Optimality and Duality for Nonsmooth Semi-infinite B-invex Multi-objective Programming with Support Functions

Tarek Emam[1,2]
[1]Department of Mathematics, Faculty of Science, Jouf University, KSA
[2]Department of Mathematics, Faculty of Science, Suez University, Egypt
E-mail: drtemam@yahoo.com

Abstract: In this paper, we study a nonsmooth semi-infinite multi-objective B-invex programming problem involving support functions. We derive sufficient optimality conditions for the primal problem. We formulate Mond-Weir type dual for the primal problem and establish weak and strong duality theorems under various generalized B-invexity assumptions.
Keywords: Nonsmooth Semi-infinite Multi-objective Optimization, Generalized B-invexity, Duality

1. Introduction

Semi-infinite multi-objective programming consider several conflicting and noncommensurate objective functions have to be optimized over a feasible set described by infinite number of inequality constraints. Multi-objective programming problems have been an active research topic due to their applications in several areas such as in engineering design, robotics, and economics, etc. In economics, many problems involve multi-objectives along with constraints on what combinations of those objectives are attainable. For example, consumer's demand for various goods is determined by the process of maximization of the utilities derived from those goods, subject to a constraint based on how much income is available to spend on those goods and on the prices of those goods. This constraint allows more of one good to be purchased only at the sacrifice of consuming less of another good; therefore, the various objectives are in conflict with each other. Another example involves the production possibilities frontier, which specifies what combinations of various types of goods can be produced by a society with certain amounts of various resources. The frontier specifies the trade-offs that the society is faced with-if the society is fully utilizing its resources, more of one good can be produced only at the expense of producing less of another good. Moreover, multi-objective optimization application in the economic field can be utilized to optimize the fisheries bioeconomic model. This model can be used as an optimal estimation tool on resource exploitation and effectiveness of management plan. The basis of the fisheries bioeconomic model is derived from the theory of open access economy or public property, which is based on the population growth logistics model. For example, a model for the North Sea fisheries with four objectives to be considered: maximizing profits, maintaining relatively historical quota shares among countries, maintaining jobs in the industry, and minimizing waste [20]. Explores the use of a multi-objective programming approach as a method for supplier selection in a just-in-time (JIT) setting. Based on a case study, develops a model of JIT supplier selection that allows for simultaneous trade-offs of price, delivery and quality criteria. The analysis occurs in a decision support system environment. A multi-objective programming decision support system is seen as advantageous because such an environment allows for judgement in decision making while simultaneously trading off key supplier selection criteria [27, 28].In addition to the above, macroeconomic policy-making is a context requiring multi-objective programming problems. Typically, a central bank must choose a stance for monetary policy that balances competing objectives-low inflation, low unemployment, low balance of trade deficit, etc. To do this, the central bank uses a model of the economy that quantitatively describes the various causal linkages in the economy; it simulates the model repeatedly under various possible stances of monetary policy, in order to obtain a menu of possible predicted outcomes for the various variables of interest.

Optimality conditions and duality results for semi-infinite programming problems have been studied see [6, 10, 14, 15, 17, 21, 25, 26]. Caristi et al. [4] obtained optimality and duality results for semi-infinite multi-objective program-ming problems that involved differentiable functions. Kanzi and Nobakhtian [16] obtained several kinds of constraints qualifications, necessary

DOI: https://doi.org/10.37256/ges.112020107.1-11

and suffcient optimality conditions for nonsmooth semi-infinite multi-objective programming problems. Optimality conditions and duality results for nonlinear programming problems containing the square root of a positive semidefinite quadratic function have been discussed by many authors, for example, Mishra et al. [22] proved necessary and suffcient optimality conditions for nondifferential semi-infinite programming problems involving square root of quadratic functions, for more details see [24]. Furthermore, the term with the square root of a positive semidefinite quadratic function has been replaced by a more general function, namely, the support function of a compact convex set, whose the subdifferential can be simply expressed. Mond and Schechter [23] have constructed symmetric duality of both Wolfe and Mond-Weir types for nonlinear programming problems where the objective contains the support function. Husain et al. [13] have obtained optimality and duality for a nondifferentiable nonlinear programming problem involving support function, see for more details [1, 12, 18, 19] and references therein. In other hand, convexity and their generalizations play an important role in optimization theory. The class of B-invex functions was introduced by Bector [3] as a generalization of invexity [9, 11]. Later, other generalizations of B-invex functions have been introduced, for details see [7, 8] and references therein.

This paper is organized as follows: In Section 2, we mention some definitions and preliminaries. In Section 3, the sucient optimality conditions for multi-objective semi-infinite B-invex programming problems involving support functions are established. In Section 4, we formulate Mond-Weir type dual for multi-objective semi-infinite B-invex programming problems involving support functions and establish weak, strong and strict-converse duality theorems under generalized B-invexity assumptions.

2. Definitions and preliminaries

In this section, we present some definitions and results, which will be needed in this article. Let R^n be the n-dimensional Euclidean space and R^n_+ be the nonnegative orthant of R^n. Let $\langle \cdot,\cdot \rangle$ denotes the Euclidean inner product and $\|\cdot\|$ be Euclidean norm in R^n. Given a nonempty set $D \subseteq R^n$, we denote the closure of D by $\bar{D}$ and convex cone (containing origin) by cone(D). The native polar cone and the strictly negative polar cone are defined respective by

$$D^{<} := \left\{ d \in R^n \,\middle|\, \langle x, d \rangle < 0, \forall x \in D \right\}.$$

$$D^{\leq} := \left\{ d \in R^n \,\middle|\, \langle x, d \rangle \leq 0, \forall x \in D \right\},$$

Definition 1 [5] *Let $D \subseteq R^n$. The contingent cone $T(D, x)$ at $\bar{x} \in \bar{D}$ is defined by*

$$T(D, \bar{x}) := \left\{ d \in R^n | \exists t_k \downarrow 0, \exists d_k \to d : \bar{x} + t_k d_k \in D, \forall k \in N \right\}.$$

Definition 2 [5] *A function $f : R^n \to R$ is said to be Lipschitz near $x \in R^n$, if there exist a positive constant K and a neighborhood N of x such that for any $y,\ z \in N$ we have*

$$|f(y) - f(z)| \leq K \| y - z \|.$$

The function f is said to be locally Lipschitz on R^n if it is Lipschitz near x for every $x \in R^n$.

Definition 3 [5] *The Clarke generalized directional derivative of a locally Lipschitz function f at $x \in R^n$ in the direction $d \in R^n$, denoted by $f^{\circ}(x, d)$, is defined as*

$$f^{\circ}(x, d) = \lim_{t \uparrow 0, y \to x} \frac{sup\left(f(y + td) - f(y) \right)}{t},$$

where $y \in R^n$.

Definition 4 [5] *The Clarke generalized subdifferential of f at $x \in R^n$ is denoted by ∂cf(x), defined as*

$$\partial cf(x) = \{ \xi \in R^n : f^{\circ}(x, d) \geq \langle \xi, d \rangle, \forall d \in R^n \}.$$

Definition 5 [7] *Let $y \in M \subset R^n$. The set M is said to be B-invex at y with respect to $\eta : R^n \times R^n \to R^n$ if there exists an n-dimensional vector valued function $b(x,y) : R^n \times R^n \to R_+$ such that $y + \lambda b\eta(x,y) \in M$, for each $x \in M$, $0 \leq \lambda \leq 1$.*

M is said to be B-invex set with respect to η if M is B-invex at each y ∈ M with respect to the same η.

Definition 6 *A locally Lipschitz function* $f : R^n \to R$ *is said to be B-invex with respect to* $\eta : R^n \times R^n \to R^n$ *at* $x^* \in R^n$ *if there exists an n-dimensional vector valued function* $b : R^n \times R^n \to R_+$ *such that*

$$b\left(x,x^*\right)[f\left(x\right)-f\left(x^*\right)] \geq \left\langle \xi, b\left(x,x^*\right)\eta^T\left(x,x^*\right)\right\rangle$$

for each $x \in R^n$ *and every* $\xi \in \partial cf\left(x^*\right)$.

The function f is said to be B-invex near $x^* \in R^n$ if it is B-invex at each point of neighborhood of $x^* \in R^n$.

Example 1 *Let* $M = (0, \frac{\pi}{2})$ *and* $f : M \to R$ *be defined as* $f(x) = x + \sin x$. *Define* $\eta : M \times M \to R$ *as* $\eta(x,y) = 2(\sin x - \sin y)/\cos y$ *and* $b : M \times M \to R_+$ *as* $b(x,y) = 2$. f *is B-invex, with respect to* η, *but it is not invex with respect to same* η *because* $\left\langle \xi, \eta(x,y)\right\rangle > [f(x)-f(y)],\ \xi \in \partial cf(y)$ *at* $x = \frac{\pi}{4}, y = \frac{\pi}{6}$.

Remark 1

1- Every invex function, with respect to η *is B-invex function with respect to same* η, *where* $b(x,y) = 1$.

2- Every B-invex function, with respect to η, *where* $b(x,y) > 1$ *for each* $x, y \in M$, *is invex function with respect to some* $\overline{\eta}$, *where* $\overline{\eta}(x,y) = \eta(x,y)/b(x,y)$.

Definition 7 *A locally Lipschitz function* $f : R^n \to R$ *is said to be strictly B-invex with respect to* $\eta : R^n \times R^n \to R^n$ *at* $x^* \in R^n$ *if there exists an n-dimensional vector valued function* $b : R^n \times R^n \to R_+$ *such that*

$$b(x,x^*)\left[f(x)-f(x^*)\right] > \left\langle \xi, b(x,x^*)\eta^T(x,x^*)\right\rangle$$

for each $x \in R^n$, $x \neq x^*$ *and every* $\xi \in \partial cf(x^*)$.

The function f is said to be strictly B-invex near $x^* \in R^n$ if it is strictly B-invex at each point of neighborhood of $x^* \in R^n$.

Proposition 1 [7] *If* $g_i : R^n \to R, i = 1,2,...,m$ *is B-invex with respect to* $\eta : R^n \times R^n \to R^n$, $i = 1,2,...,m$ *then the set* $M = \{x \in R^n : g_i(x) \leq 0, i = 1,2,...,m\}$ *is B-invex set.*

Definition 8 *A locally Lipschitz function* $f : R^n \to R$ *is said to be pseudo B-invex with respect to* $\eta : R^n \times R^n \to R^n$ *at* $x^* \in R^n$ *if there exists an n-dimensional vector valued function* $b : R^n \times R^n \to R_+$ *such that*

$$\left\langle \xi, b(x,x^*)\eta^T(x,x^*)\right\rangle \geq 0, \textit{ for some } \xi \in \partial c\ f\ (x^*) \Rightarrow b(x,x^*)f(x) \geq b(x,x^*)f(x^*),$$

for each $x \in R^n$.

Definition 9 *A locally Lipschitz function* $f : R^n \to R$ *is said to be strictly pseudo B-invex with respect to* $\eta : R^n \times R^n \to R^n$ *at* $x^* \in R^n$ *if there exists an n-dimensional vector valued function* $b : R^n \times R^n \to R_+$ *such that*

$$\left\langle \xi, b(x,x^*)\eta^T(x,x^*)\right\rangle \geq 0, \textit{ for some } \xi \in \partial c\ f\ (x^*) \Rightarrow b(x,x^*)f(x) > b(x,x^*)f(x^*),$$

for each $x \in R^n$.

Definition 10 *A locally Lipschitz function* $f : R^n \to R$ *is said to be quasi B-invex with respect to* $\eta : R^n \times R^n \to R^n$ *at* $x \in R$ *if there exists an n-dimensional vector valued function* $b : R^n \times R^n \to R_+$ *such that*

$$f(x) \leq f(x^*) \Rightarrow \left\langle \xi, b(x,x^*)\eta^T(x,x^*)\right\rangle \leq 0,$$

for each $x \in R^n$ *and every* $\xi \in \partial cf\left(x^*\right)$.

The function f is said to be quasi B-invex near $x^* \in R^n$ if it is quasi B-invex at each point of neighborhood of $x^* \in R$.

Proposition 2 [7] *If* $g_i : R^n \to R, i = 1,2,...,m$ *is quasi B-invex with respect to* $\eta : R^n \times R^n \to R^n, i = 1,2,...,m$ *then the* set $M = \{x \in R^n : g_i(x) \le 0, i = 1,2,...,m\}$ is B-invex set.

Remark 2

1- Every B-invex function is also quasi-B-invex for the same η, but not conversely.

2- Every B-invex function is also pseudo-B-invex B-invex for the same η, but not conversely.

3- Every strictly B-invex function is also strictly pseudo- B-invex for the same η, but not conversely.

Let C be a nonempty compact B-invex set in R^n. The support function $S(.|C): R^n \to R \cup \{+\infty\}$ is given by

$$S(x|C) = max\{\langle z,x \rangle : z \in C\}.$$

Example 2 [29] *If C = [0,1], then the support function* $S(.|C): R \to R \cup \{+\infty\}$ *is given by*

$$S(x|C) = max\{xz : z \in C\}$$

$$S(x|C) = \frac{x+|x|}{2}.$$

The support function, being convex and everywhere finite, has a Clark subdifferential [5], in the sense of convex analysis. Its subdifferential is given by

$$\partial S(x|C) = \{z \in C : \langle z,x \rangle = S(x|C)\}.$$

For any nonempty set $S \subseteq R^n$ the normal cone to S at the point $x \in S$ is denoted by $N_S(x)$ and defined as follows:

$$N_S(x) = \{y \in R^n : \langle y, z-x \rangle \le 0, \forall z \in S\}.$$

In this paper, we consider the following nonsmooth semi-infinite multi-objective B-invex programming problem:

$$(P) \qquad min \ \ f_j(x) + S(x|C_j), \qquad j = 1,\ldots,p,$$

subject to

$$g_i(x) \le 0, \quad i \in I, \quad x \in R^n.$$

Where I is an index set which is possibly infinite, $f_j(x), j = 1,\ldots,p$ and $g_i(x), i \in I$ are locally Lipschitz B-invex functions from R^n to $R \cup \{+\infty\}$. Let M denote the B-invex feasible set of (P).

$$M := \{x \in R^n \mid g_i(x) \le 0, \forall i \in I\}$$

Let $x^* \in M$. We denote $I(x^*) = \{i \in I | \ g_i(x^*) = 0\}$ .the index set of active constraints and let

$$F(x^*) := \bigcup_{j=1}^{p} \partial_c (f_j(x^*) + S(x^* | C_j))$$

$$G(x^*) := \bigcup_{i \in I(x^*)} \partial_c g_i(x^*)$$

The following constraint qualifications are generalization of constraint qualifications from [16] for multi-objective B-invex programming problem with support functions (P).

Definition 11 We say that:

(a) The Abedie constraint qualification (ACQ) holds at $x^* \in M$ *if* $G \le (x^*) \subseteq T(M, x^*)$.

(b) The Basic constraint qualification (BCQ) holds at $x^* \in M$ *if* $T \leq (M, x^*) \subseteq cone(G(\tilde{x}))$.

(c) The Regular constraint qualification (RCQ) holds at $\tilde{x} \in M$ *if* $F^{<}(x^*) \cap G \leq (x^*) \subseteq T(M, x^*)$.

Definition 12 *A feasible point* $x^* \in M$ *is said to be weakly efficient solution for (P) if there is no* $x \in M$ *such that*

$$f_j(x) + S(x \mid C_j) < f_j(x^*) + S(x^* \mid C_j), \quad \text{for all} \quad j = 1, \ldots, p.$$

3. Optimality conditions

In this section, we prove the sufficient optimality conditions for considered nonsmooth semi-infinite multi-objective B-invex programming problem (P) as follows:

Theorem 1 (Necessary optimality conditions) *Let* x^* *be a weakly efficient solution for (P) and assume that Basic Constraints Qualification (BCQ) of (11) holds at* x^*. *If cone* $(G(x^*))$ *is closed, then there exist* $\tau_j \geq 0, z_j \in C_j$ $(\text{for } j = 1, 2, ..., p)$ *and* $\lambda_i \geq 0$ $(\text{for } i \in I(x^*))$ *with* $\lambda_i \neq 0$ *for finitely many indices i, such that*

$$0 \in \sum_{j=1}^{p} \tau_j [\partial_c f_j(x^*) + z_j] + \sum_{I(x^*)} \lambda_i \partial_c g_i(x^*), \tag{1}$$

$$\sum_{j=1}^{p} \tau_j = 1, \tag{2}$$

$$\langle z_j, x^* \rangle = S(x^* \mid C_j), \; j = 1, 2, ..., p. \tag{3}$$

Proof: See Theorem 3.4 (ii) of Kanzi and Nobakhtian [16].

Theorem 2 (Sufficient optimality conditions) *Let* x^* *be feasible for (P) and* $I(x^*)$ *is nonempty. Assume that there exist* $\tau_j \geq 0, z_j \in C_j$ $(\text{for } j = 1, 2, ..., p)$ *and* $\lambda_i \geq 0$ $(\text{for } i \in I(x^*))$ *with* $\lambda_i \neq 0$ *for finitely many indices* i, *such that necessary optimality conditions (1)-(3) hold at* x^*. *If* $\tau_j(f_j(.) + \langle z_j, . \rangle)$ *for j = 1,2,...,p are pseudo B-invex at* x^* *and* $\lambda_i g_i(.), i \in I(x^*)$ *are quasi B-invex at* x^* *with respect to the same η. Then* x^* *is a weakly efficient solution for (P).*

Proof: Suppose, contrary to the result, that $x^* \in M$ is not a weakly efficient solution for (P). Then, there exists a feasible point $x \in M$ for (P) such that

$$f_j(x) + S(x \mid C_j) < f_j(x^*) + S(x^* \mid C_j), \quad \text{for all} \quad j = 1, \ldots, p,$$

since $\tau_j \geq 0, \text{for } j = 1, 2, ..., p$ and for $b(x, x^*) \geq 0$ we have

$$b(x, x^*) \sum_{j=1}^{p} \tau_j [f_j(x) + S(x \mid C_j)] < b(x, x^*) \sum_{j=1}^{p} \tau_j [f_j(x^*) + S(x^* \mid C_j)]. \tag{4}$$

Since $\langle z_j, x \rangle \leq S(x \mid C_j), \; j = 1, 2, ..., p$ and the assumption we have $\langle z_j, x^* \rangle = S(x^* \mid C_j), \; j = 1, 2, ..., p$

$$b(x, x^*) \sum_{j=1}^{p} \tau_j [f_j(x) + \langle z_j, x \rangle] < b(x, x^*) \sum_{j=1}^{p} \tau_j [f_j(x^*) + \langle z_j, x^* \rangle]. \tag{5}$$

Now, from equation (1), there exist $\xi_j \in \partial c f_j(x^*)$ and $\zeta_i \in \partial c g_i(x^*)$ such that

$$\sum_{j=1}^{p} \tau_j (\xi_j + z_j) + \sum_{I(x^*)} \lambda_i \zeta_i = 0, \tag{6}$$

since *x* is a feasible point for (P) and $\lambda_i g_i(x^*) = 0, i \in I(x^*)$ we have

$$\sum_{I(x^*)} \lambda_i g_i(x) \le \sum_{I(x^*)} \lambda_i g_i(x^*), \tag{7}$$

and from quasi B-invexity of $g_i(x^*) = 0, i \in I(x^*)$, we get

$$\left\langle b(x,x^*)\eta^T(x,x^*), \sum_{I(x^*)} \lambda_i \zeta_i \right\rangle \le 0, \text{ by using (6), we get } \left\langle b(x,x^*)\eta^T(x,x^*), \sum_{j=1}^{p} \tau_j(\xi_j + z_j) \right\rangle \ge 0.$$

Thus, from pseudo B-invexity of $\tau_j(f_j(.) + \langle z_j, . \rangle), \mathit{for}\ j = 1,2,...,p,$ we get

$$b(x,x^*)\sum_{j=1}^{p} \tau_j[f_j(x) + \langle z_j, x \rangle] \ge b(x,x^*)\sum_{j=1}^{p} \tau_j[f_j(x^*) + \langle z_j, x^* \rangle],$$

which contradicts (5). Thus x^* is a weakly efficient solution for (P).The following corollary is a direct consequence of Remark 1 and Theorem 2.

Corollary 1 *Let* x^* *be feasible for (P) and* $I(x^*)$ *is nonempty. Assume that there exist* $\tau_j \ge 0, z_j \in C_j$ $(\mathit{for}\ j = 1,2,...,p)$ *and* $\lambda_i \ge 0$ $(\mathit{for}\ i \in I(x^*))$ *with* $\lambda_i \ne 0$ *for finitely many indices i, such that necessary optimality conditions (1)-(3) hold at* x^* *. If* $\tau_j(f_j(.) + \langle z_j, . \rangle)$ *for* $j = 1,2,...,p$ *and* $\lambda_i g_i(.), i \in I(x^*)$ *are B-invex at* x^* *with respect to the same η. Then* x^* *is a weakly efficient solution for (P).*

Example 3 *We consider the following problem:*

(P) $\min\ (f_1(x) + S(x|C_1), f_2(x) + S(x|C_2))$

Subject to $g_i(x) \le 0, \quad i \in I, \ x \in R,$ *where* $f_1(x) = -x,\ f_2(x) = x^2,\ S(x|C_1) = S(x|C_2) = |x|\ \mathit{for}\ C_1 = C_2 = [-1,1]$

and

$$g_i(x) = \begin{cases} \frac{x}{i}, & x \ge 0; \\ x, & x < 0. \end{cases}$$

such that $I := \{2,3,\ldots\}$. *It is clear that the feasible set of (P) is for* $M := (-\infty, 0]$, *for* $y = 0 \in M, I(y) = I$ *and all defined functions are locally Lipschitz functions at y = 0 and* $\partial f_1(y) = -1, \partial f_2(y) = 0, \partial g_i(y) = [\frac{1}{i}, 1], i = 2,3,\ldots$ *Since* $\tau_j(f_j(x) + \langle z_j, x \rangle)$ *for* $j = 1,2$ *are pseudo B-invex at y and* $\lambda_i g_i(x)$ *are quasi B-invex at y with respect to η(x,y) = x − y and* $b(x,y) = \frac{1}{1-x}$, *and conditions (1)-(3) of theorem (1) holds at* $y \in M$ *as there exist* $\tau_1 = \tau_2 = \frac{1}{2}, \lambda = (1,0,0,\ldots), z_1 = -1, z_2 = 0, \xi_1 = -1, \xi_2 = 0, \zeta_i = 1, \mathit{for}\ i \in I$ *such that*

$$\sum_{j=1}^{2} \tau_j(\xi_j + z_j) + \sum_{I(y)} \lambda_i \zeta_i = \frac{1}{2}(-1-1) + 0 + 1 = 0.$$

Then there is no $x \in M$ *such that* $f_j(x)+S(x \mid C_j) < f_j(y)+S(y \mid C_j),\ j=1,2,$ *and hence y = 0 is a weakly efficient solution for (P).*

4. Duality criteria

Many authors have formulated Mond-Weir type dual and established duality results in various optimization problems with support functions; see [1, 2, 13, 21,22, 23] and the references therein. Following the above mentioned works, we formulate Mond-Weir type dual for nonsmooth semi-infinite B-invex programming problem with support function (P) and establish duality theorems.

$$\max\left(f_1(y)+\langle z_1, y\rangle, \ldots, f_p(y)+\langle z_p, y\rangle\right), \qquad \text{(D)}$$

subject to

$$0 \in \sum_{j=1}^{p} \tau_j[\partial_c f_j(y)+z_j]+\sum_{i\in I}\lambda_i \partial_c g_i(y), \tag{8}$$

$$\sum_{i\in I}\lambda_i g_i(y) \geq 0. \tag{9}$$

We now discuss the weak, strong and strict converse duality for the pair (P) and (D).

Theorem 3 (Weak Duality) *Let x be feasible for (P) and* $(y,\tau,\lambda,z_1,\ldots,z_p)$ *be feasible for (D). If* $\tau_j(f_j(.)+\langle z_j,.\rangle)$ *for* $j = 1,2,\ldots,p$ *are pseudo B-invex at y and* $\lambda_i g_i(.)$, $i \in I$ *are quasi B-invex at y with respect to the same η. Then the following cannot hold:*

$$f_j(x)+S(x \mid C_j) < f_j(y)+\langle z_j, y\rangle, \quad \textit{for all} \quad j=1,\ldots,p.$$

Proof: Let *x* be feasible for (P) and $(y,\tau,\lambda,z_1,\ldots,z_p)$ be feasible for (D), then from (8), there exist $\xi_j \in \partial c f_j(y)$ and $\zeta_i \in \partial c g_i(y)$ such that

$$\sum_{j=1}^{p} \tau_j\left(\xi_j+z_j\right)+\sum_{i\in I}\lambda_i \zeta_i = 0. \tag{10}$$

We proceed to the result of the theorem by contradiction. Assume that

$$f_j(x)+S(x \mid C_j) < f_j(y)+\langle z_j, y\rangle, \quad \textit{for all} \quad j=1,\ldots,p,$$

since $\tau_j \geq 0$, for $j = 1,2,\ldots,p$, and for $b(x,x^*) \geq 0$, then we have

$$b(x,y)\sum_{j=1}^{p}\tau_j[f_j(x)+S(x \mid C_j)] < b(x,y)\sum_{j=1}^{p}\tau_j[f_j(y)+\langle z_j, y\rangle], \tag{11}$$

and by using the inequality $\langle z, x\rangle \leq S(x \mid C)$, we get

$$b(x,y)\sum_{j=1}^{p}\tau_j[f_j(x)+\langle z_j, x\rangle] < b(x,y)\sum_{j=1}^{p}\tau_j[f_j(y)+\langle z_j, y\rangle]. \tag{12}$$

Now, since x is feasible for (P) and $(y,\tau,\lambda,z_1,...,z_p)$ is feasible for (D), we have

$$\sum_{i\in I}\lambda_i g_i(x)\le 0\le\sum_{i\in I}\lambda_i g_i(y),$$

and from definition of quasi B-invexity of $g_i(x), i\in I$ *at* $y,$ we have

$$\left\langle b(x,y)\eta^T(x,y),\sum_{i\in I}\lambda_i\zeta_i\right\rangle\le 0, \tag{13}$$

for each $x\in M$ and every and $\zeta_i\in\partial cg_i(x).$ By substituting from (10) in (13), we get

$$\left\langle b(x,y)\eta^T(x,y),\sum_{j=1}^{p}\tau_j(\xi_j+z_j)\right\rangle\ge 0,$$

for each $x\in M$ and some $\xi_j\in\partial cf_j(y).$ Thus, from the definition of pseudo B-invexity of $\tau_j(f_j(.)+\langle z_j,.\rangle)$ for j = 1,2,...,p, we have

$$b(x,y)\sum_{j=1}^{p}\tau_j[f_j(x)+\langle z_j,x\rangle]\ge b(x,y)\sum_{j=1}^{p}\tau_j[f_j(y)+\langle z_j,y\rangle],$$

which contradicts (12). Hence,

$$f_j(x)+S(x\,|\,C_j)<f_j(y)+\langle z_j,y\rangle,\ \forall\ j=1,\ldots,p,$$

cannot hold.

The following corollary is a direct consequence of Remark 1 and Theorem 3.

Corollary 2 *Let x be feasible for (P) and $(y,\tau,\lambda,z_1,...,z_p)$ be feasible for (D). If $\tau_j(f_j(.)+\langle z_j,.\rangle)$ for j = 1,2,...,p and $\lambda_i g_i(.)$, $i\in I$ are B-invex at y with respect to the same η. Then the following cannot hold:*

$$f_j(x)+S(x\,|\,C_j)<f_j(y)+\langle z_j,y\rangle,\ \forall\ j=1,\ldots,p.$$

The following example shows that the generalized B-invexity imposed in the above theorem is essential.

Example 4 *We consider the following problem:*

(P) $\quad min\left(f_1(x)+S(x\,|\,C_1),\ f_2(x)+S(x\,|\,C_2)\right)$

Subject to

$$g_i(x)\le 0,\quad i\in I$$

$$x\in R,$$

where $f_1(x)=-2x$, $f_2(x)=x^2$, $S(x|C_1)=S(x|C_2)=|x|$ for $C_1=C_2=[-1,1]$ and $g_i(x)=-i|x|$, for $i\in I:=N$. It is clear that the feasible set of (P) is $M:=R$ and for $y=1\in M$, $I(y)=I$.

Let us formulate Mond-Weir dual of (P) as follow:

(D) $\max\,(-2y+z_1, y^2+z_2y)$

Subject to

$$g_i(x)\le 0,\quad i\in I$$

$$0\in\sum_{j=1}^{2}\tau_j[\partial f_j(y)+z_j]+\sum_{i\in I}\lambda_i\partial g_i(y),$$

$$\sum_{i\in I}\lambda_i g_i(y)\ge 0,$$

where $y\in R,\tau_j\ge 0,\sum_{j=1}^{2}\tau_j=1,\lambda_i\ge 0$ *with* $\lambda=(\lambda_i)_{i\in I}\ne 0$ *for finitely many indices* $i\in N$ *and* $z_j\in C_j$ *for* $j=1,2.$

By choosing $y^*=0,\tau_1=\tau_2=\frac{1}{2}$ *,* $\lambda=(1,0,0,...),z_1=1,z_2=0$*. we have* (y,τ,λ,z_1,z_2) *be feasible for (D) satisfy the inequality*

$$f_1(x)+S(x\,|\,C_1)<f_1(y)+\langle z_1,y\rangle.$$

Because $\lambda_i g_i(.)$ *is not quasi B-invex at y with respect to* $\eta(y,y^*)=y-y^*$ *and* $b(y,y^*)=1$ *. Hence, the pseudo B-invexity and quasi B-invexity assumptions are essential for weak duality.*

The following theorem gives strong duality relation between the primal problem (P) and the dual problem (D).

Theorem 4 (Strong Duality) *Let x be a weakly efficient solution for (P) at which Abedie Constraints Qualification (ACQ) of (11) holds at* x^**. If the pseudo B-invexity and quasi B-invexity assumptions of the weak duality theorem are satisfied, then there exists* $(\tau,\lambda,z_1,...,z_p)$ *such that* $(x,\tau,\lambda,z_1,...,z_p)$ *is a weakly efficient solution for (D) and the respective objective values are equal.*

Proof: Since x is a weakly efficient solution for (P) at which the suitable constraints qualification holds and cone(G(x)) is closed, from the Kuhn-Tucker necessary conditions, there exists $(\tau,\lambda,z_1,...,z_p)$ such that $(x,\tau,\lambda,z_1,...,z_p)$ is feasible for (D).

From weak duality theorem (3), the following cannot hold for any feasible y for (D):

$$f_j(x)+S(x\,|\,C_j)<f_j(y)+\langle z_j,y\rangle,\ \ for\ j=1,\dots,p.$$

Since $\langle z,x\rangle\le S(x|C)$, we have

$$f_j(x)+\langle z_j,x\rangle<f_j(y)+\langle z_j,y\rangle,\ \ for\ j=1,\dots,p.$$

Thus, $(x,\tau,\lambda,z_1,...,z_p)$ is a weakly efficient solution for (D) and the objective values of (P) and (D) are equal at x.

The following corollary is a direct consequence of Remark 1 and Theorem 4.

Corollary 3 *Let x be a weakly efficient solution for (P) at which Abedie Constraints Qualification (ACQ) of (11) holds at* x^* *. If the B-invexity assumption of the weak duality theorem are satisfied, then there exists* $(\tau,\lambda,z_1,...,z_p)$ *such that* $(x,\tau,\lambda,z_1,...,z^p)$ *is a weakly efficient solution for (D) and the respective objective values are equal.*

The following theorem gives strict converse duality relation between the primal problem (P) and the dual problem (D).

Theorem 5 (Strict converse duality) *Let* x^* *be a weakly efficient solution for (P) at which Abedie Constraints Qualification (ACQ) of (11) holds at* x^**. Let* $\tau_j(f_j(.)+\langle z_j,.\rangle)$ *for j = 1,2,...,p be pseudo B-invex and* $\lambda_i g_i(.)$*,* $i\in I$ *be quasi B-invex with respect to the same η. If* $(\tilde{x},\tau,\lambda,z_1,\dots,z_p)$ *is a weak efficient solution for (D) and* $\tau_j(f_j(.)+\langle z_j,.\rangle)$ *for j = 1,2,...,p are strictly pseudo B-invex at* $\tilde{x}$*, then* $\tilde{x}=x^*$

Proof: We prove the result of theorem by contradiction. Assume that $\tilde{x}\ne x^*$. Then by strong duality Theorem (4) there exists $(\tau,\lambda,z_1,...,z_p)$ such that $(x^*,\tau,\lambda,z_1,...,z_p)$ is a weakly efficient solution for (P) and

$$f_j(x^*)+S(x^*\mid C_j)=f_j(\tilde{x})+\left\langle z_j,\tilde{x}\right\rangle,\ \forall\ j=1,\dots,p.$$

Using $S(x^*\mid C_j)=\left\langle z_j,x^*\right\rangle,\ j=1,\dots,p$ and $b(x^*,\tilde{x})>0,$, we have

$$b(x^*,\tilde{x})\sum_{j=1}^{p}f_j(x^*)+\left\langle z_j,x^*\right\rangle=b(x^*,\tilde{x})\sum_{j=1}^{p}f_j(\tilde{x})+\left\langle z_j,\tilde{x}\right\rangle. \tag{14}$$

Now, since x^* is a weakly efficient solution for (P), $\lambda_i\geq 0$ and $(\tilde{x},\tau,\lambda,z_1,\dots,z_p)$ is a weakly efficient solution for (D), we have

$$\sum_{i\in I}\lambda_i g_i(x^*)\leq\sum_{i\in I}\lambda_i g_i(\tilde{x}).$$

From the definition of quasi B-invexity of $\lambda_i g_i(.)$, $i\in I$

$$\left\langle b(x^*,\tilde{x})\eta^T(x^*,\tilde{x}),\sum_{i\in I}\lambda_i\zeta_i\right\rangle\leq 0, \tag{15}$$

for every $x^*\in M$ and every $\zeta_i\in\partial cg_i(\tilde{x})$. By substituting from (10) in (15), we get

$$\left\langle b(x^*,\tilde{x})\eta^T(x^*,\tilde{x}),\sum_{j=1}^{p}\tau_j(\xi_j+z_j)\right\rangle\geq 0.$$

for each $x^*\in M$ and some $\xi_j\in\partial cf_j(\tilde{x})$. Thus from strict pseudo B-invexity of $\tau_j(f_j(.)+\left\langle z_j,.\right\rangle)$ for j = 1,2,...,p at $\tilde{x}$, we get

$$b(x^*,\tilde{x})\sum_{j=1}^{p}\tau_j[f_j(x^*)+\left\langle z_j,x^*\right\rangle]>b(x^*,\tilde{x})\sum_{j=1}^{p}\tau_j[f_j(\tilde{x})+\left\langle z_j,\tilde{x}\right\rangle], \tag{16}$$

which contradicts (14). Therefore, $\tilde{x}=x^*$.

References

[1] Antczak T, Singh V. Optimality and duality for minmax fractional programming with support functions under B-(p, r)-Type I assumptions. *Mathematical and Computer Modelling*. 2013; 57:1083-1100.

[2] Bae KD, Kim DS. Optimality and duality theorems in nonsmooth multi-objective optimization. *Fixed point theory and applications*. 2011; 42. Available from: doi: 10.1186/1687-1812-2011-42.

[3] Bector CR, Sunela SK, Lalitha CS. Generalized B-vex Functions and Generalized B-vex Programming. *Journal of Optimization Theory and Applications*. 1993; 76(3):261-276.

[4] Caristi G, Ferrara M, Stefanescu A. Semi-infinite multi-objective programming with generalized invexity. *Mathematical Report*. 2010; 12:217-233.

[5] Clarke FH. Optimization and Nonsmooth Analysis. Wiley-Interscience. New York; 1983.

[6] Chuong TD, Kim SD. Nonsmooth semi-infinite multi-objective optimization problems. *Journal of Optimization Theory and Applications*. 2014; 160:748-762.

[7] Emam T. Roughly B-invex Programming Problems. *Calcolo*. 2011; 48(2):173-188.

[8] Emam T. Roughly B-invex Multi-Objective Programming Problems. *Open Journal of Optimization*. 2012; 1:1-7.

[9] Emam T. Norm–Based Approximation in Invex Multi-Objective Programming Problems. *IJET/IJENS*.2014;14(6):1-5.

[10] Emam T. Optimality for E-[0,1] convex Multi-objective Programming. *Filomat.* 2017;31(3):529-541.

[11] Hanson MA. On sufficiency of the Kuhn-Tucker conditions", Journal of Mathematical Analysis and Applications. 1981;80:545-550.

[12] Husain I, Jabeen Z. On fraction programming containing support functions. *Journal of Applied Mathematics and Computing*. 2005; 18:361-376.
[13] Husain I, Abha, Jabeen Z. On nonlinear programming with support functions. *Journal of Applied Mathematics and Computing*. 2002; 10:83-99.
[14] Kanzi N. Neccessary optimality conditions for nonsmooth semi-infinite programming problems. *Journal of Global Optimization*. 2011; 49:713-725.
[15] Kanzi N, Nobakhtian S. Optimality conditions for nonsmooth semi-infinite programming. *Optimization*. 2008; 53 717-727.
[16] Kanzi N, Nobakhtian S. Optimality conditions for nonsmooth semi-infinite multi-objective programming. *Optimization Letters*, 2014; 8:1517-1528.
[17] Kostyukova OI, Tchemisova TV. Sufficient optimality conditions for convex semi-infinite programming. *Optimization Methods and Software*. 2010; 25(2):279-297.
[18] Kim DS, Bae KD. Optimality conditions and duality for a class of nondifferentiable multi-objective programming problems. *Taiwanese Journal of Mathematics*. 2009; 13 (2):789-804.
[19] Kim DS, Lee HJ. Optimality conditions and duality in non-smooth multi-objective programs. *Journal of Inequalities and Applications*. 2010; Available from: doi:10.1155/2010/939537.
[20] Mardle SJ, Pascoe S and Tamiz M. An investigation of genetic algorithms for the optimization of multi-objective fisheries bioeconomic models. *International Transactions in Operational Research*. 2000; 7 (1):33-49.
[21] Mishra SK, Jaiswal M, Le Thi HA. Duality for nonsmooth semi-infinite programming problems. *Optimization Letter*s. 2012; 6:261-271.
[22] Mishra SK, Jaiswal M. Optimality conditions and duality for non-differential multi-objective semi-infinite programming. *Vietnam Journal of Mathematics*. 2012; 402&3:331-343.
[23] Mond B, Schechter M. Nondifferentiable symmetric duality. *Bulletin of the Australian Mathematical Society*. 1996; 53(2):177-188.
[24] Preda V, Koller E. Duality for a nondifferentiable programming problem with a square root term. *Revue Roumaine de Mathmatiques Pures et Appliques*. 2000;45:873-882.
[25] Son TQ, Strodiot JJ, Nguyen VH. ε-optimality and ε-Lagrangian duality for a nonconvex programming problem with an infinite number of constraints. *Journal of Global Optimization*. 2009; 141:389-409.
[26] Son TQ, Kim DS. ε-mixed type duality for nonconvex multi-objective programs with an infinite number of constraints. *Journal of Global Optimization*. 2013;57:447-465.
[27] Weber C and Ellram L. Supplier Selection Using Multi-objective Programming: A Decision Support System Approach. *International Journal of Physical Distribution and Logistics Management*. 1993; 23(2):3-14.
[28] Wesner N. Multi-objective Optimization via Visualization. *Economics Bulletin*. 2017; 37(2):1226-1233.
[29] Yadvendra S, Mishra SK, Lai KK. Optimality And Duality For Nons-mooth Semi-Infinite Multi-Objective Programming With Support Functions. *Yugoslav Journal of Operation Research*. 2017; 27 (2):205-218.[1]