Research Article

Effect of Magnetic and Thermal Parameters on the Reflected Waves in Generalized Magneto Thermoelastic Materials

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Abstract: The present paper is concerned with the problem of reflection of homogeneous plane waves from the free surface of a generalized magneto thermoelastic half-space. The phase speeds of corresponding to longitudinal, transverse and thermal waves are obtained and are independent of angle of propagation. The amplitude and energy ratios corresponding to the reflected waves are obtained theoretically and numerically with the help of boundary conditions. The effects of magnetic and thermal parameters on the reflected waves are examined. The results hold the conservation law of energy.

Keywords: generalized magneto thermoelastic material; plane wave; amplitude and energy ratio; phase velocity

1. Introduction

The coupled theory of thermoelasticity introduced by Biot [1] enables to eliminate the paradox that the elastic changes have no effect on temperature. This theory gives infinite phase speeds for heat waves which is different from the reality. Lord and Shulman [2] introduced the concept of relaxation time and the acceleration of heat flux in the generalized theory of thermoelasticity thereby representing finite phase speeds for the heat waves. Green and Lindsay [3] used entropy production inequality proposed by Green and Laws [4] to present an alternate generalized thermoelastic theory. These theories have been established using modifying Fourier’s heat conduction equation or correcting the energy equation and Neuman-Duhamel relation. McCarthy [5] obtained two purely mechanical transverse and two longitudinal waves in generalized thermoelastic continuum. He presented the governing equations for the propagation of acceleration waves of arbitrary shape and strength. Nowacki [6] discussed different problems of thermoelasticity and gave the Duhamel-Newmann relations for an anisotropic body. Green and Naghdi [7, 8] also explained thermoelastic theory using the concept of the propagation of heat as like a wave. Youssef [9] proposed a new thermoelasticity theory considering the heat conduction equation with fractional order and proved the uniqueness theorem.

transformation to obtain the expressions for magnetothermodynamic stress and perturbation response of magnetic field in the orthotropic cylinder. Ezzat and Ellall [16] introduced the modified Ohm’s law, the temperature gradient and charge density effects in the linear theory of generalized magneto thermoelasticity and obtained the kinematic variables using normal mode analysis. Sarkar and Lahiri [17] used the normal mode analysis and eigenvalue approach techniques to obtain the kinematic variables in the three dimensional thermoelastic problem without energy dissipation.

The subject of wave and vibrations in different elastic continuum has been a matter of concern for many researchers since long. It is used in various field like in the exploration of mines and petroleum, earthquake engineering, Seismology, Geophysics, etc. More specific, wave phenomena in thermoelasticity materials has wide applications in engineering. Othman and Song [18] studied the reflection of plane waves from an isotropic magneto-thermoelastic rotative half space and obtained the reflection coefficients of the reflected waves. Lotfy et al. [19] discussed the problem of the magnetic and rotational effects on a fiber-reinforced thermoelastic solid based on two coupled thermoelastic theory with gravitational effect using normal mode analysis-sis. Zoramuna and Singh [20] introduced the magnetic and rotational effects on the propagation of waves in transversely isotropic fiber-reinforced thermoelastic material with the help of Lord Shulman theory. Kumar and Tomar [21] investigated the effect of induced electric field due to applied magnetic field on the reflection and transmission coefficients of the elastic waves in the half-spaces of magneto-elastic materials with voids. Abo-Daheb [22] solved the governing equation of generalized magneto thermoelastic materials using Lame’s potentials method in the context of classical dynamical and Lord-Shulman theories. Othman and Kumar [23] presented the model equations of generalized magneto-thermoelasticity in an isotropic perfectly conducting elastic medium. This model is applied to four theories of the generalized thermo-elasticity: Lord-Shulman, Green-Naghdi theory, Chandrasekharahia-Tzou theory and coupled theory. Abd-Alla et al. [24] studied the problem of reflection of plane harmonic waves from a semi-infinite elastic solid under the effect of magnetic field and obtained the reflection coefficient of the reflected waves. The problems related with wave propagation in different thermoelas-tic materials have been studied since long and they are in open literature, i.e., Ben-Menahem and Singh [25], Hetnaski and Ignaczak [26], Ezzat et al. [27], Ezzat [28], Othman [29], Ezzat and Yousef [30], Dang and Wang [31], Singh [32], Othman and Lotfy [33], Singh [34], Sarkar [35], Sarkar and De [36], Singh et al. [37], Dhaliwal and Shereif [38], Yousef [39], Ignaczak [40], Abbas and Yousef [41], Abbas et al. [42], Abo-Daheb and Lotfy [43] Abbas [44] and Yu et al. [45].

In the present work, we are concerned with the magnetic and thermal effects on the reflected waves in generalized magneto thermoelastic materials. The phase velocities corresponding to longitudinal, transverse and thermal waves are obtained. The amplitude and energy ratios are derived analytically and computed numerically. We verify our results through the conservation law of energy and known results of Achenbach [46]. The paper is structured starting with introduction which discusses the various aspect of the problem and review of literature. It is followed by the section 2 which is basic equation and it discusses the constitutive relation and governing equations of the problem. Section 3 is the wave propagation consisting of the solution of governing equation. Section 4 is appropriate boundary conditions and it contains the derivation of amplitude ratios. The partition of energy in Section 5 discusses the energy distribution among the incident and reflected waves. Special cases of the problem is discussed in Section 6. The numerical discussion consists the numerical results of the phase velocity, amplitude and energy ratios in Section 7. The findings of the paper is given in the section 8 as conclusion of the problem.

2. Basic Equations

The Maxwell’s equations corresponding to linear electrodynamics for a homogeneous, thermally and electrically conducting elastic solid are given by Ezzat and Ellall [16]

\[
\text{curl } \mathbf{h} = \mathbf{J} + \mathbf{D}, \quad \text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \text{div } \mathbf{B} = 0, \quad \text{div } \mathbf{D} = \rho, \quad \mathbf{B} = \mu_0 \mathbf{H}, \quad \mathbf{D} = \varepsilon_0 \mathbf{E}.
\]  

(1)

The modified Ohm’s law for finite conductivity is

\[
\mathbf{J} = \sigma_0 \left[ \mathbf{E} + \mu_0 \mathbf{u} \times \mathbf{H} \right] - \kappa_0 \nabla T,
\]  

(2)

<table>
<thead>
<tr>
<th>Symbols</th>
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<tbody>
<tr>
<td>( \lambda, \mu )</td>
<td>Lame’s constants</td>
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<tr>
<td>( C_p )</td>
<td>specific heat</td>
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<tr>
<td>( T )</td>
<td>absolute temperature</td>
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<tr>
<td>( \tau_0 )</td>
<td>Maxwell’s tensor</td>
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<td>( \sigma_0 )</td>
<td>strain tensor</td>
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<td>( \varepsilon_0 )</td>
<td>displacement component</td>
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<td>( \rho )</td>
<td>density</td>
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<tr>
<td>( t )</td>
<td>time</td>
</tr>
<tr>
<td>( T_0 )</td>
<td>reference temperature</td>
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\( \kappa \) thermal conductivity  \( \mu_0 \) magnetic permeability  
\( \tau \) relaxation time  \( \epsilon_0 \) electric permeability 
\( \alpha_T \) lineal thermal expansion  \( \Sigma \) cubical dilatation 
\( H_0 \) initial magnetic field  \( \text{D} \) electric displacement 
\( E \) induced electric field  \( \text{h} \) induced magnetic field 
\( \sigma_0 \) electric conductivity  \( F_i \) Lorentz’s force 
\( \delta_0 \) Kronecker’s delta  \( \gamma \) \((3\lambda + 2\mu)\alpha_T\) 
\( \epsilon_i \) \(\gamma T_0/\rho C(\lambda + 2\mu)\alpha_T\)  
\( \beta_i \) \(\sigma_0\mu_0\eta\alpha_T\)  
\( \eta \) \(\rho C_0/\epsilon_0\)  
\( \epsilon_i^2 \) \(\beta^2\)  
\( \beta \) \(\mu\)  
\( \theta_i \) incident angle  \( \theta_i \) reflected angle \(i = 1, 2, 3\)  
\( c_i^2 \) \(1/\mu_0\epsilon_0\omega\) angular frequency 
\( J_i \) components of current density  \( E_i \) components of electric field 
\( u \) displacement  \( B \) magnetic field 

Table 1. Nomenclature

where \( \kappa_0 \) is the coefficient connecting the temperature gradient and the electric current density. The constitutive relations for the theory of generalized magneto thermoelastic continua are

\[
\sigma_{ij} = 2\mu e_{ij} + \lambda e_{ij} - \gamma \delta_{ij} \hat{T},
\]

\[
\tau_{ij} = \mu_0 (H_i h_j + H_j h_i - H_k h_k \sigma_{ij}),
\]

where

\[
T = T - T_0, \quad T_{0} \ll 1, \quad e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad (i, j, k = 1, 2, 3).
\]

The equation of motion in the absence of body force and heat source for linearly homogeneous magneto thermoelastic materials are

\[
\sigma_{ij, j} + F_i = \rho \ddot{u}_i,
\]

where \( F_i = \mu_0 (J \times \mathbf{H})_i \).

The heat conduction equation for such a medium is

\[
\kappa T_{,i,j} = \rho C (T_{,i} + \tau T_{,i}) + \gamma T_0 (\sum \gamma_{,i} + \tau \sum_0),
\]

where \( \Sigma = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} \).

We consider a two dimensional problem in \( xy \)-plane of the Cartesian system so that \( u_t = (u_1, u_2, 0) \) and \( \frac{\partial}{\partial z} = 0 \). The stress tensors are given by

\[
\sigma_{11} = (\lambda + 2\mu)u_{1,1} + \lambda u_{2,2} - \gamma \hat{T}, \quad \sigma_{22} = (\lambda + 2\mu)u_{2,2} + \lambda u_{1,1} - \gamma \hat{T}, \quad \sigma_{12} = \mu \left(u_{1,y} + u_{2,x}\right).
\]

The components of magnetic intensity vector in the medium are represented by

\[
H_1 = 0, \quad H_2 = 0, \quad H_3 = H_0 + h(x, y, t).
\]

The current density \( J \) is parallel to induced electric field \( E \). Thus, the linearized components of \( J \) after using (2) are given by

\[
J_1 = \sigma_0 \left( E_1 + \mu_0 H_0 \frac{\partial u_2}{\partial t} \right) - \kappa_0 \frac{\partial \hat{T}}{\partial x},
\]

\[
J_2 = \sigma_0 \left( E_2 + \mu_0 H_0 \frac{\partial u_1}{\partial t} \right) - \kappa_0 \frac{\partial \hat{T}}{\partial y}, \quad J_3 = 0.
\]

Equations (1) and (2) give the following three equations
\[
\frac{\partial h}{\partial y} = \sigma_0 \left( E_1 + \mu_0 H_0 \frac{\partial u_z}{\partial t} \right) - \kappa_0 \frac{\partial T}{\partial x} - \varepsilon_0 \frac{\partial E_1}{\partial t}, \\
\frac{\partial h}{\partial x} = -\sigma_0 \left( E_2 - \mu_0 H_0 \frac{\partial u_1}{\partial t} \right) + \kappa_0 \frac{\partial T}{\partial y} - \varepsilon_0 \frac{\partial E_2}{\partial t}, \\
\frac{\partial E_1}{\partial y} - \frac{\partial E_2}{\partial x} = \mu_0 \frac{\partial h}{\partial t}.
\]

Using Equations (8) and (9), we get
\[
F_1 = \sigma_0 \mu_0 H_0 \left( E_2 - \mu_0 H_0 \frac{\partial u_1}{\partial t} \right) - \kappa_0 H_0 \mu_0 \frac{\partial T}{\partial y}, \\
F_2 = -\sigma_0 \mu_0 H_0 \left( E_1 + \mu_0 H_0 \frac{\partial u_2}{\partial t} \right) + \kappa_0 H_0 \mu_0 \frac{\partial T}{\partial x}, \\
F_3 = 0.
\]

Putting the values of \( \sigma_0 \) and \( F_1 \) in Eqs. (7) and (13) into (5), we get
\[
(\lambda + \mu) \frac{\partial}{\partial x} + \mu \nu \frac{\partial}{\partial y} \left( \nu \frac{\partial u_1}{\partial x} - \gamma \frac{\partial T}{\partial x} + \mu \nu \sigma_0 H_0 \left( E_2 - \mu_0 H_0 \frac{\partial u_1}{\partial t} \right) - \kappa_0 H_0 \mu_0 \frac{\partial T}{\partial y} \right) = \rho \frac{\partial^2 u_1}{\partial t^2}, \\
(\lambda + \mu) \frac{\partial}{\partial y} + \mu \nu \frac{\partial}{\partial x} \left( \nu \frac{\partial u_2}{\partial y} - \gamma \frac{\partial T}{\partial y} - \mu \nu \sigma_0 H_0 \left( E_1 + \mu_0 H_0 \frac{\partial u_2}{\partial t} \right) + \kappa_0 H_0 \mu_0 \frac{\partial T}{\partial x} \right) = \rho \frac{\partial^2 u_2}{\partial t^2}.
\]

The following non-dimensional variables are used
\[
x' = c_1 \eta x, \quad y' = c_1 \eta y, \quad u_1' = c_1 \eta u_1, \quad u_2' = c_1 \eta u_2, \quad v' = c_1 \eta v, \quad \tau' = c_1 \eta \tau, \quad \psi = \frac{y' T}{\lambda + 2u}, \\
\sigma_j' = \frac{\sigma_j}{\mu}, \quad h' = \frac{\eta h}{\sigma_0 \mu_0 H_0}, \quad E_j' = \frac{\eta E_j}{\sigma_0 \mu_0^2 H_0 c_1}, \quad \kappa' = \frac{\mu_0 H_0^2}{\gamma}.
\]

Using these non-dimensional terms, in the equations of motion are represented by
\[
(\beta^2 - 1) \sum_{x} + \nu^2 u_1 - \beta^2 \psi, \beta^2 \kappa_0 \psi, + \beta^2 \alpha \beta_1 \left( \beta \nu E_2 - u_1 \right) = \beta^2 u_{1,\psi}, \\
(\beta^2 - 1) \sum_{y} + \nu^2 u_2 - \beta^2 \psi, - \beta^2 \kappa_0 \psi, + \beta^2 \alpha \beta_1 \left( \beta \nu E_1 + u_2 \right) = \beta^2 u_{2,\psi}, \\
\nabla^2 \psi = \left( \psi, + r \psi, \right) + c_1 \left( \sum_\psi + r \sum_\psi \right).
\]

3. Wave Propagation

For a plane harmonic wave, the displacement components and temperature field which are the solution of Eqs. (16)–(18) take the following form (see Ben-Menahem and Singh [25])
\[
\{u, v, \psi\} = \{A, B, C\} e^{i \left[ \nu \left( t - k \left( x \sin \theta_0 + y \cos \theta_0 \right) \right) \right]},
\]
where \( A, B, C \) are amplitudes, \( k \) is dimensionless complex wavenumber, \( \omega \) is dimensionless angular frequency. If we take \( k = \mathcal{R} + \mathcal{I} \), then \( \mathcal{R}(k) \) and \( \mathcal{I}(k) \) denote the real and imaginary parts of \( k \) respectively. For the existence of waves in real:
\[
\mathcal{R}(k) > 0, \quad \mathcal{I}(k) \leq 0.
\]

It may be noted that the phase speed of the wave will be represented by \( \frac{\omega}{\mathcal{R}(k)} \), while \( \mathcal{I}(k) \) gives attenuation coefficient.

Using Eqs. (16)–(18), we get
\[
\left( k^2 a_{11} + a_{12} \right) A + k^2 b_{11} B + k c_{11} C = 0, \\
k^2 b_{11} A + \left( k^2 b_{21} + a_{12} \right) B + k c_{21} C = 0, \\
k a_{11} A + k b_{11} B + \left( k^2 + a_{11} \right) C = 0,
\]
where
\[
a_{11} = - \left( \beta^2 \sin^2 \theta_0 + \cos^2 \theta_0 \right), \quad a_{12} = \omega \beta^2 \left( \tau \alpha \beta_1^2 H_0 \mu_0 - \tau \alpha \beta_1 + \omega \right), \quad b_{11} = \sin \theta_0 \cos \theta_0 \left( 1 - \beta^2 \right).
\]
\[ c_{11} = \beta^2 (\sin \theta_0 + \kappa_0 \cos \theta_0), \quad b_{21} = -\left(\sin^2 \theta_0 + \beta^2 \cos^2 \theta_0 \right), \quad c_{21} = \beta^2 \left(\cos \theta_0 - \kappa_0 \sin \theta_0 \right), \]
\[ a_1 = -i\varepsilon_1 \sin \theta_0 \alpha_0, \quad b_3 = -i\varepsilon_1 \cos \theta_0 \alpha_0, \quad \alpha_0 = \omega - \tau \omega^2. \]

The non-singular solutions of Eqs. (20)–(22) give
\[ k^6 g_0 + k^4 g_1 + k^2 g_2 + g_3 = 0, \quad (23) \]
where
\[ g_0 = b_2 a_{11} - b_1^2, \quad g_2 = a_{11} a_{11} + b_2 a_{11} + a_{12} - b_3 c_{21} a_{11} - c_{11} a_{12} - b_1 a_{11} - b_1 a_{12} - b_1^2 \alpha_0 + b_1 a_{12} + c_{11} a_{12} - a_{11} b_1 c_{11}. \]

If \( V_1, V_2 \) and \( V_3 \) are the phase velocities corresponding to longitudinal, transverse and thermal waves respectively, then these values will satisfy Eq. (23) and \( V_1 > V_2 > V_3. \)

We consider a generalized magneto-thermoelastic half-space, \((\varepsilon \gg 0)\) with a constant magnetic intensity, \( H = (0, 0, H_0) \) which acts parallel to the bounding plane. The full form of displacement components and temperature field are given by
\[ (u_x, u_y, \psi) = (1, f_0, r_0) A_0 e^{i [\omega - k_0 (\sin \theta_0 + \gamma \cos \theta_0)]} + \sum_{i=1}^3 (1, f_i, r_i) A_i e^{i [\omega - k_i (\sin \theta_0 - \gamma \cos \theta_0)]}, \quad (24) \]
where \( A_i \) is amplitude constant, \( f_i \) and \( r_i \) are the coupling parameters whose expressions are given below
\[ f_0 = \frac{\left(k^2 + \alpha_0\right)\left(k^2 a_{11} + a_{12} a_{12} - k^2 b_1 b_1 + k^2 b_2 a_{12} + a_{12} b_1 c_{21}\right)}{b_1 \left(k^2 + \alpha_0\right)\left(k^2 b_{21} + a_{12}\right) - k^2 b_1 c_{21}}, \]
\[ f_i = \frac{-\left(k^2 + \alpha_0\right)\left(k^2 a_{11} + a_{12} a_{12} - k^2 b_1 b_1 + k^2 b_2 a_{12} + a_{12} b_1 c_{21}\right)}{b_1 \left(k^2 + \alpha_0\right)\left(k^2 b_{21} + a_{12}\right) - k^2 b_1 c_{21}}, \]
\[ r_0 = \frac{-\left(k^2 + \alpha_0\right)\left(k^2 b_{11} + a_{11}\right)}{\left(k^2 + \alpha_0\right)\left(k^2 b_{21} + a_{12}\right) + k^2 b_1 c_{21}}, \quad r_i = \frac{k^2 b_{11} - k^2 a_{11} b_{12}}{\left(k^2 + \alpha_0\right)\left(k^2 b_{21} + a_{12}\right) - k^2 b_1 c_{21}}. \]

The Snell’s law, for the problem, is given by
\[ k_0 \sin \theta_0 = k_i \sin \theta_i, \quad (i = 1, 2, 3). \quad (25) \]

### 4. Appropriate Boundary Conditions

The normal and tangential stress tensors and gradient of temperature field at \( y = 0 \) vanish. These conditions may be written as (i) \( \sigma_{12}+\tau_{22} = 0 \), (ii) \( \sigma_{21}+\tau_{11} = 0 \) and (iii) Temperature gradient at \( y = 0 \) which can be written as
\[ \left(\mu_0 H_0^2 + \beta^2 - 2\right) u_{1,x} + \left(\mu_0 H_0^2 + \beta^2\right) u_{2,y} - \beta^2 \psi = 0, \quad (26) \]
\[ \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} = 0, \quad \frac{\partial \psi}{\partial y} = 0. \quad (27) \]

Using Eqs. (24) and (25) into (26) and (27), we get a matrix equation
\[ \left[ B_0 \right] \left[ Z \right] = \left[ N \right], \quad (i, j = 1, 2, 3) \quad (28) \]
where \( B_0 \) is \( 3 \times 3 \) matrix, \( Z = \left[Z_1, Z_2, Z_3\right]^\top \) is the amplitude ratio and \( N = \left[N_1, N_2, N_3\right]^\top \). The expression of \( Z_i = \frac{A_i}{A_0} \) are given by
\[ Z_1 = \frac{N_1 \left(B_{11} B_{33} - B_{13} B_{31}\right) - B_{21} \left(2 N_1 B_{33} - N_1 B_{22}\right) + B_{31} \left(N_1 B_{23} - N_3 B_{22}\right)}{B_{11} \left(B_{13} B_{33} - B_{33} B_{33}\right) - B_{12} \left(B_{13} B_{33} - B_{33} B_{31}\right) + B_{13} \left(B_{13} B_{33} - B_{31} B_{33}\right)}, \]
\[ Z_2 = \frac{B_{11} \left(N_1 B_{33} - N_3 B_{33}\right) - N_1 \left(B_{11} B_{33} - B_{13} B_{33}\right) + B_{31} \left(N_1 B_{23} - B_{13} N_3\right)}{B_{11} \left(B_{23} B_{33} - B_{33} B_{33}\right) - B_{12} \left(B_{23} B_{33} - B_{33} B_{31}\right) + B_{13} \left(B_{23} B_{33} - B_{31} B_{33}\right)}. \]
\[ Z_i = \frac{B_{11}(B_{22}N_3 - B_{23}N_2) - B_{21}(B_{12}N_3 - B_{13}N_2) + N_1(B_{12}B_{23} - B_{13}B_{22})}{B_{11}(B_{22}B_{33} - B_{23}B_{32}) - B_{12}(B_{12}B_{33} - B_{13}B_{32}) + B_{13}(B_{12}B_{23} - B_{13}B_{22})}, \]  

(29)

where the expressions of \( B_{ij} \) and \( N_i \) are given below

\[ B_{ij} = i\left\{ (\mu_i N_{0}^2 + \beta^2 - 2)k_0 \sin \theta_i - (\mu_i N_{0}^2 + \beta^2 - 2)k_0 \cos \theta_i \right\} + \beta^2 r_i, \]  

\[ B_{3j} = r_j k_j \cos \theta_j, \]  

\[ N_1 = -i\left\{ (\mu_0 N_{0}^2 + \beta^2 - 2)k_0 \sin \theta_0 + f_0 \left( \mu_0 N_{0}^2 + \beta^2 - 2 \right)k_0 \cos \theta_0 \right\} - \beta^2 r_0, \]  

\[ N_2 = k_0 \left( \cos \theta_0 + f_0 \sin \theta_0 \right), \]  

\[ N_3 = r_0 k_0 \cos \theta_0. \]

It may be noted that these ratios depend on elastic constants, angle of propagation, magnetic and thermal parameters.

5. Energy Partition

The incident energy is distributed among the reflected waves at the boundary surface. The power per unit area represents the energy flux across the surface element and is given by Achenbach [46] as

\[ P_{ij} = \frac{1}{2} \Re \left\{ (\sigma_{22} + \tau_{22})i, (\hat{u}_i^j)^* \right\} + \frac{1}{2} \Re \left\{ (\sigma_{21} + \tau_{21})i, (\hat{u}_i^j)^* \right\}, i, j = (0,1,2,3), \]

(30)

where \((^*\)) denotes complex conjugate of ()

There are bulk energy and interacting energy between the incident and reflected waves. This energy can be obtained together from the energy matrix, \( \mathcal{E}_{ij} \) in the form of ratios as

\[ \mathcal{E}_{ij} = \frac{-P_{ij}}{P_{00}}, \approx - (i, j = 1, 2, 3) \]

(31)

where

\[ P_{00} = \frac{\left| A_{ii} \right|^2}{2} \omega k_0 \left\{ \cos \theta_i \left( f_0 ^* \left( \mu_0 N_{0}^2 + \beta^2 \right) + 1 \right) - \sin \theta_i \left( f_0 ^* \left( \mu_0 N_{0}^2 + \beta^2 - 2 \right) + f_i \right) \right\}, \]

\[ P_{ij} = \frac{\left| A_{ij} \omega k_0 \right|}{2} \left\{ \cos \theta_i \left( f_j ^* f_i + 1 \right) - \sin \theta_i \left( f_j ^* \left( \mu_0 N_{0}^2 + \beta^2 - 2 \right) - f_i \right) \right\}. \]

The sum of the net interacting energy, \( \mathcal{E}^{\text{int}} \) and bulk energy ratios, \( \mathcal{E}^{\text{bulk}} \) due to the interactions of incident wave with the three dissimilar reflected waves can be obtained respectively from

\[ \mathcal{E}^{\text{int}} = \sum_{i=1}^{3} \left( \mathcal{E}_{ii} + \mathcal{E}_{0i} \right) + \sum_{i=1}^{3} \left( \sum_{j=1}^{3} \mathcal{E}_{ij} - \mathcal{E}_{ii} \right), \quad \mathcal{E}^{\text{bulk}} = \sum_{i=1}^{3} \mathcal{E}_{ii}. \]

(32)

Thus, the energy ratios corresponding to reflected longitudinal, transverse and thermal waves are respectively represented by \( E_1, E_2 \) and \( E_3 \), which are given by

\[ E_i = \mathcal{E}^{\text{int}}_i + \mathcal{E}^{\text{bulk}}_i. \]

(33)

The sum of energy ratios at the boundary surface for the three reflected waves must be equal to unity.

\[ \mathcal{E}^{\text{bulk}} + \mathcal{E}^{\text{int}} = \sum_{i=1}^{3} E_i = 1. \]

(34)

This shows the conservation law of energy for this specific problem.

6. Special cases

Case 1: If we neglect magnetic effect, the problem reduces to wave propagation in generalized thermoelastic material. In this case, \( h = H_o = \alpha = 0 \) and \( a_{12} = \omega^2 \beta^2 \). The amplitude and energy ratios are given by Eqs. (29) and (31) respectively with the following modified values

\[ N_1 = -i\left\{ (\sin \theta_0 k_0 \left( \beta^2 - 2 \right) + f_0 k_0 \cos \theta_0 \beta^2) - \beta^2 r_0 \right\}, B_{11} = i\left( \beta^2 - 2 \right) k_0 \sin \theta_0 - \beta^2 f_0 \cos \theta_0 k_0 + \beta^2 r_0, \]

\[ B_{12} = i\left( \beta^2 - 2 \right) k_0 \sin \theta_0 - \beta^2 f_0 \cos \theta_0 k_0 + \beta^2 r_0, B_{13} = i\left( \beta^2 - 2 \right) k_0 \sin \theta_0 - \beta^2 f_0 \cos \theta_0 k_0 + \beta^2 r_0, \]

\[ P_{00} = \frac{|A_0|^2 \omega k_0}{2} \left( \cos \theta_0 \left( |f_0|^2 \beta^2 + 1 \right) - \sin \theta_0 \left( f_0^* \left( \beta^2 - 2 \right) + f_0 \right) \right), \]
\[ P_{ij} = \frac{A_i A_j \omega k_0}{2} \left( \cos \theta_i \left( \beta^2 f_j^* f_i + 1 \right) - \sin \theta_i \left( f_j^* \left( \beta^2 - 2 \right) - f_j \right) \right). \]

**Case 2:** If we neglect thermal effect, the problem reduces to wave propagation in isotropic material. In this case, \( \kappa = \eta = C_\varepsilon = \epsilon_1 = \beta = \gamma = \alpha_\varepsilon = 0 \) and \( f_i \) is taken so that:
\[
\left( \sin \theta_i u_i^{(0)}, \cos \theta_i u_i^{(0)} \right), \left( \sin \theta_i u_i^{(1)}, -\cos \theta_i u_i^{(1)} \right), \left( \cos \theta_i u_i^{(2)}, \sin \theta_i u_i^{(2)} \right)
\]
are components of the displacement. The amplitude and energy ratios are respectively given by
\[
Z_1 = \frac{\sin 2\theta_0 \sin 2\theta_2 - \beta^2 \cos^2 2\theta_2}{\sin 2\theta_0 \sin 2\theta_2 + \beta^2 \cos^2 2\theta_2}, \quad Z_2 = \frac{2\beta \sin 2\theta_0 \cos 2\theta_2}{\sin 2\theta_0 \sin 2\theta_2 + \beta^2 \cos^2 2\theta_2}, \quad \text{(35)}
\]
\[
E_1 = Z_1^2, \quad E_2 = \frac{\cos \theta_0 Z_2^2}{\beta^2 \cos \theta_0}. \quad \text{(36)}
\]

These results exactly match with those of Achenbach [46] for classical elasticity.

### 7. Numerical discussion

We have computed the amplitude and energy ratios of reflected waves for the incident longitudinal wave through Matlab programming. The following relevant values of parameters for the generalized thermoelastic solid are considered in Ezzat and Youssef [30].

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Values</th>
<th>Parameters</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )</td>
<td>386 N / Ks</td>
<td>( \alpha_r )</td>
<td>1.78 \times 10^{-2} K^{-1}</td>
</tr>
<tr>
<td>( C_\varepsilon )</td>
<td>383.1 m² / Ks²</td>
<td>( \eta )</td>
<td>8886.73 m / s²</td>
</tr>
<tr>
<td>( \mu )</td>
<td>3.86 \times 10^{10} N / m²</td>
<td>( \lambda )</td>
<td>7.76 \times 10^{10} N / m²</td>
</tr>
<tr>
<td>( c_1 )</td>
<td>1.39 \times 10^{-3}</td>
<td>( \rho )</td>
<td>8954 kg / m³</td>
</tr>
<tr>
<td>( \mu_0 )</td>
<td>4\pi \times 10^{-7} Nms² / C²</td>
<td>( \epsilon_0 )</td>
<td>10^{-9} / 36\pi C² / Nm²</td>
</tr>
<tr>
<td>( H_0 )</td>
<td>10³ C / ms</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Figure 1.** Variation of \( V_i \) for different values of (a) \( H_0 (\text{C/ms}) \) and (b) \( C_\varepsilon (\text{m²/Ks²}) \)

It is interesting to note the effects of initial magnetic field \( (H_0) \) and specific heat \( (C_\varepsilon) \) on the phase velocity, amplitude and energy ratios of the reflected waves. These effects lead to understand the physical properties of the materials. The variation of phase velocities of longitudinal, transverse and thermal waves with angular frequency are computed for different values of \( H_0 \) and \( C_\varepsilon \) in Figures 1, 2 and 3 respectively. In Figure
1, the phase velocity of longitudinal wave increases with the increase of $\omega$ thereby making a parabolic profile. It is observed that the values of $V_1$ decrease with the increase of $H_0$ and $C_e$. The phase velocity of the transverse wave in Figure 2 with the angular frequency shows a parabolic-type decreasing profile until it reaches a minimum value located in the range of $30Hz - 50Hz$ to then grow again.

Figure 2. Variation of $V_2$ for different values of (a) $H_0$ (C/ms) and (b) $C_e$ ($m^2/ks^2$)

Figure 3. Variation of $V_3$ for different values of (a) $H_0$ (C/ms) and (b) $C_e$ ($m^2/ks^2$)

Figure 4. Variation of $Z_1$ for different values of (a) $H_0$ (C/ms) and (b) $C_e$ ($m^2/ks^2$)
Figure 5. Variation of $Z_2$ for different values of (a) $H_0$ (C/ms) and (b) $C_e$ (m$^2$/Ks$^2$)

Figure 6. Variation of $Z_3$ for different values of (a) $H_0$ (C/ms) and (b) $C_e$ (m$^2$/Ks$^2$)

Figure 7. Variation of $E_1$ for different values of (a) $H_0$ (C/ms) and (b) $C_e$ (m$^2$/Ks$^2$)

The minimum effect of $H_0$ is observed in the lower values of $\omega$. Figure 3 shows that $V_3$ decreases with the increase of $\omega$. It is observed that the values of $V_3$ decrease with the increase of $C_e$ and $H_0$.

The variation of amplitude and energy ratios with angle of incidence are plotted for different values of $H_0$ and $C_e$ in Figures 4–10. In Figure 4, the amplitude ratio, $Z_1$ corresponding to longitudinal reflected wave starts from certain value and decreases up to certain value of $\theta_0$ which increases thereafter with the increase of $\theta_0$. The variation of $Z_2$ is similar with $Z_1$. In Figure 6, $Z_3$ starts from certain value which decreases followed by increasing and then decreasing with the increase of $\theta_0$. We have seen small effect of $H_0$ on $Z_i$ and the effect of $C_e$ on $Z_i$ are minimum near the normal angle of incidence. In Figures 7 and 8, the energy ratios $E_1$ and $E_2$ start from certain values which decrease to zero at certain values of $\theta_0 = \theta_s$ followed by increasing thereafter with the increase of $\theta_0$. The value of $\theta_s$ for $E_1$ and $E_2$ are different for different values of $H_0$ and $C_e$. The values of
\( \theta \) for different values of \( H_0 \) and \( C_e \) on \( E_1 \) are near to 64° and 70° respectively. This value is near 70° for the energy ratio, \( E_2 \). In Figure 9, \( E_1 \) starts from certain value which decreases to zero making a parabolic curves in the regions \( I : 31^0 \leq \theta_0 \leq 74^0, II : 31^0 \leq \theta_0 \leq 76^0, III : 32^0 \leq \theta_0 \leq 78^0 \) in Figure 9 (a) and \( I : 32^0 \leq \theta_0 \leq 75^0, II : 34^0 \leq \theta_0 \leq 82^0, III : 35^0 \leq \theta_0 \leq 80^0 \) in Figure 9 (b). Then, the graph of \( E_3 \) increases with the increase of \( \theta_0 \). Interestingly, these parabolic regions are different for different values of \( H_0 \) and \( C_e \). The effect of \( H_0 \) and \( C_e \) on \( E_1 \) are found to be minimum near grazing angle of incidence. In Figure 10, we have seen that sum of energy ratios of the reflected waves is closed to unity. This shows the conservation law of energy for this problem.

![Figure 8](image_url)

Figure 8. Variation of \( E_1 \) for different values of (a) \( H_0 \) (C/ms) and (b) \( C_e \) (m²/ks²)

![Figure 9](image_url)

Figure 9. Variation of \( E_1 \) for different values of (a) \( H_0 \) (C/ms) and (b) \( C_e \) (m²/ks²)
8. Conclusions

We have investigated the problem of reflection of waves from a half-space of generalized magneto thermoelastic materials. The expressions of the amplitude and energy ratios of reflected waves are found by using the appropriate boundary conditions. These ratios and phase velocities of the longitudinal, transverse and thermal waves are calculated numerically and the results are plotted graphically. We have observed that the present study confirms the significant effects of the magnetic field and thermal parameter on the wave propagation. The following points are the concluding remarks:

(i) The amplitude and energy ratios are functions of angle of incidence, magnetic, thermal and elastic parameters.

(ii) The phase velocity of longitudinal wave increases with the increase of $H_0$ and $C_e$.

(iii) The phase velocity corresponding to thermal wave increases with the increase of $\omega$.

(iv) The variations of $Z_l$ and $Z_t$ are similar in nature.

(v) The effect of $H_0$ on $Z_t$ is very small.

(vi) The effects of $H_0$ and $C_e$ on $E$ are minimum near the grazing angle of incidence.

(vii) The sum of energy ratios of reflected waves are close to one.

The new direction of the present work: The problem of reflection and refraction of elastic waves in the half-spaces of generalized magneto thermoelastic materials can be investigated. In this problem, the amplitude and energy ratios of the reflected and refracted waves could be one of the interesting area to discuss the various effects of thermal and magnetic fields of the wave propagation.

References


