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Research Article

Analysis of a Long Wave of Depression from the Point of View of Energy Motion

Sergei Sokolov

Polytechnical Institute, Surgut State University, Surgut, Russia
Email: sokolovsb@list.ru

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Abstract: The paper considers the problem of unsteady water flow in the form of a long wave with a decrease in the water level. The flow is generated by a moving wave-maker in a horizontal, infinitely long channel. Under the action of constant external force, the flow tends to a steady and uniform movement, which parameters are of particular interest. It derives equations to calculate mean velocity, depth, and water discharge, utilizing the principles from the theory of energy motion. Also, it identifies the conditions under which maximum water discharge or flow power occurs. The case of uniform flow of fluids in a horizontal channel revealed a principal contradiction between the theory of energy motion and the theory of uniform flow.

***Keywords*:** long wave, wave of depression, unsteady water flow, energy flow, water discharge

1. Introduction

The paper considers the problem of unsteady water flow in the form of a long wave with a decrease in the water level. It continues the approach based on the principles of the theory of energy motion (TEM) in the analysis of water flow. The aim of the paper is to advance TEM into fluid mechanics and to demonstrate solutions derived from this approach. Previously, this new approach has been used to analyze how the influence of channel cross-section geometry on the propagation of long waves of elevation [1], [2]. Analytical solutions obtained for wave velocity have shown encouraging agreement with both known experimental results [3] and field observations [4]. Although further experimental validation across a wider range of parameters is needed, these results reinforce the validity of the approach and its potential for broader application. One such application is the analysis of unsteady flow with a decreasing water level.

According to the classification of rapidly varied flows described in [5], this scenario corresponds to a retreating upstream flow or a “negative wave”, also referred to as a demand surge. These flows occur in practical situations, such as power channels serving hydraulic turbines if the demand suddenly increases. Stoker [6] considered this type of flow under conditions most convenient for analysis when the flow was created by a wave maker, as shown in Figure 1. He called the flow a simple wave of depression. A defining characteristic of this type of wave is that the wave disturbance and the associated water flow move in opposite directions, in contrast to waves of elevation, where both motions align.

Compared to waves of elevation, which are widely studied due to their relevance in scenarios such as flash floods and dam-break waves, waves of depression have received less attention. Research has predominantly focused on their

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existence as solitary waves, beginning with Russell's original experiments [3] and the theoretical analysis of Korteweg and de Vries in their early paper [7]. Stoker's work appears to be the sole study of the wave of depression as a current. He employed the method of characteristics for analysis. No significant follow-up studies in this specific case are known.

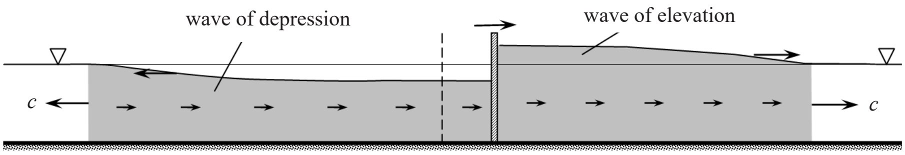

Figure 1. Formation of long waves in a channel by a wavemaker

This paper reexamines the same scenario addressed by Stoker but employs TEM, a classical physical theory [8]. TEM conceptualizes energy as a "substance-like quantity" localized within a medium, possessing bulk density and propagating as an energy flow. The principle of energy conservation within a unit volume of a medium is expressed mathematically as a continuity equation

$$\frac{\partial e}{\partial t} + \operatorname{div}(\mathbf{q}) = 0, \tag{1}$$

where, e represents the bulk energy density, t is time, and $\mathbf{q} = e\boldsymbol{c}$ is the energy flux density vector (the Umov-Poynting vector), where $\boldsymbol{c}$ is the velocity vector of energy motion. The energy flow is related to mechanical work according to the so-called 'second Umov theorem'

$$e\boldsymbol{c} = -\sigma\upsilon, \tag{2}$$

where, $\boldsymbol{c}$ is the component of the vector that is normal to the elementary area through which the energy flow passes in a medium, υ is the normal velocity of the elementary area, and σ is the normal stress. In fluids, the normal stress is represented by pressure p. The fundamental nature of TEM suggests its wide application in various fields of physics. However, up to now, its solutions have had limited application, for example in acoustics.

It is important to note that the primary purpose of TEM was to provide a methodology for formulating the laws of motion of a medium based on the principles of energy motion within that medium. This constitutes the direct problem of TEM. However, until recently, this approach had not been widely considered. In the field of fluid mechanics, only a few examples of its application can be found in the aforementioned papers.

According to the conventional view, a long wave is a surface disturbance that propagates along the flow, whether the water level rises or falls. The separate treatment of wave and current movement originates from Saint-Venant's original theory [9] and requires accounting for the superposition of their velocities. This approach makes the problem nonlinear and necessitates the use of complex analysis or numerical modeling. Keeping in mind the necessity of superposition, Stoker even considered an extreme scenario in which wave disturbances generated by the wave-maker fail to propagate along the channel. In this limiting case, the disturbances cannot escape the surface of the wave-maker, effectively stopping the wave formation entirely.

The main conclusion from the application of TEM is a fundamentally different understanding of the structure of a long wave and the mechanism of its propagation. It suggests that the wave can be represented as a continuous sequence of elementary waves, similar to acoustics, which is physically reliable. This idea is further justified in [10]. With this perspective, the analysis of unsteady flow becomes much simpler. It enables the derivation of equations for the main flow parameters and the identification of the conditions under which maximum water discharge and energy flow occur.

Using a wave-maker to model unsteady fluid flow allows for a reliable representation of the formation of such a flow on one hand, while also providing boundary conditions for it on the other hand. Moreover, experiments with colored water [11] have demonstrated that the mechanism of propagation of a long wave in a channel is very similar to that at the surface of the wave-maker. From the standpoint of TEM, the certainty of boundary conditions leads to accurate calculations of the work of the driving force and the energy of the flow.

2. Methodology

When addressing the problem related to energy behavior, we focus solely on the forms of mechanical energy: kinetic and potential.

2.1 *Localization of energy and propagation with long waves*

Localization of energy in fluids and its movement is a key problem when using TEM. It has a simple solution, as demonstrated in the examples with elevation waves discussed earlier. According to the definition in physics, mechanical energy reaches its minimum value when a physical system, for instance, a fluid, is at rest and in a steady equilibrium state. Any motion or change in the fluid state results in the appearance of energy and its localization. In the problem under consideration, this can be seen in the example of the water volume near the wave-maker after it is displaced, as shown in Figure 1. The deformation of the water volume and the velocity, due to boundary conditions, are signs of potential and kinetic energy localized within this volume.

The further propagation of energy along the channel depends on the nature of the wave-maker's motion. A short-term displacement creates a solitary wave, where a finite amount of energy moves at the velocity of the wave c. The continuous progressive movement of the wave-maker creates a long wave in the form of a current. As for energy localization, Russell established that during the propagation of a solitary wave, the area of motion remains within the boundaries of the wave, occupying the entire volume of water down to the bottom. In particular, Longuet-Higgins [12] considered energy localization in this way. Therefore, it is advisable to associate this region with the wave, and not just the surface disturbance.

With the continuous movement of the wave-maker, the driving force continuously performs work, and portions of energy continuously enter the water. These portions of energy then move along the channel at the velocity of the corresponding waves, successively passing through volumes of water. The energy near the wave-maker is constantly replenished in the same boundary mass of water.

2.2 *Formation of a long wave of depression*

We consider a horizontal, infinitely long channel with a regular rectangular cross-section of width B, containing water at a uniform depth h_0. A vertical plate holds the water's stationary state due to an external constant force F_0 (Figure 2a). This force ensures a steady equilibrium, with the potential energy of the system defined as zero. Consequently, the bulk energy density e is uniformly zero throughout the channel.

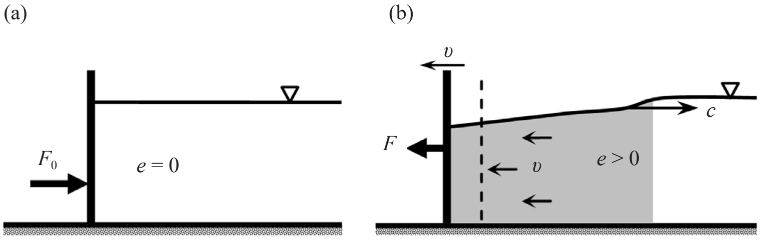

Figure 2. Wave of depression as a forced water flow: (a) the system is in equilibrium, (b) active force F is generating fluid flow

Suppose that an external force F displaces the plate, as shown in Figure 2b. This displacement induces the formation of a progressive wave of depression propagating through the initially stationary water at a velocity c. It is further assumed that the water immediately adjacent to the plate maintains contact with its surface, moving at the same velocity as the plate. Under these conditions, the plate serves as a wave-maker, and the motion of the water is entirely driven by the applied external force.

The work performed by the external force leads to an increase in both the momentum and energy of the water. As the wave propagates, the mass of the moving water grows in proportion to the wave's velocity. Similarly, the region where energy is concentrated expands as the wave advances.

The forced flow in the form of a wave of depression is symmetrical to the flow in the form of a wave of elevation. The specific flow type arises solely from a change in the direction of the driving force (Figure 3). The exact manner in which a wave of depression propagates within the channel remains unclear. However, considering its symmetry with the wave of elevation, we assume that the propagation mechanism is analogous to the process occurring near the wave-maker.

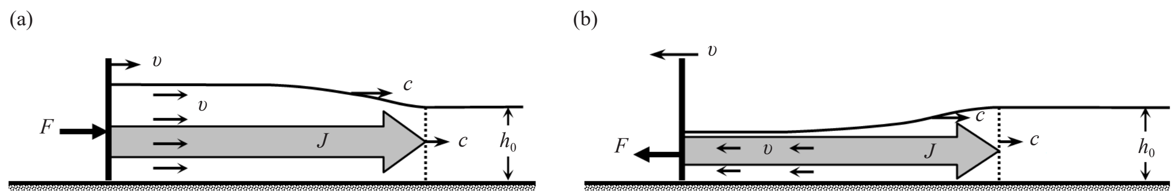

Figure 3. Water flow and the energy flow J in a wave of elevation (a) and in a wave of depression (b)

Despite the change in the direction of fluid flow, the energy flow pattern J and the propagation of long waves associated with the energy flow appear identical. This allows the analysis to be carried out in the same manner, with only the sign of the force being reversed.

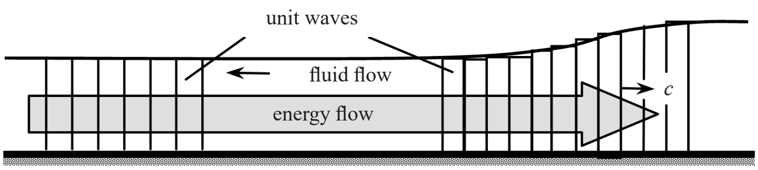

Figure 4. Approximation of a long wave of depression by continuity of unit flat waves

A long wave, as a form of energy flow, can be represented as a continuous sequence of flat elementary waves (Figure 4). These waves originate near the surface of the wave-maker and travel along the channel, carrying elementary amounts of momentum and energy with varying densities from the wave-maker into still water. Consequently, the velocity of any unit wave corresponds to the velocity of energy flow c. Given the independent motion of unit waves, the distance traveled by each wave is simply proportional to the wave velocity c and the time elapsed since its formation.

When a wave of elevation is replaced by a wave of depression, the basic quantitative relationships remain unchanged due to the mechanism of wave formation (Figure 5). If the wave-maker moves at a velocity of υ over a distance υt, the wave of depression propagates into the channel over a distance ct. Simultaneously, the water level changes from its initial value h_0 to a new value h.

The incompressibility condition requires that the volume of water in the undisturbed state, equal to h_0Bct, is conserved and matches the volume in the disturbed state, given as $hB(\upsilon + c)t$. For a channel with a rectangular cross-section, this relationship can be expressed mathematically as follows:

$$\frac{\upsilon}{c} = \frac{h - h_0}{h}. \tag{3}$$

By analogy with a wave of elevation, the difference $h - h_0 = \eta$ can be referred to as the wave height. For a wave of depression, this value is negative, reflecting the downward displacement of the water surface. In the context of the governing equation, a negative wave height implies that the directions of water flow and energy flow are opposite, with their respective signs differing accordingly.

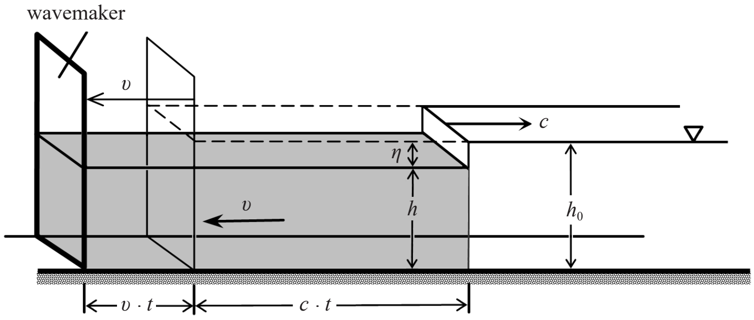

Figure 5. Definition sketch for a wave of depression formation

2.3 *Derivation of Newton's second law*

Let the mass of the wave-maker be M. Its dynamics serve as the source of both energy transfer and water flow within the channel. The flow is driven by the application of a constant external force F.

Consider the change in the momentum of the system over a time interval dt. Assume that at the initial time t_0, a long wave is already propagating through the channel, with c_0 representing the velocity of the wave front (Figure 6a). At this moment, the velocity of the wave-maker is υ_0, and the momentum within the region of moving water is given by p_0^W. The total momentum of the system at time t_0, denoted as $p(t_0)$, can be expressed as follows:

$$p\left(t_0\right) = M\upsilon_0 + p_0^W. \tag{4}$$

After the time interval dt, the velocity of the wave-maker increases to $\upsilon_t = \upsilon_0 + \mathrm{d}\upsilon$ (Figure 6b). During this time interval, a new elementary wave forms at the surface of the wave-maker in the same mass of water. According to the boundary conditions, the velocity of the water in contact with the wave-maker becomes equal to υ_t.

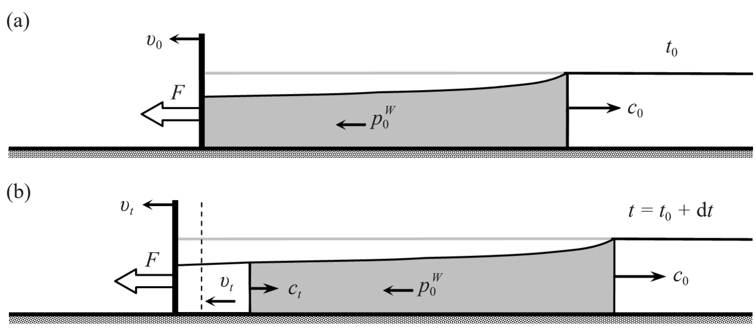

Figure 6. Development of a wave of depression at successive moments of time: (a) $t = t_0$, (b) $t = t_0 + \mathrm{d}t$

The velocity of the new wave element is c_t and its momentum is given by $\rho\upsilon_t h_0 B c_t \mathrm{d}t$, where ρ is the fluid density, and $h_0 B c_t \mathrm{d}t$ represents the volume of water occupied by this wave element.

The total momentum of the system at time $t = t_0 + \mathrm{d}t$ can be expressed as:

$$p\left(t_0 + \mathrm{d}t\right) = M\upsilon_t + p_0^W + \rho\upsilon_t h_0 B c_t \cdot \mathrm{d}t. \tag{5}$$

According to Newton's second law, the change in momentum over the time interval $\mathrm{d}t$ is equal to the force acting on the system. Consequently, the driving force F at any given time can be expressed as:

$$F = M\frac{d\upsilon}{dt} + \rho\upsilon c \cdot h_0 B, \tag{6}$$

in which the second term on the right-hand side of the equation represents Newton's law applied to fluids. By its nature, this term corresponds to a drag force, as it reduces the energy of the wave-maker. The work performed by this force over the time interval $\mathrm{d}t$ can be expressed as:

$$\mathrm{d}W = F\upsilon \cdot \mathrm{d}t = \rho\upsilon^2 \mathrm{d}V, \tag{7}$$

where $\mathrm{d}V$ represents the volume of a new unit wave, and $\rho\upsilon^2 = e$ denotes the bulk density of energy within the wave. This result aligns with the equation describing the energy of flat acoustic waves [13]. The energy is evenly divided equally between the kinetic energy $e_K = \rho\upsilon^2/2$ and the potential energy e_U, which value depends on the deformation of the water volume.

Since the bulk densities of kinetic energy e_K and potential energy e_U are equal, i.e., $e_K = e_U$, it follows from Eq. (3) that the velocity of energy (or wave) propagation is directly related to the wave energy. Moreover, the velocity can be expressed in terms of potential energy e_U and geometric parameters of the wave,

$$c = \frac{h}{h - h_0}\sqrt{\frac{2}{\rho}e_U}. \tag{8}$$

2.4 *Potential energy of deformed volume of water*

The value of e_U can be determined as follows. According to the definition in TEM, the potential energy U of a liquid volume V in a deformed state corresponds to the work W required to return the volume to its state of steady equilibrium. This work is performed by two forces. The first is the hydrostatic pressure force F_P, which acts on the boundary of the volume and depends on the liquid level h (Figure 7). The second is the external constant force F_0, which maintains the equilibrium state of the volume at the level h_0. These forces together displace the boundary from the position at the x-coordinate (deformed state) to the x_0-coordinate (equilibrium state).

Assuming the boundary moves without friction, the work W performed by these forces is given by the integral:

$$W = \int_x^{x_0} \left(F_P - F_0\right)\mathrm{d}x. \tag{9}$$

In a rectangular channel, this work is given by

$$W = \rho g \frac{\left(h - h_0\right)^2}{2h} V. \tag{10}$$

where the density of potential energy e_U in a unit wave is expressed as

$$e_U = \rho g \frac{\left(h - h_0\right)^2}{2h}. \tag{11}$$

In the system under consideration, the potential energy increases as the depth of the flow decreases.

Substituting Eq. (11) into Eq. (8), we obtain the velocity of a unit wave and energy flow, represented by the well-known Russell equation

$$c = \sqrt{g\left(h_0 + \eta\right)}. \tag{12}$$

This allows us to derive the equation for the mean velocity of water in a unit wave from Eq. (3) as

$$\upsilon = \eta \sqrt{\frac{g}{h_0 + \eta}}, \tag{13}$$

which resembles a variation of the Comoy equation [14], and describes water flow in tidal waves.

To analyze the motion of a wave of depression, we orient the x-axis along the channel toward the resting water and place the origin of the coordinate system at the initial position of the wave-maker. Under these conditions, the wave-maker moves in the negative x-direction, generating waves that propagate in the positive direction of the axis.

As the wave-maker progresses, the velocity c of individual wave elements decreases with the lowering water level h. However, due to the sequential and independent motion of the wave elements along the channel, their propagation remains strictly in the positive x-direction, contrary to the conclusions drawn by Stoker.

The motion of an individual wave element must account for the position of the wave-maker along the x-axis at the time of the element's formation. In general, this relationship is expressed by the equation:

$$x = c\left(t - t_0\right) - \int_0^{t_0} \upsilon \cdot \mathrm{d}t, \tag{14}$$

where t_0 is the moment in time when the wave-maker generates a specific unit wave. The integral represents the distance traveled by the wave-maker since the beginning of its movement.

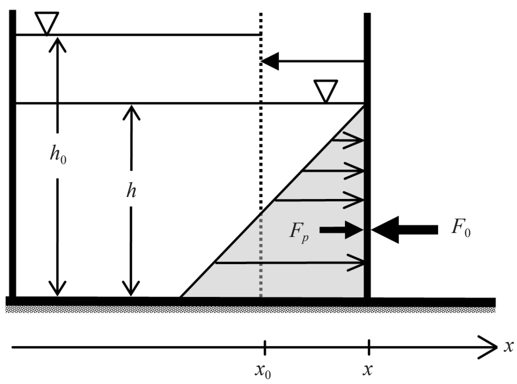

Figure 7. Sketch for estimating the potential energy of deformation of a liquid volume

3. Discussion

Considering the mechanism of a long wave propagation, it is unnecessary to superpose flow and wave velocities. Consequently, the well-known Eqs. (12)-(13) can be directly employed to compute the dynamic characteristics of the flow using only its geometric parameters, namely the flow depth h and its initial value h_0. Notably, the formulation of these equations ensures their applicability to flows of any fluid, irrespective of viscosity. They are derived from the principles of fluid incompressibility and Newton's second law, which are inherently invariant. As a result, the presence or absence of viscosity does not affect their validity.

3.1 *Uniform flow of water*

In the present analysis, the case of steady and uniform water flow is of particular interest, as it allows for a theoretical estimation of water discharge. Eq. (6) governs the motion of a body subjected to a constant force and resistance. Over time, the drag force on the right-hand side of the equation continuously increases, causing the acceleration of the wave-maker to diminish until it approaches zero. At this point, the wave-maker's motion becomes uniform, imparting equal amounts of momentum and energy to unit waves. Consequently, the water flow achieves a steady and uniform state, characterized by a maximum velocity and a corresponding minimum flow depth.

The water flow discharge can then be calculated as:

$$Q = \upsilon h B. \tag{15}$$

Let's make some preliminary additions. It is followed from Eqs. (12) and (13) that

$$\upsilon c = g\eta. \tag{16}$$

Then, Eq. (6), can also be written as

$$F = M\frac{d\upsilon}{dt} + \rho g\eta \cdot h_0 B. \tag{17}$$

It is evident that the drag force is equivalent to the pressure force exerted by the water layer η, acting on the initial section h_0B of the water body in the channel.

Since the driving force F is constant and $d\upsilon/dt$ is zero at the stage of uniform motion of the wave-maker, Eq. (6) provides an expression for the maximum velocity of uniform flow of water

$$F = \rho \upsilon_{max} c \cdot h_0 B. \tag{18}$$

Solving for the maximum velocity, we get:

$$\upsilon_{max} = \frac{F}{\rho c h_0 B}. \tag{19}$$

Here, the velocity of wave elements c is at its minimum. As expected, the water flow velocity depends on the initial conditions in the channel and the value of the driving force.

Similarly, from Eq. (17), we can express the maximum wave height, which corresponds to the drop in water level

$$\eta_{max} = \frac{F}{\rho g h_0 B}. \tag{20}$$

3.2 *Water discharge in a uniform flow of water*

We make use of Eq. (13) for water discharge in Eq. (15) to obtain the following expression

$$Q = \eta\sqrt{g\left(h_0 + \eta\right)}B. \tag{21}$$

To demonstrate how the water flow discharge at the stage of uniform flow depends on the driving force, we transform Eq. (21) as a function of η/h_0. The new expression is

$$Q = K_1 \frac{\eta}{h_0}\sqrt{1 + \frac{\eta}{h_0}}. \tag{22}$$

Here, $K_1 = (c_0^3/g)B$ is the constant coefficient for the given conditions; $c_0 = \sqrt{gh_0}$. The equation is applicable for both waves of elevation and waves of depression.

The analysis indicates that for a wave of depression, there is a maximum water discharge, which occurs at the flow depth of $h = h_0/3$. This discharge should be equal to $Q = 0.38K_1$. Accordingly, the driving force should be as $F = 2/3\rho g h_0^2 B$.

The diagram showing the relationship for water discharge is presented in Figure 8 as $Q/K_1 = f(\eta/h_0)$. Values are provided both for waves of elevation and waves of depression. The discharge in the case of a wave of depression is indicated with a positive sign.

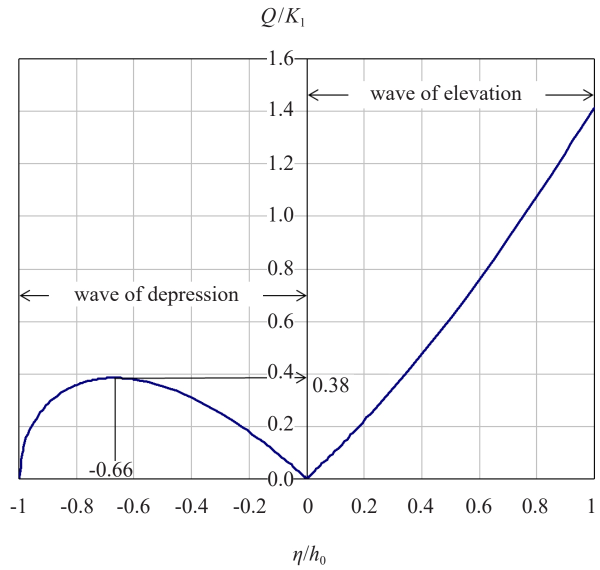

Figure 8. Water discharge Q in a uniform flow of a long wave in dependence on η/h_0 ($K_1 = const$)

3.3 *Energy flow in a uniform flow of water*

From the standpoint of TEM, the uniform motion of the wave-maker implies that the work done by the driving force is equal to the work done by the drag force of the water. This work is entirely used to increase the energy of the water. The energy flow then propagates through the water in the form of a long wave, which can be understood as the power of the water flow.

The energy flow will be expressed as follows:

$$J = \rho \upsilon^2 chB = \rho \eta^2 \sqrt{g\left(h_0 + \eta\right)}hB. \tag{23}$$

We express this formula as a function of η/h_0 similar to Eq. (22)

$$J = K_2 \left(\frac{\eta}{h_0}\right)^2 \sqrt{1 + \frac{\eta}{h_0}}. \tag{24}$$

Here, $K_2 = \rho(c_0^5/g)B$ is the constant for given conditions similar to K_1 in Eq. (17).

This relationship is shown in Figure 9 as $J/K_2 = f(\eta/h_0)$. It also exhibits a local maximum, which is associated with the flow where the depth is $h = h_0/5$.

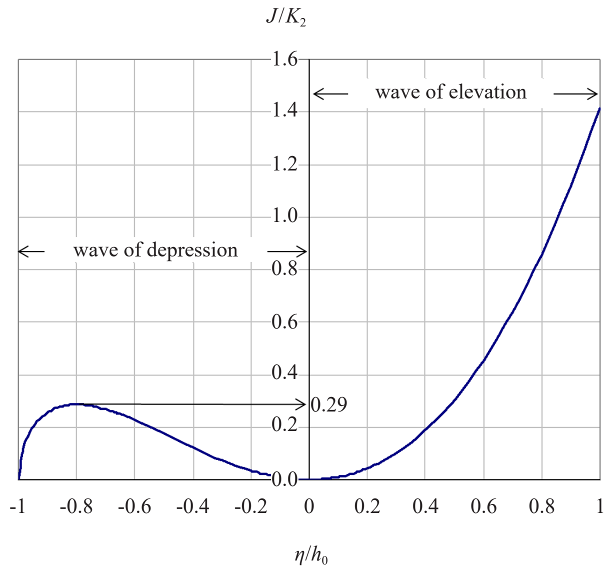

Figure 9. Energy flow J in a uniform flow of a long wave in dependence on η/h_0 ($K_2 = const$)

The results obtained are novel and require experimental verification. It is assumed that the solutions derived for a wave of depression with forced water flow are also applicable to free flow and hold relevance in natural streams. Additionally, it is important to validate the approach used. If proven reliable, the results will require interpretation and explanation.

3.4 *Additional notes on the uniform flow of water in a horizontal channel*

In addition to the solutions obtained, the assumption of uniform water flow as a consequence of the uniform motion of the wave-maker is of particular interest. The idea that such a flow should occur in a horizontal channel contradicts the fundamental principles of the theory of uniform flow in fluids. According to traditional fluid dynamics, uniform flow occurs when the driving force of the flow is balanced by the drag force [5]. In free-surface flows, the driving force is a component of gravity directed along the channel, which is absent in a horizontal channel. At the same time, there is certainly a drag force, caused by the viscosity and roughness of the channel bed. In a horizontal channel, there is no component of gravity directed along the channel, from which it follows that uniform flow of a viscous fluid in such a channel is impossible.

However, in this article, uniform flow in a horizontal channel seems intuitively possible if the channel is infinite or sufficiently long. This is also conceivable from the perspective of TEM. Such a flow reflects the principle of energy conservation, as outlined in Eq. (1), where the energy inflow into a section of the channel is equal to the energy outflow.

In summary, the theory of uniform flow suggests that energy losses should occur in regions with uniform flow and do not allow such flow in a horizontal channel. However, from the standpoint of TEM, the flow is considered acceptable, as it implies the absence of energy losses in this case.

Since the proposed uniform flow of water is still theoretical and challenges the basic principles of one of the key theories in hydraulics, this hypothesis must be experimentally verified.

4. Conclusions

The method of modeling water flow as movement induced by a wave-maker is particularly effective for simplifying the analysis of energy motion. It introduces a fresh perspective on the structure of long waves, where wave elements move in sequence, unlike traditional models. This approach simplifies the study of these processes and ensures that the derived relationships also apply to cases of free flow.

This approach is advantageous because it:

- Separates energy transfer into discrete, manageable units for analytical purposes;
- Simplifies calculations by treating energy motion and wave propagation as connected;
- Extends to free-flow scenarios.

The kinematic characteristics of the flow correspond to the well-known equations of Russell and Comoy, which are commonly used for wave phenomena. This paper demonstrates that these equations are substantiated from the perspective of TEM, and their applicability extends beyond wave phenomena to the flow of fluids. These equations reveal the relationship between the geometric parameters of the flow and the characteristics of its motion, which is crucial from a methodological standpoint.

The solutions obtained are independent of fluid properties such as viscosity.

For the case of a horizontal channel with a rectangular cross-section, it was determined that the maximum water discharge in a wave of depression occurs at a flow depth equal to one-third of the initial depth, while the greatest power does at a flow depth equal to one-fifth of the initial depth.

The problem explored in this paper reveals a contradiction between traditional hydraulic theory and physical theory regarding the possibility of uniform fluid flow in a horizontal channel. These theories differ in explaining both the causes of such a flow and its feasibility. Nevertheless, uniform flow appears intuitively possible. This issue requires a critical analysis of the foundational assumptions of hydraulic theory regarding uniform fluid flow.

The solutions presented here are novel and warrant experimental verification.

Conflict of interest

The author declares no competing financial interest.

Reference

[1] E. E. Egorov and S. B. Sokolov, “Elevation wave velocity in a trapezoidal channel,” *Power Technology and Engineering*, vol. 50, no.4, pp. 361-364, 2016.

[2] S. Sokolov, “Approximate technique for calculation the celerity of long wave in channels with complex cross section,” *SN Applied Sciences*, vol. 2, no. 2, pp. 231, 2020.

[3] J. S. Russell, “Report on waves,” In 14th Meeting British Association for the Advancement of Science. John Murray, London, 1845, pp. 311-390.

[4] D. Reungoat, P. Lubin, X. Leng, and H. Chanson, “Tidal bore hydrodynamic and sediment processes: 2010-2016 field observation in France,” *Coastal Engineering Journal*, vol. 60, no. 4, pp. 484-498, 2018.

[5] D. J. Chow, *Open-channel Hydraulics*. Caldwell, New Jersey: Blackburn Press, 2009.

[6] J. Stoker, *Water Waves. Mathematical Theory with Application*. New York: Interscience Publishers, Inc., 1957.

[7] D. J. Korteweg and G. De Vries, "On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves," *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, vol. 39, no. 240, pp. 422-443, 1895.
[8] N. Umow, "Ableitung der bewegungsgleichungen der energie in continuirlichen körpern [Derivation of the equations of motion of energy in continuous bodies]," *Zeitschrift fur Angewandte Mathematik und Physik*, vol. 19, pp. 418-431, 1874.
[9] A. J. C. Saint-Venant, "Théorie du mouvement non permanent des eaux, avec application aux crues des rivières et a l'introduction des marées dans leurs lit [Theory of the non-permanent movement of waters with application to the floods of rivers and to the introduction of tides within their beds]," *Comptes Rendus de l'Académie des Sciences*, vol. 73, pp. 147-154, 1871.
[10] S. Sokolov, "To the problem of long waves in shallow water," In Proceedings of the 39th IAHR World Congress, Granada, Spain, 2022, pp. 1788-1796.
[11] V. A. Arkhangelsky, *Raschety Neustanovivshegosia Dvizheniia v Otkrytykh Vodotokakh [Calculations of Unsteady Flow in Open Channels]*. Academy Sciences, U.S.S.R., 1947.
[12] M. S. Longuet-Higgins, "On the mass, momentum, energy and circulation of a solitary wave," *Proceedings of the Royal Society A*, vol. 337, no. 1605, pp. 1-13, 1974.
[13] L. D. Landau and E. M. Lifshitz, *Fluid Mechanics*, 2nd ed., Oxford: Pergamon Press, 1987.
[14] G. E. Comoy, *Etude Pratique sur les Marées Fluviales et Notamment sur le Mascaret: Application Aux Travaux de la Partie Maritime des Fleuves [Practical Study on River Tides and in Particular on the Tidal Bore: Application to Work on the Maritime Part of Rivers]*. Gauthier-Villars, Paris, 1881.