Consequences of Holographic Dark Energy Cosmological Model in Gravity

S. H. Shekh¹, V.R. Chirde², S. Raut³

¹Department of Mathematics, S.P.M. Science and Gilani Art, Commerce College, Ghatanji-445301. India.
²Department of Mathematics, G. S. G. Mahavidyalaya, Umarkhed-445206. India.
³Department of Mathematics, PLITMS, Buldhana-443001. India.
E-mail: - vrchirde333@rediffmail.com; da_salim@rediff.com.

Abstract: We have investigated the dynamics of spatially homogeneous Bianchi type-I (LRS) space-time filled with two minimally interacting fields—matter and holographic dark energy components with volumetric power laws expansion towards the gravitational field equations for the linear form of gravity. Solving the set of field equation we obtained the exact solution and observed that the mode of expansion of the model is accelerating throughout the evolution due to destructive assessment of deceleration parameter. Also it has found that the Gauss-Bonnet invariant and the function of Gauss-Bonnet invariant both are not occur for , the equation of state parameter admits the different values for different values of n which are relevant in the standard range provided by recent theoretical and experimental data along with the model has a Big-Bang type of singularity at singular point.

Keywords: Bianchi typespace-time, holographic dark energy, gravity

1. Introduction

An awesome abundance of observational evidence (SNe-Ia Supernova, CMBR, LSS and WMAP) favor the universe is spatially flat and late-time accelerating expansion which is not fit within the framework of Einstein’s General Theory of Relativity. The proposals that have been put forward to explain this observed phenomenon can basically be classified into two categories. One, an exotic component with negative pressure called mysterious energy or Dark Energy (DE) introduce in to Einstein’s General Theory of Relativity (GTR). The dynamical dark energy models are classified into two different categories: (a) the scalar fields including Quintessence, Phantom, Quintom, K-essence, Tachyon, Dilaton, and so forth. (b) The interacting models of dark energy such as the Chaplygin gas models, Braneworld models, Holographic and Agegraphic models.

For a good review of the dynamics of different dark energy models, Mishra and Biswal [1] have constructed a self-consistent system of Bianchi Type-VI₀ cosmology in five dimensions with a binary mixture of perfect fluid and dark energy where the dark energy is chosen to be either the quintessence or Chaplygin gas using solutions to the corresponding Einstein’s field equations as a quadrature and observed that the equation of state parameter for dark energy is found to be consistent with the recent observations (SNe-Ia Supernova with CMBR) and Galaxy Clustering Statistics. Chirde and Shekh [2-4] investigated interacting two-fluid viscous dark energy and magnetized dark energy cosmological models in self creation cosmology and Lyra geometry respectively. The same authors [5, 6] studied plane symmetric dark energy cosmological model in the form of wet dark fluid in f(R,T) gravity. Dark energy dominated Bianchi type-VI₀ cosmological model with hybrid law of expansion in f(R,T) gravity investigated by Bhoyar et al. [7]. Recently, Mishra et al. [8] have investigated the anisotropic behavior of the accelerating universe in Bianchi type-V space time in the frame work of General Theory of Relativity considering the matter field is of two non-interacting fluids i.e. the usual string fluid and dark energy fluid and the skewness parameters are introduced along three different spatial directions in order to represent the pressure anisotropy. In recent years, an interesting observation is made to determine the nature of dark energy in quantum gravity which is termed as Holographic dark energy [9, 10]. Recently, Singh and Kumar [11] study non-viscous and viscous Holographic dark energy models for a homogeneous and isotropic flat Friedmann-Robertson-Walker Universe and find
that the Hubble horizon as an IR cut-off is suitable for both the models to explain the recent accelerated expansion of the Universe. Very recently, Bhoyar et al. \cite{12} obtained a spatially homogeneous and anisotropic Bianchi type-I Space-time in \( f(G) \) gravity theory using the interaction and non-interaction between Holographic dark energy and dark matter. Chirde and Shekh \cite{13} inspected the dynamics of spatially homogeneous Bianchi type-I space-time filled with two minimally interacting fields matter and Holographic dark energy components with volumetric power and exponential expansion laws towards the gravitational field equations for the linear form of \( f(T) \) gravity and observed that both models at late times turn out to be flat Universe whereas the power law model has an initially singular and stable but with expansion it is unstable while exponential model is free from any type of singularity and stable throughout the expansion and predicts that the anisotropy of the Universe will damp out and the Universe will become isotropic.

Another categories to obtain an accelerating expansion of the Universe is the change in the gravity law through the modification of action in General Theory of Relativity. Various modification in the action are present, out of which one replaces the Ricci scalar \( R \) in the Einstein-Hilbert action by an arbitrary function of \( R \) belongs to the well-known \( f(R) \) modified gravity. Vacuum solution in cylindrically symmetric space time in the same gravity studied by Azadi et al. \cite{14}. Bianchi type-cosmological models with bulk viscosity in \( f(R) \) theory investigated by Katore and Shaikh \cite{15} along with many authors have discussed some features of same gravity in [16-19]. Another generalization is the gravitational action includes an arbitrary function of the Ricci scalar and trace of the stress energy tensor known as \( f(R,T) \) gravity. Several authors have investigated the aspect of cosmological models in this gravity [20-22]. Among the various modifications of Einstein’s theory, another one way to look at the theory beyond GR is the Teleparallel Gravity (TG) which uses the Weitzenbock connection in place of the Levi-Civita connection and so it has no curvature but has torsion which is responsible for the acceleration of the Universe. Some relevant works in this gravity are presented in [23-26].

Modified Gauss–Bonnet gravity is another theory that has gained popularity in the last few years [27-28]. It is also known as the \( f(G) \) theory of gravity, where \( f(G) \) is a generic function of the Gauss-Bonnet invariant \( G \).

Modified Gauss–Bonnet gravity is described by the action

\[
S = \frac{1}{2\pi} \int d^4x \sqrt{-g} \left[ R + f(G) \right] + S_M(g^{\mu\nu},\psi),
\]

where \( x \) is the coupling constant, \( g \) is the determinant of the metric tensor \( g_{\mu\nu} \) and \( S_M(g^{\mu\nu},\psi) \) is the matter action, in which matter is minimally coupled to the metric tensor and \( \psi \) denotes the matter fields. This coupling of matter to the metric tensor suggests that \( f(G) \) gravity is a purely metric theory of gravity. The \( f(G) \) is an arbitrary function of the Gauss–Bonnet invariant \( G \).

\[
G \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta},
\]

where \( R \) is the Ricci scalar and \( R_{\mu\nu} \) and \( R_{\mu\nu\alpha\beta} \) denote the Ricci and Riemann tensors. Gravitational field equations are obtained by varying the action in equation (1) with respect to the metric tensor:

\[
\nabla^\alpha \nabla^\nu F + (Gf_G - f)g_{\mu\nu} = \kappa T_{\mu\nu},
\]

where \( \nabla \) denotes the covariant derivative and \( f(G) \) represents the derivative of \( f \) with respect to \( G \).

The Gauss–Bonnet term plays an important role because it may allow avoiding ghost contributions and is helpful in regularizing the gravitational action [29]. It has been suggested that this theory may describe the late-time cosmic acceleration. Moreover, the theory also passes the solar system tests for some specific choices of \( f(G) \) gravity models. Some interesting work has been done so far in this theory. Nojiri and Odintsov [30] developed the reconstruction techniques for \( f(G) \) gravity and it was demonstrated that how cosmological sequence of matter dominance, deceleration-acceleration transition and acceleration era could emerge by using a modified theory. Garciaet al. [31] explored energy condition to find the viability of some specific choices of \( f(G) \) gravity models. Fayaz et al. [32] investigated power law solutions with anisotropic background in \( f(G) \) gravity and it was concluded that Bianchi type-Ipower law solutions only existed for some
special choices of $f(G)$ gravity models. Abbas et al.\textsuperscript{[33]} gave the possibility for the existence of anisotropic compact stars in $f(G)$ gravity. Sharif and Fatima\textsuperscript{[34]} argued the role of Gauss–Bonnet term in the late time accelerated phases of the universe.

Incited by the above discussion, we focus our attention to investigate $f(G)$ gravity in anisotropic background with Holographic dark energy. For this purpose, we consider Locally Rotationally Symmetric (LRS) Bianchi type-I space-time. We explore the exact solutions of the LRS Bianchi type-I field equations in modified $f(G)$ gravity. A well-known $f(G)$ gravity model has been used to solve the set of differential equations.

2. Holographic dark energy model in Bianchi type-I space-time

The line element for a spatially homogeneous, anisotropic LRS Bianchi type-I space-time is given by

$$ds^2 = dt^2 - A^2 dx^2 - B^2 (dy^2 + dz^2),$$

where $A$ and $B$ are the directional scale factors which is a function of cosmic time $t$.

The corresponding Ricci scalar and Gauss–Bonnet invariant for the space-time (4) are turn out to be

$$R = -2 \left[ \frac{\dot{A}}{A} + 2 \frac{\dot{B}}{B} + 2 \left( \frac{\dot{A}}{A} \frac{\dot{B}}{B} + \frac{\dot{A}}{A} \right) \right],$$

$$G = 8 \left[ \frac{\dot{A} \dot{B}^2}{A B^2} + 2 \left( \frac{\dot{A}}{A} \frac{\dot{B}}{B} \right) \right],$$

where the dot denotes the differentiation with respect to cosmic time.

Let us consider that the matter content is dark matter and Holographic dark energy such that the energy momentum tensor $T^\nu_\mu$ are respectively given by

$$T^\nu_\mu = \rho u^\nu u_\mu,$$

$$\overline{T}^\nu_\mu = (\rho_m + p_m) u^\nu u_\mu - p_m g^\nu_\mu,$$

together with comoving coordinates

$$u^\nu = (0, 0, 0, 1) \text{and } u^\nu u_\nu = 1,$$

where $u^\nu$ is the four-velocity vector of the fluid, $\rho$ and $\rho_m$ be the pressure and energy density of the fluid respectively.

From the equation of motion (3), Bianchi type-I space-time (4) for the fluid of stress energy tensor (7) and (8) can be written as

$$-\frac{\dot{B}}{B} - \frac{\dot{B}^2}{B^2} - 16 \frac{\dot{B}}{B} \frac{\dot{B}}{B} f_G + 8 \frac{\dot{B}^2}{B^2} f_G - G f_G + f = k(p_m),$$

$$-\frac{\dot{A}}{A} - \frac{\dot{A}^2}{A^2} + 8 \left( \frac{\dot{A}}{A} \frac{\dot{B}}{B} + \frac{\dot{A}}{A} \frac{\dot{A}}{A} \right) f_G + 8 \frac{\dot{A} \dot{B}}{A B} f_G - G f_G + f = k(p_m),$$

$$\frac{\dot{B}^2}{B^2} + 2 \frac{\dot{A} \dot{B}}{A B} - 24 \frac{\dot{A}^2 \dot{B}^2}{A B^2} f_G + G f_G - f = k(\rho_m + \rho),$$

where the dot (\dot{\cdot}) denotes the derivative with respect to time $t$. 


3. Exact Matter Dominated Solution of the Field Equations

Finally, here we have three differential equations (10) – (12) with six unknowns namely $A, B, \phi, \rho_\perp, \rho_\parallel, \rho_m$. Now to solve system of equations completely, we assume shear scalar is proportional to the expansion scalar (which gives a linear relationship between the directional Hubble’s parameters) this assumption gives an anisotropic relation between the scale factors $A$ and $B$ as follows:

$$A = B^n,$$

where $n > 1$ is an arbitrary constant. If $n = 1$ the matter distribution in the Universe is allover same, hence the model becomes an isotropic otherwise it turn out to be anisotropic.

Following the work of Granda and Oliveros [35] and Sarkar [36], the Holographic dark energy density is given by

$$\rho_\perp = 3(\alpha H^2 + \beta \dot{H}) \text{with} M_{\rho}^2 = 8\pi G = 1,$$

(14)

The equation of state for Holographic dark energy is

$$p_\perp = \omega_\perp \rho_\perp,$$

(15)

Before finding the solution of these field equations, consider some kinematical quantities of the space-time such as Average scale factor and Spatial volume respectively as

$$a = V^{1/3}, V = AB^2.$$

(16)

Another dimensionless kinematical quantity is the mean deceleration parameter which tells whether the Universe exhibits accelerating volumetric expansion or not is

$$q = -1 + \frac{\dot{a}}{a} \left(\frac{1}{H}\right),$$

(17)

for $-1 \leq q < 0$, $q > 0$, and $q = 0$ the Universe exhibit accelerating volumetric expansion, decelerating volumetric expansion and expansion with constant-rate respectively.

The mean Hubble parameter, which expresses the volumetric expansion rate of the Universe, given as

$$H = \frac{1}{3}(H_1 + H_2 + H_3),$$

(18)

where $H_1, H_2$ and $H_3$ are the directional Hubble’s parameter in the direction of x, y and z-axis respectively.

Using equations (16) and (18), we obtain

$$H = \frac{1}{3} \ddot{a} \frac{1}{a}.$$ 

(19)

To discuss whether the Universe either approach isotropy or not, we define an anisotropy parameter as

$$A_m = \frac{1}{3} \sum_{i=1}^{3} \left( \frac{H_i - H}{H} \right)^2.$$ 

(20)

The expansion scalar and shear scalar are defined as follows
\[ \theta = u; \quad \mu = \frac{\dot{A}}{A} + 2 \frac{\dot{B}}{B}, \]  \hspace{1cm} (21) \\

\[ \sigma^2 = \frac{3}{2} H^2 A_m. \] \hspace{1cm} (22)

Subtracting equation (11) from equation (10), we obtained

\[ \frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} + \frac{\dot{A} \dot{B}}{AB} = 0, \] \hspace{1cm} (23)

Using equation (13) and (23), we get the directional scale factor as

\[ A = (ct - d) \left(1 + n - n^2\right), \] \hspace{1cm} (24)

\[ B = (ct - d) \left(\frac{n}{1 + n - n^2}\right), \] \hspace{1cm} (25)

where \( c = c' \left(1 + n - n^2\right) \) and \( d = d' \left(1 + n - n^2\right). \)

Hence the model (4) becomes

\[ ds^2 = dt^2 - (ct - d)^2 \left(1 + n - n^2\right) dx^2 - (ct - d)^2 \left(1 + n - n^2\right) (dy^2 + dz^2). \] \hspace{1cm} (26)

From the above model (26), it is observed that the metric potentials are different hence it represent anisotropic model but for the constant \( n \) if these are identical which represents an isotropic model. Also, at \( t = t_\star = \frac{d}{c} \) the metric potential in the model vanishes hence the model represent singular model. Also, there is no such relation between the constants in the model for which the model shows isotropy.

**Some kinematical parameters:**

The kinematical parameters such as the Hubble parameter, the anisotropic parameter, the shear scalar, the expansion scalar and the spatial volume of model (26), which are of cosmological importance, are respectively given by

\[ H = \frac{c(1 + 2n)}{3(1 + n - n^2)} \frac{1}{(ct - d)}, \] \hspace{1cm} (27)

\[ A_m = \frac{2n(1 - n)}{(1 + 2n)^2}. \] \hspace{1cm} (28)

\[ \sigma^2 = \frac{nc^2(n - 1)}{(n^2 - n - 1)(ct - d)^2}, \] \hspace{1cm} (29)

\[ \theta = \frac{(1 + 2n)c}{(1 + n - n^2)} \frac{1}{ct - d}. \] \hspace{1cm} (30)

\[ V = (ct - d)^{\frac{1 + 2n}{1 + n - n^2}}. \] \hspace{1cm} (31)
We observe that the spatial volume is constant at $t \to 0$. Therefore, the model starts evolving with constant volume at $t = 0$ and expands with cosmic time along with other parameters such as expansion scalar, shear scalar and Hubble’s parameter are constants but at a singular point $t_s$, the spatial volume vanishes and other parameters are diverge. Hence the model (26) has a Big-Bang type of singularity at singular point $t_s$.

Deceleration parameter is

$$ q = -\frac{(3n^2 - n - 3)}{(1 + 2n)}. $$

(32)

From equation (32), it is observed that the deceleration parameter is constant with negative sign and not associated with expansion. Hence, the mode of expansion of the model is accelerating and throughout the evolution, the deceleration parameter is constant. For $-1 \leq q < 0$, $q > 0$, and $q = 0$, the Universe exhibit accelerating volumetric expansion, decelerating volumetric expansion and expansion with constant-rate respectively. For, the model is isotropic and accelerating and for $q > 0$, the model is anisotropic which is also an accelerating.

Gauss–Bonnet invariant is defined as

$$ f(G) = G = \frac{8nc^4(n^3 + n^2 - 2)}{(1 + n - n^2)^4} \frac{1}{(ct - d)^4}. $$

(33)

Equation (33) represents the Gauss–Bonnet and function of Gauss–Bonnet invariants which is a function of cosmic time and depends on $n$. Our derived model gives two types of expansions isotropic and anisotropic for $n = 1$ and $n > 1$ respectively. But it is seen that in an isotropic expansion the Gauss–Bonnet invariant and function both are zero hence it seems that in an isotropic expansion the Gauss–Bonnet invariant and function both are not exist.

Pressure is defined as

$$ P_n = \frac{1}{k} \left[ \frac{c^2(1+n)^2}{(1 + n - n^2)^2 (ct - d)^2} - \frac{c^2(1+2n)}{(1 + n - n^2)^2 (ct - d)^2} \right]. $$

(34)

Energy density of Holographic dark energy is

$$ P_n = \frac{3c^2(2+n)\left[\alpha(n+2) - \beta(1+n-n^2)\right]}{(1 + n - n^2)^2} \frac{1}{(ct - d)^2}. $$

(35)

It is observed that the energy density is a function of time, for $q > 0$, it is zero and for $q$ it always decrease positively with the expansion. At the initial stage $t \to 0$ the universe has infinite large energy density $\rho \to \infty$ but with the expansion of the universe it declines and at large $t \to \infty$ it is null $\rho \to 0$. The behavior is clearly depicted in the following figures.
Energy density versus cosmic time for $n = 4$

Energy density versus cosmic time for $n = 5$

Energy density of matter is

$$\rho_m = \frac{c^2 (2+n)[(n-3\alpha(n+2)+3\beta)]}{(1+n-n^2)^2} \frac{1}{(ct-d)^2}.$$  \quad (36)

Energy density of matter shows the same deeds as that of energy density of Holographic dark energy. Equation of state parameter for Holographic dark energy is

$$\omega_\alpha = \frac{n^3}{(n+2)(n-3n\alpha-6\alpha+3\beta)}.$$  \quad (37)

From equation (37), it is observed that the Equation of state parameter of Holographic dark energy is independent of cosmic time (constant) hence which is not deflected throughout the expansion of the model. The results from SNe-Ia data collaborated with CMBR anisotropy and galaxy clustering statistics yield $\omega_\alpha$ as $\omega_\alpha = -0.97$ (WMAP, SNe-Ia results) at 68% confidence level for dark energy. These results are consistent with time variable equation of state parameter and also for time free $\omega_\alpha$. The quintessence models, ($\omega_\alpha > -1$) (explanation of observations of accelerating universe) involving scalar field and phantom model, ($\omega_\alpha < -1$) (expansion of universe increases to infinite degree in finite time) give rise to the time-dependent parameter $\omega_\alpha$. Some other limits of equation of state parameter is obtained from observational results that came from SNe-Ia data and a combination of SNe-Ia data with CMB anisotropy and Galaxy clustering statistics are $1.67 < \omega_\alpha < 0.62$ and $-1.33 < \omega_\alpha < -0.79$ respectively. The latest result in 2009, obtained after a combination of cosmological data sets coming from CMB anisotropy, luminosity distances of high redshift SNe-Ia, and galaxy clustering constrain shows the range of the dark energy equation of state is $-1.44 < \omega_\alpha < -0.92$. In an isotropic model, the equation of state parameter admits the value -0.1666 corresponding to $n = 1$, also it has found that for the value of constant $n$ in the range $[2, 5]$ the equation of state parameter admits a value $-0.5$, $-0.9$, $-1.333$, $-1.777$ which is exist in the standard range given above.

$$\mathcal{F} = \frac{3\left[\alpha(n+2) - \beta(1+n-n^2)\right]}{(n-3n\alpha-6\alpha+3\beta)}.$$  \quad (38)

It is observed that the coincidence parameter is independent of cosmic time, throughout i.e. at very early to large stage of evolution it is constant and remains constant throughout the evolution, by this means avoiding the coincidence problem (unlike $\Lambda$CDM).

The density parameter of dark matter and Holographic dark energy are as follows:

$$\Omega_m = \frac{(n+2)(n-3\alpha(n+2)+3\beta)}{(1+2n)^2},$$  \quad (39)
\[ \Omega_n = \frac{(n+2)(\alpha(n+2) - \beta(1+n+n^2))}{(1+2n)^2}. \]  

The overall density parameter as

\[ \Omega = \frac{(n+2)(n-2n\alpha - 4\alpha + 2\beta - n\beta + n^2\beta)}{(1+2n)^2}. \]

From the above equation one can observe that the sum of the energy density parameter approaches to constant value.

So, from initial to late time the Universe becomes flat. Therefore, for sufficiently large time, this model predicts that the anisotropy of the Universe will damp out and the Universe will become isotropic. This result also shows that in the early Universe, i.e. during the radiation and matter-dominated era the Universe was anisotropic and the Universe approaches isotropy as dark energy starts to dominate the energy density of the Universe.

4. Conclusions

In the investigation of the spatially homogeneous Bianchi type-I space-time with two minimally interacting fields matter and holographic dark energy components with volumetric power laws expansion towards the gravitational field equations for the linear form of \( f(G) \) gravity it is observed that, the metric potentials are altered hence it represents an anisotropic model but for the constant \( n \) if these are identical which represents an isotropic model. Also, at , both are constant but at a specific time \( t = t_s = \frac{d}{c} \) the metric potential in the model vanishes hence the model represents a singular model. Also, except for there is no such relation between the constants in the model for which it shows isotropy. The spatial volume is constant at \( t \to 0 \). Therefore, the model starts evolving with constant volumetric expansion with cosmic time along with other parameters such as expansion scalar, shear scalar and Hubble’s parameter are constants but at a singular point \( t_s \) the spatial volume vanishes and other parameters are diverge. Hence the derived model has a Big-Bang type of singularity at singular point \( t_s \).

The deceleration parameter is constant with negative sign and not associated with expansion. Hence, the mode of expansion of the model is accelerating throughout the evolution, the deceleration parameter is constant. For \(-1 \leq q < 0\), \( q > 0 \), and \( q = 0 \) the Universe exhibit accelerating volumetric expansion, decelerating volumetric expansion and expansion with constant-rate respectively. For, the model is isotropic and accelerating and for, the model is anisotropic which is also an accelerating.

The Gauss–Bonnet invariant and function of Gauss–Bonnet invariants both are the functions of cosmic time and depends on \( n \). Our derived model gives two types of expansions isotropic and anisotropic for \( n = 1 \) and \( n > 1 \) respectively. But it is seen that in an isotropic expansion the Gauss–Bonnet invariant and function both are zero hence it looks that in an isotropic expansion the Gauss–Bonnet invariant and function both are not exist. The energy density is a function of time \( t \) and always decrease positively with the expansion. At the initial stage \( t \to 0 \) the universe has infinitesimal large energy density \( \rho \to 0 \) but with the expansion of the universe it declines and at large \( t \to \infty \) it is null \( \rho \to 0 \).

It is observed that the Equation of state parameter of Holographic dark energy is independent of cosmic time (constant) hence which is not deflected throughout the expansion of the model. In an isotropic model, the equation of state parameter admits the value -0.1666 corresponding to \( n = 1 \), also it has found that for the value of constant \( n \) in the range \([2, 5]\) the equation of state parameter admits a value -0.5, -0.9, -1.333, -1.777 which is exist in the standard range given by recent theoretical experiments along with the coincidence parameter is also independent of cosmic time, throughout i.e. at very early stage of evolution it is constant and remains constant throughout the evolution, by this means avoiding the coincidence problem (unlike \( \Lambda CDM \)). The energy density parameter approaches to constant value. So, from initial to late time the Universe becomes flat. Therefore, for sufficiently large time, this model predicts that the anisotropy of the Universe will damp out and the Universe will become isotropic. This result also shows that in the early Universe, i.e. during the radiation and matter-dominated era the Universe was anisotropic and the Universe approaches isotropy as dark energy starts to dominate the energy density of the Universe.
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