



Approximate Solutions for Solving Fractional-order Painlevé Equations

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Abstract: In this work, Chebyshev orthogonal polynomials are employed as basis functions in a collocation scheme to solve the nonlinear Painlevé initial value problems known as the first and second Painlevé equations. Using the collocation points, representing the solution and its fractional derivative (in the Caputo sense) in matrix forms, and the matrix operations, the proposed technique transforms a solution of the initial-value problem for the Painlevé equations into a system of nonlinear algebraic equations. To get ride of nonlinearity, the technique of quasi-linearization is also applied, which converts the equations into a sequence of linear algebraic equations. The accuracy and efficiency of the presented methods are investigated by some test examples and a comparison has been made with some existing available numerical schemes.

Keywords: Caputo fractional derivative, Chebyshev functions, Collocation method, Painlevé equations

1. Introduction

The fractional Painlevé models can be obtained by using the fractional derivative operator on the classical Painlevé equations. Historically, the origin of the Painlevé equations traced back to more than one century ago^[18]. Indeed, Painlevé equations are nonlinear second-order ordinary differential equations that satisfy the so-called Painlevé properties (the general solutions are free from movable branch points) and their solutions known as Painlevé transcendents. Painlevé equations appear in many important physical applications. Among others, we emphasize as a model for describing the electric field in a semiconductor^[14], quantum gravity^[9], and random matrix theory^[26]. Furthermore, the exact solutions to many nonlinear partial differential equations such as Korteweg-de Vries (KdV), cylindrical KdV and Bussiness equations can be written in terms of Painlevé transcendent^[1, 25]. The mathematical theory of some of the classical Painlevé differential equations along with some applications is considered in^[5, 10, 15].

In the current study, we consider the fractional-order Painlevé models of the forms

$$\begin{cases} D^{(\nu)}Y(t) = 6Y^2(t) + t\mu_1, & 0 \leq t \leq 1, \\ Y(0) = \gamma_0, \quad Y'(0) = \gamma_1, \end{cases} \quad (1)$$

and

$$\begin{cases} D^{(\nu)}Y(t) = 2Y^3(t) + tY(t) + \mu_2, & 0 \leq t \leq 1, \\ Y(0) = \gamma_0, \quad Y'(0) = \gamma_1, \end{cases} \quad (2)$$

where μ_1 , μ_2 , and γ_0 , γ_2 are real constants. Here, $D^{(\nu)}$ is the standard Caputo fractional derivative operator and $1 < \nu \leq 2$. There has been significant interest in developing analytical, approximative as well as numerical schemes for the solution of the classical and fractional Painlevé differential equations (1)-(2). The most significant analytical schemes include Adomian's decomposition (ADM) and homotopy perturbation methods (HPM)^[7], variational iteration method (VIM) and HPM^[12], homotopy analysis method^[8, 11], sinc collocation method and VIM^[23], optimal homotopy asymptotic method^[16], to name but a few. On the other hand, numerical techniques such as Chebyshev series^[6, 13], computational intelligence technique based on neural networks and particle swarm optimization^[21, 22], and reproducing kernel Hilbert space algorithms^[3] have been developed in the past to solve the nonlinear equation (1)-(2).

In this note, we take a further step towards proposing approximation methods for solving (1)-(2). As a generalization,

we consider the following form

$$\begin{aligned} D^{(\nu)}Y(t) &= c_0(t)Y^3(t) + c_1(t)Y^2(t) + c_2(t)Y(t) + c_3(t), \quad 0 \leq t \leq 1, \\ Y(0) &= \gamma_0, \quad Y'(0) = \gamma_1, \end{aligned} \quad (3)$$

which describes completely the first and second Painlevé equations. We use the shifted Chebyshev polynomials to get an approximate solution of (3) accurately. The main idea of the proposed technique based on using these (orthogonal) functions along with collocation points is that it converts the differential or integral operator involved in (3) to an algebraic form, thus greatly reducing the computational effort. As an efficient version of the previous scheme, we develop a Chebyshev-QLM algorithm as a combination of the Chebyshev collocation scheme and quasi-linearization technique. In the latter method, we solve (1) and (2) as a sequence of linear equations via the Chebyshev collocation scheme rather than one single nonlinear equation.

The rest of this paper is organized as follows. In Section 2, some definitions and mathematical preliminaries of fractional calculus are presented. Hence, a brief review of the properties of the (shifted) Chebyshev polynomials is outlined. Section 3 is devoted to the presentation of the proposed collocation scheme applied to the nonlinear initial value problem (3). The error analysis technique based on the residual function is also developed for the present method. The technique of quasi-linearization is briefly described for the model problems in Section 4. In computational Section 5, we apply the proposed methods to some test problems and report our numerical findings. We end the paper with a few concluding remarks in Section 6.

2. Basic definitions

In this section, first some properties of the fractional calculus theory is presented. Hence, the definition of (shifted) Chebyshev polynomials and some of their properties are mentioned.

2.1 Fractional calculus

Definition 2.1 Suppose that $f(t)$ is m -times continuously differentiable. The fractional derivative $D^{(\nu)}$ of $f(t)$ of order $\nu > 0$ in the Caputo's sense is defined as

$$D^{(\nu)}f(t) = \begin{cases} I^{m-\nu}f^{(m)}(t) & \text{if } m-1 < \nu < m, \\ f^{(m)}(t), & \text{if } \nu = m, \quad m \in \mathbb{N}, \end{cases} \quad (4)$$

where

$$I^\nu f(t) = \frac{1}{\Gamma(\nu)} \int_0^t \frac{f(s)}{(t-s)^{1-\nu}} ds, \quad t > 0.$$

The properties of the operator $D^{(\nu)}$ can be found in [20]. We make use of the followings

$$D^{(\nu)}(C) = 0 \quad (C \text{ is a constant}), \quad (5)$$

$$D^{(\nu)}t^\gamma = \begin{cases} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\nu)} t^{\gamma-\nu}, & \text{for } \gamma \in \mathbb{N}_0 \text{ and } \gamma \geq \lceil \nu \rceil \text{ or } \gamma \notin \mathbb{N}_0 \text{ and } \gamma > \lceil \nu \rceil \\ 0, & \text{for } \gamma \in \mathbb{N}_0 \text{ and } \gamma < \lceil \nu \rceil \end{cases} \quad (6)$$

where $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ denote the ceiling and floor functions respectively.

2.2 Chebyshev functions

It is known that the classical Chebyshev polynomials are defined on $[-1,1]$. Starting with $T_0(z) = 1$ and $T_1(z) = z$, these polynomials satisfy the following recurrence relation [2]

$$T_{n+1}(z) = 2zT_n(z) - T_{n-1}(z), \quad n = 1, 2, \dots$$

By introducing the change of variable $z=1-2t/R$ one obtains the shifted version of the polynomials defined on $[0, R]$ and will be denoted by $T_n^*(t) = T_n(z)$. The explicit analytical form of $T_n^*(t)$ of degree n is given for $n = 0, 1, \dots$

$$T_n^*(t) = \sum_{k=0}^n c_{n,k} t^k, \quad c_{n,k} = (-1)^k \frac{n2^{2k}(n+k-1)!}{(n-k)!R^k(2k)!}, \quad k = 0, 1, \dots, n, \quad (7)$$

with $c_{0,k}=1$ for all $k = 0; 1, \dots, n$. It is proved that the set of fractional polynomial functions $\{T_0^*, T_1^*, \dots\}$ is orthogonal on $[0, R]$ with respect to the weight function $w(t) = \frac{t^{-1/2}}{\sqrt{R-t}}$; i.e.

$$\int_0^R T_n^*(t) T_m^*(t) w(t) dt = \frac{\pi}{2} d_n \delta_{mn}, \quad n, m \geq 0.$$

Here, δ_{mn} is Kronecker delta function, $d_0 = 2$ while $d_n = 1$ for $n \geq 1$. Our aim is to find an approximate solution of model (3) expressed in the truncated Chebyshev series form (12)

$$Y_N(t) = \sum_{n=0}^N a_n T_n^*(t), \quad 0 \leq t \leq R, \quad (8)$$

where the unknown coefficients $a_n, n=0, 1, \dots, N$ are sought. To proceed, we write $T_n^*(t), n=0, 1, \dots, N$ in the matrix form as follows

$$T_N(t) = B_N(t) D^t \Leftrightarrow T_N^t(t) = D B_N^t(t), \quad (9)$$

here, a superscript t denotes the matrix transpose operation and

$$T_N(t) = [T_0^*(t) \quad T_1^*(t) \quad \dots \quad T_N^*(t)], \quad B_N(t) = [1 \quad t \quad t^2 \quad \dots \quad t^N].$$

The $(N+1) \times (N+1)$ matrix D takes the form

$$D = \begin{bmatrix} 1 & 0 & 0 & \dots & \dots & \dots & 0 \\ 1 & c_{1,1} & 0 & 0 & \dots & \dots & 0 \\ 1 & c_{2,1} & c_{2,2} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 1 & c_{N-1,1} & c_{N-1,2} & \dots & c_{N-1,N-2} & c_{N-1,N-1} & 0 \\ 1 & c_{N,1} & c_{N,2} & \dots & c_{N,N-2} & c_{N,N-1} & c_{N,N} \end{bmatrix}.$$

By means of (9) one can write the relation (8) in the matrix form

$$Y_N(t) = B_N(t) D^t A, \quad (10)$$

where the vector of unknown is $A = [a_0 \ a_1 \ \dots \ a_N]^t$. Finally, to obtain a solution in the form (10) of the problem (3) on the interval $0 < t \leq R$, we will use the spectral collocation points defined by

$$t_i = \frac{R}{2}(1-x_i), \quad i = 0, 1, \dots, N, \quad (11)$$

where x_i are the zeros of the usual Chebyshev polynomials of degree $N+1$ on $(-1, 1)$.

3. The method of solution

We are aiming to approximate the solution $Y(t)$ of the nonlinear Painlevé equation (3) in terms of Chebyshev polynomial series denoted by $Y_N(t)$ on the interval $[0, R]$. As previously mentioned in (10), in the vector form one may write

$$Y(t) \approx Y_N(t) = B_N(t)D^t A. \quad (12)$$

By substituting the collocation points (11) into (12), we get the following system of matrix equations

$$Y_N(t_i) = B_N(t_i)D^t A, \quad i = 0, 1, \dots, N.$$

These equations can be written in a single and compact representation as follows

$$Y = BD^t A, \quad Y = \begin{bmatrix} Y_N(t_0) \\ Y_N(t_1) \\ \vdots \\ Y_N(t_N) \end{bmatrix}, \quad B = \begin{bmatrix} B_N(t_0) \\ B_N(t_1) \\ \vdots \\ B_N(t_N) \end{bmatrix}. \quad (13)$$

By taking the fractional derivative of order ν from the both sides of (12), we get

$$D^{(\nu)}Y_N(t) = D^{(\nu)}B_N(t)D^t A. \quad (14)$$

The calculation of $D^{(\nu)}B_N(t)$ can be easily obtained via the property (5) and (6) as follows

$$B_N^{(\nu)}(t) := D^{(\nu)}B_N(t) = [0 \quad D^{(\nu)}t \quad \dots \quad D^{(\nu)}t^N].$$

To obtain a system of matrix equations for the fractional derivative, we insert the collocation points (11) into (14) to get

$$D^{(\nu)}Y_N(t_i) = B_N^{(\nu)}(t_i)D^t A, \quad i = 0, 1, \dots, N,$$

which can be written in the matrix form

$$Y^{(\nu)} = B^{(\nu)}D^t A, \quad Y^{(\nu)} = \begin{bmatrix} D^{(\nu)}Y_N(t_0) \\ D^{(\nu)}Y_N(t_1) \\ \vdots \\ D^{(\nu)}Y_N(t_N) \end{bmatrix}, \quad B^{(\nu)} = \begin{bmatrix} B_N^{(\nu)}(t_0) \\ B_N^{(\nu)}(t_1) \\ \vdots \\ B_N^{(\nu)}(t_N) \end{bmatrix}. \quad (15)$$

To continue, we approximate the nonlinear term $Y^2(t)$. By substituting the collocation points into $Y_N^2(t)$ we arrive at the following matrix representation

$$Y^2 = \begin{bmatrix} Y_N^2(t_0) \\ Y_N^2(t_1) \\ \vdots \\ Y_N^2(t_N) \end{bmatrix} = \begin{bmatrix} Y_N(t_0) & 0 & \dots & 0 \\ 0 & Y_N(t_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & Y_N(t_N) \end{bmatrix} \begin{bmatrix} Y_N(t_0) \\ Y_N(t_1) \\ \vdots \\ Y_N(t_N) \end{bmatrix} = \hat{Y}Y. \quad (16)$$

Moreover, the matrix \hat{Y} can be written as a product of three block diagonal matrices as

$$\hat{Y} = \hat{D} \hat{B} \hat{A} \quad (17)$$

where

$$\hat{B} = \begin{bmatrix} B_N(t_0) & 0 & \dots & 0 \\ 0 & B_N(t_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B_N(t_N) \end{bmatrix}, \hat{D} = \begin{bmatrix} D' & 0 & \dots & 0 \\ 0 & D' & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & D' \end{bmatrix}, \hat{A} = \begin{bmatrix} A & 0 & \dots & 0 \\ 0 & A & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A \end{bmatrix}.$$

Similarly, by inserting the collocation points (11) into the $Y^3(t)$ we arrive at the following matrix representation

$$Y^3 = \begin{bmatrix} Y_N^3(t_0) \\ Y_N^3(t_1) \\ \vdots \\ Y_N^3(t_N) \end{bmatrix} = \begin{bmatrix} Y_N^2(t_0) & 0 & \dots & 0 \\ 0 & Y_N^2(t_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & Y_N^2(t_N) \end{bmatrix} \begin{bmatrix} Y_N(t_0) \\ Y_N(t_1) \\ \vdots \\ Y_N(t_N) \end{bmatrix} = (\hat{Y})^2 Y, \quad (18)$$

where \hat{Y} is defined in (17).

Now, we are able to compute the Chebyshev solutions of (3). The collocation procedure is based on calculating these polynomial coefficients by means of collocation points defined in (11). To proceed, insert the collocation points into the fractional differential equation to get the system

$$D^{(\nu)}Y(t_i) = c_0(t_i)Y^3(t_i) + c_1(t_i)Y^2(t_i) + c_2(t_i)Y(t_i) + c_3(t_i), \quad i = 0, 1, \dots, N.$$

In the matrix form we may write the above equations as

$$Y^{(\nu)} - C_0 Y^3 - C_1 Y^2 - C_2 Y = C_3, \quad (19)$$

where the coefficient matrices C_i , $i = 0, 1, 2$ of size $(N + 1) \times (N + 1)$ and the vector C_3 of size $(N + 1) \times 1$ have the following forms

$$C_i = \begin{bmatrix} c_i(t_0) & 0 & \dots & 0 \\ 0 & c_i(t_1) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & c_i(t_N) \end{bmatrix}, \quad i = 0, 1, 2, \quad C_3 = \begin{bmatrix} c_3(t_0) \\ c_3(t_1) \\ \vdots \\ c_3(t_N) \end{bmatrix}.$$

By substituting the relations (13), (15), and (16), (18) into (19), the fundamental matrix equation is obtained

$$W A = C_3, \quad W := B^{(\nu)} D' - C_0 (\hat{B} \hat{D} \hat{A})^2 B D' - C_1 \hat{B} \hat{D} \hat{A} B D' - C_2 B D'. \quad (20)$$

Obviously, (20) is a nonlinear matrix equation with a_n , $n = 0, 1, \dots, N$, being the unknowns Chebyshev coefficients. To take into account the initial condition $Y(0) = \gamma_0$, we tend $t \rightarrow 0$ in (12) to get the following matrix representation

$$\tilde{Y}_0 A = \gamma_0, \quad \tilde{Y}_0 := B_N(0) D' = [Y_{00} \quad Y_{01} \quad \dots \quad Y_{0N}]'.$$

Analogously, to hold the initial condition $Y'(0) = \gamma_1$, we differentiate (12) with respect to the variable t to get

$$Y_N'(t) = B_N'(t) D' A. \quad (21)$$

On the other hand, it is not a difficult task to show that the relationship between $B_N(t)$ and its derivative is

$$B'_N(t) = B_N(t)Q', \quad Q' = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \dots & N \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}.$$

Consequently, by tending $t \rightarrow 0$ in (21) we obtain

$$\tilde{Y}_1 A = \gamma_1, \quad \tilde{Y}_1 := B_N(0)Q' D' = [Y_{10} \quad Y_{11} \quad \dots \quad Y_{1N}]'.$$

Now, by replacing two rows of the augmented matrix $[W; C_3]$ by the row matrices $[\tilde{Y}_0; \gamma_0]$ and $[\tilde{Y}_1; \gamma_1]$, we arrive at the nonlinear algebraic system

$$\tilde{W} A = \tilde{C}. \quad (22)$$

For convenience, the first and last rows are replaced. Thus, the unknown Chebyshev coefficients in (12) will be calculated via solving this nonlinear system of equations. This task can be performed using for instance Newton's iterative method.

3.1 Error function

Since the exact solutions of the fractional Painlevé differential equations are not known in general we require to measure the accuracy of the proposed collocation scheme. Due to the fact that the truncated Chebyshev series (8) are the approximate solutions of (3), we expect that the residual obtained by inserting the computed approximated solutions $Y_N(t)$ into the differential equation becomes approximately small. This implies that for $t = t_s \in [0, R]$, $s = 0, 1, \dots$

$$E_N(t_s) = D^{(\nu)} Y_N(t_s) - c_0(t_s) Y_N^3(t_s) - c_1(t_s) Y_N^2(t_s) - c_2(t_s) Y_N(t_s) - c_3(t_s) \cong 0, \quad (23)$$

and $E_N(t_s) \leq 10^{-k_s}$ (k_s is positive integer). If $\max 10^{-k_s} \leq 10^{-k}$ (k positive integer) is prescribed, then the truncation limit N is increased until the difference $E_N(t_s)$ at each of the points becomes smaller than the prescribed 10^{-k} , see [4, 27]. Here, we note that the ν th-order fractional derivative of the approximate solution (23) is computed by using the properties (5) - (6). As the error function is clearly zero at the collocation points (11), we expect that $E_N(t)$ tends to zero as N increased.

4. Quasi-linearization technique

Clearly, solving the nonlinear system (22) using iterative procedures like Newton's methods is very time-consuming and even sometimes is inefficient when N is getting large. To overcome this difficulty we may first convert the original equation (3) into a sequence of linear equations and then apply the aforementioned Chebyshev collocation scheme through an iterative process. For this purpose, we describe briefly the quasilinearization method (QLM) as a generalized Newton-Raphson scheme for functional equations, see [17, 19, 24].

Let us consider the general form of nonlinear differential equation (3),

$$D^{(\nu)} Y(t) = f(t, Y(t)), \quad (24)$$

with the initial conditions $Y(0) = \gamma_0$, $Y'(0) = \gamma_1$. Here f is a function of $Y(t)$. To start the computation, we need to choose an initial approximation of the function $Y(t)$. Assuming that $Y_0(t) = \gamma_0$ as an initial guess, the QLM iteration for (24) is determined as follows [24]

$$D^{(\nu)} Y_{r+1}(t) = f(t, Y_r(t)) + (Y_{r+1}(t) - Y_r(t)) f_{Y_r}(t, Y_r(t)), \quad (25)$$

with the same initial conditions $Y_{r+1}(0) = \gamma_0$, $Y'_{r+1}(0) = \gamma_1$ and $r = 0, 1, \dots$ and the function $f_Y = \partial f / \partial Y$ denotes the functional

derivative of $f(t, Y(t))$. By applying the QLM technique on the first and second Painlevé equations (1)-(2) we get respectively

$$D^{(\nu)}Y_{r+1}(t) = t\mu_1 - 6Y_r^2(t) + 12Y_r(t)Y_{r+1}(t),$$

$$D^{(\nu)}Y_{r+1}(t) = \mu_2 - 4Y_r^3(t) + (t + 6Y_r^2(t))Y_{r+1}(t),$$

with the corresponding initial conditions. Therefore, instead of applying the Chebyshev collocation scheme directly to Painlevé equations we solve a sequence of linear equations by the collocation method, which is referred to as the Chebyshev-QLM.

5. Numerical Applications

In this section, we illustrate the accuracy and effectiveness of the proposed Chebyshev collocation and Chebyshev-QLM methods numerically when applied to the first and second Painlevé equations. Comparisons with existing numerical schemes are also made to solve these equations.

5.1 The first Painlevé equation

To start, we take $\nu=2$ and $\mu_1=1$ in (1) and set $N=8$ as the number of basis functions. In this case, the initial conditions are given as $\gamma_0=0$ and $\gamma_1=1$. The approximate solutions $Y_8(t)$ of this model problem using Chebyshev basis functions in the interval $0 \leq t \leq 1$ are obtained as follows

$$Y_8(t) = 2.22851774309907 t^8 - 6.43647638938382 t^7 + 8.39819213815701 t^6 - 5.62894940522023 t^5 + 2.6409234138635 t^4 - 0.280581190647174 t^3 + 0.0468676171123739 t^2 + 1.0 t - 3.27971157176865 \times 10^{-18}.$$

The corresponding approximation by means of Chebyshev-QLM using $r=5$ iterations takes the form

$$Y_{8,5}(t) = 2.22851774373322 t^8 - 6.43647639147174 t^7 + 8.39819214094082 t^6 - 5.62894940713877 t^5 + 2.64092341459117 t^4 - 0.280581190795731 t^3 + 0.0468676171271075 t^2 + 1.0 t + 3.73778698964378 \times 10^{-17}.$$

We plot the above approximations in Fig. 1. We further visualize the numerical solution after $r=20$ iterations. The comparison shows that using a higher number of iterations in Chebyshev-QLM does not necessarily improve the quality approximations significantly. In the next experiment, we utilize various numbers of basis functions $N=10$, and $N=15$. A comparison between numerical solutions obtained via Chebyshev collocation and its variant Chebyshev-QLM using $\nu=2$ are reported in Table 1.

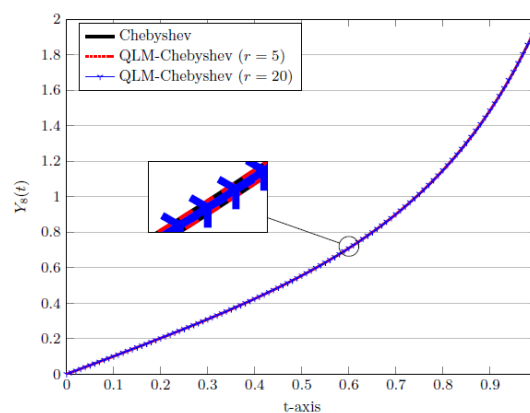


Figure 1 The approximated Chebyshev (dotted) and Chebyshev-QLM ($r=5, 20$) series solutions $Y_8(t)$ using $\nu=2$.

Table 1 Comparison of numerical solutions in Chebyshev and Chebyshev-QLM methods for $N=10$, and $\nu=2$

t	Chebyshev		Chebyshev-QLM	
	$N = 10$	$N = 15$	$N = 10$	$N = 15$
0.0	0.000000000000000	0.000000000000000	0.000000000000000	0.000000000000000
0.1	0.100233440934837	0.100216721768474	0.100233440934855	0.100216721768474
0.2	0.202172942865099	0.202139399406387	0.202172942865131	0.202139399406387
0.3	0.308683817506714	0.308630666356972	0.308683817506759	0.308630666356972
0.4	0.424058632355808	0.423986175756277	0.424058632355862	0.423986175756276
0.5	0.554437431225822	0.554339966343574	0.554437431225889	0.554339966343573
0.6	0.708590867689984	0.708461888417664	0.708590867690066	0.708461888417663
0.7	0.899419130011388	0.899249671519371	0.899419130011497	0.899249671519370
0.8	1.146763765000132	1.146531366011453	1.146763765000316	1.146531366011452
0.9	1.482848083001009	1.482523924102483	1.482848083001331	1.482523924102482
1.0	1.963603214288424	1.963127474136290	1.963603214289592	1.963127474136286

In fact, using $N = 15$ one gets the following approximations

$$\begin{aligned}
 Y_{15}(t) = & 1.50620193639481 t^{15} - 9.56034539304587 t^{14} \\
 & + 27.8574174502187 t^{13} - 48.7421802970722 t^{12} + 56.8749191569847 t^{11} - 46.4261553570832 t^{10} \\
 & + 27.2406403711492 t^9 - 11.5176160218311 t^8 + 3.65173794319553 t^7 \\
 & - 0.690095137856233 t^6 + 0.112223278875648 t^5 + 0.489088705028293 t^4 \\
 & + 0.167311194424251 t^3 - 0.0000203552461724901 t^2 + 1.0 t - 9.41105146 \times 10^{-17},
 \end{aligned}$$

and

$$\begin{aligned}
 Y_{15,5}(t) = & 1.50620193470276 t^{15} - 9.56034538088735 t^{14} \\
 & + 27.8574174110793 t^{13} - 48.7421802225348 t^{12} + 56.8749190635703 t^{11} - 46.4261552759972 t^{10} \\
 & + 27.24064032118 t^9 - 11.5176159997833 t^8 + 3.6517379362573 t^7 - 0.690095136322924 t^6 \\
 & + 0.112223278644379 t^5 + 0.489088705051029 t^4 + 0.167311194422898 t^3 - 0.0000203552461297526 t^2 \\
 & + 1.0 t - 2.28193592 \times 10^{-17}.
 \end{aligned}$$

The estimated errors calculated via (23) related to the results shown in Table 1, i.e., $N = 10, 15$ and also $N = 5, 20$ are depicted in Fig. 2. It can be clearly seen from Fig. 2 that the errors are exponentially decreased while N is increased. Note that we only considered here the Chebyshev-QLM as the efficient version of the Chebyshev collocation scheme.

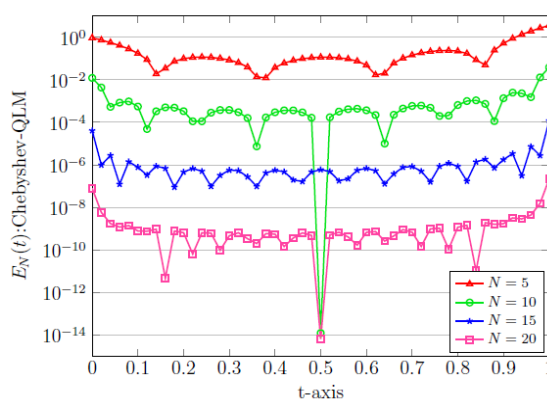


Figure 2 Comparison of the error functions obtained in Chebyshev-QLM with $\nu = 2$, and various $N = 5, 10, 15, 20$.

To further show the advantage of the Chebyshev-QLM proposed in this paper and validate our results, we now present comparison experiments for the first Painlevé equation at various t in $[0, 1]$. For comparison, the following numerical methods are used. These include variational iteration method (VIM) [12], homotopy perturbation method (HPM) [12], homotopy analysis method (HAM) [12], particle swarm optimization algorithm (PSOA) [21], neural networks algorithm (NNA) [22], and reproducing kernel algorithm (RKA) [3]. The numerical results obtained by (23) while using $N = 15$ and $N = 20$ are shown in Table 2.

Now, we turn to the fractional-order case and see its impact on the numerical solutions as well as errors. We fix $N = 15$ and set $\nu = 17/10$. The approximate solutions $Y_N(t)$ obtained via the Chebyshev as well as Chebyshev-QLM for the model problem (1) are respectively as

$$\begin{aligned}
 Y_{15}(t) = & 398.145987794885 t^{15} - 2602.61360067137 t^{14} + 7690.06850481191 t^{13} \\
 & - 13541.7460708438 t^{12} + 15799.4426992446 t^{11} - 12851.8879326914 t^{10} \\
 & + 7474.55821939334 t^9 - 3137.87669789055 t^8 + 949.99018787205 t^7 \\
 & - 204.772868308747 t^6 + 31.0812727192556 t^5 - 2.6142642158395 t^4 \\
 & + 0.680283786274891 t^3 + 0.00401406232082111 t^2 + 1.0 t + 8.168510458 \times 10^{-17},
 \end{aligned}$$

Table 2 Comparison of error functions in Chebyshev-QLM for $N = 15, 20$ and absolute errors in various numerical schemes when $\nu = 2$

t	$N = 15$	$N = 20$	RKA	VIM	HPM	HAM	PSOA	NNA
0.0	4.07 _{.05}	7.85 _{.08}	0	0	0	0	0	0
0.1	7.70 _{.07}	7.73 _{.10}	1.32 _{.10}	1.35 _{.08}	7.96 _{.10}	8.00 _{.10}	1.05 _{.03}	6.15 _{.06}
0.2	4.61 _{.07}	6.44 _{.10}	4.74 _{.09}	1.85 _{.06}	4.88 _{.09}	1.19 _{.09}	8.05 _{.04}	2.58 _{.06}
0.3	5.63 _{.07}	4.60 _{.10}	1.38 _{.08}	3.20 _{.05}	2.22 _{.07}	5.62 _{.09}	6.71 _{.04}	2.00 _{.06}
0.4	5.60 _{.07}	5.49 _{.10}	2.48 _{.08}	2.45 _{.04}	3.94 _{.06}	1.12 _{.08}	6.39 _{.04}	2.21 _{.06}
0.5	5.93 _{.07}	6.39 _{.15}	4.42 _{.08}	1.20 _{.03}	3.79 _{.05}	5.31 _{.08}	6.79 _{.04}	1.17 _{.06}
0.6	6.84 _{.07}	6.64 _{.10}	7.41 _{.08}	4.50 _{.03}	2.45 _{.04}	6.38 _{.07}	7.72 _{.04}	4.55 _{.06}
0.7	8.42 _{.07}	6.76 _{.10}	1.22 _{.07}	1.40 _{.03}	1.21 _{.03}	7.55 _{.06}	9.10 _{.04}	4.05 _{.06}
0.8	8.50 _{.07}	1.16 _{.09}	2.06 _{.07}	3.84 _{.02}	4.97 _{.03}	6.89 _{.05}	1.07 _{.03}	8.42 _{.06}
0.9	1.77 _{.06}	1.71 _{.09}	3.84 _{.07}	9.63 _{.02}	1.78 _{.02}	5.02 _{.04}	1.29 _{.03}	8.85 _{.06}
1.0	1.18 _{.04}	2.18 _{.07}	9.14 _{.07}	2.27 _{.01}	5.74 _{.02}	3.07 _{.03}	1.98 _{.03}	4.13 _{.05}

$$\begin{aligned}
 Y_{15,5}(t) = & 398.145987890918 t^{15} - 2602.61360127502 t^{14} + 7690.06850650495 t^{13} - 13541.7460736445 t^{12} \\
 & + 15799.4427022878 t^{11} - 12851.8879349803 t^{10} + 7474.55822061666 t^9 - 3137.87669835973 t^8 \\
 & + 949.990188000809 t^7 - 204.772868333644 t^6 + 31.0812727225441 t^5 - 2.61426421612067 t^4 \\
 & + 0.680283786289027 t^3 + 0.00401406232048115 t^2 + 1.0 t - 1.706196465 \times 10^{-16}.
 \end{aligned}$$

Looking at the above approximations reveals that the coefficients of the two polynomials are in excellent agreement. The next plot, i.e., Fig. 3 (left picture) also confirms that $Y_8(t)$ and $Y_{8,5}(t)$ are very close together and indistinguishable. The corresponding errors obtained by means of (23) are also depicted on the same figure, right picture. In Table 3, we report the numerical values of $Y_{11,5}(t)$ correspond to $N = 11$ obtained by the Chebyshev-QLM using various fractional orders $\nu = 17/10, 18/10, 19/10$ at some points $t \in [0, 1]$.

5.2 The second Painlevé equation

In the second experiment for model (2), we have the initial conditions as $\gamma_0 = 1$ while $\gamma_1 = 0$. Also we set the parameter μ_2 equals 2. Since the Chebyshev collocation scheme does not produce satisfactory results for the second equation, we only consider the Chebyshev-QLM for (2). Contrary to the first Painlevé equation, in this case also we need a higher number of basis functions to get reasonable solutions. Let us consider the approximate solutions $Y_{25,5}(t)$ obtained via (22) of the model (2)

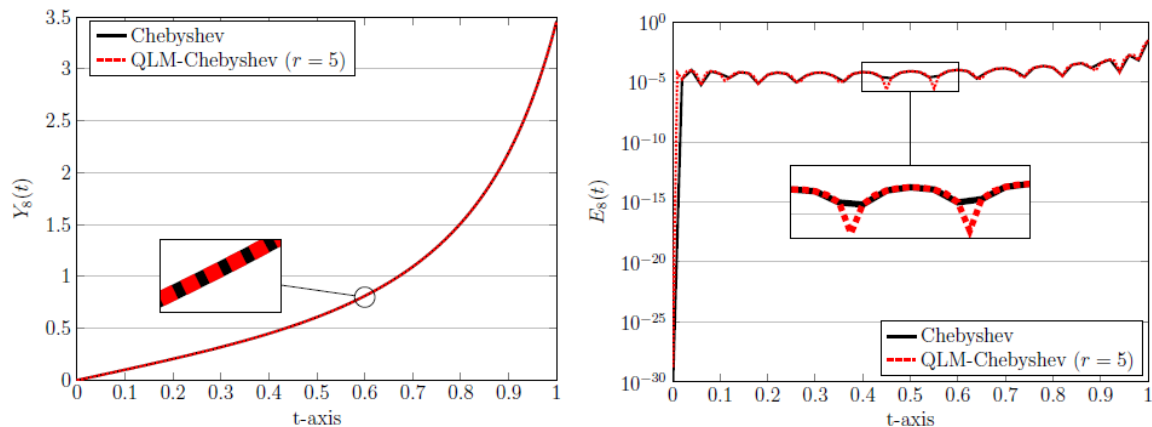


Figure 3 Comparison of numerical solutions (left) and the corresponding error functions using Chebyshev-QLM (right) with $v = 17/10$, $N = 15$.

Table 3 Comparison of numerical solutions in Chebyshev-QLM methods for $N = 15$, and different $v = 16/10, \dots, 19/10$.

t	$v = 16/10$	$v = 17/10$	$v = 18/10$	$v = 19/10$
0.0	0.000000000000001	0.000000000000002	0.000000000000000	0.000000000000001
0.1	0.100892616382118	0.100634992664078	0.100445623652112	0.100311337412375
0.2	0.206903949357798	0.205172396211206	0.203861932319962	0.202877485877063
0.3	0.324376542817710	0.318819715470020	0.314522459778674	0.311200925590407
0.4	0.462644362307263	0.449127531994382	0.438634726451574	0.430432599468519
0.5	0.636289468981412	0.607413255330861	0.585246346970207	0.567965708773963
0.6	0.870266887810312	0.811742862530598	0.768092086765636	0.734628015439952
0.7	1.211748352389076	1.093334413918111	1.009325719773671	0.946988294802117
0.8	1.762169037309381	1.511534676366765	1.347434154302437	1.231916981358393
0.9	2.780639665326436	2.193457202781560	1.855512561074200	1.636134934820037
1.0	5.134504937036170	3.459735062980938	2.688919524403786	2.247985351657967

in the interval $[0, 1]$. This polynomial using $v = 2$ is obtained as follows

$$\begin{aligned}
 Y_{25,5}(t) = & 1519002836.01173 t^{25} - 18080410039.9662 t^{24} - 101106571617.261 t^{23} - 353052320743.744 t^{22} \\
 & + 863054465356.402 t^{21} - 1569476097075.06 t^{20} + 2202391660196.4 t^{19} - 2441573668839.93 t^{18} \\
 & + 2171599684326.47 t^{17} - 1565068824322.46 t^{16} + 919324983031.743 t^{15} - 441268413906.854 t^{14} \\
 & + 173027389594.549 t^{13} - 55261564362.6884 t^{12} + 14294473217.4143 t^{11} - 2968999255.35433 t^{10} \\
 & + 489245755.616686 t^9 - 62933223.2590893 t^8 + 6184153.53742734 t^7 - 450959.300520993 t^6 \\
 & + 23456.0987336942 t^5 - 821.876608946902 t^4 + 18.0827293922669 t^3 + 1.78823593787783 t^2 \\
 & + 1.455230555470150 \times 10^{-14} t + 0.13999999999999999.
 \end{aligned}$$

In the next experiments, we fix $v = 2$ and use $r = 5$ iterations. We employ a different number of basis functions $N = 20, 25, 30, 35$ as well as various values for the parameter $\mu = 0, 1, 2$. Table 4 demonstrates the numerical solutions obtained via Chebyshev-QLM at some points $t \in [0, 1]$. The corresponding error functions computed by (23) are visualized in Fig. 4.

Table 4 Comparison of numerical solutions in Chelyshkov-QLM for (2) with different $N = 20, 25, 30, 35$ and $\mu_2 = 0, 1, 2$ correspond to $\nu = 2$

t	$\mu_2 = 0$	$\mu_2 = 1$	$\mu_2 = 2$		
	$N=20$	$N=25$	$N=25$	$N=30$	$N=35$
0.05	1.0025243121	1.0037744428	1.0049726166	1.0050276267	1.0050271476
0.15	1.0233279062	1.0347052861	1.0459237592	1.0460935475	1.0460920616
0.25	1.0672389433	1.0995613103	1.1316308785	1.1319275134	1.1319249238
0.35	1.1383346324	1.2040949994	1.2696552351	1.2701036913	1.2700997878
0.45	1.2432461518	1.3583538131	1.4740080405	1.4746553679	1.4746497372
0.55	1.39276539852	1.5800019500	1.7710392397	1.7719761450	1.7719679609
0.65	1.60509287070	1.9017048216	2.2136772266	2.2150950273	2.2150864314
0.75	1.91293999401	2.3894830576	2.9213692185	2.9237046597	2.9237254327
0.85	2.38100259580	3.1969283471	4.2224361670	4.2270675845	4.2277048651
0.95	3.15700129479	4.7760636696	7.4336889987	7.4479446895	7.4499032984

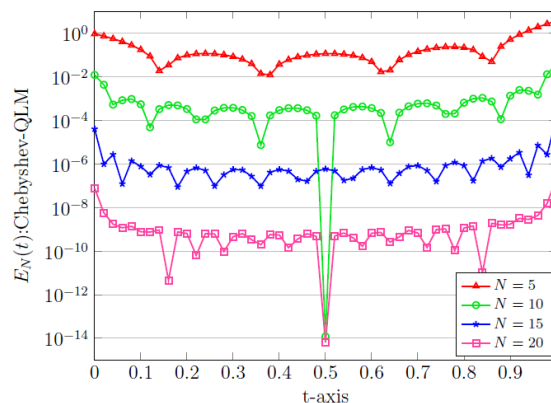


Figure 4 Comparison of the error functions obtained in Chebyshev-QLM with $\nu = 2$, and various $N = 20, 25, 30$ and $\mu_2 = 0, 1, 2$.

To validate the results shown in Table 4, we report the numerical solutions obtained by the well-known computational procedures when both ν and μ_2 equal 2. We compare our results with methods such as RKA^[3], Adomian decomposition method (ADM) and HPM^[7], sinc-collocation method (SCM) and VIM^[23] in Table 5.

Table 5 Comparison of various numerical results for (2) for $\nu = 2$ and $\mu_2 = 2$.

t	RKA	ADM & HPM	LTM	SCM	VIM
0.05	1.005027078	1.005027146	1.004529011	1.005027405	1.005027146
0.15	1.046091419	1.046092056	1.045688050	1.046092872	1.046092056
0.25	1.131923096	1.131924915	1.133214910	1.131925931	1.131924915
0.35	1.270096323	1.270099775	1.268383414	1.270101106	1.270099772
0.45	1.474644843	1.474649720	1.476560547	1.474651851	1.474649662
0.55	1.771962921	1.771968010	1.770023756	1.771971939	1.771967255
0.65	2.215079232	2.215083211	2.217051331	2.215088621	2.215076626
0.75	2.923716297	2.923717805	2.921567093	2.923725942	2.923673264
0.85	4.227126140	4.227163801	4.229171242	4.227190830	4.226911437
0.95	7.449529101	7.442209560	7.449037963	7.447975354	7.446337458

Finally, to see the impact of using fractional values of ν , we fix $N = 15$ and $\mu_2 = 2$ in the next simulation. In Table. 6 we use the values of $\nu = 17/10, 19/10$ and compute the numerical solutions at some points in $[0, 1]$. We compare the computed solutions in this table with RKA^[3].

Table 6 Comparison of numerical results for (2) using $\nu = 17/10, 19/10, N = 15$, and $\mu_2 = 2$.

t	$\nu = 1.7$	RKA ($\nu = 1.7$)	$\nu = 1.9$	RKA ($\nu = 1.9$)
0.05	0.926621336903589	1.006295442147268	0.989258867769277	1.005406191355046
0.15	0.892791245029791	1.054748889714946	1.009418349575315	1.048817224525932
0.25	0.946907962894924	1.152633223268984	1.082338726187619	1.138490294848368
0.35	1.060304207373517	1.308490392008869	1.208886883271213	1.282284848051581
0.45	1.216478398438600	1.541332880816618	1.403046773920370	1.495623382131538
0.55	1.512400984081957	1.886348302942640	1.693080019362010	1.807434835830699
0.65	1.888326503708029	2.402691429231555	2.131623836595632	2.272926226379522
0.75	2.714347440156015	3.284399889818568	2.863799893884206	3.031323582840315
0.85	4.251023034117291	5.382442497422057	4.287921852183477	4.537849120152092
0.95	11.92306812387751	12.63073471169516	8.537201082107739	8.581981145359327

6. Conclusions

In this paper, a collocation approach based on the well-known (shifted) Chebyshev polynomials is developed for numerical solutions of fractional-order Painlevé differential equations arising in several areas of mathematics as well as physics. Using the Chebyshev basis functions with together the spectral collocation points one converts the differential equations into an algebraic system of nonlinear equations. To get rid of the nonlinearity a quasi-linearization approach (called Chebyshev-QLM) is also developed to solve the initial-value problems efficiently. Numerical simulations are performed to illustrate the efficiency and accuracy of the proposed methods and the performance of these two approaches has been assessed for the first Painlevé equation. Moreover, a comparison between them and other well-established numerical procedures is made when applied to the first and second Painlevé model problems. Furthermore, the reliability of the proposed techniques is checked through defining the residual error functions.

Looking at tables and figures one can infer that the proposed (linearized) scheme is easy to implement a procedure for obtaining the approximate solutions of Painlevé differential equations. From comparisons, it is observed that this method (in particular when applied to the first equation) achieved a comparable or even better accuracy than numerical results of other existing well-known numerical methods.

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