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Research Article

Effect of Magnetic and Thermal Parameters on the Reflected Waves in Generalized Magneto Thermoelastic Materials

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Abstract: The present paper is concerned with the problem of reflection of homogeneous plane waves from the free surface of a generalized magneto thermoelastic half-space. The phase speeds of corresponding to longitudinal, transverse and thermal waves are obtained and are independent of angle of propagation. The amplitude and energy ratios corresponding to the reflected waves are obtained theoretically and numerically with the help of boundary conditions. The effects of magnetic and thermal parameters on the reflected waves are examined. The results hold the conservation law of energy.

***Keywords*:** generalized magneto thermoelastic material; plane wave; amplitude and energy ratio; phase velocity

1. Introduction

The coupled theory of thermoelasticity introduced by Biot [1] enables to eliminate the paradox that the elastic changes have no effect on temperature. This theory gives infinite phase speeds for heat waves which is different from the reality. Lord and Shulman [2] introduced the concept of relaxation time and the acceleration of heat flux in the generalized theory of thermoelas-ticity thereby representing finite phase speeds for the heat waves. Green and Lindsay [3] used entropy production inequality proposed by Green and Laws [4] to present an alternate generalized thermoelastic theory. These theories have been established using modifying Fourier's heat conduction equation or correcting the energy equation and Neuman-Duhamel relation. McCarthy [5] obtained two purely mechanical transverse and two longitudinal waves in gener-alized theromoelastic continuum. He presented the governing equations for the propagation of acceleration waves of arbitrary shape and strength. Nowacki [6] discussed different problems of thermoelasticity and gave the Duhamel-Newmann relations for an anisotropic body. Green and Naghdi [7, 8] also explained thermoelastic theory using the concept of the propagation of heat as like a wave. Youssef [9] proposed a new thermoelasticity theory considering the heat conduction equation with fractional order and proved the uniqueness theorem.

Paria [10] discussed the interacting effects of applied magnetic field in the theories of magneto-elasticity and magneto-thermo-elasticity of a solid body. Chaudhuri and Debnath [11] investi-gated the propagation of plane harmonic waves in an infinite conducting thermo-elastic solid permeated by a primary uniform magnetic field when the entire elastic medium is rotating with a uniform angular velocity. Sharma and Chand [12] studied the problem of distribution of deformation, temperature and magnetic field in the generalized magneto thermoelastic half-space. Massalas [13] discussed the magnetothermoelastic interactions in ferromagnetic mate-rial in the generalized theory of thermoelasticity. Sherief and Helmy [14] attempted the two dimensional problem of electro-magneto-thermoelastic material with thermal relaxation and obtained the distributions of temperature, displacement, stress, magnetic and electric fields. Wang and Dai [15] used the finite Hankle integral

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transformation to obtain the expressions for magnetothermodynamic stress and perturbation response of magnetic field in the orthotropic cylinder. Ezzat and Elall [16] introduced the modified Ohm's law, the temperature gradient and charge density effects in the linear theory of generalized magneto thermoelasticity and obtained the kinematic variables using normal mode analysis. Sarkar and Lahiri [17] used the normal mode analysis and eigenvalue approach techniques to obtain the kinematic variables in the three dimensional thermoelastic problem without energy dissipation.

The subject of wave and vibrations in different elastic continuum has been a matter of concern for many researchers since long. It is used in various field like in the exploration of mines and petroleum, earthquake engineering, Seismology, Geophysics, etc. More specific, wave phenom-ena in thermoelasticity materials has wide applications in engineering. Othman and Song [18] studied the reflection of plane waves from an isotropic magneto-thermoelastic rotating half space and obtained the reflection coefficients of the reflected waves. Lotfy et al. [19] discussed the problem of the magnetic and rotational effects on a fiber-reinforced thermoelastic solid based on two coupled thermoelastic theory with gravitational effect using normal mode analy-sis. Zoramuana and Singh [20] introduced the magnetic and rotational effects on the propogation of waves in transversly isotropic fiber-reinforced thermoelastic material with the help of Lord Shulman theory. Kumar and Tomar [21] investigated the effect of induced electric field due to applied magnetic field on the reflection and transmission coefficients of the elastic waves in the half-spaces of magneto-elastic materials with voids. Abo-Daheb [22] solved the governing equation of generalized magneto thermoelastic materials using Lame's potentials method in the context of classical dynamical and Lord-S¸hulman theories. Othman and Kumar [23] pre-sented the model equations of generalized magneto-thermoelasticity in an isotropic perfectly conducting elastic medium. This model is applied to four theories of the generalized thermoe-lasticity: Lord-Shulman, Green-Naghdi theory, Chandrasekharaiah-Tzou theory and coupled theory. Abd-Alla et al. [24] studied the problem of reflection of plane harmonic waves from a semi-infinite elastic solid under the effect of magnetic field and obtained the reflection co-efficient of the reflected waves. The problems related with wave propagation in different thermoelas-tic materials have been studied since long and they are in open literature, i.e., Ben-Menahem and Singh [25], Hetnaski and Ignaczak [26], Ezzat et al. [27], Ezzat [28], Othman [29], Ezzat and Youssef [30], Dai and Wang [31], Singh [32], Othman and Lotfy [33], Singh [34], Sarkar [35], Sarkar and De [36], Singh et al. [37], Dhaliwal and Sherief [38], Youssef [39], Ignaczack [40], Abbas and Youssef [41], Abbas et al.[42], Abo-Daheb and Lotfy [43] Abbas [44] and Yu et al. [45].

In the present work, we are concerned with the magnetic and thermal effects on the reflected waves in generalized magneto thermoelastic materials. The phase velocities corresponding to longitudinal, transverse and thermal waves are obtained. The amplitude and energy ratios are derived analytically and computed numerically. We verify our results through the conservation law of energy and known results of Achenbach [46]. The paper is structured starting with introduction which discusses the various aspect of the problem and review of literature. It is followed by the section 2 which is basic equation and it discusses the constitutive relation and governing equations of the problem. Section 3 is the wave propagation consisting of the solution of governing equation. Section 4 is appropriate boundary conditions and it contains the derivation of amplitude ratios. The partition of energy in Section 5 discusses the energy distribution among the incident and reflected waves. Special cases of the problem is discussed in Section 6. The numerical discussion consists the numerical results of the phase velocity, amplitude and energy ratios in Section 7. The findings of the paper is given in the section 8 as conclusion of the problem.

2. Basic Equations

The Maxwell's equations corresponding to linear electrodynamics for a homogeneous, thermally and electrically conducting elastic solid are given by Ezzat and Ellal [16]

$$\text{curl}\,\mathbf{h} = \mathbf{J} + \dot{\mathbf{D}},\ \text{curl}\,\mathbf{E} = -\dot{\mathbf{B}},\ \text{div}\,\mathbf{B} = 0,\ \text{div}\,\mathbf{D} = \rho_e,\ \mathbf{B} = \mu_0\mathbf{H},\ \mathbf{D} = \varepsilon_0\mathbf{E}. \tag{1}$$

The modified Ohm's law for finite conductivity is

$$\mathbf{J} = \sigma_0[\mathbf{E} + \mu_0\dot{\mathbf{u}} \times \mathbf{H}] - \kappa_0\nabla T, \tag{2}$$

Symbols		Symbols	
λ, μ	Lame's constants	ρ	density
C_e	specific heat	t	time
T	absolute temperature	T_0	reference temperature
τ_{ij}	Maxwell's tensor	σ_{ij}	stress tensor
e_{ij}	strain tensor	u_i	displacement component

κ	thermal conductivity	**J**	current density
μ_0	magnetic permeability	ϵ_0	electric permeability
τ	relaxation time	$\sum$	cubical dilatation
α_T	linear thermal expansion	**H**	magnetic intensity
H_0	initial magnetic field	**D**	electric displacement
E	induced electric field	h	induced magnetic field
σ_0	electric conductivity	F_i	Lorentz's force
δ_{ij}	Kronecker's delta	γ	$(3\lambda + 2\mu)\alpha_T$
ϵ_1	$\gamma^2 T_0/\rho C_e(\lambda + 2\mu)$	α	$\mu_0 H_0^2/(\lambda + 2\mu)$
β_1	$\sigma_0\mu_0/\eta$	ϵ_2	c_1^2/c^2
η	$\rho C_e/\kappa$	β^2	$(\lambda + 2\mu)/\mu$
c_1^2	$(\lambda + 2\mu)/\rho$	c_2^2	$\frac{\mu}{\rho}$
θ_0	incident angle	θ_i	reflected angle (i = 1,2,3)
c^2	$1/\mu_0\epsilon_0$	ω	angular frequency
J_i	components of current density	E_i	components of electric field
u	displacement	**B**	magnetic field

Table 1. Nomenclature

where κ_0 is the coefficient connecting the temperature gradient and the electric current density. The constitutive relations for the theory of generalized magneto thermoelastic continua are

$$\sigma_{ij} = 2\mu e_{ij} + \lambda e\delta_{ij} - \gamma\delta_{ij}\hat{T}, \tag{3}$$

$$\tau_{ij} = \mu_0(H_i h_j + H_j h_i - H_k h_k \sigma_{ij} \tag{4}$$

where

$$T = T - T_0, \frac{T}{T_0} \ll 1, e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), (i, j, k = 1, 2, 3).$$

The equation of motion in the absence of body force and heat source for linearly homogeneous magneto thermoelastic materials are

$$\sigma_{ij,j} + F_i = \rho\ddot{u}_i, \tag{5}$$

where $F_i = \mu_0(\mathbf{J} \times \mathbf{H})_i$.

The heat conduction equation for such a medium is

$$\kappa T,_{ii} = \rho C_e(T,_t + \tau T,_{tt}) + \gamma T_0(\sum,_t + \tau\sum,_{tt}), \tag{6}$$

where $\Sigma = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}$.

We consider a two dimensional problem in *xy*-plane of the Cartesian system so that $u_i = (u_1, u_2, 0)$ and $\frac{\partial}{\partial_z} \equiv 0$. The stress tensors are given by

$$\sigma_{11} = (\lambda + 2\mu)u_{1,x} + \lambda u_{2,y} - \gamma\overset{\frown}{T}, \sigma_{22} = (\lambda + 2\mu)u_{2,y} + \lambda u_{1,x} - \gamma\overset{\frown}{T}, \sigma_{12} = \mu\left(u_{1,y} + u_{2,x}\right). \tag{7}$$

The components of magnetic intensity vector in the medium are represented by

$$H_1 = 0, H_2 = 0, H_3 = H_0 + h\left(x, y, t\right).$$

The current density **J** is parallel to induced electric field **E**. Thus, the linearized components of **J** after using (2) are given by

$$J_1 = \sigma_0\left(E_1 + \mu_0 H_0\frac{\partial u_2}{\partial t}\right) - \kappa_0\frac{\partial T}{\partial x}, \tag{8}$$

$$J_2 = \sigma_0\left(E_2 + \mu_0 H_0\frac{\partial u_1}{\partial t}\right) - \kappa_0\frac{\partial T}{\partial y}, J_3 = 0. \tag{9}$$

Equations (1) and (2) give the following three equations

$$\frac{\partial h}{\partial y}=\sigma_0\left(E_1+\mu_0H_0\frac{\partial u_2}{\partial t}\right)-\kappa_0\frac{\partial T}{\partial x}+\varepsilon_0\frac{\partial E_1}{\partial t}, \tag{10}$$

$$\frac{\partial h}{\partial x}=-\sigma_0\left(E_2-\mu_0H_0\frac{\partial u_1}{\partial t}\right)+\kappa_0\frac{\partial T}{\partial y}-\varepsilon_0\frac{\partial E_2}{\partial t}, \tag{11}$$

$$\frac{\partial E_1}{\partial y}-\frac{\partial E_2}{\partial x}=\mu_0\frac{\partial h}{\partial t}. \tag{12}$$

Using Equations (8) and (9), we get

$$F_1=\sigma_0\mu_0H_0\left(E_2-\mu_0H_0\frac{\partial u_1}{\partial t}\right)-\kappa_0H_0\mu_0\frac{\partial T}{\partial y},F_2=-\sigma_0\mu_0H_0\left(E_1+\mu_0H_0\frac{\partial u_2}{\partial t}\right)+\kappa_0H_0\mu_0\frac{\partial T}{\partial x},F_3=0. \tag{13}$$

Putting the values of σ_{ij} and F_i in Eqs. (7) and (13) into (5), we get

$$(\lambda+\mu)\frac{\partial\Sigma}{\partial x}+\mu\nabla^2u_1-\gamma\frac{\partial T}{\partial x}+\mu_0\sigma_0H_0\left(E_2-\mu_0H_0\frac{\partial u_1}{\partial t}\right)-\kappa_0H_0u_0\frac{\partial T}{\partial y}=\rho\frac{\partial^2u_1}{\partial t^2}, \tag{14}$$

$$(\lambda+\mu)\frac{\partial\Sigma}{\partial y}+\mu\nabla^2u_2-\gamma\frac{\partial T}{\partial y}-\mu_0\sigma_0H_0\left(E_1+\mu_0H_0\frac{\partial u_2}{\partial t}\right)+\kappa_0H_0u_0\frac{\partial T}{\partial x}=\rho\frac{\partial^2u_2}{\partial t^2}. \tag{15}$$

The following non-dimensional variables are used

$$x'=c_1\eta x,\ y'=c_1\eta y,\ u'_1=c_1\eta u_1,\ u'_2=c_1\eta u_2,\ t'=c_1^2\eta t,\ \tau'=c_1^2\eta\tau,\ \psi=\frac{\gamma\widehat{T}}{\lambda+2u},$$

$$\sigma'_{ij}=\frac{\sigma_{ij}}{\mu},\ h'=\frac{\eta h}{\sigma_0\mu_0H_0},\ E'_i=\frac{\eta E_i}{\sigma_0\mu_0^2H_0c_1},\ \kappa'_0=\frac{\mu_0H_0\kappa}{\gamma}.$$

Using these non-dimensional terms, in the equations of motion are represented by

$$\left(\beta^2-1\right)\Sigma,_x+\nabla^2u_1-\beta^2\psi,_x-\beta^2\kappa_0\psi,_y+\beta^2\alpha\beta_1\left(\beta_1E_2-u_{1,t}\right)=\beta^2u_{1,tt}, \tag{16}$$

$$\left(\beta^2-1\right)\Sigma,_y+\nabla^2u_2-\beta^2\psi,_y+\beta^2\kappa_0\psi,_x-\beta^2\alpha\beta_1\left(\beta_1E_1+u_{2,t}\right)=\beta^2u_{2,tt}, \tag{17}$$

$$\nabla^2\psi=\left(\psi_{,t}+\tau\psi_{,tt}\right)+\varepsilon_1\left(\Sigma_{,t}+\tau\Sigma_{,tt}\right). \tag{18}$$

3. Wave Propagation

For a plane harmonic wave, the displacement components and temperature field which are the solution of Eqs. (16)–(18) take the following form(see Ben-Menahem and Singh [25])

$$\{u,v,\psi\}=\{A,B,C\}e^{\iota\{wt-k(x\sin\theta_0+y\cos\theta_0)\}}, \tag{19}$$

where A, B, C are amplitudes, k is dimensionless complex wavenumber, ω is dimensionless angular frequency. If we take $k=\mathscr{R}+\iota\mathscr{J}$, then $\mathscr{R}$(k) and $\mathscr{J}$(k) denote the real and imaginary parts of k respectively. For the existence of waves in real:

$$\mathscr{R}(k)>0,\ \mathscr{J}(k)\leqslant 0.$$

It may be noted that the phase speed of the wave will be represented by $\frac{\omega}{\mathcal{R}(k)}$, while $\mathscr{J}(k)$ gives attenuation coefficient.

Using Eqs. (16)–(19), we get

$$\left(k^2a_{11}+a_{12}\right)A+k^2b_{11}B+kc_{11}C=0, \tag{20}$$

$$k^2b_{11}A+\left(k^2b_{21}+a_{12}\right)B+kc_{21}C+0, \tag{21}$$

$$ka_3A+kb_3B+\left(k^2+\alpha_0\right)C=0, \tag{22}$$

where

$$a_{11}=-\left(\beta^2\sin^2\theta_0+\cos^2\theta_0\right),\ a_{12}=\omega\beta^2\left(\iota\alpha\beta_1^2H_0\mu_0-\iota\alpha\beta_1+\omega\right),\ b_{11}=\sin\theta_0\cos\theta_0\left(1-\beta^2\right),$$

$$c_{11} = \iota\beta^2\left(\sin\theta_0 + \kappa_0\cos\theta_0\right),\ b_{21} = -\left(\sin^2\theta_0 + \beta^2\cos^2\theta_0\right),\ c_{21} = \iota\beta^2\left(\cos\theta_0 - \kappa_0\sin\theta_0\right),$$

$$a_3 = -\iota\varepsilon_1\sin\theta_0\alpha_0,\ b_3 = -\iota\varepsilon_1\cos\theta_0\alpha_0,\ \alpha_0 = \iota\omega - \tau\omega^2.$$

The non-singular solutions of Eqs. (20)–(22) give

$$k^6 g_0 + k^4 g_1 + k^2 g_2 + g_3 = 0, \tag{23}$$

where

$$g_0 = b_{21}a_{11} - b_{11}^2,\ g_2 = a_{12}a_{11}\alpha_0 + b_{21}\alpha_0 a_{12} + a_{12}^2 - b_3c_{21}a_{21} - c_{11}a_3a_{12},\ g_3 = a_{12}^2\alpha_0,$$

$$g_1 = b_{12}\alpha_0 a_{11} + a_{11}a_{12} - b_3c_{21}a_{11} + b_{21}a_{12} - b_{11}^2\alpha_0 + b_{11}a_3c_{21} + c_{11}b_3b_{11} - a_3b_{21}c_{11}.$$

If V_1, V_2 and V_3 are the phase velocities corresponding to longitudinal, transverse and thermal waves respectively, then these values will satisfy Eq. (23) and $V_1 > V_2 > V_3$.

We consider a generalized magneto-thermoelastic half-space, ($x \geqslant 0$) with a constant mag-netic intensity, $H = (0, 0, H_0)$ which acts parallel to the bounding plane. The full form of displacement components and temperature field are given by

$$\left(u_1, u_2, \psi\right) = \left(1,\ f_0, r_0\right) A_0 e^{\iota\left\{wt - k_0\left(x\sin\theta_0 + y\cos\theta_0\right)\right\}} + \sum_{i=1}^{3}\left(1,\ f_i, r_i\right) A_i e^{\iota\left\{wt - k_i\left(x\sin\theta_i - y\cos\theta_i\right)\right\}}, \tag{24}$$

where A_i is amplitude constant, f_i and r_i are the coupling parameters whose expressions are given below

$$f_0 = \frac{\left(k_i^2 + \alpha_0\right)\left(k_i a_3 b_{21} + a_{12}a_3 - k_i^2 b_3 b_{11} - k_i^2 b_{21}a_3b_3 - a_3b_3a_{12}\right) + a_3b_3k_i^2c_{21}}{b_3\left\{(k_i^2 + \alpha_0)\left(k_i^2 b_{21} + a_{12}\right) - k_i^2 b_3 c_{21}\right\}},$$

$$f_i = -\frac{\left(k_i^2 + \alpha_0\right)\left(k_i a_3 b_{21} + a_{12}a_3 - k_i^2 b_3 b_{11} - k_i^2 b_{21}a_3b_3 - a_3b_3a_{12}\right) + a_3b_3k_i^2c_{21}}{b_3\left\{(k_i^2 + \alpha_0)\left(k_i^2 b_{21} + a_{12}\right) - k_i^2 b_3 c_{21}\right\}},$$

$$r_0 = -\frac{k_i^3 b_3 b_{11} + k_i a_3\left(k_i^2 b_{21} + a_{12}\right)}{\left(k_i^2 + \alpha_0\right)\left(k_i^2 b_{21} + a_{12}\right) + k_i^2 b_3 c_{21}},\ r_i = \frac{k_i^3 b_3 b_{11} - k_i a_3\left(k_i^2 b_{21} + a_{12}\right)}{\left(k_i^2 + \alpha_0\right)\left(k_i^2 b_{21} + a_{12}\right) - k_i^2 b_3 c_{21}}.$$

The Snell's law, for the problem, is given by

$$k_0\sin\theta_0 = k_i\sin\theta_i,\ \left(i = 1, 2, 3\right). \tag{25}$$

4. Appropriate Boundary Conditions

The normal and tangential stress tensors and gradient of temperature field at $y = 0$ vanish. These conditions may be written as (i) $\sigma_{22}+\tau_{22} = 0$, (ii) $\sigma_{21} + \tau_{21} = 0$ and (iii)Temperature gradient=0 at y = 0 which can be written as

$$\left(\mu_0 H_0^2 + \beta^2 - 2\right)u_{1,x} + \left(\mu_0 H_0^2 + \beta^2\right)u_{2,y} - \beta^2\psi = 0, \tag{26}$$

$$\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} = 0,\ \frac{\partial\psi}{\partial y} = 0. \tag{27}$$

Using Eqs. (24) and (25) into (26) and (27), we get a matrix equation

$$\left[B_{ij}\right]\left[Z\right] = \left[N\right],\ \left(i, j = 1, 2, 3\right) \tag{28}$$

where B_{ij} is 3 × 3 matrix, $Z = [Z_1, Z_2, Z_3]^t$ is the amplitude ratio and $N = [N_1, N_2, N_3]^t$. The expression of $Z_i = \dfrac{A_i}{A_0}$ are given by

$$Z_1 = \frac{N_1\left(B_{11}B_{33} - B_{23}B_{32}\right) - B_{21}\left(N_2B_{33} - N_3B_{22}\right) + B_{31}\left(N_2B_{23} - N_3B_{22}\right)}{B_{11}\left(B_{22}B_{33} - B_{32}B_{23}\right) - B_{12}\left(B_{21}B_{33} - B_{31}B_{23}\right) + B_{13}\left(B_{21}B_{32} - B_{31}B_{22}\right)},$$

$$Z_2 = \frac{B_{11}\left(N_2B_{33} - N_3B_{32}\right) - N_1\left(B_{12}B_{33} - B_{13}B_{32}\right) + B_{31}\left(B_{12}N_3 - B_{13}N_2\right)}{B_{11}\left(B_{22}B_{33} - B_{32}B_{23}\right) - B_{12}\left(B_{21}B_{33} - B_{31}B_{23}\right) + B_{13}\left(B_{21}B_{32} - B_{31}B_{22}\right)},$$

$$Z_3 = \frac{B_{11}\left(B_{22}N_3 - B_{23}N_2\right) - B_{21}\left(B_{12}N_3 - B_{13}N_2\right) + N_1\left(B_{12}B_{23} - B_{13}B_{22}\right)}{B_{11}\left(B_{22}B_{33} - B_{32}B_{23}\right) - B_{12}\left(B_{21}B_{33} - B_{31}B_{23}\right) + B_{13}\left(B_{21}B_{32} - B_{31}B_{22}\right)}, \tag{29}$$

where the expressions of B_{ij} and N_i are given below

$$B_{1j} = \iota\left\{\left(\mu_0 H_0^2 + \beta^2 - 2\right)k_0 \sin\theta_0 - \left(\mu_0 H_0^2 + \beta^2\right) f_i k_j \cos\theta_j\right\} + \beta^2 r_i,\ B_{2j} = k_j \cos\theta_j - f_j k_0 \sin\theta_0,$$

$$B_{3j} = r_j k_j \cos\theta_j,\ N_1 = -\iota\left\{\left(\mu_0 H_0^2 + \beta^2 - 2\right)k_0 \sin\theta_0 + f_0\left(\mu_0 H_0^2 + \beta^2\right)k_0 \cos\theta_0\right\} - \beta^2 r_0,$$

$$N_2 = k_0\left(\cos\theta_0 + f_0 \sin\theta_0\right),\ N_3 = r_0 k_0 \cos\theta_0.$$

It may be noted that these ratios depend on elastic constants, angle of propagation, magnetic and thermal parameters.

5. Energy Partition

The incident energy is distributed among the reflected waves at the boundary surface. The power per unit area represents the energy flux across the surface element and is given by Achenbach [46] as

$$\mathcal{P}_{ij} = \frac{1}{2}\mathcal{R}\left(< (\sigma_{22} + \tau_{22})^i, \left(\dot{u}_2^j\right)^* >\right) + \frac{1}{2}\mathcal{R}\left(< (\sigma_{21} + \tau_{21})^i, \left(\dot{u}_1^j\right)^* >\right), i, j = (0,1,2,3), \tag{30}$$

where $()^*$ denotes complex conjugate of ().

There are bulk energy and interacting energy between the incident and reflected waves. This energy can be obtained together from the energy matrix, $\mathcal{E}_{ij}$ in the form of ratios as

$$\mathcal{E}_{ij} = -\frac{\mathcal{P}_{ij}}{\mathcal{P}_{00}}, \approx \sim (i, j = 1,2,3) \tag{31}$$

where

$$\mathcal{P}_{00} = \frac{\left|A_0\right|^2 \omega k_0}{2}\left(\cos\theta_0\{\left|f_0\right|^2\left(\mu_0 H_0^2 + \beta^2\right) + 1\} - \sin\theta_0\{f_0^*\left(\mu_0 H_0^2 + \beta^2 - 2\right) + f_0\}\right),$$

$$\mathcal{P}_{ij} = \frac{A_i A_j^* \omega k_i}{2}\left(\cos\theta_i\{\left(\mu_0 H_0^2 + \beta^2\right) f_j^* f_i + 1\} - \sin\theta_i\{f_j^*\left(\mu_0 H_0^2 + \beta^2 - 2\right) - f_i\}\right).$$

The sum of the net interacting energy, $\mathcal{E}^{int}$ and bulk energy ratios, $\mathcal{E}^{bulk}$ due to the interactions of incident wave with the three dissimilar reflected waves can be obtained respectively from

$$\mathcal{E}^{int} = \sum_{i=1}^{3}\left(\mathcal{E}_{0i} + \mathcal{E}_{i0}\right) + \sum_{i=1}^{3}\left(\sum_{j=1}^{3}\mathcal{E}_{ij} - \mathcal{E}_{ii}\right),\quad \mathcal{E}^{bulk} = \sum_{i=1}^{3}\mathcal{E}_{ii}. \tag{32}$$

Thus, the energy ratios corresponding to reflected longitudinal, transverse and thermal waves are respectively represented by E_1, E_2 and E_3 which are given by

$$E_i = \mathcal{E}_i^{int} + \mathcal{E}_i^{bulk}. \tag{33}$$

The sum of energy ratios at the boundary surface for the three reflected waves must be equal to unity.

$$\mathcal{E}^{bulk} + \mathcal{E}^{int} = \sum_{i=1}^{3} E_i = 1. \tag{34}$$

This shows the conservation law of energy for this specific problem.

6. Special cases

Case 1: If we neglect magnetic effect, the problem reduces to wave propagation in generalized thermoelastic material. In this case, $h = H_0 = \alpha = 0$ and $a_{12} = \omega^2\beta^2$. The amplitude and energy ratios are given by Eqs. (29) and (31) respectively with the following modified values

$$N_1 = -\iota\{\sin\theta_0 k_0\left(\beta^2 - 2\right) + f_0 k_0 \cos\theta_0 \beta^2\} - \beta^2 r_0, B_{11} = \iota\{\left(\beta^2 - 2\right)k_0 \sin\theta_0 - \beta^2 f_1 \cos\theta_1 k_1\} + \beta^2 r_1,$$

$$B_{12} = \iota\{\left(\beta^2 - 2\right)k_0 \sin\theta_0 - \beta^2 f_2 \cos\theta_2 k_2\} + \beta^2 r_2, B_{13} = \iota\{\left(\beta^2 - 2\right)k_0 \sin\theta_0 - \beta^2 f_3 \cos\theta_3 k_3\} + \beta^2 r_3,$$

$$\mathcal{P}_{00} = \frac{|A_0|^2 \omega k_0}{2}\left(\cos\theta_0\left(|f_0|^2\beta^2+1\right)-\sin\theta_0\{f_0^*\left(\beta^2-2\right)+f_0\}\right),$$

$$\mathcal{P}_{ij} = \frac{A_i A_j^* \omega k_i}{2}\left(\cos\theta_i\left(\beta^2 f_j^* f_i+1\right)-\sin\theta_i\{f_j^*\left(\beta^2-2\right)-f_i\}\right).$$

Case 2: If we neglect thermal effect, the problem reduces to wave propagation in isotropic material. In this case, $\kappa = \eta = C_e = \epsilon_1 = \beta_1 = \gamma = \alpha_T = 0$ and f_i is taken so that:

$$\left(\sin\theta_0 u_1^{(0)}, \cos\theta_0 u_2^{(0)}\right),\ \left(\sin\theta_1 u_1^{(1)}, -\cos\theta_1 u_2^{(1)}\right),\ \left(\cos\theta_2 u_1^{(2)}, \sin\theta_2 u_2^{(2)}\right)$$

are components of the displacement. The amplitude and energy ratios are respectively given by

$$Z_1 = \frac{\sin 2\theta_0 sin2\theta_2 - \beta^2\cos^2 2\theta_2}{\sin 2\theta_0 \sin 2\theta_2 + \beta^2\cos^2 2\theta_2},\quad Z_2 = \frac{2\beta\sin 2\theta_0\cos 2\theta_2}{\sin 2\theta_0\sin 2\theta_2 + \beta^2\cos^2 2\theta_2}, \tag{35}$$

$$E_1 = Z_1^2,\quad E_2 = \frac{\cos\theta_2 Z_2^2}{\beta^2\cos\theta_0}. \tag{36}$$

These results exactly match with those of Achenbach [46] for classical elasticity.

7. Numerical discussion

We have computed the amplitude and energy ratios of reflected waves for the incident longitudinal wave through Matlab programming. The following relevant values of parameters for the generalized thermoelastic solid are considered in Ezzat and Youssef [30].

Table 2. Values of the parameters

Parameters	Values	Parameters	Values
k	$386\ N/Ks$	α_T	$1.78\times10^{-5}K^{-1}$
C_e	$383.1m^2/Ks^2$	ŋ	$8886.73m/s^2$
μ	$3.86\times10^{10}N/m^2$	λ	$7.76\times10^{10}N/m^2$
c_1	1.39×10^{-5}	ρ	$8954kg/m^3$
μ_0	$4\pi\times10^{-7}Nms^2/C^2$	ϵ_0	$10^{-9}/36\pi C^2/Nm^2$
H_0	$10^3\ C/ms$		

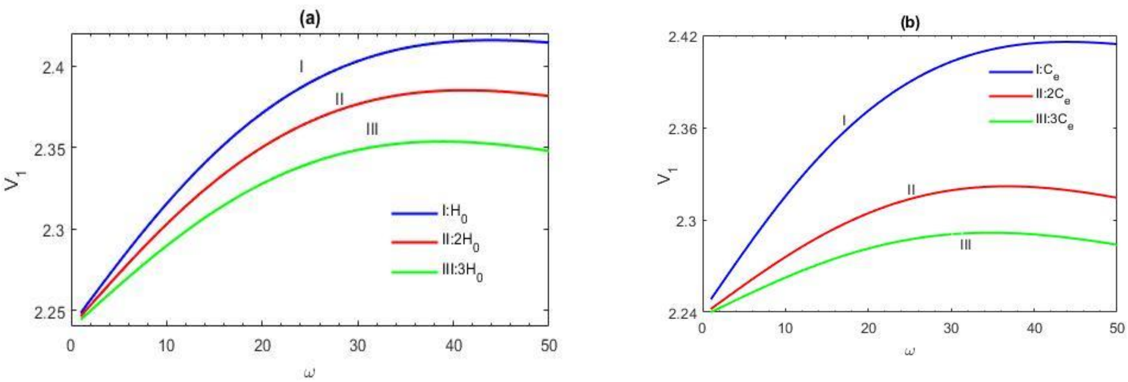

Figure 1. Variation of V_1 for different values of (a) H_0 (C/ms) and (b) C_e (m^2/Ks^2)

It is interesting to note the effects of initial magnetic field (H_0) and specific heat (C_e) on the phase velocity, amplitude and energy ratios of the reflected waves. These effects lead to understand the physical properties of the materials. The variation of phase velocities of longitudinal, transverse and thermal waves with angular frequency are computed for different values of H_0 and C_e in Figures 1, 2 and 3 respectively. In Figure

1, the phase velocity of longitudinal wave increases with the increase of ω thereby making a parabolic profile. It is observed that the values of V_1 decrease with the increase of H_0 and C_e. The phase velocity of the transverse wave in Figure 2 with the angular frequency shows a parabolic-type decreasing profile until it reaches a minimum value located in the range of $30Hz - 50Hz$ to then grow again.

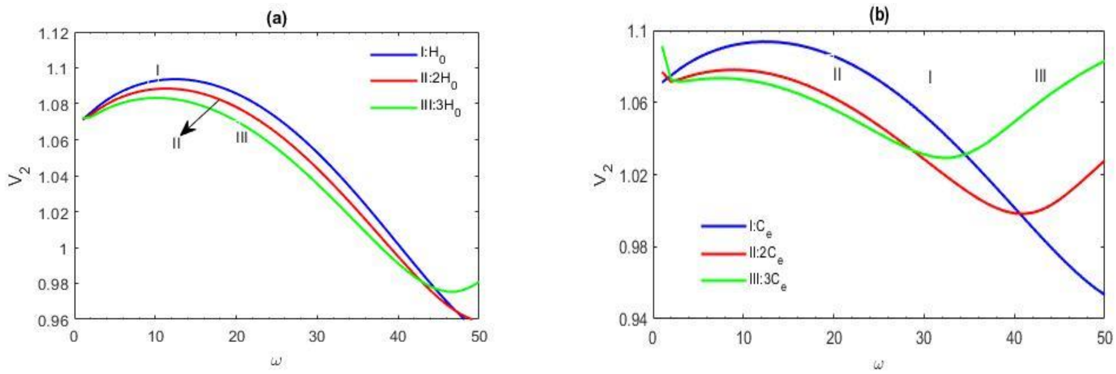

Figure 2. Variation of V_2 for different values of (a) H_0 (*C/ms*) and (b) C_e (m^2/Ks^2)

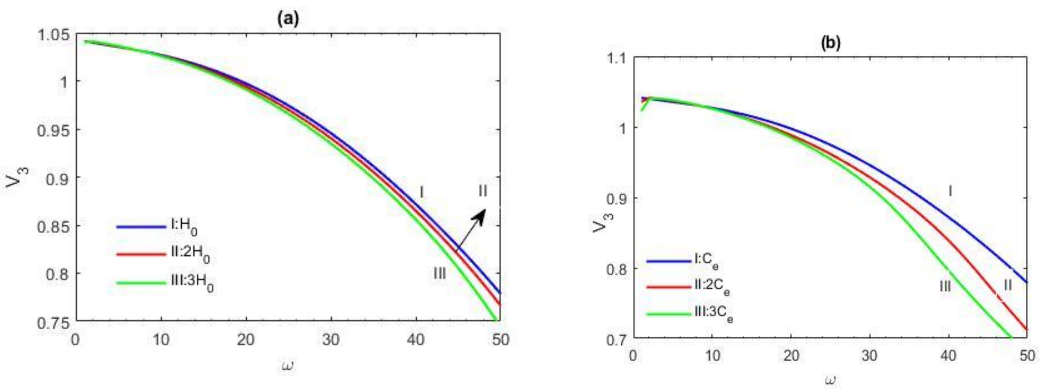

Figure 3. Variation of V_3 for different values of (a) H_0 (*C/ms*) and (b) C_e (m^2/Ks^2)

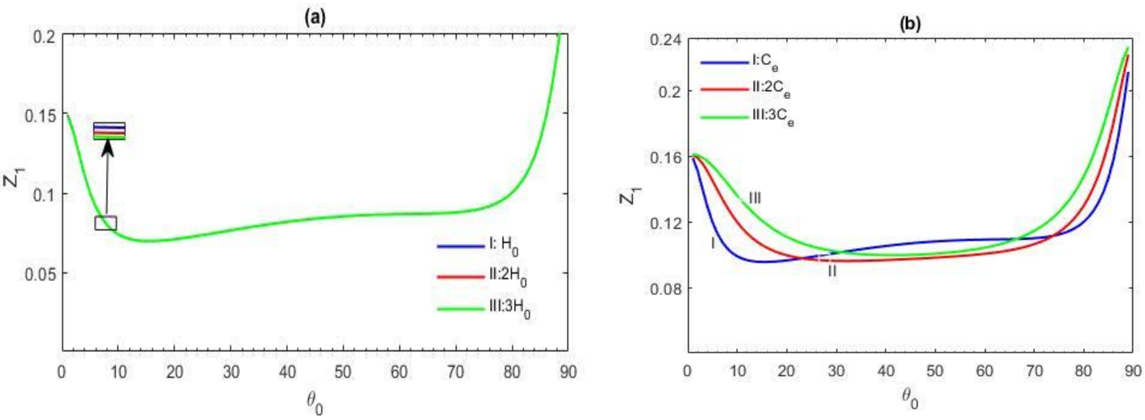

Figure 4. Variation of Z_1 for different values of (a) H_0 (*C/ms*) and (b) C_e (m^2/Ks^2)

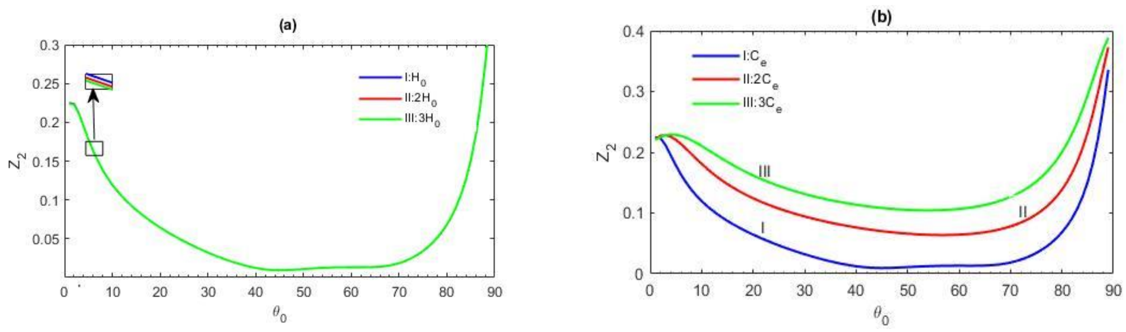

Figure 5. Variation of Z_2 for different values of (a) H_0 (*C/ms*) and (b) C_e (m^2/Ks^2)

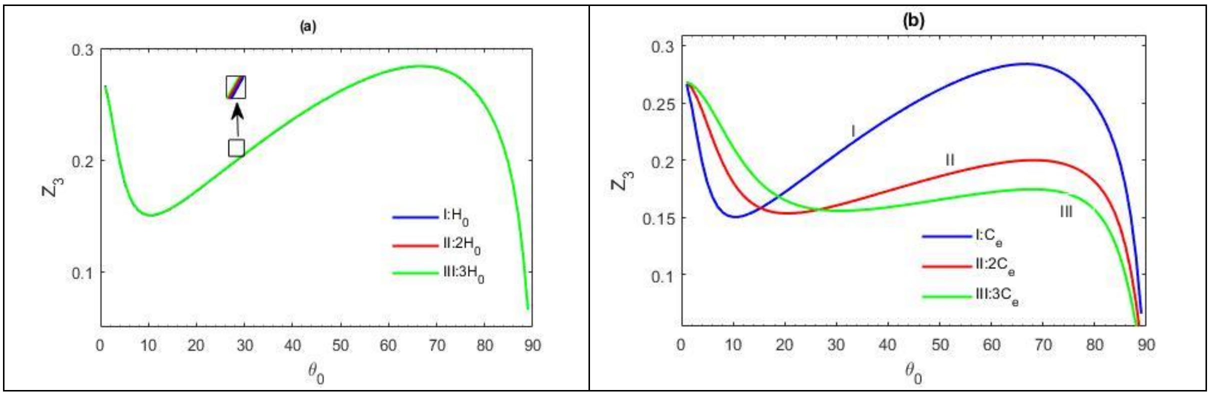

Figure 6. Variation of Z_3 for different values of (a) H_0 (*C/ms*) and (b) C_e (m^2/Ks^2)

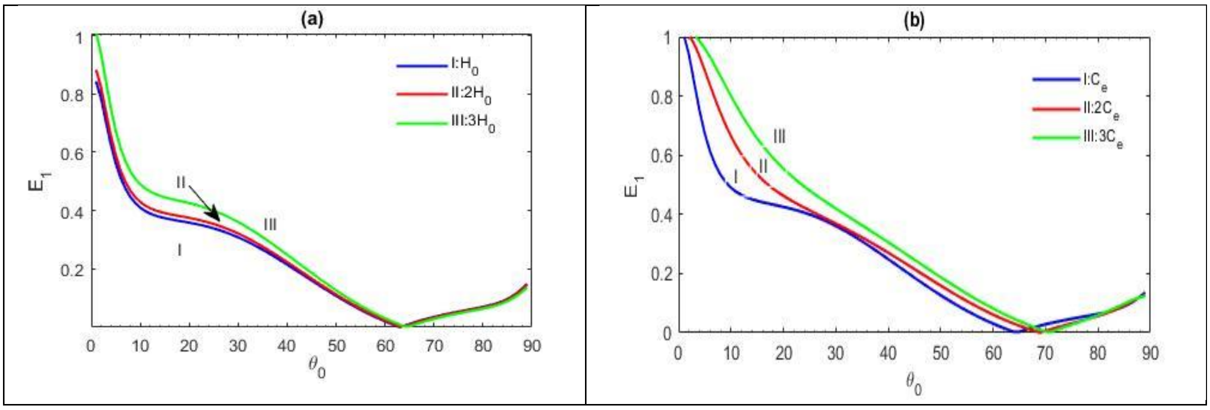

Figure 7. Variation of E_1 for different values of (a) H_0 (*C/ms*) and (b) C_e (m^2/Ks^2)

The minimum effect of H_0 is observed in the lower values of ω. Figure 3 shows that V_3 decreases with the increase of ω. It is observed that the values of V_3 decrease with the increase of C_e and H_0.

The variation of amplitude and energy ratios with angle of incidence are plotted for different values of H_0 and C_e in Figures 4–10. In Figure 4, the amplitude ratio, Z_1 corresponding to longitudinal reflected wave starts from certain value and decreases up to certain value of θ_0 which increases thereafter with the increase of θ_0. The variation of Z_2 is similar with Z_1. In Figure 6, Z_3 starts from certain value which decreases followed by increasing and then decreasing with the increase of θ_0. We have seen small effect of H_0 on Z_i and the effect of C_e on Z_i are minimum near the normal angle of incidence. In Figures 7 and 8, the energy ratios E_1 and E_2 start from certain values which decrease to zero at certain values of $\theta_0 = \theta_g$ followed by increasing thereafter with the increase of θ_0. The value of θ_g for E_1 and E_2 are different for different values of H_0 and C_e. The values of

θ_g for different values of H_0 and C_e on E_1 are near to 64° and 70° respectively. This value is near 70° for the energy ratio, E_2. In Figure 9, E_3 starts from certain value which decreases to zero making a parabolic curves in the regions $I:31^0 \le \theta_0 \le 74^0, II:31^0 \le \theta_0 \le 76^0, III:32^0 \le \theta_0 \le 78^0$ in Figure 9 (a) and $I:32^0 \le \theta_0 \le 75^0, II:34^0 \le \theta_0 \le 82^0, III:35^0 \le \theta_0 \le 80^0$ in Figure 9 (b). Then, the graph of E_3 increases with the increase of θ_0. Interestingly, these parabolic regions are different for different values of H_0 and C_e. The effect of H_0 and C_e on E_i are found to be minimum near grazing angle of incidence. In Figure 10, we have seen that sum of energy ratios of the reflected waves is closed to unity. This shows the conservation law of energy for this problem.

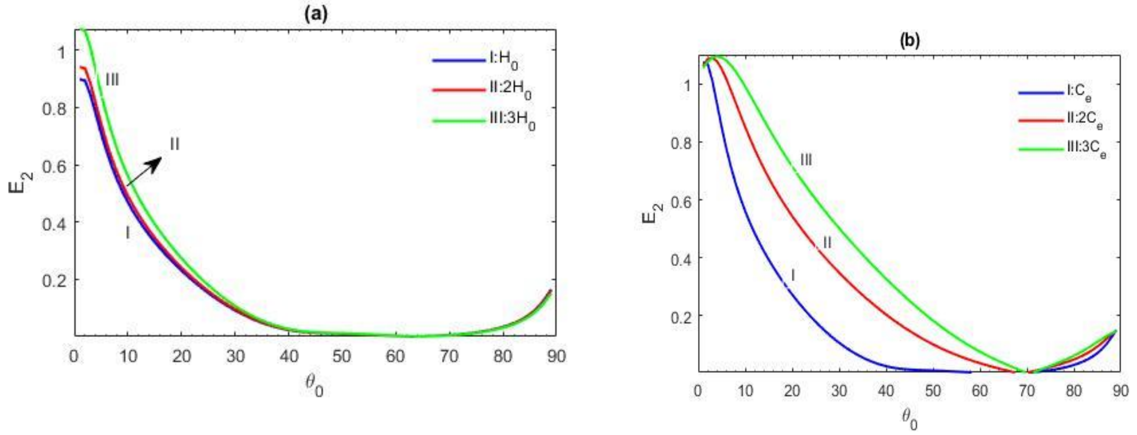

Figure 8. Variation of E_2 for different values of (a) H_0 (*C/ms*) and (b) C_e (m^2/Ks^2)

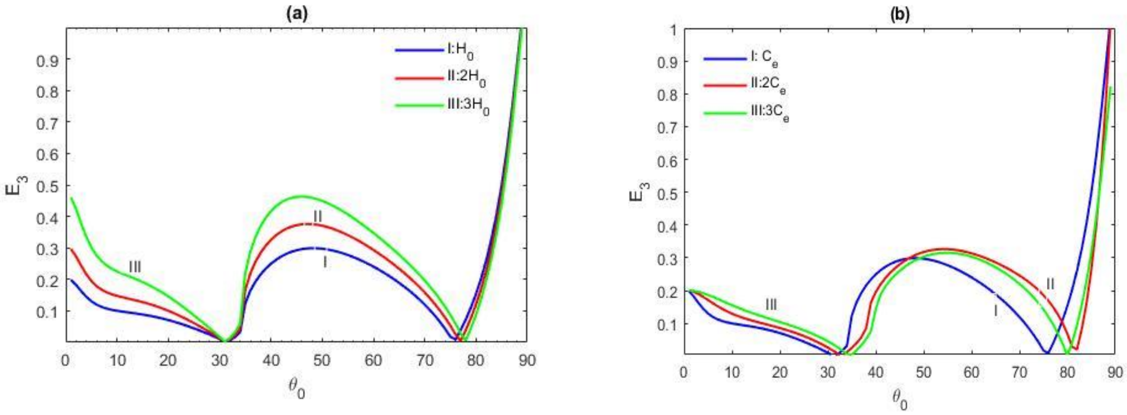

Figure 9. Variation of E_3 for different values of (a) H_0 (*C/ms*) and (b) C_e (m^2/Ks^2)

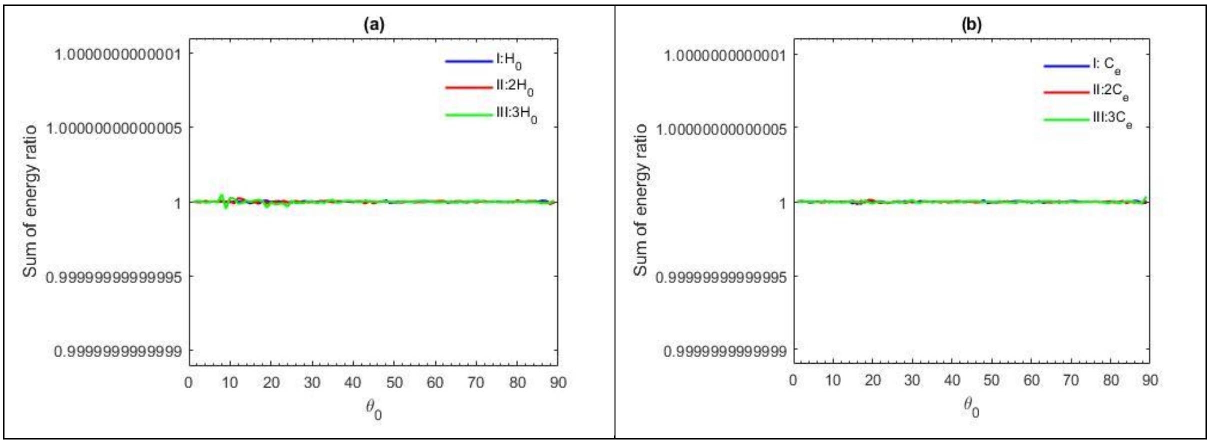

Figure 10. Sum of energy ratios for different values of (a) H_0 (*C/ms*) and (b) C_e (m^2/Ks^2)

8. Conclusions

We have investigated the problem of reflection of waves from a half-space of generalized magneto thermoelastic materials. The expressions of the amplitude and energy ratios of reflected waves are found by using the appropriate boundary conditions. These ratios and phase velocities of the longitudinal, transverse and thermal waves are calculated numerically and the results are plotted graphically. We have observed that the present study confirms the significant effects of the magnetic field and thermal parameter on the wave propagation. The following points are the concluding remarks:

(i) The amplitude and energy ratios are functions of angle of incidence, magnetic, thermal and elastic parameters.

(ii) The phase velocity of longitudinal wave increases with the increase of H_0 and C_e.

(iii) The phase velocity corresponding to thermal wave increases with the increase of ω.

(iv) The variations of Z_i and Z_2 are similar in nature.

(v) The effect of H_0 on Z_i is very small.

(vi) The effects of H_0 and C_e on E_i are minimum near the grazing angle of incidence.

(vii) The sum of energy ratios of reflected waves are close to one.

The new direction of the present work: The problem of reflection and refraction of elastic waves in the half-spaces of generalized magneto thermoelastic materials can be investigated. In this problem, the amplitude and energy ratios of the reflected and refracted waves could be one of the interesting area to discuss the various effects of thermal and magnetic fields of the wave propagation.

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