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Research Article

Fractional Bullen-Mercer Type Inequalities via Majorization: A Conticrete Approach with Numerical Validation and Applications

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Abstract: This article establishes new Bullen-Mercer type inequalities through a unified framework combining majorization theory, convexity, and Riemann-Liouville fractional integrals. A generalized conticrete integral identity is first constructed, which act as the foundation for deriving estimates of the Bullen-Mercer gap in conticrete settings. These estimates are obtained under the convexity assumptions of $|f'|$, $|f'|^q\,(1<q)$, utilizing properties of the modulus, Hölder, power mean, and Young's inequalities. Each theoretical result is accompanied by numerical validation tables for varying fractional orders, as well as 2-D and 3-D graphical illustrations that confirm their validity. Remarks accompanying each result highlight new and previously known inequalities as special cases of the major findings. To demonstrate practical relevance, the main findings are applied to modified Bessel functions of the first kind and to special means, with supporting 2-D visualizations. In last, the work is summed up in a thorough conclusion.

Keywords: Riemann-Liouville fractional integrals, Bullen-Mercer inequality, Hermite-Hadamard inequality, majorization, conticrete approach, numerical validatio

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1. Introduction

One of the most persistent and significant ideas in mathematical analysis is the concept of convexity [1, 2]. It has remained a valuable tool for mathematicians since its discovery over a century ago. It enables the study of functional relationships, optimization problems, and structural properties of mappings [3]. Convexity, on one hand, is simple to allow elegant theoretical refinement, while on the other, it is flexible enough to have meaningful applications across Engineering disciplines [4], Mathematical Statistics [5], Economics [6], Information theory [7], and Physical sciences [8]. Due to this versatile nature, it is treated as an active area of investigation. It continues to evolve through different generalizations and refinements that enhance its reach into new mathematical domains. Convex function [9] is an important aspect of this theory which is defined by the property that the line segment connecting any two points on its graph lies above the graph itself. The analytic condition that has developed into the fundamental characteristic of convexity comes from this geometric concept. This idea has resulted in many noteworthy extensions over time. These include s-convex functions [10], geometric-arithmetic (GA)-convex functions [11], m-convex functions [12], η-convex functions [13], exponentially convex functions [14], co-ordinate convex functions [15] and many others. These classes have opened up new ways for research.

There exist a firm linkage between the areas of convexity and inequalities. There is some property of convex functions that makes them particularly useful in working with inequalities. Whether one is looking to bound a quantity, estimate an integral, or make a comparison between functional values, a convex function is always the appropriate choice. This is the primary reason a number of classical inequalities have been established, and which now serve as one of the main sources of inspiration for current research. Jensen's inequality [7] is the most classical of these, and most of the generalizations that have been obtained through it, capture the essence of convex functions. Fejér's inequality [16] is one of the first to incorporate the use of weighting functions into the convex function framework. Meanwhile, Slater's inequality [17] addresses convex functions from a different, yet complementary, perspective. Additionally, Ostrowski's inequality [18] and the theory of majorization [19] have been useful in supporting the theory of convex functions, and each of them yields important results related to the structure and behavior of convex functions.

The Hermite-Hadamard [20] inequality holds a special place in the list of classical inequalities, described above. This inequality provides an upper and lower bound to the integrative average of convex function. Additionally, it has been the focus of certain remarkable generalizations which show a specific aspect of convexity. One of these was Bullen [21], contribution, in which he was able to formulate an inequality that associate the integral mean of a convex function with a mixture of its mid and end point averages. To be more specific, Bullen stated that given a convex function f defined on $[h_3, h_4]$, the inequality looks as follows:

$$\frac{1}{h_4 - h_3} \int_{h_3}^{h_4} f(u)\, du \leq \frac{1}{2} \left[f\left(\frac{h_3 + h_4}{2} \right) + \frac{f(h_3) + f(h_4)}{2} \right]. \tag{1}$$

This Bullen inequality has drawn significant interest because of its application to quadrature theory and its ability to give more sophisticated error estimates for numerical integration methods. The inequality is situated between the classical Hermite-Hadamard bounds and provides an alternative view of convex functions in relation to the averaging of their values. It is obvious from the literature that the influence of Bullen's inequality spans multiple areas of analysis. Erden and Sarikaya [22] obtained Bullen-type inequalities in the context of fractal geometry. Cakmak [23, 24] investigated Bullen-type variants with the aid of conformable fractional operators. A significant advancement was made in [25], where unified fractional operators were utilized to determine new bounds for Bullen-type results. Hezenci and collaborators [26] studied Bullen-type inequalities in connection with conformable integral operators. Zhao et al. [27] presented fractional analogues of Bullen's inequality by constructing a different general identity and presenting visual interpretations of their findings.

More recently, Mercer's refinement of Jensen's inequality has opened new directions for investigation. By establishing bounds for convex functions evaluated at points symmetric with respect to the interval endpoints, Mercer's inequality has proven remarkably fruitful in generating new results across various classes of convex functions. The combination of Mercer's ideas with other classical inequalities has emerged as a particularly productive line of inquiry, yielding what are now known as Jensen-Mercer [28] and Hermite-Hadamard-Mercer type [29, 30] results.

The present work adopts a combined approach of fractional calculus and majorization theory to determine the Bullen-Mercer type results in compact settings. This approach follows an emerging trend in contemporary research, where investigators attempt to develop inequalities that unify both discrete and continuous forms into a single compact framework. Usually, generalized integral operators, advanced concepts of convexity, or a calculated combination of both are used to obtain such unified results [31–33]. In [33], the term "conticrete" has been introduced for such compact inequalities. It is formed according to the English linguistic rules of word blending or coining. For instance, the word "brunch" is obtained by combining "breakfast" and "lunch".

The theory of majorization is a key mathematical tool that makes this kind of unification possible. Majorization defines a partial order between two tuples or vectors that shows the comparison of the spread out or the degree of the balance among the elements of the tuples. This idea is very helpful for transforming hard optimization problems into easier and more manageable ones [34, 35]. In addition to its theoretical interests , it has several practical uses in the areas of communication theory and signal processing [36, 37]. For some more study related to the concept of majorization, one can see [38, 39].

Fractional calculus has become a significant field of mathematical analysis. To put it another way, it is an extension of the classical concepts of differentiation and integration to non-integer orders. It began in the late ninetieth century, as an answer to a major question concerning the fractional order of derivative. It is now a developed field of mathematics that is widely applicable in the efforts of various mathematicians. Fractional calculus has been applied in various fields of study in the recent years including geophysics, control theory, biology, medicine, and engineering [40–44]. As a result of such applications, fractional calculus has been used in numerous studies, particularly those that utilize it to elucidate different concepts or processes occurring within them. Its ability to model the memory effects and long-range dependencies, makes it useful in modeling complex systems where history behavior still influences the current dynamics. For some more recent applications of fractional calculus, the reader is advised to see [45–47]. The historical development of fractional calculus is characterized by the introduction of a number of foundational operators. Among the first and most important was the Riemann-Liouville fractional integrals, allowing generalizations [48] that came later. Based on this, scientists have devised a wide selection of the fractional operators, each with advantages specific to their applications. The most commonly used formulations include the Caputo, Caputo-Fabrizio, Katugampola, Hadamard and Atangana-Baleanu operators and Riemann-Liouville fractional type Schurer operators [49–54]. All these tools enable mathematicians and scientists to be more flexible and accurate in dealing with problems that lie outside the limits of classical calculus.

The rest of this paper is structured in the following way.

• Section 2 recalls the fundamental concepts and inequalities that form the basis of our work.

• These preliminaries are then employed in Section 3 to establish the main results.

• Each result is accompanied by numerical validation tables and graphical illustrations, which are presented in a dedicated subsection.

• Section 4 discusses applications of the primary findings to modified Bessel functions and special means.

• The concluding remarks are presented in Section 5.

2. Preliminaries

This section recalls the fundamental concepts and inequalities needed for the subsequent development of our results.

2.1 *Convex functions and foundational inequalities*

Convex Function [9]:

A function $f : [h_3, h_4] \to \mathbb{R}$ is termed convex provided that for arbitrary $\xi, \zeta \in [h_3, h_4]$, and $\sigma \in [0, 1]$, one always has

$$f(\sigma\xi + (1-\sigma)\zeta) \leq \sigma f(\xi) + (1-\sigma) f(\zeta). \tag{2}$$

Jensen-Mercer Inequality [28]:

Let $f : [h_3, h_4] \to \mathbb{R}$ be a convex mapping. Given $\zeta_j \in [h_3, h_4]$, $0 \leq \sigma_j$ $(j = 1, 2, \ldots, m)$ with $\sum_{j=1}^{m} \sigma_j = 1$, one obtains

$$f\left(h_3 + h_4 - \sum_{j=1}^{m} \sigma_j \zeta_j\right) \leq f(h_3) + f(h_4) - \sum_{j=1}^{m} \sigma_j f(\zeta_j). \tag{3}$$

Hermite-Hadamard Inequality [20]:

Let $f : [h_3, h_4] \to \mathbb{R}$ represents a convex function. Then

$$f\left(\frac{h_3+h_4}{2}\right) \leq \frac{1}{h_4-h_3}\int_{h_3}^{h_4} f(u)du \leq \frac{f(h_3)+f(h_4)}{2}. \tag{4}$$

2.2 *Majorization and generalized Jensen-Mercer inequality*

Majorization [19]:

Let $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_m)$ and $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \ldots, \zeta_m)$ be two vectors or tuples whose entries are arranged in non-increasing order. We say that $\boldsymbol{\beta}$ majorizes $\boldsymbol{\zeta}$, denoted by $\boldsymbol{\zeta} \prec \boldsymbol{\beta}$, provided that

$$\sum_{j=1}^{k} \beta_{[j]} \geq \sum_{j=1}^{k} \zeta_{[j]} \quad for\, k = 1, 2, \ldots, m-1,$$

and

$$\sum_{j=1}^{m} \beta_j = \sum_{j=1}^{m} \zeta_j.$$

The framework of majorization allows us to extend the Jensen-Mercer inequality in the following manner.

Theorem 1 [55] Let f be a convex mapping on the interval I. Also, take an $n \times m$ matrix with entries y_{ij}, $(i = 1, 2, \ldots, n\ , j = 1, 2, \ldots, m)$, a tuple $\boldsymbol{\rho} = (\rho_1, \rho_2, \ldots, \rho_m)$, and $0 \leq \sigma_i$ with $\sum_{i=1}^{n} \sigma_i = 1$. If every row of the matrix is majorized by the tuple $\boldsymbol{\rho}$, then

$$f\left(\sum_{j=1}^{m} \rho_j - \sum_{j=1}^{m-1}\sum_{i=1}^{n} \sigma_i y_{ij}\right) \leq \sum_{j=1}^{m} f(\rho_j) - \sum_{j=1}^{m-1}\sum_{i=1}^{n} \sigma_i f(y_{ij}). \tag{5}$$

2.3 *Fractional integral operators*

We now recall the definition of the fractional operator that will be employed throughout this work.

Definition 1 Let $f \in L_1[h_3, h_4]$. The Riemann-Liouville integrals $I^{\alpha}_{h_3+}f$ and $I^{\alpha}_{h_4-}f$ of order $0 < \alpha$ with $h_3 \geq 0$ are defined by

$$I^{\alpha}_{h_3+}f(\xi) = \frac{1}{\Gamma(\alpha)}\int_{h_3}^{\xi} (\xi-\eta)^{\alpha-1} f(\eta)d\eta, \quad \xi > h_3$$

and

$$I^{\alpha}_{h_4-}f(\xi) = \frac{1}{\Gamma(\alpha)}\int_{\xi}^{h_4} (\eta-\xi)^{\alpha-1} f(\eta)d\eta, \quad \xi < h_4$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $I^{0}_{h_3+}f(\xi) = I^{0}_{h_4-}f(\xi) = f(\xi)$.

2.4 *Beta and incomplete Beta function*

The beta and incomplete functions are given by

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt, \qquad x>0,\ y>0.$$

$$B_z(x,y)=\int_0^z t^{x-1}(1-t)^{y-1}\,dt, \qquad 0\le z\le 1,$$

where $x>0$ and $y>0$.

3. Main results

In this section, we first establish a key integral identity and then based on this identity, we establish various estimates for the Bullen-Mercer gap.
Throughout this paper, we use the following notation.

Define $M=\{1,2,\ldots,m\}$ and $M^*=\{1,2,\ldots,m-1\}$. Then:

We write $\sum\limits_{j=1}^{m}$ as $\sum\limits_{j\in M}$ and $\sum\limits_{j=1}^{m-1}$ as $\sum\limits_{j\in M^*}$.

3.1 *Formulation of the fundamental integral identity*

Lemma 1 Let the function $f:I\to\mathbb{R}$ be differentiable on its domain I. Let the tuples $\boldsymbol{\lambda},\boldsymbol{\beta},\boldsymbol{\zeta}\in I^m$ be defined by $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\ldots,\lambda_m)$, $\boldsymbol{\beta}=(\beta_1,\beta_2,\ldots,\beta_m)$ and $\boldsymbol{\zeta}=(\zeta_1,\zeta_2,\ldots,\zeta_m)$ where $\lambda_j,\beta_j,\zeta_j\in I$ for all $j=1,2,\ldots,m$. Also, suppose that $\sum_{j\in M}\lambda_j=\sum_{j\in M}\beta_j$ $\sum_{j\in M}\lambda_j=\sum_{j\in M}\zeta_j$, $\beta_m>\zeta_m$, $\alpha>0$, and $f'\in L(I)$. Then

$$\begin{aligned}\xi(\lambda_j,\beta_j,\zeta_j,f)=\frac{\sum\limits_{j\in M^*}(\zeta_j-\beta_j)}{8}\Bigg[&\int_0^1(1-2\omega^\alpha)f'\Bigg(\sum_{j\in M}\lambda_j-\frac{1-\omega}{2}\sum_{j\in M^*}\beta_j-\frac{1+\omega}{2}\sum_{j\in M^*}\zeta_j\Bigg)d\omega\\ &-\int_0^1(1-2\omega^\alpha)f'\Bigg(\sum_{j\in M}\lambda_j-\frac{1-\omega}{2}\sum_{j\in M^*}\zeta_j-\frac{1+\omega}{2}\sum_{j\in M^*}\beta_j\Bigg)d\omega\Bigg],\end{aligned}\tag{6}$$

where

$$\begin{aligned}\xi(\lambda_j,\beta_j,\zeta_j,f)=&\frac{1}{2}\Bigg[\frac{1}{2}\Bigg\{f\Bigg(\sum_{j\in M}\lambda_j-\sum_{j\in M^*}\beta_j\Bigg)+f\Bigg(\sum_{j\in M}\lambda_j-\sum_{j\in M^*}\zeta_j\Bigg)\Bigg\}+f\Bigg(\sum_{j\in M}\lambda_j-\sum_{j\in M^*}\Bigg(\frac{\beta_j+\zeta_j}{2}\Bigg)\Bigg)\Bigg]\\ &-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\Bigg(\sum\limits_{j\in M^*}(\zeta_j-\beta_j)\Bigg)^\alpha}\Bigg[I^\alpha_{\left(\sum\limits_{j\in M}\lambda_j-\sum\limits_{j\in M^*}\zeta_j\right)^+}f\Bigg(\sum_{j\in M}\lambda_j-\sum_{j\in M^*}\Bigg(\frac{\beta_j+\zeta_j}{2}\Bigg)\Bigg)\end{aligned}$$

$$+I^{\alpha}_{\left(\sum\limits_{j\in M}\lambda_j-\sum\limits_{j\in M^*}\beta_j\right)^-}f\left(\sum_{j\in M}\lambda_j-\sum_{j\in M^*}\left(\frac{\beta_j+\zeta_j}{2}\right)\right)\Bigg].$$

Proof. We begin by assuming that

$$I=\int_0^1\left(1-2\omega^{\alpha}\right)f'\left(\sum_{j\in M}\lambda_j-\frac{1-\omega}{2}\sum_{j\in M^*}\beta_j-\frac{1+\omega}{2}\sum_{j\in M^*}\zeta_j\right)d\omega+\int_0^1\left(2\omega^{\alpha}-1\right)$$

$$\times f'\left(\sum_{j\in M}\lambda_j-\frac{1-\omega}{2}\sum_{j\in M^*}\zeta_j-\frac{1+\omega}{2}\sum_{j\in M^*}\beta_j\right)d\omega$$

$$=I_1+I_2. \tag{7}$$

The expression for I_1 and I_2 are obtained through integration by parts as follows:

$$I_1=\int_0^1\left(1-2\omega^{\alpha}\right)f'\left(\sum_{j\in M}\lambda_j-\frac{1-\omega}{2}\sum_{j\in M^*}\beta_j-\frac{1+\omega}{2}\sum_{j\in M^*}\zeta_j\right)d\omega$$

$$=\left(1-2\omega^{\alpha}\right)\frac{f\left(\sum\limits_{j\in M}\lambda_j-\frac{1-\omega}{2}\sum\limits_{j\in M^*}\beta_j-\frac{1+\omega}{2}\sum\limits_{j\in M^*}\zeta_j\right)}{-\frac{1}{2}\sum\limits_{j\in M^*}\left(\zeta_j-\beta_j\right)}\Bigg|_0^1-\frac{2\alpha}{\frac{1}{2}\sum\limits_{j\in M^*}\left(\zeta_j-\beta_j\right)}$$

$$\times\int_0^1\omega^{\alpha-1}f\left(\sum_{j\in M}\lambda_j-\frac{1-\omega}{2}\sum_{j\in M^*}\beta_j-\frac{1+\omega}{2}\sum_{j\in M^*}\zeta_j\right)d\omega. \tag{8}$$

$$I_1=\frac{1}{\frac{1}{2}\sum\limits_{j\in M^*}\left(\zeta_j-\beta_j\right)}\left[f\left(\sum_{j\in M}\lambda_j-\sum_{j\in M^*}\zeta_j\right)+f\left(\sum_{j\in M}\lambda_j-\sum_{j\in M^*}\left(\frac{\beta_j+\zeta_j}{2}\right)\right)\right]-\frac{2\alpha}{\frac{1}{2}\sum\limits_{j\in M^*}\left(\zeta_j-\beta_j\right)}$$

$$\times\int_0^1\omega^{\alpha-1}f\left(\sum_{j\in M}\lambda_j-\frac{1-\omega}{2}\sum_{j\in M^*}\beta_j-\frac{1+\omega}{2}\sum_{j\in M^*}\zeta_j\right)d\omega. \tag{9}$$

And

$$I_2=\int_0^1\left(2\omega^{\alpha}-1\right)f'\left(\sum_{j\in M}\lambda_j-\frac{1-\omega}{2}\sum_{j\in M^*}\zeta_j-\frac{1+\omega}{2}\sum_{j\in M^*}\beta_j\right)d\omega$$

$$
= \left(2\omega^{\alpha}-1\right)\frac{f\left(\sum\limits_{j\in M}\lambda_j-\frac{1-\omega}{2}\sum\limits_{j\in M^*}\zeta_j-\frac{1+\omega}{2}\sum\limits_{j\in M^*}\beta_j\right)}{\frac{1}{2}\sum\limits_{j\in M^*}\left(\zeta_j-\beta_j\right)}\Bigg|_0^1-\frac{2\alpha}{\frac{1}{2}\sum\limits_{j\in M^*}\left(\zeta_j-\beta_j\right)}
$$

$$
\times\int_0^1\omega^{\alpha-1}f\left(\sum_{j\in M}\lambda_j-\frac{1-\omega}{2}\sum_{j\in M^*}\zeta_j-\frac{1+\omega}{2}\sum_{j\in M^*}\beta_j\right)d\omega. \tag{10}
$$

$$
I_2=\frac{1}{\frac{1}{2}\sum\limits_{j\in M^*}\left(\zeta_j-\beta_j\right)}\left[f\left(\sum_{j\in M}\lambda_j-\sum_{j\in M^*}\beta_j\right)+f\left(\sum_{j\in M}\lambda_j-\sum_{j\in M^*}\left(\frac{\beta_j+\zeta_j}{2}\right)\right)\right]-\frac{2\alpha}{\frac{1}{2}\sum\limits_{j\in M^*}\left(\zeta_j-\beta_j\right)}
$$

$$
\times\int_0^1\omega^{\alpha-1}f\left(\sum_{j\in M}\lambda_j-\frac{1-\omega}{2}\sum_{j\in M^*}\zeta_j-\frac{1+\omega}{2}\sum_{j\in M^*}\beta_j\right)d\omega. \tag{11}
$$

By making a suitable substitution in (9) and (11), we have

$$
I_1=\frac{1}{\frac{1}{2}\sum\limits_{j\in M^*}\left(\zeta_j-\beta_j\right)}\left[f\left(\sum_{j\in M}\lambda_j-\sum_{j\in M^*}\zeta_j\right)+f\left(\sum_{j\in M}\lambda_j-\sum_{j\in M^*}\left(\frac{\beta_j+\zeta_j}{2}\right)\right)\right]
$$

$$
-\frac{2\alpha}{\left(\frac{1}{2}\sum\limits_{j\in M^*}\left(\zeta_j-\beta_j\right)\right)^{\alpha+1}}\int_{\sum\limits_{j\in M}\lambda_j-\sum\limits_{j\in M^*}\zeta_j}^{\sum\limits_{j\in M}\lambda_j-\sum\limits_{j\in M^*}\left(\frac{\beta_j+\zeta_j}{2}\right)}\left(\left(\sum_{j\in M}\lambda_j-\sum_{j\in M^*}\left(\frac{\beta_j+\zeta_j}{2}\right)\right)-u\right)^{\alpha-1}f(u)du, \tag{12}
$$

and

$$
I_2=\frac{1}{\frac{1}{2}\sum\limits_{j\in M^*}\left(\zeta_j-\beta_j\right)}\left[f\left(\sum_{j\in M}\lambda_j-\sum_{j\in M^*}\beta_j\right)+f\left(\sum_{j\in M}\lambda_j-\sum_{j\in M^*}\left(\frac{\beta_j+\zeta_j}{2}\right)\right)\right]
$$

$$
-\frac{2\alpha}{\left(\frac{1}{2}\sum\limits_{j\in M^*}\left(\zeta_j-\beta_j\right)\right)^{\alpha+1}}\int_{\sum\limits_{j\in M}\lambda_j-\sum\limits_{j\in M^*}\left(\frac{\beta_j+\zeta_j}{2}\right)}^{\sum\limits_{j\in M}\lambda_j-\sum\limits_{j\in M^*}\beta_j}\left(u-\left(\sum_{j\in M}\lambda_j-\sum_{j\in M^*}\left(\frac{\beta_j+\zeta_j}{2}\right)\right)\right)^{\alpha-1}f(u)du. \tag{13}
$$

Before applying the definition of the fractional integral in (12) and (13), it is necessary to establish the following inequalities:

$$
\sum_{j\in M}\lambda_j-\sum_{j\in M^*}\zeta_j<\sum_{j\in M}\lambda_j-\sum_{j\in M^*}\left(\frac{\beta_j+\zeta_j}{2}\right), \tag{14}
$$

and

$$\sum_{j\in M}\lambda_j-\sum_{j\in M^*}\left(\tfrac{\beta_j+\zeta_j}{2}\right)\ <\ \sum_{j\in M}\lambda_j-\sum_{j\in M^*}\beta_j. \tag{15}$$

Since

$$\sum_{j\in M}\lambda_j=\sum_{j\in M}\beta_j \quad \text{and} \quad \sum_{j\in M}\lambda_j=\sum_{j\in M}\zeta_j,$$

it follows that

$$\sum_{j\in M^*}\zeta_j-\sum_{j\in M^*}\beta_j=\beta_m-\zeta_m.$$

Now, using the assumption $\beta_m>\zeta_m$, we obtain

$$\sum_{j\in M^*}\zeta_j-\sum_{j\in M^*}\beta_j>0\ \Rightarrow\ -\sum_{j\in M^*}\zeta_j<-\sum_{j\in M^*}\beta_j.$$

Consequently,

$$\sum_{j\in M^*}\zeta_j-2\sum_{j\in M^*}\zeta_j<-\sum_{j\in M^*}\beta_j\ \Rightarrow\ -\sum_{j\in M^*}\zeta_j<-\sum_{j\in M^*}\left(\tfrac{\beta_j+\zeta_j}{2}\right).$$

By adding $\sum_{j\in M}\lambda_j$ to both sides, we arrive at

$$\sum_{j\in M}\lambda_j-\sum_{j\in M^*}\zeta_j<\sum_{j\in M}\lambda_j-\sum_{j\in M^*}\left(\tfrac{\beta_j+\zeta_j}{2}\right).$$

Similarly, one can show that

$$\sum_{j\in M}\lambda_j-\sum_{j\in M^*}\left(\tfrac{\beta_j+\zeta_j}{2}\right)<\sum_{j\in M}\lambda_j-\sum_{j\in M^*}\beta_j.$$

Now, (12) and (13) implies

$$I_1=\frac{1}{\frac{1}{2}\sum\limits_{j\in M^*}(\zeta_j-\beta_j)}\left[f\left(\sum_{j\in M}\lambda_j-\sum_{j\in M^*}\zeta_j\right)+f\left(\sum_{j\in M}\lambda_j-\sum_{j\in M^*}\left(\frac{\beta_j+\zeta_j}{2}\right)\right)\right]$$

$$-\frac{2\Gamma(\alpha+1)}{\left(\frac{1}{2}\sum\limits_{j\in M^*}\left(\zeta_j-\beta_j\right)\right)^{\alpha+1}} I^{\alpha}_{\left(\sum\limits_{j\in M}\lambda_j-\sum\limits_{j\in M^*}\zeta_j\right)^+} f\left(\sum_{j\in M}\lambda_j-\sum_{j\in M^*}\left(\frac{\beta_j+\zeta_j}{2}\right)\right), \tag{16}$$

and

$$I_2=\frac{1}{\frac{1}{2}\sum\limits_{j\in M^*}\left(\zeta_j-\beta_j\right)}\left[f\left(\sum_{j\in M}\lambda_j-\sum_{j\in M^*}\beta_j\right)+f\left(\sum_{j\in M}\lambda_j-\sum_{j\in M^*}\left(\frac{\beta_j+\zeta_j}{2}\right)\right)\right]$$

$$-\frac{2\Gamma(\alpha+1)}{\left(\frac{1}{2}\sum\limits_{j\in M^*}\left(\zeta_j-\beta_j\right)\right)^{\alpha+1}} I^{\alpha}_{\left(\sum\limits_{j\in M}\lambda_j-\sum\limits_{j\in M^*}\beta_j\right)^-} f\left(\sum_{j\in M}\lambda_j-\sum_{j\in M^*}\left(\frac{\beta_j+\zeta_j}{2}\right)\right). \tag{17}$$

By inserting (16) and (17) in (7), we get

$$I=\frac{4}{\sum\limits_{j\in M^*}\left(\zeta_j-\beta_j\right)}\left[\frac{1}{2}\left\{f\left(\sum_{j\in M}\lambda_j-\sum_{j\in M^*}\beta_j\right)+f\left(\sum_{j\in M}\lambda_j-\sum_{j\in M^*}\zeta_j\right)\right\}+f\left(\sum_{j\in M}\lambda_j-\sum_{j\in M^*}\left(\frac{\beta_j+\zeta_j}{2}\right)\right)\right]$$

$$-\frac{4\times 2^{\alpha}\Gamma(\alpha+1)}{\left(\sum\limits_{j\in M^*}\left(\zeta_j-\beta_j\right)\right)^{\alpha+1}}\left[I^{\alpha}_{\left(\sum\limits_{j\in M}\lambda_j-\sum\limits_{j\in M^*}\zeta_j\right)^+} f\left(\sum_{j\in M}\lambda_j-\sum_{j\in M^*}\left(\frac{\beta_j+\zeta_j}{2}\right)\right)\right.$$

$$\left.+I^{\alpha}_{\left(\sum\limits_{j\in M}\lambda_j-\sum\limits_{j\in M^*}\beta_j\right)^-} f\left(\sum_{j\in M}\lambda_j-\sum_{j\in M^*}\left(\frac{\beta_j+\zeta_j}{2}\right)\right)\right]. \tag{18}$$

Now, by multiplying both sides of (18) with $\frac{\sum_{j\in M^*}\left(\zeta_j-\beta_j\right)}{8}$, we get (6). $\square$

Remark 1

1. By setting $m=2$ in (6), we arrive at the following equality, which represents a new contribution to the literature of mathematical inequalities:

$$\frac{1}{2}\left[\frac{1}{2}\left\{f\left(\lambda_1+\lambda_2-\beta_1\right)+f\left(\lambda_1+\lambda_2-\zeta_1\right)\right\}+f\left(\lambda_1+\lambda_2-\tfrac{\beta_1+\zeta_1}{2}\right)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left(\zeta_1-\beta_1\right)^{\alpha}}$$

$$\times\left[I^{\alpha}_{\left(\lambda_1+\lambda_2-\zeta_1\right)^+} f\left(\lambda_1+\lambda_2-\tfrac{\beta_1+\zeta_1}{2}\right)+I^{\alpha}_{\left(\lambda_1+\lambda_2-\beta_1\right)^-} f\left(\lambda_1+\lambda_2-\tfrac{\beta_1+\zeta_1}{2}\right)\right]$$

$$=\frac{\zeta_1-\beta_1}{8}\left[\int_0^1\left(1-2\omega^{\alpha}\right)f'\left(\lambda_1+\lambda_2-\frac{1-\omega}{2}\beta_1-\frac{1+\omega}{2}\zeta_1\right)d\omega\right.$$

$$-\int_0^1 (1-2\omega^\alpha) f'(\lambda_1+\lambda_2-\frac{1-\omega}{2}\zeta_1-\frac{1+\omega}{2}\beta_1)d\omega\Bigg]. \tag{19}$$

2. Furthermore, for the special case $m=2$ and $\alpha=1$ in (6), we recover the inequality previously established in [56]:

$$\frac{1}{2}\left[\frac{1}{2}\{f(\lambda_1+\lambda_2-\beta_1)+f(\lambda_1+\lambda_2-\zeta_1)\}+f(\lambda_1+\lambda_2-\tfrac{\beta_1+\zeta_1}{2})\right]-\frac{1}{\zeta_1-\beta_1}\int\limits_{\lambda_1+\lambda_2-\zeta_1}^{\lambda_1+\lambda_2-\beta_1} f(u)du$$

$$=\frac{\zeta_1-\beta_1}{8}\Bigg[\int_0^1 (1-2\omega) f'(\lambda_1+\lambda_2-\frac{1-\omega}{2}\beta_1-\frac{1+\omega}{2}\zeta_1)d\omega$$

$$-\int_0^1 (1-2\omega) f'(\lambda_1+\lambda_2-\frac{1-\omega}{2}\zeta_1-\frac{1+\omega}{2}\beta_1)d\omega\Bigg]. \tag{20}$$

3.2 *Estimates based on the integral identity*

Using Lemma 1, we now arrive at the following conclusions:

Theorem 2 Let the function $f: I \to \mathbb{R}$ be differentiable on its domain I. Let the tuples $\boldsymbol{\lambda}, \boldsymbol{\beta}, \boldsymbol{\zeta} \in I^m$ be defined by $\boldsymbol{\lambda}=(\lambda_1, \lambda_2, \ldots, \lambda_m)$, $\boldsymbol{\beta}=(\beta_1, \beta_2, \ldots, \beta_m)$ and $\boldsymbol{\zeta}=(\zeta_1, \zeta_2, \ldots, \zeta_m)$ with $\beta_m > \zeta_m$ and $\lambda_j, \beta_j, \zeta_j \in I$ for all $j=1,2,\ldots,m$. Suppose further that $\boldsymbol{\beta} \prec \boldsymbol{\lambda}$ and $\boldsymbol{\zeta} \prec \boldsymbol{\lambda}$, $\alpha>0$, and $|f'|$ is convex function. Then

$$|\xi(\lambda_j,\beta_j,\zeta_j,f)| \le \frac{\sum\limits_{j\in M^*}|\zeta_j-\beta_j|(2^\alpha-1)}{2^{\alpha+1}(\alpha+1)}\left[\sum_{j\in M}|f'(\lambda_j)|-\frac{1}{2}\left\{\sum_{j\in M^*}|f'(\beta_j)|+\sum_{j\in M^*}|f'(\zeta_j)|\right\}\right]. \tag{21}$$

Proof. Taking modulus of Lemma 1, we have

$$|\xi(\lambda_j,\beta_j,\zeta_j,f)| = \Bigg|\frac{\sum\limits_{j\in M^*}(\zeta_j-\beta_j)}{8}\Bigg[\int_0^1 (1-2\omega^\alpha) f'\Bigg(\sum_{j\in M}\lambda_j-\frac{1-\omega}{2}\sum_{j\in M^*}\beta_j-\frac{1+\omega}{2}\sum_{j\in M^*}\zeta_j\Bigg)d\omega$$

$$-\int_0^1 (1-2\omega^\alpha) f'\Bigg(\sum_{j\in M}\lambda_j-\frac{1-\omega}{2}\sum_{j\in M^*}\zeta_j-\frac{1+\omega}{2}\sum_{j\in M^*}\beta_j\Bigg)d\omega\Bigg]\Bigg|. \tag{22}$$

By implementing modulus property in (22), we get

$$|\xi(\lambda_j,\beta_j,\zeta_j,f)| \le \frac{\sum\limits_{j\in M^*}|\zeta_j-\beta_j|}{8}\Bigg[\int_0^1 |(1-2\omega^\alpha)|\Bigg|f'\Bigg(\sum_{j\in M}\lambda_j-\frac{1-\omega}{2}\sum_{j\in M^*}\beta_j-\frac{1+\omega}{2}\sum_{j\in M^*}\zeta_j\Bigg)\Bigg|d\omega$$

$$+\int_0^1 |(1-2\omega^\alpha)|\Bigg|f'\Bigg(\sum_{j\in M}\lambda_j-\frac{1-\omega}{2}\sum_{j\in M^*}\zeta_j-\frac{1+\omega}{2}\sum_{j\in M^*}\beta_j\Bigg)\Bigg|d\omega\Bigg]. \tag{23}$$

$$= \frac{\sum\limits_{j\in M^*}|\zeta_j-\beta_j|}{8}\int_0^1 \left|\left(1-2\omega^{\alpha}\right)\right|\left[\left|f'\left(\sum_{j\in M}\lambda_j-\frac{1-\omega}{2}\sum_{j\in M^*}\beta_j-\frac{1+\omega}{2}\sum_{j\in M^*}\zeta_j\right)d\omega\right|\right.$$

$$\left.+\left|f'\left(\sum_{j\in M}\lambda_j-\frac{1-\omega}{2}\sum_{j\in M^*}\zeta_j-\frac{1+\omega}{2}\sum_{j\in M^*}\beta_j\right)d\omega\right|\right]. \tag{24}$$

$$\left|\xi(\lambda_j,\beta_j,\zeta_j,f)\right| \leq \frac{\sum\limits_{j\in M^*}|\zeta_j-\beta_j|}{8}\times\left[\int_0^{\frac{1}{2}}\left(1-2\omega^{\alpha}\right)\left\{\left|f'\left(\sum_{j\in M}\lambda_j-\frac{1-\omega}{2}\sum_{j\in M^*}\beta_j-\frac{1+\omega}{2}\sum_{j\in M^*}\zeta_j\right)\right|\right.\right.$$

$$\left.+\left|f'\left(\sum_{j\in M}\lambda_j-\frac{1-\omega}{2}\sum_{j\in M^*}\zeta_j-\frac{1+\omega}{2}\sum_{j\in M^*}\beta_j\right)\right|\right\}d\omega$$

$$+\int_{\frac{1}{2}}^1\left(2\omega^{\alpha}-1\right)\left\{\left|f'\left(\sum_{j\in M}\lambda_j-\frac{1-\omega}{2}\sum_{j\in M^*}\beta_j-\frac{1+\omega}{2}\sum_{j\in M^*}\zeta_j\right)\right|\right.$$

$$\left.\left.+\left|f'\left(\sum_{j\in M}\lambda_j-\frac{1-\omega}{2}\sum_{j\in M^*}\zeta_j-\frac{1+\omega}{2}\sum_{j\in M^*}\beta_j\right)\right|\right\}d\omega\right]. \tag{25}$$

Using Theorem 1 for the choices $\sigma_1=\frac{1-\omega}{2}$, $n=2$ and $\sigma_2=\frac{1+\omega}{2}$ in (25), we get

$$\left|\xi(\lambda_j,\beta_j,\zeta_j,f)\right| \leq \frac{\sum\limits_{j\in M^*}|\zeta_j-\beta_j|}{8}$$

$$\times\left[\int_0^{\frac{1}{2}}\left(1-2\omega^{\alpha}\right)\left\{\sum_{j\in M}\left|f'(\lambda_j)\right|-\frac{1-\omega}{2}\sum_{j\in M^*}\left|f'(\beta_j)\right|-\frac{1+\omega}{2}\sum_{j\in M^*}\left|f'(\zeta_j)\right|\right.\right.$$

$$\left.+\sum_{j\in M}\left|f'(\lambda_j)\right|-\frac{1-\omega}{2}\sum_{j\in M^*}\left|f'(\zeta_j)\right|-\frac{1+\omega}{2}\sum_{j\in M^*}\left|f'(\beta_j)\right|\right\}d\omega$$

$$+\int_{\frac{1}{2}}^1\left(2\omega^{\alpha}-1\right)\left\{\sum_{j\in M}\left|f'(\lambda_j)\right|-\frac{1-\omega}{2}\sum_{j\in M^*}\left|f'(\beta_j)\right|-\frac{1+\omega}{2}\sum_{j\in M^*}\left|f'(\zeta_j)\right|\right.$$

$$\left.\left.+\sum_{j\in M}\left|f'(\lambda_j)\right|-\frac{1-\omega}{2}\sum_{j\in M^*}\left|f'(\zeta_j)\right|-\frac{1+\omega}{2}\sum_{j\in M^*}\left|f'(\beta_j)\right|\right\}d\omega\right]. \tag{26}$$

$$= \frac{\sum\limits_{j\in M^*}|\zeta_j-\beta_j|}{8}\left[\int_0^{\frac{1}{2}}\left(1-2\omega^{\alpha}\right)\left\{2\sum_{j\in M}\left|f'(\lambda_j)\right|-\sum_{j\in M^*}\left|f'(\beta_j)\right|-\sum_{j\in M^*}\left|f'(\zeta_j)\right|\right\}d\omega\right.$$

$$+\int_{\frac{1}{2}}^{1}\left(2\omega^{\alpha}-1\right)\left\{2\sum_{j\in M}\left|f'(\lambda_j)\right|-\sum_{j\in M^*}\left|f'(\beta_j)\right|-\sum_{j\in M^*}\left|f'(\zeta_j)\right|\right\}d\omega\Bigg]. \tag{27}$$

Now, evaluating the integrals, we obtain

$$\int_0^{\frac{1}{2}}\left(1-2\omega^{\alpha}\right)d\omega=\frac{1}{2}-\frac{2}{\alpha+1}\left(\frac{1}{2}\right)^{\alpha+1} \quad and \quad \int_{\frac{1}{2}}^{1}\left(2\omega^{\alpha}-1\right)d\omega=\frac{2}{\alpha+1}-\frac{2}{\alpha+1}\left(\frac{1}{2}\right)^{\alpha+1}-\frac{1}{2}.$$

Substituting these values into (27), we immediately arrive at (21). □

Remark 2

1. If we substitute $m=2$ and $\alpha=1$ in (21), then we get the following result of [56].

$$\left|\frac{1}{2}\left[\frac{1}{2}\left\{f\left(\lambda_1+\lambda_2-\beta_1\right)+f\left(\lambda_1+\lambda_2-\zeta_1\right)\right\}+f\left(\lambda_1+\lambda_2-\tfrac{\beta_1+\zeta_1}{2}\right)\right]-\frac{1}{\zeta_1-\beta_1}\int_{\lambda_1+\lambda_2-\zeta_1}^{\lambda_1+\lambda_2-\beta_1}f(u)du\right|$$

$$\leq\frac{|\zeta_1-\beta_1|}{8}\left[|f'(\lambda_1)|+|f'(\lambda_2)|-\frac{1}{2}\left\{|f'(\beta_1)|+|f'(\zeta_1)|\right\}\right]. \tag{28}$$

2. If we substitute $m=2$, $\alpha=1$, $\beta_1=\lambda_1$, $\zeta_1=\lambda_2$ in (21), then we get the following result acquired in [57].

$$\left|\frac{1}{2}\left[\frac{1}{2}\left\{f(\lambda_2)+f(\lambda_1)\right\}+f\left(\tfrac{\lambda_1+\lambda_2}{2}\right)\right]-\frac{1}{\zeta_1-\beta_1}\int_{\lambda_1}^{\lambda_2}f(u)du\right|\leq\frac{|\zeta_1-\beta_1|}{16}\left(|f'(\lambda_1)|+|f'(\lambda_2)|\right). \tag{29}$$

3.2.1 *Numerical and graphical validation*

The numerical validation and graphical illustrations of Theorem 2 are presented in Table 1 and Figures 1 and 2, respectively.

Table 1. Numerical validation of Theorem 2 for $f(t)=\frac{t^3}{3}$, $\lambda_1=1$, $\lambda_2=4$, $x_1=2$, $y_1=3$, and $0<\alpha\leq 1$

Values of α	Values of left side term	Values of right side term
0.1	0.0039	0.2476
0.2	0.0148	0.4660
0.3	0.0317	0.6609
0.4	0.0535	0.8358
0.5	0.0793	0.9935
0.6	0.1083	1.1365
0.7	0.1397	1.2667
0.8	0.1730	1.3857
0.9	0.2076	1.4949
1.0	0.2431	1.5955

3.2.1.1 *Numerical simulation*

For Figures 1–8 and Tables 1–4, the sentence is mentioned in each subsection “Numerical and graphical validation” while for Figures 9–12, the sentence is mentioned in propositions 1–4.

3.2.1.2 *Graphical interpretation*

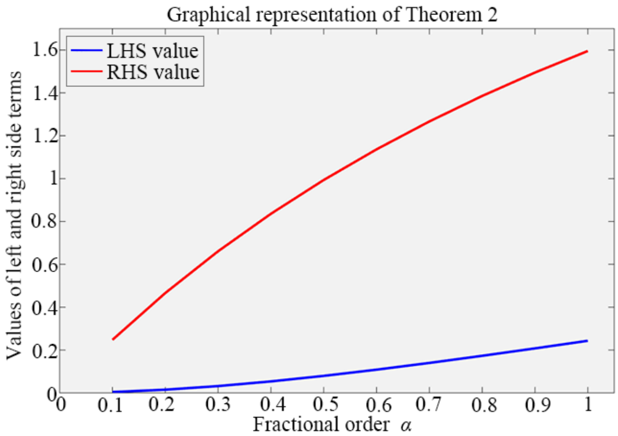

Figure 1. Graphical representation of Theorem 2 for $f(t)=\frac{t^3}{3}$, $f'(t)=t^2$, $m=2$, $\lambda_1=1$, $\lambda_2=4$, $\beta_1=2$ and $\zeta_1=3$, $\alpha\in(0,1]$

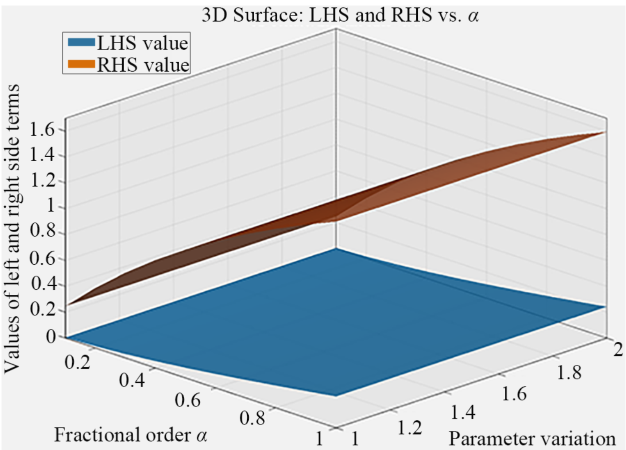

Figure 2. 3-D Graphical representation of Theorem 2

Theorem 3 Let the function $f: I\to\mathbb{R}$ be differentiable on its domain I. Let the tuples $\boldsymbol{\lambda},\boldsymbol{\beta},\boldsymbol{\zeta}\in I^m$ be defined by $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\ldots,\lambda_m)$, $\boldsymbol{\beta}=(\beta_1,\beta_2,\ldots,\beta_m)$ and $\boldsymbol{\zeta}=(\zeta_1,\zeta_2,\ldots,\zeta_m)$ with $\beta_m>\zeta_m$ and $\lambda_j,\beta_j,\zeta_j\in I$ for all $j=1,2,\ldots,m$. Suppose further that $\boldsymbol{\beta}\prec\boldsymbol{\lambda}$ and $\boldsymbol{\zeta}\prec\boldsymbol{\lambda}$, $\alpha>0$, and $q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$, $|f'|^q$ is convex function. Then

$$\left|\xi(\lambda_j,\beta_j,\zeta_j,f)\right|\leq\frac{\sum\limits_{j\in M^*}|\zeta_j-\beta_j|}{8}\left[\beta\, 2^{-\beta}\left(B(\beta,p+1)+B_{1/2}(p+1,-p-\beta)\right)\right]^{\frac{1}{p}}$$

$$\times\left[\left(\sum_{j\in M}|f'(\lambda_j)|^q-\frac{1}{4}\sum_{j\in M^*}|f'(\beta_j)|^q-\frac{3}{4}\sum_{j\in M^*}|f'(\zeta_j)|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\sum_{j\in M}|f'(\lambda_j)|^q-\frac{1}{4}\sum_{j\in M^*}|f'(\zeta_j)|^q-\frac{3}{4}\sum_{j\in M^*}|f'(\beta_j)|^q\right)^{\frac{1}{q}}\right], \tag{30}$$

where $B(.,.)$ and $B_{1/2}(.,.)$ represent Beta and incomplete Beta functions respectively.

Proof. The modulus property, when applied to Lemma 1, yields:

$$\left|\xi(\lambda_j,\beta_j,\zeta_j,f)\right|\leq\frac{\sum_{j\in M^*}|\zeta_j-\beta_j|}{8}\left[\int_0^1\left|(1-2\omega^\alpha)\right|\left|f'\left(\sum_{j\in M}\lambda_j-\frac{1-\omega}{2}\sum_{j\in M^*}\beta_j-\frac{1+\omega}{2}\sum_{j\in M^*}\zeta_j\right)\right|d\omega\right.$$

$$\left.+\int_0^1\left|(1-2\omega^\alpha)\right|\left|f'\left(\sum_{j\in M}\lambda_j-\frac{1-\omega}{2}\sum_{j\in M^*}\zeta_j-\frac{1+\omega}{2}\sum_{j\in M^*}\beta_j\right)\right|d\omega\right]. \tag{31}$$

By employing Hölder inequality, we get

$$\left|\xi(\lambda_j,\beta_j,\zeta_j,f)\right|\leq\frac{\sum_{j\in M^*}|\zeta_j-\beta_j|}{8}$$

$$\times\left[\left(\int_0^1\left|(1-2\omega^\alpha)\right|^p d\omega\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\sum_{j\in M}\lambda_j-\frac{1-\omega}{2}\sum_{j\in M^*}\beta_j-\frac{1+\omega}{2}\sum_{j\in M^*}\zeta_j\right)\right|^q d\omega\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1\left|(1-2\omega^\alpha)\right|^p d\omega\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\sum_{j\in M}\lambda_j-\frac{1-\omega}{2}\sum_{j\in M^*}\zeta_j-\frac{1+\omega}{2}\sum_{j\in M^*}\beta_j\right)\right|^q d\omega\right)^{\frac{1}{q}}\right]. \tag{32}$$

$$=\frac{\sum_{j\in M^*}|\zeta_j-\beta_j|}{8}\left(\int_0^1\left|(1-2\omega^\alpha)\right|^p d\omega\right)^{\frac{1}{p}}$$

$$\times\left[\left(\int_0^1\left|f'\left(\sum_{j\in M}\lambda_j-\frac{1-\omega}{2}\sum_{j\in M^*}\beta_j-\frac{1+\omega}{2}\sum_{j\in M^*}\zeta_j\right)\right|^q d\omega\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1\left|f'\left(\sum_{j\in M}\lambda_j-\frac{1-\omega}{2}\sum_{j\in M^*}\zeta_j-\frac{1+\omega}{2}\sum_{j\in M^*}\beta_j\right)\right|^q d\omega\right)^{\frac{1}{q}}\right]. \tag{33}$$

Applying Theorem 1 with the parameters $n=2$ with $\sigma_1=\frac{1-\omega}{2}$ and $\sigma_2=\frac{1+\omega}{2}$ in (33), along with the convexity assumption on $|f'|^q$, we have:

$$\left|\xi(\lambda_j,\beta_j,\zeta_j,f)\right| \leq \frac{\sum\limits_{j\in M^*}|\zeta_j-\beta_j|}{8}\left(\int_0^1\left|\left(1-2\omega^\alpha\right)\right|^p d\omega\right)^{\frac{1}{p}}$$

$$\times\left[\left(\int_0^1\left(\sum_{j\in M}|f'(\lambda_j)|^q-\frac{1-\omega}{2}\sum_{j\in M^*}|f'(\beta_j)|^q-\frac{1+\omega}{2}\sum_{j\in M^*}|f'(\zeta_j)|^q\right)d\omega\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1\left(\sum_{j\in M}|f'(\lambda_j)|^q-\frac{1-\omega}{2}\sum_{j\in M^*}|f'(\zeta_j)|^q-\frac{1+\omega}{2}\sum_{j\in M^*}|f'(\beta_j)|^q\right)d\omega\right)^{\frac{1}{q}}\right]. \tag{34}$$

We now evaluate the integral

$$\int_0^1|1-2\omega^\alpha|^p\,d\omega.$$

Using the substitution $u=\omega^\alpha$, we obtain

$$\int_0^1|1-2\omega^\alpha|^p\,d\omega=\beta\int_0^1|1-2u|^p\,u^{\beta-1}\,du,\quad \beta=\tfrac{1}{\alpha}.$$

Splitting at $u=\frac{1}{2}$ yields

$$\int_0^1|1-2\omega^\alpha|^p\,d\omega=\beta\Big(I_1+I_2\Big),$$

where

$$I_1=\int_0^{1/2}(1-2u)^p u^{\beta-1}\,du,\quad I_2=\int_{1/2}^1(2u-1)^p u^{\beta-1}\,du.$$

Evaluation of I_1. With the change of variable $v=2u$, we get

$$I_1=2^{-\beta}\int_0^1 v^{\beta-1}(1-v)^p\,dv=2^{-\beta}B(\beta,p+1).$$

Evaluation of I_2. With the substitution $w=2u-1$, we obtain

$$I_2=2^{-\beta}\int_0^1 w^p(1+w)^{\beta-1}\,dw.$$

Applying $w = \frac{\omega}{1-\omega}$ transforms this into an incomplete Beta integral:

$$I_2 = 2^{-\beta} B_{1/2}(p+1, -p-\beta).$$

Therefore,

$$\int_0^1 |1-2\omega^\alpha|^p \, d\omega = \beta\, 2^{-\beta} \Big[B(\beta, p+1) + B_{1/2}(p+1, -p-\beta) \Big]. \tag{35}$$

Also,

$$\int_0^1 \frac{1-\omega}{2} d\omega = \frac{1}{4}, \int_0^1 \frac{1+\omega}{2} dt = \frac{3}{4}. \tag{36}$$

Substituting (35) and (36) in (34), we obtain (30). □

Remark 3

1. If we substitute $m = 2$ and $\alpha = 1$ in (30), then we get the following result achieved in [56].

$$\left| \frac{1}{2} \left[\frac{1}{2} \left\{ f\big(\lambda_1+\lambda_2-\beta_1\big) + f\big(\lambda_1+\lambda_2-\zeta_1\big) \right\} + f\big(\lambda_1+\lambda_2-\tfrac{\beta_1+\zeta_1}{2}\big) \right] - \frac{1}{\zeta_1-\beta_1} \int\limits_{\lambda_1+\lambda_2-\zeta_1}^{\lambda_1+\lambda_2-\beta_1} f(u)du \right|$$

$$\leq \frac{|\zeta_1-\beta_1|}{8} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left[\left(|f'(\lambda_1)|^q + |f'(\lambda_2)|^q - \frac{1}{4}|f'(\beta_1)|^q - \frac{3}{4}|f'(\zeta_1)|^q \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(|f'(\lambda_1)|^q + |f'(\lambda_2)|^q - \frac{1}{4}|f'(\zeta_1)|^q - \frac{3}{4}|f'(\beta_1)|^q \right)^{\frac{1}{q}} \right]. \tag{37}$$

2. By setting $m = 2$ in (30), we arrive at the following inequality, which represents a new contribution to the literature of mathematical inequalities:

$$\left| \frac{1}{2} \left[\frac{1}{2} \left\{ f\big(\lambda_1+\lambda_2-\beta_1\big) + f\big(\lambda_1+\lambda_2-\zeta_1\big) \right\} + f\big(\lambda_1+\lambda_2-\tfrac{\beta_1+\zeta_1}{2}\big) \right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{\big(\zeta_1-\beta_1\big)^\alpha} \right.$$

$$\left. \times \left[I^\alpha_{\left(\lambda_1+\lambda_2-\zeta_1\right)^+} f\big(\lambda_1+\lambda_2-\tfrac{\beta_1+\zeta_1}{2}\big) + I^\alpha_{\left(\lambda_1+\lambda_2-\beta_1\right)^-} f\big(\lambda_1+\lambda_2-\tfrac{\beta_1+\zeta_1}{2}\big) \right] \right|$$

$$\leq \frac{|\zeta_1-\beta_1|}{8} \left[\beta\, 2^{-\beta} \Big(B(\beta, p+1) + B_{1/2}(p+1, -p-\beta) \Big) \right]^{\frac{1}{p}}$$

$$\times \left[\left(|f'(\lambda_1)|^q + |f'(\lambda_2)|^q - \frac{1}{4}|f'(\beta_1)|^q - \frac{3}{4}|f'(\zeta_1)|^q \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(|f'(\lambda_1)|^q + |f'(\lambda_2)|^q - \frac{1}{4}|f'(\zeta_1)|^q - \frac{3}{4}|f'(\beta_1)|^q \right)^{\frac{1}{q}} \right]. \tag{38}$$

3.2.2 *Numerical and graphical validation*

The numerical validation and graphical illustrations of Theorem 3 are presented in Table 2 and Figures 3 and 4, respectively.

Table 2. Numerical validation of Theorem 3 for $f(t) = t^3/3$, $\lambda_1 = 1$, $\lambda_2 = 4$, $x_1 = 2$, $y_1 = 3$, $q = 2.0$ ($p = 2.0$), and $0 < \alpha \leq 1$

Value of α	Value of the left hand side (LHS)	Value of the right hand side (RHS)
0.1	0.0039	3.0114
0.2	0.0148	2.6077
0.3	0.0317	2.3510
0.4	0.0535	2.2377
0.5	0.0793	2.2089
0.6	0.1083	2.2021
0.7	0.1397	2.1893
0.8	0.1730	2.1638
0.9	0.2076	2.1271
1.0	0.2431	2.0826

3.2.2.1 *Numerical simulation*

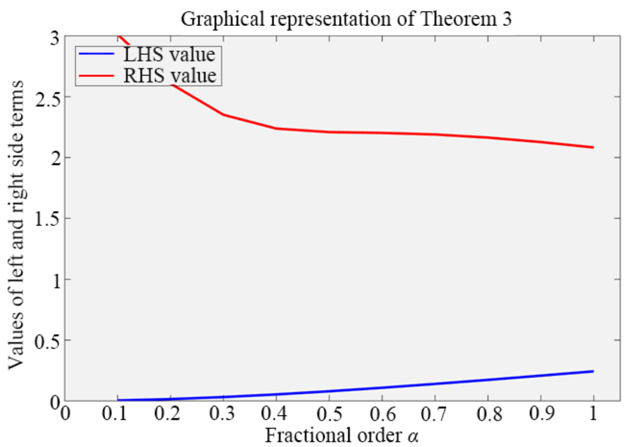

Figure 3. Graphical representation of Theorem 3 for $f(t) = \frac{t^3}{3}$, $f'(t) = t^2$, $m = 2$, $\lambda_1 = 1$, $\lambda_2 = 4$, $\beta_1 = 2$ and $\zeta_1 = 3$, $q = 2$, $\alpha \in (0, 1]$

3.2.2.2 *Graphical interpretation*

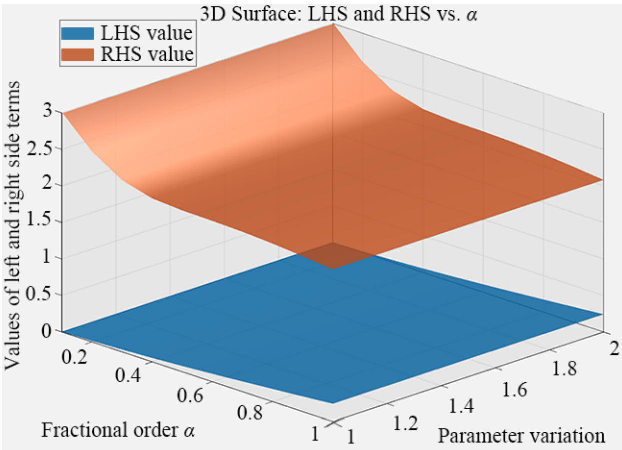

Figure 4. 3-D Graphical representation of Theorem 3

Theorem 4 Let the function $f : I \to \mathbb{R}$ be differentiable on its domain I. Let the tuples $\boldsymbol{\lambda}, \boldsymbol{\beta}, \boldsymbol{\zeta} \in I^m$ be defined by $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_m)$, $\boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_m)$ and $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \ldots, \zeta_m)$ with $\beta_m > \zeta_m$ and $\lambda_j, \beta_j, \zeta_j \in I$ for all $j = 1, 2, \ldots, m$. Suppose further that $\boldsymbol{\beta} \prec \boldsymbol{\lambda}$ and $\boldsymbol{\zeta} \prec \boldsymbol{\lambda}$, $\alpha > 0$, and , $|f'|^q$ $(1 < q)$ is convex function. Then

$$\left|\xi(\lambda_j, \beta_j, \zeta_j, f)\right| \leq \frac{\sum_{j\in M^*} |\zeta_j - \beta_j|}{8} \left(\frac{2(1-2^{-\alpha})}{\alpha+1}\right)^{1-\frac{1}{q}}$$

$$\times \left[\left(\left(\frac{2(1-2^{-\alpha})}{\alpha+1}\right) \sum_{j\in M} |f'(\lambda_j)|^q - \left(\frac{1}{8} + \frac{1-2^{-\alpha}}{\alpha+1} - \frac{1-2^{-(\alpha+1)}}{\alpha+2}\right) \sum_{j\in M^*} |f'(\beta_j)|^q \right.\right.$$

$$\left. - \left(\frac{1-2^{-\alpha}}{\alpha+1} + \frac{1-2^{-(\alpha+1)}}{\alpha+2} - \frac{1}{8}\right) \sum_{j\in M^*} |f'(\zeta_j)|^q \right)^{\frac{1}{q}}$$

$$+ \left(\left(\frac{2(1-2^{-\alpha})}{\alpha+1}\right) \sum_{j\in M} |f'(\lambda_j)|^q - \left(\frac{1}{8} + \frac{1-2^{-\alpha}}{\alpha+1} - \frac{1-2^{-(\alpha+1)}}{\alpha+2}\right) \sum_{j\in M^*} |f'(\zeta_j)|^q \right.$$

$$\left.\left. - \left(\frac{1-2^{-\alpha}}{\alpha+1} + \frac{1-2^{-(\alpha+1)}}{\alpha+2} - \frac{1}{8}\right) \sum_{j\in M^*} |f'(\beta_j)|^q \right)^{\frac{1}{q}} \right]. \tag{39}$$

Proof. The modulus property, when applied to Lemma 1, yields:

$$\left|\xi(\lambda_j, \beta_j, \zeta_j, f)\right| \leq \frac{\sum_{j\in M^*} |\zeta_j - \beta_j|}{8} \left[\int_0^1 \left|(1-2\omega^\alpha)\right| \left| f'\left(\sum_{j\in M} \lambda_j - \frac{1-\omega}{2} \sum_{j\in M^*} \beta_j - \frac{1+\omega}{2} \sum_{j\in M^*} \zeta_j \right) \right| d\omega \right.$$

$$+\int_0^1 \left|(1-2\omega^{\alpha})\right| \left| f'\left(\sum_{j\in M} \lambda_j - \frac{1-\omega}{2} \sum_{j\in M^*} \zeta_j - \frac{1+\omega}{2} \sum_{j\in M^*} \beta_j \right) \right| d\omega \Bigg]. \tag{40}$$

By implementing power mean inequality, we get

$$\left|\xi(\lambda_j,\beta_j,\zeta_j,f)\right| \leq \frac{\sum\limits_{j\in M^*} |\zeta_j-\beta_j|}{8} \left(\int_0^1 |1-2\omega^{\alpha}| d\omega \right)^{1-\frac{1}{q}}$$

$$\times \left[\left(\int_0^1 |1-2\omega^{\alpha}| \left| f'\left(\sum_{j\in M} \lambda_j - \frac{1-\omega}{2} \sum_{j\in M^*} \beta_j - \frac{1+\omega}{2} \sum_{j\in M^*} \zeta_j \right) \right|^q d\omega \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\int_0^1 |1-2\omega^{\alpha}| \left| f'\left(\sum_{j\in M} \lambda_j - \frac{1-\omega}{2} \sum_{j\in M^*} \zeta_j - \frac{1+\omega}{2} \sum_{j\in M^*} \beta_j \right) \right|^q d\omega \right)^{\frac{1}{q}} \right]. \tag{41}$$

Applying Theorem 1 with the parameters $n=2$, $\sigma_1 = \frac{1-\omega}{2}$ and $\sigma_2 = \frac{1+\omega}{2}$ in (41) along with convexity assumption on $|f'|^q$, we achieve

$$\left|\xi(\lambda_j,\beta_j,\zeta_j,f)\right| \leq \frac{\sum\limits_{j\in M^*} |\zeta_j-\beta_j|}{8} \left(\int_0^1 |1-2\omega^{\alpha}| d\omega \right)^{1-\frac{1}{q}}$$

$$\times \left[\left(\int_0^1 |1-2\omega^{\alpha}| \left(\sum_{j\in M} |f'(\lambda_j)|^q - \frac{1-\omega}{2} \sum_{j\in M^*} |f'(\beta_j)|^q - \frac{1+\omega}{2} \sum_{j\in M^*} |f'(\zeta_j)|^q \right) d\omega \right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\int_0^1 |1-2\omega^{\alpha}| \left(\sum_{j\in M} |f'(\lambda_j)|^q - \frac{1-\omega}{2} \sum_{j\in M^*} |f'(\zeta_j)|^q - \frac{1+\omega}{2} \sum_{j\in M^*} |f'(\beta_j)|^q \right) d\omega \right)^{\frac{1}{q}} \right]. \tag{42}$$

$$\left|\xi(\lambda_j,\beta_j,\zeta_j,f)\right| \leq \frac{\sum\limits_{j\in M^*} |\zeta_j-\beta_j|}{8} \left(\int_0^1 |1-2\omega^{\alpha}|\, d\omega \right)^{1-\frac{1}{q}} \left[I_1^{1/q} + I_2^{1/q} \right], \tag{43}$$

where

$$I_1 = \int_0^1 |1-2\omega^{\alpha}| \left(\sum_{j\in M} |f'(\lambda_j)|^q - \frac{1-\omega}{2} \sum_{j\in M^*} |f'(\beta_j)|^q - \frac{1+\omega}{2} \sum_{j\in M^*} |f'(\zeta_j)|^q \right) d\omega,$$

$$I_2 = \int_0^1 |1-2\omega^{\alpha}| \left(\sum_{j\in M} |f'(\lambda_j)|^q - \frac{1-\omega}{2} \sum_{j\in M^*} |f'(\zeta_j)|^q - \frac{1+\omega}{2} \sum_{j\in M^*} |f'(\beta_j)|^q \right) d\omega.$$

After a direct computation, these integrals reduce to

$$I_1 = \left(\tfrac{2(1-2^{-\alpha})}{\alpha+1}\right) \sum_{j\in M} |f'(\lambda_j)|^q - \left(\tfrac{1}{8} + \tfrac{1-2^{-\alpha}}{\alpha+1} - \tfrac{1-2^{-(\alpha+1)}}{\alpha+2}\right) \sum_{j\in M^*} |f'(\beta_j)|^q$$

$$- \left(\tfrac{1-2^{-\alpha}}{\alpha+1} - \tfrac{1-2^{-(\alpha+1)}}{\alpha+2} - \tfrac{1}{8}\right) \sum_{j\in M^*} |f'(\zeta_j)|^q, \tag{44}$$

and

$$I_2 = \left(\tfrac{2(1-2^{-\alpha})}{\alpha+1}\right) \sum_{j\in M} |f'(\lambda_j)|^q - \left(\tfrac{1}{8} + \tfrac{1-2^{-\alpha}}{\alpha+1} - \tfrac{1-2^{-(\alpha+1)}}{\alpha+2}\right) \sum_{j\in M^*} |f'(\zeta_j)|^q$$

$$- \left(\tfrac{1-2^{-\alpha}}{\alpha+1} - \tfrac{1-2^{-(\alpha+1)}}{\alpha+2} - \tfrac{1}{8}\right) \sum_{j\in M^*} |f'(\beta_j)|^q. \tag{45}$$

Finally, substituting the above expressions for I_1 and I_2 into (43) yields the desired result (39). □

Remark 4

1. By setting $m = 2$ in (39), we arrive at the following inequality, which represents a new contribution to the literature of mathematical inequalities:

$$\left| \frac{1}{2}\left[\frac{1}{2}\left\{f\left(\lambda_1+\lambda_2-\beta_1\right)+f\left(\lambda_1+\lambda_2-\zeta_1\right)\right\}+f\left(\lambda_1+\lambda_2-\tfrac{\beta_1+\zeta_1}{2}\right)\right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left(\zeta_1-\beta_1\right)^{\alpha}} \right.$$

$$\left. \times \left[I^{\alpha}_{\left(\lambda_1+\lambda_2-\zeta_1\right)^+} f\left(\lambda_1+\lambda_2-\tfrac{\beta_1+\zeta_1}{2}\right) + I^{\alpha}_{\left(\lambda_1+\lambda_2-\beta_1\right)^-} f\left(\lambda_1+\lambda_2-\tfrac{\beta_1+\zeta_1}{2}\right)\right] \right|$$

$$\leq \frac{|\zeta_1-\beta_1|}{8}\left(\tfrac{2(1-2^{-\alpha})}{\alpha+1}\right)^{1-\frac{1}{q}}$$

$$\times \left[\left(\left(\tfrac{2(1-2^{-\alpha})}{\alpha+1}\right)\left\{|f'(\lambda_1)|^q + |f'(\lambda_2)|^q\right\} - \left(\tfrac{1}{8} + \tfrac{1-2^{-\alpha}}{\alpha+1} - \tfrac{1-2^{-(\alpha+1)}}{\alpha+2}\right)|f'(\beta_1)|^q \right.\right.$$

$$\left. - \left(\tfrac{1-2^{-\alpha}}{\alpha+1} + \tfrac{1-2^{-(\alpha+1)}}{\alpha+2} - \tfrac{1}{8}\right)|f'(\zeta_1)|^q \right)^{\frac{1}{q}}$$

$$+ \left(\left(\tfrac{2(1-2^{-\alpha})}{\alpha+1}\right)\left\{|f'(\lambda_1)|^q + |f'(\lambda_1)|^q\right\} - \left(\tfrac{1}{8} + \tfrac{1-2^{-\alpha}}{\alpha+1} - \tfrac{1-2^{-(\alpha+1)}}{\alpha+2}\right)|f'(\zeta_1)|^q \right.$$

$$\left.\left. - \left(\tfrac{1-2^{-\alpha}}{\alpha+1} + \tfrac{1-2^{-(\alpha+1)}}{\alpha+2} - \tfrac{1}{8}\right)|f'(\beta_1)|^q \right)^{\frac{1}{q}} \right].$$

2. By setting $m=2$ and $\alpha=1$ in (39), then we get the following result achieved in [56].

$$\left|\frac{1}{2}\left[\frac{1}{2}\left\{f(\lambda_1+\lambda_2-\beta_1)+f(\lambda_1+\lambda_2-\zeta_1)\right\}+f\left(\lambda_1+\lambda_2-\tfrac{\beta_1+\zeta_1}{2}\right)\right]-\frac{1}{\zeta_1-\beta_1}\int\limits_{\lambda_1+\lambda_2-\zeta_1}^{\lambda_1+\lambda_2-\beta_1} f(u)du\right|$$

$$\leq \frac{|\zeta_1-\beta_1|}{8}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{2}\left[|f'(\lambda_1)|^q+|f'(\lambda_2)|^q\right]-\frac{1}{8}|f'(\beta_1)|^q-\frac{3}{8}|f'(\zeta_1)|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{1}{2}\left[|f'(\lambda_1)|^q+|f'(\lambda_2)|^q\right]-\frac{1}{8}|f'(\zeta_1)|^q-\frac{3}{8}|f'(\beta_1)|^q\right)^{\frac{1}{q}}\right]. \tag{46}$$

3.2.3 *Numerical and graphical validation*

The numerical validation and graphical illustrations of Theorem 4 are presented in Table 3 and Figures 5 and 6, respectively.

3.2.3.1 *Numerical simulation*

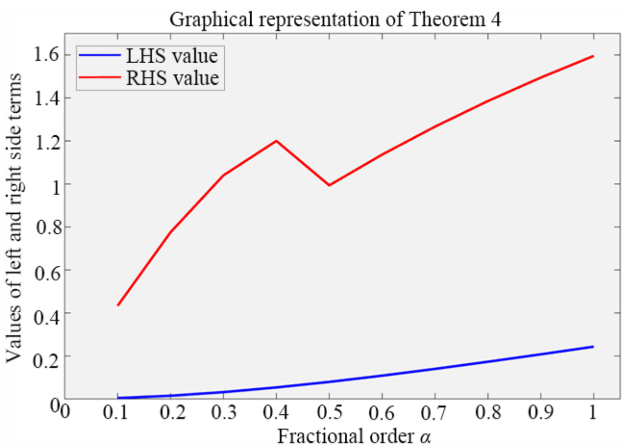

Figure 5. Graphical representation of Theorem 4 for $f(t)=\frac{t^3}{3}$, $f'(t)=t^2$, $m=2$, $\lambda_1=1$, $\lambda_2=4$, $\beta_1=2$ and $\zeta_1=3$, $q=2$, $\alpha\in(0,1]$

3.2.3.2 *Graphical interpretation*

Theorem 5 Let the function $f: I\to\mathbb{R}$ be differentiable on its domain I. Let the tuples $\boldsymbol{\lambda},\boldsymbol{\beta},\boldsymbol{\zeta}\in I^m$ be defined by $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\ldots,\lambda_m)$, $\boldsymbol{\beta}=(\beta_1,\beta_2,\ldots,\beta_m)$ and $\boldsymbol{\zeta}=(\zeta_1,\zeta_2,\ldots,\zeta_m)$ with $\beta_m>\zeta_m$ and $\lambda_j,\beta_j,\zeta_j\in I$ for all $j=1,2,\ldots,m$. Suppose further that $\boldsymbol{\beta}\prec\boldsymbol{\lambda}$ and $\boldsymbol{\zeta}\prec\boldsymbol{\lambda}$, $\alpha>0$, and $q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$, $|f'|^q$ is convex function. Then

$$|\xi(\lambda_j,\beta_j,\zeta_j,f)|\leq\frac{\sum\limits_{j\in M^*}|\zeta_j-\beta_j|}{8}\left[\frac{2}{p}\left\{\beta\,2^{-\beta}\left[B(\beta,p+1)+B_{1/2}(p+1,-p-\beta)\right]\right\}\right.$$

$$+\frac{2}{q}\sum_{j\in M}\left|f'(\lambda_j)\right|^q-\frac{1}{q}\left\{\sum_{j\in M^*}\left|f'(\beta_j)\right|^q+\sum_{j\in M^*}\left|f'(\zeta_j)\right|^q\right\}\Bigg], \tag{47}$$

where $B(.,.)$ and $B_{1/2}(.,.)$ represent Beta and incomplete Beta functions respectively.

Table 3. Numerical validation of Theorem 4 for $f(t)=\frac{t^3}{3}$, $f'(t)=t^2$, $m=2$, $q=2$, $\lambda_1=1$, $\lambda_2=4$, $x_1=2$, and $y_1=3$, $0<\alpha\leq 1$

Values of α	Values of left side term	Values of right side term
0.1	0.0039	0.4333
0.2	0.0148	0.7753
0.3	0.0317	1.0400
0.4	0.0535	1.2464
0.5	0.0793	1.4077
0.6	0.1083	1.5335
0.7	0.1397	1.6310
0.8	0.1730	1.7057
0.9	0.2076	1.7621
1.0	0.2431	1.8036

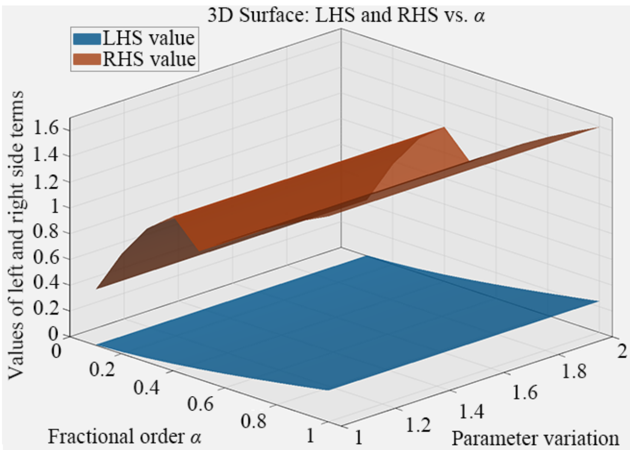

Figure 6. 3-D Graphical representation of Theorem 4

Proof. The modulus property, when applied to Lemma 1, yields:

$$\left|\xi(\lambda_j,\beta_j,\zeta_j,f)\right|\leq\frac{\sum_{j\in M^*}|\zeta_j-\beta_j|}{8}\left[\int_0^1\left|(1-2\omega^\alpha)\right|\left|f'\left(\sum_{j\in M}\lambda_j-\frac{1-\omega}{2}\sum_{j\in M^*}\beta_j-\frac{1+\omega}{2}\sum_{j\in M^*}\zeta_j\right)\right|d\omega\right.$$

$$\left.+\int_0^1\left|(1-2\omega^\alpha)\right|\left|f'\left(\sum_{j\in M}\lambda_j-\frac{1-\omega}{2}\sum_{j\in M^*}\zeta_j-\frac{1+\omega}{2}\sum_{j\in M^*}\beta_j\right)\right|d\omega\right]. \tag{48}$$

By implementing Young's inequality, we get

$$\left|\xi(\lambda_j,\beta_j,\zeta_j,f)\right| \leq \frac{\sum\limits_{j\in M^*}|\zeta_j-\beta_j|}{8}$$

$$\times\left[\left(\frac{1}{p}\int_0^1|1-2\omega^\alpha|^p\,d\omega+\frac{1}{q}\int_0^1\left|f'\left(\sum_{j\in M}\lambda_j-\frac{1-\omega}{2}\sum_{j\in M^*}\beta_j-\frac{1+\omega}{2}\sum_{j\in M^*}\zeta_j\right)\right|^q d\omega\right)\right.$$

$$\left.+\left(\frac{1}{p}\int_0^1|1-2\omega^\alpha|\,d\omega+\frac{1}{q}\int_0^1\left|f'\left(\sum_{j\in M}\lambda_j-\frac{1-\omega}{2}\sum_{j\in M^*}\zeta_j-\frac{1+\omega}{2}\sum_{j\in M^*}\beta_j\right)\right|^q d\omega\right)\right]. \qquad (49)$$

Applying Theorem 1 with the parameters $n=2$, $\sigma_1=\frac{1-\omega}{2}$ and $\sigma_2=\frac{1+\omega}{2}$ in (41), along with the convexity assumption on $|f'|^q$, we obtain

$$\left|\xi(\lambda_j,\beta_j,\zeta_j,f)\right| \leq \frac{\sum\limits_{j\in M^*}|\zeta_j-\beta_j|}{8}\times\left[\frac{2}{p}\int_0^1|1-2\omega^\alpha|^p\,d\omega\right.$$

$$+\frac{1}{q}\left(\sum_{j\in M}\left|f'(\lambda_j)\right|^q\int_0^1 1\,d\omega-\int_0^1\frac{1-\omega}{2}d\omega\sum_{j\in M^*}\left|f'(\beta_j)\right|^q-\int_0^1\frac{1+\omega}{2}d\omega\sum_{j\in M^*}\left|f'(\zeta_j)\right|^q\right)$$

$$\left.+\frac{1}{q}\left(\int_0^1 1\,d\omega\sum_{j\in M}\left|f'(\lambda_j)\right|^q-\int_0^1\frac{1-\omega}{2}d\omega\sum_{j\in M^*}\left|f'(\zeta_j)\right|^q-\int_0^1\frac{1+\omega}{2}d\omega\sum_{j\in M^*}\left|f'(\beta_j)\right|^q\right)\right]. \qquad (50)$$

By substituting (35) and (36), we obtain

$$\left|\xi(\lambda_j,\beta_j,\zeta_j,f)\right| \leq \frac{\sum\limits_{j\in M^*}|\zeta_j-\beta_j|}{8}\times\left[\frac{2}{p}\left\{\beta\,2^{-\beta}\left[B(\beta,p+1)+B_{1/2}(p+1,-p-\beta)\right]\right\}\right.$$

$$\left.+\frac{2}{q}\sum_{j\in M}\left|f'(\lambda_j)\right|^q-\frac{1}{q}\left\{\sum_{j\in M^*}\left|f'(\beta_j)\right|^q+\sum_{j\in M^*}\left|f'(\zeta_j)\right|^q\right\}\right]. \qquad (51)$$

□

Remark 5

1. By setting $m=2$ in (47), we arrive at the following inequality, which represents a new contribution to the literature of mathematical inequalities:

$$\left|\frac{1}{2}\left[\frac{1}{2}\left\{f\left(\lambda_1+\lambda_2-\beta_1\right)+f\left(\lambda_1+\lambda_2-\zeta_1\right)\right\}+f\left(\lambda_1+\lambda_2-\tfrac{\beta_1+\zeta_1}{2}\right)\right]-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{\left(\zeta_1-\beta_1\right)^\alpha}\right.$$

$$\times\left[I^{\alpha}_{\left(\lambda_1+\lambda_2-\zeta_1\right)^{+}}f\left(\lambda_1+\lambda_2-\tfrac{\beta_1+\zeta_1}{2}\right)+I^{\alpha}_{\left(\lambda_1+\lambda_2-\beta_1\right)^{-}}f\left(\lambda_1+\lambda_2-\tfrac{\beta_1+\zeta_1}{2}\right)\right]\Bigg|$$

$$\leq\frac{|\zeta_1-\beta_1|}{8}\left[\frac{2}{p}\left\{\beta\, 2^{-\beta}\left[B(\beta,p+1)+B_{1/2}(p+1,-p-\beta)\right]\right\}\right.$$

$$\left.+\frac{2}{q}\left\{\left|f'(\lambda_1)\right|^q+\left|f'(\lambda_2)\right|^q\right\}-\frac{1}{q}\left\{\left|f'(\beta_1)\right|^q+\left|f'(\zeta_1)\right|^q\right\}\right]. \tag{52}$$

2. If we substitute $m=2$ and $\alpha=1$ in (47), then we get the following result achieved in [56].

$$\left|\frac{1}{2}\left[\frac{1}{2}\left\{f\left(\lambda_1+\lambda_2-\beta_1\right)+f\left(\lambda_1+\lambda_2-\zeta_1\right)\right\}+f\left(\lambda_1+\lambda_2-\tfrac{\beta_1+\zeta_1}{2}\right)\right]-\frac{1}{\zeta_1-\beta_1}\int_{\lambda_1+\lambda_2-\zeta_1}^{\lambda_1+\lambda_2-\beta_1}f(u)du\right|$$

$$\leq\frac{|\zeta_1-\beta_1|}{8}\left[\frac{2}{p(p+1)}+\frac{2}{q}\left\{\left|f'(\lambda_1)\right|^q+\left|f'(\lambda_2)\right|^q\right\}-\frac{1}{q}\left\{\left|f'(\beta_1)\right|^q+\left|f'(\zeta_1)\right|^q\right\}\right]. \tag{53}$$

3.2.4 *Numerical and graphical validation*

The numerical validation and graphical illustrations of Theorem 5 are presented in Table 4 and Figures 7 and 8, respectively.

3.2.4.1 *Numerical simulation*

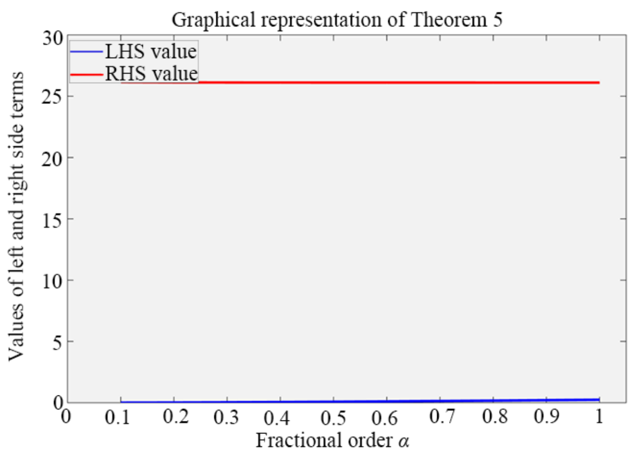

Figure 7. Graphical representation of Theorem 5 for $f(t)=\frac{t^3}{3}$, $f'(t)=t^2$, $m=2$, $\lambda_1=1$, $\lambda_2=4$, $\beta_1=2$ and $\zeta_1=3$, $q=2$, $\alpha\in(0,1]$

3.2.4.2 *Graphical Interpretation*

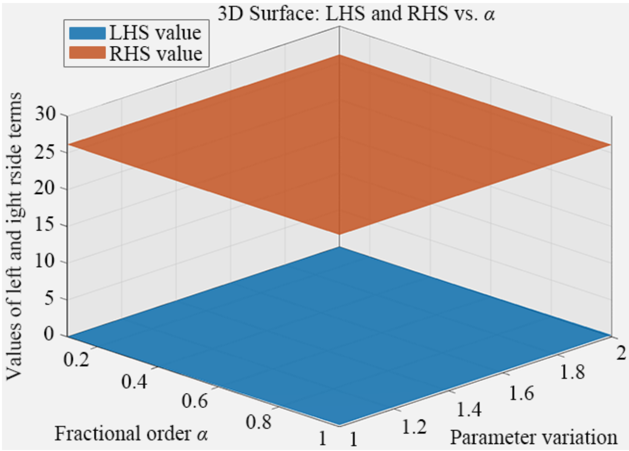

Figure 8. 3-D Graphical representation of Theorem 5

Table 4. Numerical validation of Theorem 5 for $f(t) = t^3/3$, $\lambda_1 = 1$, $\lambda_2 = 4$, $x_1 = 2$, $y_1 = 3$, $q = 2.0$ ($p = 2.0$), and $0 < \alpha \leq 1$

Value of α	Value of the LHS	Value of the RHS
0.1	0.0039	26.1496
0.2	0.0148	26.1278
0.3	0.0317	26.1155
0.4	0.0535	26.1106
0.5	0.0793	26.1093
0.6	0.1083	26.1090
0.7	0.1397	26.1085
0.8	0.1730	26.1074
0.9	0.2076	26.1059
1.0	0.2431	26.1041

4. Applications of the major results

4.1 *Applications to modified Bessel functions*

The modified Bessel function of the first kind is given by the series representation [58]

$$I_b(\Theta) = \sum_{j=0}^{\infty} \frac{(\Theta/2)^{b+2j}}{j!\Gamma(b+j+1)}. \tag{54}$$

Following Watson [58], we consider the auxillary function $J_b : \mathbb{R} \to \infty$ defined by

$$\mathscr{J}_b(\Theta) = 2^{\Theta}\Gamma(b+1)\Theta^{-v}I_b(\Theta), \quad \Theta \in \mathbb{R}, \tag{55}$$

From above, the first two derivatives of $\mathscr{I}_b$ are obtained as

$$\mathscr{I}_b'(\Theta) = \frac{\Theta}{2(b+1)} \mathscr{I}_{b+1}(\Theta), \tag{56}$$

$$\mathscr{I}_b''(\Theta) = \frac{1}{4(b+1)} \left[\frac{\Theta^2}{b+2} \mathscr{I}_{b+2}(\Theta) + 2\mathscr{I}_{b+1}(\Theta) \right]. \tag{57}$$

These derivative expressions are now considered to establish the below results.

Example 1 Set $f(x) = \mathscr{I}_b'(x)$ in Theorem 2. Then, employing, (56), (57), and the condition $\alpha = 1$, we obtain:

$$\Bigg| \frac{1}{2} \Bigg[\frac{1}{4(b+1)} \Big\{ \big(\sum_{j\in M} \lambda_j - \sum_{j\in M^*} x_j \big) \mathscr{I}_{b+1} \big(\sum_{j\in M} \lambda_j - \sum_{j\in M^*} x_j \big) + \big(\sum_{j\in M} \lambda_j - \sum_{j\in M^*} y_j \big) \mathscr{I}_{b+1} \big(\sum_{j\in M} \lambda_j - \sum_{j\in M^*} y_j \big) \Big\}$$

$$+ \frac{1}{2(b+1)} \big(\sum_{j\in M} \lambda_j - \sum_{j\in M^*} \tfrac{x_j+y_j}{2} \big) \mathscr{I}_{b+1} \big(\sum_{j\in M} \lambda_j - \sum_{j\in M^*} \tfrac{x_j+y_j}{2} \big) \Bigg]$$

$$- \frac{1}{\sum_{j\in M^*}(y_j - x_j)} \Big[\mathscr{I}_b \big(\sum_{j\in M} \lambda_j - \sum_{j\in M^*} x_j \big) - \mathscr{I}_b \big(\sum_{j\in M} \lambda_j - \sum_{j\in M^*} y_j \big) \Big] \Bigg|$$

$$\leq \frac{\sum_{j\in M^*} |\zeta_j - \beta_j|}{8} \Bigg[\sum_{j\in M} \Big| \tfrac{1}{4(b+1)} \Big(\tfrac{\lambda_j^2}{b+2} \mathscr{I}_{b+2}(\lambda_j) + 2\mathscr{I}_{b+1}(\lambda_j) \Big) \Big| - \tfrac{1}{2} \Bigg\{ \sum_{j\in M^*} \Big| \tfrac{1}{4(b+1)} \Big(\tfrac{\beta_j^2}{b+2} \mathscr{I}_{b+2}(\beta_j) + 2\mathscr{I}_{b+1}(\beta_j) \Big) \Big|$$

$$+ \sum_{j\in M^*} \Big| \tfrac{1}{4(b+1)} \Big(\tfrac{\zeta_j^2}{b+2} \mathscr{I}_{b+2}(\zeta_j) + 2\mathscr{I}_{b+1}(\zeta_j) \Big) \Big| \Bigg\} \Bigg].$$

Example 2 Set $f(x) = \mathscr{I}_b'(x)$ in Theorem 3. Then, employing, (56), (57), and the condition $\alpha = 1$, we obtain:

$$\Bigg| \frac{1}{2} \Bigg[\frac{1}{4(b+1)} \Big\{ \big(\sum_{j\in M} \lambda_j - \sum_{j\in M^*} x_j \big) \mathscr{I}_{b+1} \big(\sum_{j\in M} \lambda_j - \sum_{j\in M^*} x_j \big) + \big(\sum_{j\in M} \lambda_j - \sum_{j\in M^*} y_j \big) \mathscr{I}_{b+1} \big(\sum_{j\in M} \lambda_j - \sum_{j\in M^*} y_j \big) \Big\}$$

$$+ \frac{1}{2(b+1)} \big(\sum_{j\in M} \lambda_j - \sum_{j\in M^*} \tfrac{x_j+y_j}{2} \big) \mathscr{I}_{b+1} \big(\sum_{j\in M} \lambda_j - \sum_{j\in M^*} \tfrac{x_j+y_j}{2} \big) \Bigg]$$

$$- \frac{1}{\sum_{j\in M^*}(y_j - x_j)} \Big[\mathscr{I}_b \big(\sum_{j\in M} \lambda_j - \sum_{j\in M^*} x_j \big) - \mathscr{I}_b \big(\sum_{j\in M} \lambda_j - \sum_{j\in M^*} y_j \big) \Big] \Bigg|$$

$$\leq \frac{\sum_{j\in M^*} |y_j - x_j|}{8} (p+1)^{-1/p} \Bigg\{ \Bigg[\sum_{j\in M} \Big| \frac{1}{4(b+1)} \Big(\frac{\lambda_j^2}{b+2} \mathscr{I}_{b+2}(\lambda_j) + 2\mathscr{I}_{b+1}(\lambda_j) \Big) \Big|^q$$

$$-\frac{1}{4}\sum_{j\in M^*}\left|\frac{1}{4(b+1)}\left(\frac{x_j^2}{b+2}\mathscr{I}_{b+2}(x_j)+2\mathscr{I}_{b+1}(x_j)\right)\right|^q-\frac{3}{4}\sum_{j\in M^*}\left|\frac{1}{4(b+1)}\left(\frac{y_j^2}{b+2}\mathscr{I}_{b+2}(y_j)+2\mathscr{I}_{b+1}(y_j)\right)\right|^q\Bigg]^{1/q}$$

$$+\Bigg[\sum_{j\in M}\left|\frac{1}{4(b+1)}\left(\frac{\lambda_j^2}{b+2}\mathscr{I}_{b+2}(\lambda_j)+2\mathscr{I}_{b+1}(\lambda_j)\right)\right|^q-\frac{1}{4}\sum_{j\in M^*}\left|\frac{1}{4(b+1)}\left(\frac{y_j^2}{b+2}\mathscr{I}_{b+2}(y_j)+2\mathscr{I}_{b+1}(y_j)\right)\right|^q$$

$$-\frac{3}{4}\sum_{j\in M^*}\left|\frac{1}{4(b+1)}\left(\frac{x_j^2}{b+2}\mathscr{I}_{b+2}(x_j)+2\mathscr{I}_{b+1}(x_j)\right)\right|^q\Bigg]^{1/q}\Bigg\}. \tag{58}$$

4.2 *Applications to special means*

Arithmetic mean:

$$A(\Theta_1,\Theta_2)=\frac{\Theta_1+\Theta_2}{2},\quad \Theta_1,\Theta_2\in\mathbb{R}, \tag{59}$$

Logarithmic mean:

$$\bar{L}(\Theta_1,\Theta_2)=\frac{\Theta_2-\Theta_1}{\ln|\Theta_2|-\ln|\Theta_1|},\qquad ,\Theta_1,\Theta_2\in\mathbb{R}\setminus\{0\} \tag{60}$$

Generalized log-mean:

$$L_n(\Theta_1,\Theta_2)=\left(\frac{\Theta_2^{n+1}-\Theta_1^{n+1}}{(n+1)(\Theta_2-\Theta_1)}\right)^{\frac{1}{n}},\quad n\in\mathbb{R}\backslash\{-1,0\},\Theta_1<\Theta_2. \tag{61}$$

Proposition 1 By the same assumptions as in Theorem 2, we have the following inequality.

$$\left|\frac{1}{2}\Big(A\big(U^n,V^n\big)+A\big(U,V\big)\Big)-L_n^n\big(U,V\big)\right|\leq\frac{|n|}{8}\sum_{j\in M^*}|\zeta_j-\beta_j|A\Big(\sum_{j\in M}|\lambda_j|^{n-1}-\sum_{j\in M^*}|\beta_j|^{n-1},\sum_{j\in M}|\lambda_j|^{n-1}-\sum_{j\in M^*}|\zeta_j|^{n-1}\Big),$$

where

$$U:=\sum_{j\in M}\lambda_j-\sum_{j\in M^*}x_j,\quad V:=\sum_{j\in M}\lambda_j-\sum_{j\in M^*}y_j.$$

Proof. The result is obtained directly from Theorem 2 by setting $\alpha=1$ and taking $f(x)=x^n$, where $x\in\mathbb{R}$. □

The graphical illustration of Proposition 1 is presented in Figure 9.

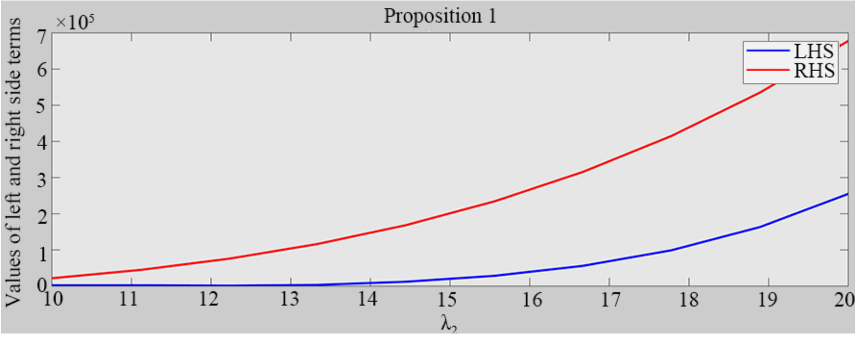

Figure 9. Graphical representation of Proposition 1 for $m = 2$, $\alpha = 1$ with parameters $n = 5$, $\lambda_1 = 1$, $\lambda_2 \in [10, 20]$, $\beta_1 = 10$ and $\zeta_1 = 3$, showing left side (blue curve) and right side (red curve) of the inequality

Proposition 2 By the same assumptions as in Theorem 3, we have the following inequality.

$$\left|\frac{1}{2}\Big(A\big(U^n, V^n\big) + A\big(U, V\big)\Big) - L_n^n(U, V)\right|$$

$$\leq \frac{|n|}{8}\left(\tfrac{1}{p+1}\right)^{\frac{1}{p}} \sum_{j\in M^*} |\zeta_j - \beta_j| \left[\left\{\sum_{j\in M} |\lambda_j|^{(n-1)q} - \tfrac{1}{2}A\Big(\sum_{j\in M^*} |\beta_j|^{(n-1)q},\ 3\sum_{j\in M^*} |\zeta_j|^{(n-1)q}\Big)\right\}^{\frac{1}{q}}\right.$$

$$\left. + \left\{\sum_{j\in M} |\lambda_j|^{(n-1)q} - \tfrac{1}{2}A\Big(3\sum_{j\in M^*} |\beta_j|^{(n-1)q},\ \sum_{j\in M^*} |\zeta_j|^{(n-1)q}\Big)\right\}^{\frac{1}{q}}\right].$$

Proof. The result is obtained directly from Theorem 3 by setting $\alpha = 1$ and taking $f(x) = x^n$, where $x \in \mathbb{R}$. □

The graphical illustration of Proposition 2 is presented in Figure 10.

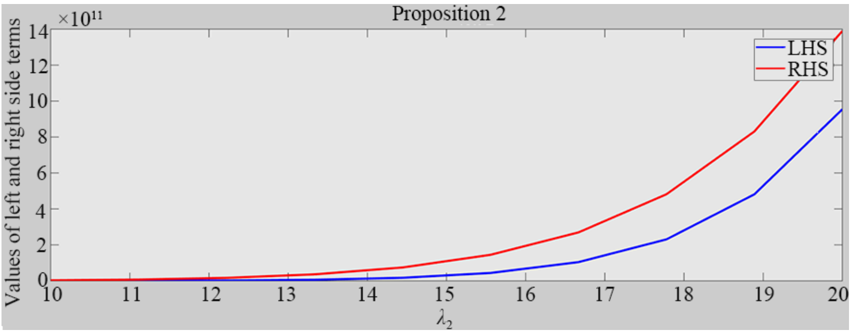

Figure 10. Graphical representation of Proposition 2 for $m = 2$, $\alpha = 1$ with parameters $n = 10$, $\lambda_1 = 1$, $\lambda_2 \in [10, 20]$, $\beta_1 = 5$ and $\zeta_1 = 3$, $q = 3$, $p = \frac{3}{2}$, showing left side (blue curve) and right side (red curve) of the inequality

Proposition 3 By the same assumptions as in Theorem 4, we have the following inequality.

$$\left|\frac{1}{2}\Big(A\big(U^n, V^n\big) + A\big(U, V\big)\Big) - L_n^n(U, V)\right|$$

$$\leq \frac{|n|}{8}\left(\frac{1}{2}\right)^{1-\frac{1}{q}} \sum_{j\in M^*}|\zeta_j-\beta_j|\left[\left\{\tfrac{1}{2}\sum_{j\in M}|\lambda_j|^{(n-1)q}-\tfrac{1}{4}A\left(\sum_{j\in M^*}|\beta_j|^{(n-1)q},\ 3\sum_{j\in M^*}|\zeta_j|^{(n-1)q}\right)\right\}^{\frac{1}{q}}\right.$$

$$\left.+\left\{\tfrac{1}{2}\sum_{j\in M}|\lambda_j|^{(n-1)q}-\tfrac{1}{4}A\left(3\sum_{j\in M^*}|\beta_j|^{(n-1)q},\ \sum_{j\in M^*}|\zeta_j|^{(n-1)q}\right)\right\}^{\frac{1}{q}}\right].$$

Proof. The result is obtained directly from Theorem 4 by setting $\alpha = 1$ and taking $f(x) = x^n$, where $x \in \mathbb{R}$. □

The graphical illustration of Proposition 3 is presented in Figure 11.

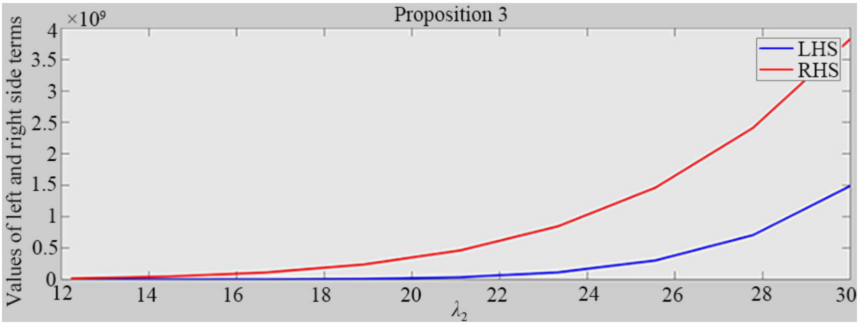

Figure 11. Graphical representation of Proposition 3 for $m = 2$, $\alpha = 1$ with parameters $n = 7$, $\lambda_1 = 1$, $\lambda_2 \in [12, 30]$, $\beta_1 = 11$, $\zeta_1 = 5$, and $q = 3$, showing left side (blue curve) and right side (red curve) of the inequality

Proposition 4 By the same assumptions as in Theorem 5, we have the following inequality.

$$\left|\frac{1}{2}\Big(A\big(U^n,V^n\big)+A\big(U,V\big)\Big)-L_n^n(U,V)\right|$$

$$\leq \frac{1}{8}\sum_{j\in M^*}|\zeta_j-\beta_j|\left[\left\{\frac{2}{p(p+1)}+\frac{2|n|}{q}\sum_{j\in M}|\lambda_j|^{(n-1)q}-\frac{2|n|}{q}A\left(\sum_{j\in M^*}|\beta_j|^{(n-1)q},\ \sum_{j\in M^*}|\zeta_j|^{(n-1)q}\right)\right\}\right].$$

Proof. The result is obtained directly from Theorem 5 by setting $\alpha = 1$ and taking $f(x) = x^n$, where $x \in \mathbb{R}$. □

The graphical illustration of Proposition 4 is presented in Figure 12.

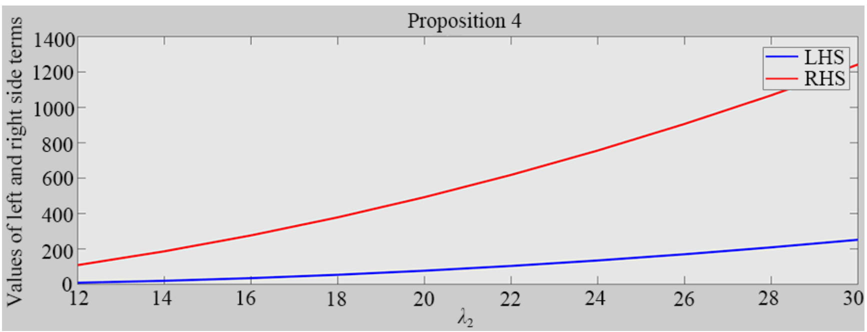

Figure 12. Graphical representation of Proposition 4 for $m = 2$, $\alpha = 1$ with parameters $n = 2$, $\lambda_1 = 1$, $\lambda_2 \in [12, 30]$, $\beta_1 = 11$, $\zeta_1 = 5$, $q = 2$, and $p = 2$, showing left side (blue curve) and right side (red curve) of the inequality

5. Conclusion

The combination between majorization theory, fractional calculus, and convex functions has provided a rich foundation for this investigation. By merging the classical Bullen inequality with Mercer's framework, a family of new estimates has been developed that bridges discrete and continuous settings into a single unified form. The establishment of a suitable integral identity for differentiable functions worked as the pivotal step that enabled the derivation of several bounds under the convexity conditions on the absolute value of the first derivative. The graphical representations and Numerical experiments verified the established results. They also, showed their behavior over various values of the fractional order. The link made between the recently discovered inequalities and previous findings is a prominent aspect of this research. Several classical inequalities are recovered by choosing particular values of the parameters, which consequently links this work to the larger body of literature. Concrete applications to modified Bessel functions of the first kind and various special means were used to further illustrate the practical significance of the findings, with graphical support enhancing the conclusions achieved. The findings discovered here not only broaden current understanding but also provide a number of new directions for further investigation. These may include exploring alternative convexity classes, and employing other fractional integral operators.

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Conflict of interest

The author declare no competing financial interest.

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