{\partial \zeta} = 0$, we get

$$\zeta = \frac{25\mu^3}{2304(\mu^2-3)} = \zeta_1.$$

Then $\zeta_1 \in (0,1)$ if $\mu > \mu_0$, with $\mu_0 \approx 1.748721115$. Also

$$\frac{\partial I_2}{\partial \mu} = 93750\mu^5 - 24883200\mu^3 + \left(4-\mu^2\right)\left(3240000\mu^2\zeta - 99532800\mu\zeta^2 + 24883200\mu\right)$$

$$-2160000\mu^4\zeta + \mu\left(99532800\mu^2 - 298598400\right)\zeta^2. \tag{41}$$

Plugging the value of ζ in (41) along with $\frac{\partial I_2}{\partial \mu} = 0$, we attain

$$\frac{\partial I_2}{\partial \mu} = 75\mu\left(3125\mu^8 - 2671708\mu^6 + 21256164\mu^4 - 55738368\mu^2 + 47775744\right) = 0. \tag{42}$$

Based on a numerical calculation, the solution to (42) in $(0,2)$ is $\mu \approx 1.41421356$. This implies $I_2(\mu,\zeta)$ has no maxima in $(0,2)\times(0,1)$.

Selecting $\kappa = 1$ gives

$$I_3(\mu,\zeta) = Y(\mu,1,\zeta) = 437305\mu^6 - 34242240\mu^4 + 93358080\mu^2 + 147456000. \tag{43}$$

Solving $\frac{\partial I_3}{\partial \mu} = 0$, we get critical point at $\mu \approx 1.1835494$. Thus $I_3(\mu,\zeta)$ acquires its maxima at μ. Then

$$Y(\mu,1,\zeta) \leq 212242506.7.$$

If $\zeta = 0$, then

$$\begin{aligned} I_4(\mu,\kappa) = Y(\mu,\kappa,0) = {} & 622080\mu^6\kappa^4 - 230400\mu^6\kappa^3 + 180000\mu^6\kappa^2 - 4976640\mu^4\kappa^4 - 150000\mu^6\kappa \\ & - 41195520\mu^4\kappa^3 + 15625\mu^6 - 16041600\mu^4\kappa^2 + 9953280\mu^2\kappa^4 + 40413120\mu^4\kappa \\ & + 290856960\mu^2\kappa^3 - 12441600\mu^4 + 61286400\mu^2\kappa^2 - 318504960\mu^2\kappa \\ & - 489553920\kappa^3 + 49766400\mu^2 + 637009920\kappa. \end{aligned}$$

A computational analysis illustrates that there is no solution in $(0,2)\times(0,1)$ to the equations $\frac{\partial I_4}{\partial \kappa} = 0$ and $\frac{\partial I_4}{\partial \mu} = 0$. Setting $\zeta = 1$, it gives

$$\begin{aligned} I_5(\mu,\kappa) = Y(\mu,\kappa,1) = {} & 622080\mu^6\kappa^4 - 230400\mu^6\kappa^3 - 2488320\mu^5\kappa^4 + 180000\mu^6\kappa^2 \\ & + 2073600\mu^5\kappa^3 - 7464960\mu^4\kappa^4 - 150000\mu^6\kappa + 3568320\mu^5\kappa^2 \\ & + 11059200\mu^4\kappa^3 + 19906560\mu^3\kappa^4 + 15625\mu^6 - 2073600\mu^5\kappa \\ & - 63319680\mu^4\kappa^2 + 33177600\mu^3\kappa^3 + 29859840\mu^2\kappa^4 - 1080000\mu^5 \\ & - 11841600\mu^4\kappa - 24226560\mu^3\kappa^2 - 77414400\mu^2\kappa^3 - 39813120\mu\kappa^4 \\ & + 37324800\mu^4 - 33177600\mu^3\kappa + 389744640\mu^2\kappa^2 - 165888000\mu\kappa^3 \\ & - 39813120\kappa^4 + 4320000\mu^3 + 49766400\mu^2\kappa + 39813120\mu\kappa^2 \\ & + 147456000\kappa^3 - 298598400\mu^2 + 165888000\mu\kappa - 557383680\kappa^2 \\ & + 597196800. \end{aligned}$$

The system of equations $\frac{\partial I_5}{\partial \kappa} = 0$ and $\frac{\partial I_5}{\partial \mu} = 0$ in $(0,2)\times(0,1)$ does not have a solution.

3. On the Edges of Cuboid Ξ :

If $\zeta = 0$, it gives

$$Y(\mu,0,0) = 15625\mu^6 - 12441600\mu^4 + 49766400\mu^2 = I_6(\mu).$$

Clearly $I_6'(\mu) = 0$ for $\mu \approx 1.41689538$ in $[0,2]$, where maxima for $I_6(\mu)$ is acquired at μ. Then

$$Y(\mu,0,0) \leq 49892111.78.$$

Assuming $\zeta = 1$, it yields

$$Y(\mu,0,1) = 93750\mu - 1080000\mu^5 + 37324800\mu^4 + 4320000\mu^3 - 298598400\mu^2 + 597196800 = I_7(\mu).$$

Since $I_7'(\mu) < 0$. It reveals that $I_7(\mu)$ decreases as μ decreases. Thus

$$I_7(\mu) \leq 597196800.$$

Setting $\mu = 0$, it yields

$$Y(0,0,\zeta) = 597196800\zeta^2 = I_{70}(\zeta).$$

As $I_{70}'(\zeta) > 0$. It is shown that $I_7(\zeta)$ increases in a continuous manner with ζ, then

$$I_{70}(\zeta) \leq 597196800.$$

We see that (43) is independent of ζ, it follows

$$Y(\mu,1,0) = Y(\mu,1,1) = 437305\mu^6 - 34242240\mu^4 + 93358080\mu^2 + 147456000 = I_8(\mu).$$

Now $I_8'(\mu) = 0$ gives $\mu \approx 1.18354949$ within $[0,2]$ at which $I_8(\mu)$ acquire its optima at μ, that is

$$I_8(\mu) \leq 212242506.71.$$

When $\mu = 0$, then

$$Y(0,1,\zeta) = 147456000.$$

Given that (40) is not dependent on μ, ζ and κ, it derives

$$Y(2,0,\zeta) = Y(2,\kappa,0) = Y(2,1,\zeta) = Y(2,\kappa,1) = 1000000.$$

Considering $\zeta = 1$ gives

$$I_9(\kappa) = -39813120\kappa^4 + 147456000\kappa^3 - 557383680\kappa^2 + 597196800.$$

It is established that $I_9'(\kappa) < 0$ on $[0,1]$, confirming that $I_9(\kappa)$ acquires its optima at $\kappa = 0$. Then

$$I_9(\kappa) \leq 597196800.$$

Selecting $\zeta = 0$, it becomes

$$I_{10}(\kappa) = -489553920\kappa^3 + 637009920\kappa.$$

We determine $\kappa \approx 0.722315118$ for $[0,1]$ upon solving $I_{10}'(\kappa) = 0$. Thereby, the maxima of $I_{10}(\kappa)$ are located at κ as indicated by

$$I_{10}(\kappa) \leq \frac{2548039680}{83}\sqrt{83}.$$

Consequently, the aforementioned information enables us to deduce that

$$Y(\mu,\kappa,\zeta) \leq 597196800 \quad \text{on} \quad [0,2] \times [0,1] \times [0,1].$$

From (37), we find that

$$|H_{2,2}(G_g/2)| \leq \frac{1}{55037657088}(Y(\mu,\kappa,\zeta)) \leq \frac{25}{2304} \approx 0.01085.$$

The use of (3)–(5) and (32) yields a precise estimate for the current determinant if $g \in \mathscr{R}_{4,\ell}$. □

5. Conclusions

Geometric Function Theory is one of the numerous fields of mathematics where inequalities have a significant role in resolving mathematical issues. The Hankel determinant is the most important function theory problem that is still under investigation due to its widespread applications in non-stationary signals, Markov processes, and other fields. We have investigated the precise estimates of logarithmic coefficients, the Krushkal, Fekete-Szegö and Zalcman inequalities for the function of bounded turning related to a four-leaf-shaped domain in this paper. More importantly, we have investigated the sharp estimates of the second Hankel determinants by using logarithmic coefficients as entries.

All of the achieved outcomes have been verified as the best possible. The goal of this work is to draw more attention to coefficient-related issues regarding logarithmic inverse and inverse functions for particular families of holomorphic functions. For example, Majorization problems, Toeplitz determinants, partial sum inequalities and convolution characteristics are merely some of the topics in which scholars can demonstrate their expertise [38–41].

Author contributions

Muhammad Abbas developed the idea for this article, and Muhammad Arif enhanced it further. Muhammad Abbas and Reda Elbarougy performed the computation and finalized the writing. Alina Alb Lupas and Muhammad Imran Faisal examined and confirmed all the results. All authors reviewed the article and gave their approval for the final version.

Conflict of interest

The authors declare no competing financial interest.

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