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Higher Order Finite Element Methods for Some One-dimensional Boundary Value Problems

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Abstract: In this paper, third-order compact and fourth-order finite element methods (FEMs) based on simple modifications of traditional FEMs are proposed for solving one-dimensional Sturm-Liouville boundary value problems (BVPs). The key idea is based on interpolation error estimates. A simple posterior error analysis of the original piecewise linear finite element space leads to a third-order accurate solution in the L^2 norm, second-order in the H^1, and the energy norm. The novel idea is also applied to obtain a fourth-order FEM based on the quadratic finite element space. The basis functions in the new fourth-order FEM are more compact compared with that of the classic cubic basis functions. Numerical examples presented in this paper have confirmed the convergence order and analysis. A generalization to a class of nonlinear two-point BVPs is also discussed and tested.

Keywords: high-order compact finite element method, posterior analysis, modified basis functions, Sturm-Liouville boundary value problem

MS Subject Classification 2000: 65L20, 65L60

1. Introduction

In this paper, we discuss a new third-order compact and a fourth-order finite element method (FEM) for the following Sturm-Liouville (S-L) boundary value problem (BVP),

$$-\left(\beta(x)u'(x)\right)' + q(x)u(x) = f(x), \qquad x_l < x < x_r, \tag{1}$$

with linear boundary conditions at $x = x_l$ and $x = x_r$, such as Dirichlet, Neumann, and Robin conditions. To guarantee the well-posedness of the problem, we assume that

$$\beta(x) \in L^\infty(x_l, x_r), \quad \beta(x) \ge \beta_0 > 0, \quad q(x) \in L^\infty(x_l, x_r), \quad q(x) \ge 0, \quad f(x) \in L^2(x_l, x_r), \tag{2}$$

where β_0 is a positive constant. If the solution is specified at $x = x_l$ or $x = x_r$, or $q(x) > 0$, then the problem has a unique

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solution $u(x) \in H^2(x_l, x_r)$ from the Lax-Milgram lemma, see for example, [1, 2].

It is often difficult to obtain the analytic solution to the BVP. In many situations, it is easier to get a numerical solution using modern computers. Finite difference and finite element methods have been developed to solve the BVP. In a finite difference or a finite element method, often a mesh is constructed so that the infinite-dimensional problem becomes a finite one, and an approximation to the true solution can be obtained. The most popular FEM is the one using piecewise linear basis functions (P_1) over a mesh. Such a P_1 FEM is second-order accurate in the L^2 norm, and first-order accurate in the H^1 norm and the energy norm, see for example, [3, 4]. High-order FEMs such as P_2 or P_3 have been developed with increased complexity and denser coefficient matrices. In the literature, there is a rich collection of different numerical methods such as spline collocation methods, discontinuous Galerkin (DG) methods, techniques for singular two-point BVPs, and nonlinear two-point BVPs, see [5-17] for an incomplete list.

Naturally, we hope to have better than second-order methods, like third-order ($O(h^3)$) or higher without too much additional cost and effort, here h is the mesh size. There are many high-order compact finite difference methods in the literature, but almost none of the high-order compact FEMs, except the techniques developed in [18, 19].

In this paper, we first propose a third-order accurate *compact* FEM based on the P_1 finite element space in Section 2. The new idea is based on the interpolation theory to offset the leading error terms with added local compact supports to the basis functions so that the interpolation functions are third-order accurate. In this way, we obtain a new third-order accurate finite element solution. A generalization to a class of *nonlinear* two-point value problems is discussed. Then, we apply the new idea to the P_2 (piecewise quadratic functions in H^1) FEM to obtain a new fourth-order accurate method in Section 3. While the new fourth-order FEM is not compact, the bandwidth of the coefficient matrix is reduced by two compared with the classic fourth-order FEM. We present numerical examples including a nonlinear one in Section 4. We conclude in the last section.

2. A third-order compact FEM using the P_1 finite element

In the classical FEM using a piecewise linear function to approximate the solution to (1), a mesh

$$x_l = x_0 < x_1 < x_2 < \cdots < x_i < \cdots < x_{n-1} < x_n = x_r, \tag{3}$$

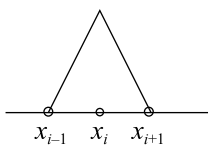
will be used, where n is an integer and the mesh parameter is defined as $h = \max_{0 \le i \le n-1} |x_{i+1} - x_i|$. Note that $h \sim O(1/n)$. In the classic FEM, the weak solution form for (1) is defined by

$$\begin{aligned} a(u,v) &= L(v), \quad \forall v(x) \in H_0^1(x_l, x_r), \\ a(u,v) &= \int_{x_l}^{x_r} \big(\beta(x)u'(x)v'(x) + q(x)u(x)v(x)\big)\,dx + R_1(u,v), \\ L(v) &= \int_{x_l}^{x_r} fv\,dx + R_2(v), \end{aligned} \tag{4}$$

where R_1 and R_2 are contributions from two boundary conditions, $a(u,v)$ is a bilinear form, and $L(v)$ is a linear form. For homogenous Dirichlet boundary conditions, that is, $u(x_l) = u(x_r) = 0$, we have $R_1 = R_2 = 0$.

Once we have a mesh, we construct a set of basis functions based on the mesh, namely the hat functions,

$$\phi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h} & \text{if } x_{i-1} \le x < x_i, \\ \dfrac{x_{i+1} - x}{h} & \text{if } x_i \le x < x_{i+1}, \\ 0 & \text{otherwise,} \end{cases}$$



$i = 1, 2, \cdots, n-1$, see the right diagram for a hat function. We denote all the piecewise linear functions over the mesh as a finite element space V_h,

$$V_h = \{v_h(x), \quad v_h(x) = \sum_{j=1}^{M} c_j \phi_j(x), \quad c_j \in \mathbf{R}, \quad j = 1, 2, \cdots M\}, \tag{5}$$

where M is $n - 1$, or n, or $n + 1$, depending on the boundary conditions. The finite element solution then is a special combination of the basis functions

$$u_h(x) = \sum_{j=1}^{M} c_j \phi_j(x), \tag{6}$$

that minimizes the error in the energy norm in the space V_h. The coefficients c_j's are determined from the linear system of equations $A\mathbf{U} = \mathbf{F}$ with

$$A = \{a_{ij}\}, \quad a_{ij} = a(\phi_i(x), \phi_j(x)), \quad \mathbf{F} = \{F_i\}, \quad F_i = L(\phi_i(x)), \tag{7}$$

and $\mathbf{U} = [c_1, c_2, \cdots, c_M]^T$.

It is well known that the finite element solution converges to the true solution as $h \to 0$ according to the following error estimates

$$\| u - u_h \|_{L^2} \leq Ch^2, \quad \| u - u_h \|_{H^1} \leq Ch, \quad \| u - u_h \|_e \leq Ch, \tag{8}$$

where C is a generic error constant. That is, the *FEM* is second-order accurate in the L^2 norm and first-order accurate in the H^1 norm.

2.1 *A new third-order compact FEM based on a posterior error analysis*

We start with the special case in which $\beta(x) = \beta$, a constant, and $q(x) = 0$. In this case, the ordinary differential equation (ODE) is simply $u''(x) = -f(x)/\beta$. It is well-known that the finite element solution is exact at nodal points if the integrals of the load vector can be computed exactly. Thus, the interpolation function is the same as the finite element solution,

$$u_I^h(x) = \sum_{j=1}^{M} u(x_j)\phi_j(x) = \sum_{j=1}^{M} c_j \phi_j(x) = u_h(x), \tag{9}$$

where $\{c_j\}$'s are the coefficients obtained from the FEM using P_1 elements. From the classical interpolation theory, see for example, [1, 3], we know that $u_I^h(x_i) = u(x_i)$, and on an element (x_k, x_{k+1}), we have the following,

$$\begin{aligned} u(x) &= u_I^h(x) + \frac{1}{2}(x - x_k)(x - x_{k+1})u''(x) + O(h^3) \\ &= u_h(x) - \frac{1}{2}(x - x_k)(x - x_{k+1})\frac{f(x)}{\beta} + O(h^3). \end{aligned} \tag{10}$$

Thus, we obtain a third-order method in the L^2 norm using a posterior error estimate with a simple correction term that can be easily computed, as stated in the following theorem.

Theorem 1. Let $u(x) \in H^2(x_l, x_r)$ be the solution of $u''(x) = -f(x)$, $f(x) \in H^1(x_l, x_r)$, and $u_h(x)$ be the finite element solution obtained using the P_1 finite element space, then

$$u_h^{new}(x) = u_h(x) - \frac{1}{2\beta}(x - x_k)(x - x_{k+1})f(x), \quad x_k < x < x_{k+1}, \tag{11}$$

is a third-order approximation to the true solution $u(x)$; and the following error estimates hold,

$$\| u-u_h^{new} \|_{L^2} \le Ch^3, \quad \| u-u_h^{new} \|_{H^1} \le Ch^2, \quad \| u-u_h^{new} \|_a \le Ch^2. \tag{12}$$

Note also that $u_h^{new}(x_k)=u(x_k)$ for all k's.

Proof. For this special case, we know that $u(x_i) = u_h(x_i)$. From the derivation process, we know that

$$| u_h(x)-u(x) | \le Ch^3, \tag{13}$$

where C depends on $\max |u'''(x)| = \max |f'(x)|$. Thus, we have $|u_h(x)-u(x)|_{L^2} \le Ch^3$. It follows that $|u_h(x)-u(x)|_{H^1}$ $\le Ch^2$ and $|u_h(x)-u(x)|_a \le Ch^2$.

2.2 *A third-order compact FEM for a class of nonlinear BVPs*

In this subsection, we apply the proposed third-order compact FEM to the following nonlinear two-point BVP,

$$-\left(\beta u'(x)\right)' + q(u)u(x) = f(x), \qquad x_l < x < x_r, \tag{14}$$

with linear boundary conditions at $x = x_l$ and $x = x_r$. We assume that β is a constant. With a simple transformation, the nonlinear equation (14) can be rewritten as a semi-linear form,

$$-\left(\beta u'(x)\right)' = F(u,x), \tag{15}$$

where $F(u,x) = f(x)-q(u)u(x)$. From the semi-linear equation (15), we obtain that

$$u''(x) = -\frac{F(u(x),x)}{\beta} = -\frac{1}{\beta}\left[f(x)-q(u)u(x)\right]. \tag{16}$$

For nonlinear differential equations, an approximate solution can be obtained from

$$\tilde{u}_h''(x) = -\frac{F(\tilde{u}_h(x),x)}{\beta} = -\frac{1}{\beta}\left[f(x)-q(\tilde{u}_h(x))\tilde{u}_h(x)\right]. \tag{17}$$

To solve the nonlinear equation, we use a substitution method, see for example, [20], we start with an initial guess of the solution $\tilde{u}_h^0(x)$, then update the approximation using

$$((\tilde{u}_h(x))^{l+1})'' = -\frac{F(\tilde{u}_h^l(x),x)}{\beta} = -\frac{1}{\beta}\left[f(x)-q(\tilde{u}_h^l(x))\tilde{u}_h^l(x)\right], \quad l=0,1,\cdots, \tag{18}$$

for which we use the new third-order FEM until the sequence $\left\{\tilde{u}_h^l(x)\right\}$ convergence. In other words, we approximate the nonlinear term using the previous guess so that the problem is linearized. A numerical example is shown in Section 4.

2.3 *A new third-order compact FEM for the S-L BVPs with variable coefficients*

The third-order method based on a posterior error analysis discussed in the previous sub-section is valid only if the coefficient β is a constant and the integrals of the load vector can be computed by a numerical integration formula with at least third-order accuracy, which may not be true in general if $\beta(x)$ is a variable function. Note that, for the problems with variable coefficients $\beta(x)$ and $q(x)$, the P_2 FEM (piecewise quadratic) with added auxiliary points is not *compact*. In this section, we propose a new third-order compact FEM based on the interpolation theory to construct bubble functions that have compact supports.

Note that the differential equation with a variable coefficient is equivalent to

$$u''(x) = -\frac{\beta'(x)}{\beta(x)}u'(x) + \frac{q(x)}{\beta(x)}u(x) - \frac{f(x)}{\beta(x)}. \tag{19}$$

We use a new estimate to construct the basis functions and the FEM solution. On an element (x_k, x_{k+1}), we know that

$$\begin{aligned} u(x) &= u_I^h(x) + \frac{1}{2}u''(x)(x-x_k)(x-x_{k+1}) + O(h^3) \\ &= u_I^h(x) - \frac{1}{2}\frac{\beta'(x)}{\beta(x)}(x-x_k)(x-x_{k+1})\frac{\mathrm{d}}{\mathrm{d}x}u_I^h(x) + \frac{1}{2}\frac{q(x)}{\beta(x)}(x-x_k)(x-x_{k+1})u_I^h(x) - \frac{1}{2}\frac{f(x)}{\beta(x)}(x-x_k)(x-x_{k+1}) + O(h^3), \end{aligned}$$

where $u_I^h(x) = \sum_{j=1}^{M} u(x_j)\phi_j(x)$ is a piecewise linear interpolation function based on the linear finite element space. Note that the added terms have compact supports.

The new modified finite element solution in (x_k, x_{k+1}) is defined as

$$\begin{aligned} u_h^{new}(x) &= c_k\phi_k(x) + c_{k+1}\phi_{k+1}(x) - \frac{1}{2}\frac{f(x)}{\beta(x)}(x-x_k)(x-x_{k+1}) - \frac{1}{2}\frac{\beta'(x)}{\beta(x)}(x-x_k)(x-x_{k+1})(c_k\phi_k'(x) + c_{k+1}\phi_{k+1}'(x)) \\ &+ \frac{1}{2}\frac{q(x)}{\beta(x)}(x-x_k)(x-x_{k+1})(c_k\phi_k(x) + c_{k+1}\phi_{k+1}(x)). \end{aligned} \tag{20}$$

To determine the coefficients $\{c_k\}$'s, we apply the original weak form

$$a(u_h^{new}, \phi_i) = L(\phi_i), \quad i = 1, 2, \cdots, M. \tag{21}$$

In this way, we get a linear system of equations for the coefficients.

For the new finite elements, which is not a traditional FEM, but rather a Petrov-Galerkin FEM, the convergence proof is often a challenge. We have the following conjecture that has been approved by numerical experiments.

Conjecture 1. Assume that $u(x) \in H^3(x_l, x_r)$ be the solution to the BVP with $f(x) \in H^1(x_l, x_r)$, $\beta(x) \in H^2(x_l, x_r)$, $q(x) \in H^1(x_l, x_r)$. Let $u_h^{new}(x)$ be the finite element solution obtained using the new FEM, then $u_h^{new}(x)$ is a third-order approximation to $u(x)$ with the following error estimate

$$\| u - u_h^{new} \|_{L^2} \le Ch^3, \quad \| u - u_h^{new} \|_{H^1} \le Ch^2, \quad \| u - u_h^{new} \|_{e} \le Ch^2. \tag{22}$$

3. A fourth-order FEM for S-L BVPs

In Section 2.3, a third-order compact FEM based on the linear interpolation function and bubble functions with compact supports has been proposed. In this section, we expand the idea to obtain a fourth-order FEM using quadratic interpolation functions and corresponding bubble functions. The new fourth-order FEM is not compact but has less support than that of the traditional fourth-order FEM.

For a reference quadratic element (x_{k-1}, x_{k+1}), the quadratic interpolation function of $u(x)$ can be written as

$$u_I^h(x) = u(x_{k-1})\varphi_{k-1}(x) + u(x_k)\varphi_k(x) + u(x_{k+1})\varphi_{k+1}(x), \tag{23}$$

in which three basis functions are defined below

$$\begin{cases} \varphi_{k-1}(x) = \dfrac{1}{2h^2}(x-x_k)(x-x_{k+1}), \\ \varphi_k(x) = -\dfrac{1}{h^2}(x-x_{k-1})(x-x_{k+1}), \\ \varphi_{k+1}(x) = \dfrac{1}{2h^2}(x-x_{k-1})(x-x_k). \end{cases} \tag{24}$$

According to the finite element theory, see for example, [1, 3], we have the following error estimate of the quadratic interpolation on (x_{k-1}, x_{k+1}),

$$u(x) = u_I^h(x) + \frac{1}{6}u'''(x)(x-x_{k-1})(x-x_k)(x-x_{k+1}) + O(h^4). \tag{25}$$

From the differential equation (1), we also know that

$$u'''(x) = -\frac{2\beta'(x)}{\beta(x)}u''(x) + \frac{q(x)-\beta''(x)}{\beta(x)}u'(x) + \frac{q'(x)}{\beta(x)}u(x) - \frac{f'(x)}{\beta(x)}. \tag{26}$$

On an interval (x_{k-1}, x_{k+1}), using the quadratic interpolation theory and the third derivative of $u(x)$, we further obtain

$$u(x) = u_I^h(x) + \frac{1}{6}\Psi_I^h(x)(x-x_{k-1})(x-x_k)(x-x_{k+1}) + O(h^4), \tag{27}$$

where $\Psi_I^h(x)$ is an approximation of $u'''(x)$ with first-order accuracy, as shown below

$$\Psi_I^h(x) = -\frac{2\beta'(x)}{\beta(x)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}u_I^h(x) + \frac{q(x)-\beta''(x)}{\beta(x)}\frac{\mathrm{d}}{\mathrm{d}x}u_I^h(x) + \frac{q'(x)}{\beta(x)}u_I^h(x) - \frac{f'(x)}{\beta(x)}. \tag{28}$$

Based on the derivation above, we can construct a new modified finite element solution on an element (x_{k-1}, x_{k+1}) with fourth-order accuracy

$$u_h^{new}(x) = c_{k-1}\varphi_{k-1}(x) + c_k\varphi_k(x) + c_{k+1}\varphi_{k+1}(x) + \frac{1}{6}\Psi_h(x)(x-x_{k-1})(x-x_k)(x-x_{k+1}), \tag{29}$$

where the bubble function $\Psi_h(x)$ is defined as below

$$\begin{aligned} \Psi_h(x) = &-\frac{2\beta'(x)}{\beta(x)}\left(c_{k-1}\varphi''_{k-1}(x) + c_k\varphi''_k(x) + c_{k+1}\varphi''_{k+1}(x)\right) + \frac{q(x)-\beta''(x)}{\beta(x)}\left(c_{k-1}\varphi'_{k-1}(x) + c_k\varphi'_k(x) + c_{k+1}\varphi'_{k+1}(x)\right) \\ &+ \frac{q'(x)}{\beta(x)}\left(c_{k-1}\varphi_{k-1}(x) + c_k\varphi_k(x) + c_{k+1}\varphi_{k+1}(x)\right) - \frac{f'(x)}{\beta(x)}. \end{aligned} \tag{30}$$

To determine the coefficients $\{c_k\}$'s, we apply the original weak form

$$a(u_h^{new}, \phi_i) = L(\phi_i), \quad i = 1, 2, \cdots, M. \tag{31}$$

In this way, we get a linear system of equations for the coefficients.

For the new fourth-order FEM, we have the following conjecture that has been approved by numerical experiments.

Conjecture 2. Assume that $u(x) \in H^4(x_l, x_r)$ be the solution to the BVP with $f(x) \in H^2(x_l, x_r)$, $\beta(x) \in H^3(x_l, x_r)$, $q(x) \in H^2(x_l, x_r)$. Let $u_h^{new}(x)$ be the finite element solution obtained using the new FEM, then $u_h^{new}(x)$ is a fourth-order approximation to $u(x)$ with the following error estimates

$$\| u - u_h^{new} \|_{L^2} \leq Ch^4, \quad \| u - u_h^{new} \|_{H^1} \leq Ch^3, \quad \| u - u_h^{new} \|_e \leq Ch^3. \tag{32}$$

4. Numerical experiments

In this section, we show four numerical examples for the one-dimensional S-L BVP. We present the L^2 and H^1 errors. The order of convergence is estimated using the following formula

$$Order = \left| \frac{\log(\| E_{N_1} \|_{L^2} / \| E_{N_2} \|_{L^2})}{\log(N_2 / N_1)} \right|$$

with two different N's.

Example 4.1 In this example, we show a one-dimensional Poisson equation with the Dirichlet boundary condition at the two ends. The analytic solution is the following,

$$u(x) = \sin(kx). \tag{33}$$

The force term is

$$f(x) = k^2 \sin(kx).$$

Table 1. A grid refinement analysis of the two proposed compact FEMs for Example 4.1 with $k = 5\pi$

N	Third-order FEM				Fourth-order FEM			
	$\| E \|_{L^2}$	Order	$\| E \|_{H^1}$	Order	$\| E \|_{L^2}$	Order	$\| E \|_{H^1}$	Order
8	9.4674E-01		1.5922E+01		4.2241E+00		5.3578E+01	
16	7.6248E-02	3.63	2.8671E+00	2.47	4.6240E-01	3.19	6.5758E+00	3.03
32	1.1364E-02	2.75	8.0230E-01	1.84	3.1773E-02	3.86	8.6654E-01	2.92
64	1.4521E-03	2.97	2.0772E-01	1.95	2.0416E-03	3.96	1.0966E-01	2.98
128	1.8229E-04	2.99	5.2407E-02	1.99	1.2850E-04	3.99	1.3749E-02	3.00
256	2.2810E-05	3.00	1.3132E-02	2.00	8.0467E-06	4.00	1.7199E-03	3.00
512	2.8532E-06	3.00	3.2849E-03	2.00	5.0302E-07	3.98	2.1503E-04	3.00
1024	3.7524E-07	2.93	8.2137E-04	2.00	3.1452E-08	4.00	2.6880E-05	3.00

Table 2. A grid refinement analysis of the two proposed compact FEMs for Example 4.1 with $k = 50\pi$

N	Third-order FEM				Fourth-order FEM			
	$\|\| E \|\|_{L^2}$	Order	$\|\| E \|\|_{H^1}$	Order	$\|\| E \|\|_{L^2}$	Order	$\|\| E \|\|_{H^1}$	Order
64	7.7242E+00		3.8035E+02		1.0786E+01		1.4341E+03	
128	1.2320E-01	5.97	4.4494E+01	3.10	1.1264E+00	3.26	1.2701E+02	3.50
256	2.1751E-02	2.50	1.2243E+01	1.86	7.6181E-02	3.89	1.6789E+01	2.92
512	2.8265E-03	2.94	3.2238E+00	1.93	4.9613E-03	3.94	2.1375E+00	2.97
1024	3.5579E-04	2.99	8.1745E-01	1.98	3.1337E-04	3.98	2.6839E-01	2.99
2048	4.4562E-05	3.00	2.0510E-01	1.99	1.9730E-05	3.99	3.3587E-02	3.00
4096	5.8690E-06	2.92	5.1322E-02	2.00	1.2280E-06	4.00	4.2002E-03	3.00

In Tables 1 and 2, we show grid refinement results with $k = 5\pi$ and $k = 50\pi$, respectively. In the two tables, the first column is the mesh size, and the rest are the errors of the finite element solutions in the L^2 and the energy norm obtained from the third and fourth-order new FEMs. In all of the cases, we see that the computed convergence order is consistent with our analysis.

Example 4.2 In this example, we show an example for one-dimensional elliptic problem, with the following analytic solution

$$u(x) = \sin(k_1 x)\cos(k_2 x), \tag{34}$$

with Dirichlet boundary conditions at the two ends. The coefficients $\beta(x)$ and $q(x)$ are chosen as

$$\beta(x) = e^x, \qquad q(x) = x^2, \tag{35}$$

and the force term is

$$\begin{aligned} f(x) = {} & (k_1^2 + k_2^2)e^x \sin(k_1 x)\cos(k_2 x) + 2k_1 k_2 e^x \cos(k_1 x)\sin(k_2 x) \\ & - e^x \left(k_1 \cos(k_1 x)\cos(k_2 x) - k_2 \sin(k_1 x)\sin(k_2 x)\right) + x^2 \sin(k_1 x)\cos(k_2 x). \end{aligned}$$

In Tables 3 to 6, we show grid refinement results for the problem with different k_1 and k_2. In all of the cases, we see a clean third-order convergence in the solution obtained by the third-order FEM and fourth-order convergence for the fourth-order FEM.

Table 3. A grid refinement analysis of the two proposed compact FEMs for Example 4.2 with $k_1 = 5\pi$ and $k_2 = 0$

N	Third-order FEM				Fourth-order FEM			
	$\|\| E \|\|_{L^2}$	Order	$\|\| E \|\|_{H^1}$	Order	$\|\| E \|\|_{L^2}$	Order	$\|\| E \|\|_{H^1}$	Order
8	1.5393E+00		1.7273E+01		4.9180E+00		8.1012E+01	
16	9.5685E-02	4.01	3.0023E+00	2.52	4.4777E-01	3.46	7.0704E+00	3.52
32	1.2644E-02	2.92	8.0654E-01	1.90	1.9163E-02	4.55	1.1222E+00	2.66
64	1.5437E-03	3.03	2.0909E-01	1.95	1.0508E-03	4.19	1.5039E-01	2.90
128	1.8828E-04	3.04	5.2686E-02	1.99	6.3291E-05	4.05	1.9090E-02	2.98
256	2.3235E-05	3.02	1.3196E-02	2.00	3.9192E-06	4.01	2.3961E-03	2.99
512	2.8846E-06	3.01	3.3004E-03	2.00	2.4916E-07	3.98	2.9983E-04	3.00
1024	3.6254E-07	2.99	8.2518E-04	2.00	1.5265E-08	4.03	3.7489E-05	3.00

Table 4. A grid refinement analysis of the two proposed compact FEMs for Example 4.2 with $k_1 = 50\pi$ and $k_2 = 0$

N	Third-order FEM				Fourth-order FEM			
	$\|\| E \|\|_{L^2}$	Order	$\|\| E \|\|_{H^1}$	Order	$\|\| E \|\|_{L^2}$	Order	$\|\| E \|\|_{H^1}$	Order
64	7.9628E+00		3.8417E+02		9.9800E+00		1.0512E+03	
128	1.2781E-01	5.96	4.4780E+01	3.10	7.2003E-01	3.79	1.6024E+02	2.71
256	2.2066E-02	2.53	1.2249E+01	1.87	4.1550E-02	4.12	2.2609E+01	2.83
512	2.8471E-03	2.95	3.2242E+00	1.93	2.4616E-03	4.08	2.9234E+00	2.95
1024	3.5711E-04	3.00	8.1755E-01	1.98	1.5132E-04	4.02	3.6857E-01	2.99
2048	4.4621E-05	3.00	2.0512E-01	1.99	9.4154E-06	4.00	4.6174E-02	3.00
4096	5.6445E-06	2.98	5.1325E-02	2.00	5.8785E-07	4.00	5.7748E-03	3.00

Table 5. A grid refinement analysis of the two proposed compact FEMs for Example 4.2 with $k_1 = 5\pi$ and $k_2 = 5\pi$

N	Third-order FEM				Fourth-order FEM			
	$\|\| E \|\|_{L^2}$	Order	$\|\| E \|\|_{H^1}$	Order	$\|\| E \|\|_{L^2}$	Order	$\|\| E \|\|_{H^1}$	Order
8	9.5172E+00		1.2581E+02		3.6864E+01		4.7307E+02	
16	6.2881E-01	3.92	1.6649E+01	2.92	1.7357E+00	4.41	5.6043E+01	3.08
32	4.3173E-02	3.86	2.9322E+00	2.51	1.4622E-01	3.57	8.6727E+00	2.69
64	6.0059E-03	2.85	8.0410E-01	1.87	8.3826E-03	4.12	1.1780E+00	2.88
128	7.4785E-04	3.01	2.0788E-01	1.95	5.0187E-04	4.06	1.5042E-01	2.97
256	9.2580E-05	3.01	5.2486E-02	1.99	3.1037E-05	4.02	1.8956E-02	2.99
512	1.1502E-05	3.01	1.3149E-02	2.00	1.9385E-06	4.00	2.3727E-03	3.00
1024	1.4323E-06	3.01	3.2889E-03	2.00	1.2080E-07	4.00	2.9670E-04	3.00

Table 6. A grid refinement analysis of the two proposed compact FEMs for Example 4.2 with $k_1 = 50\pi$ and $k_2 = 50\pi$

N	Third-order FEM				Fourth-order FEM			
	$\|\| E \|\|_{L^2}$	Order	$\|\| E \|\|_{H^1}$	Order	$\|\| E \|\|_{L^2}$	Order	$\|\| E \|\|_{H^1}$	Order
128	3.9235E+00		3.8229E+02		4.9680E+00		1.0506E+03	
256	6.2757E-02	5.97	4.4637E+01	3.10	3.5870E-01	3.79	1.6018E+02	2.71
512	1.0954E-02	2.52	1.2245E+01	1.87	2.0751E-02	4.11	2.2605E+01	2.83
1024	1.4185E-03	2.95	3.2238E+00	1.93	1.2305E-03	4.08	2.9230E+00	2.95
2048	1.7822E-04	2.99	8.1748E-01	1.98	7.5722E-05	4.02	3.6855E-01	2.99
4096	2.2338E-05	3.00	2.0511E-01	1.99	4.7072E-06	4.00	4.6168E-02	3.00
8192	3.0299E-06	2.90	5.1324E-02	2.06	2.9459E-07	4.00	5.7741E-03	3.00

Example 4.3 In this example, we use the two proposed FEMs to solve the following one-dimensional generalized Helmholtz equation

$$-u''(x) + k^2 u(x) = f(x),$$

with Dirichlet boundary conditions at the two ends using the exact solution. The analytic solution is defined below

$$u(x) = \sin(kx). \tag{36}$$

The force term is

$$f(x) = 2k^2 \sin(kx).$$

In Tables 7 and 8, we show grid refinement results for Example 4.3 with different k's. In all of the cases, the numerical results are consistent with our analysis for the proposed third-order and fourth-order FEMs, respectively.

Table 7. A grid refinement analysis of the two proposed compact FEMs for Example 4.3 with $k = 5\pi$

N	Third-order FEM				Fourth-order FEM			
	$\|\| E \|\|_{L^2}$	Order	$\|\| E \|\|_{H^1}$	Order	$\|\| E \|\|_{L^2}$	Order	$\|\| E \|\|_{H^1}$	Order
16	1.5587E-01		6.9095E+00		3.0785E-01		9.8124E+00	
32	1.1729E-02	3.73	1.0884E+00	2.67	1.6384E-02	4.23	1.1531E+00	3.09
64	1.4487E-03	3.02	2.2595E-01	2.27	9.9483E-04	4.04	1.4924E-01	2.95
128	2.2052E-04	2.72	6.0639E-02	1.90	6.1743E-05	4.01	1.8854E-02	2.98
256	2.2801E-05	3.27	1.3203E-02	2.20	3.8522E-06	4.00	2.3632E-03	3.00
512	2.8515E-06	3.00	3.2894E-03	2.01	2.4066E-07	4.00	2.9561E-04	3.00
1024	3.5649E-07	3.00	8.2162E-04	2.00	1.5039E-08	4.00	3.6958E-05	3.00

Table 8. A grid refinement analysis of the two proposed compact FEMs for Example 4.3 with $k = 50\pi$

N	Third-order FEM				Fourth-order FEM			
	$\Vert E \Vert_{L^2}$	Order	$\Vert E \Vert_{H^1}$	Order	$\Vert E \Vert_{L^2}$	Order	$\Vert E \Vert_{H^1}$	Order
128	5.5965E-01		1.5543E+02		9.0066E-01		2.2282E+02	
256	2.4290E-02	4.52	1.9117E+01	3.02	4.0951E-02	4.46	2.2124E+01	3.33
512	2.8241E-03	3.10	3.6683E+00	2.38	2.4465E-03	4.07	2.8934E+00	2.93
1024	3.5515E-04	2.99	8.4543E-01	2.12	1.5107E-04	4.02	3.6750E-01	2.98
2048	4.4517E-05	3.00	2.0685E-01	2.03	9.4102E-06	4.00	4.6133E-02	2.99
4096	5.5689E-06	3.00	5.1431E-02	2.01	5.8763E-07	4.00	5.7729E-03	3.00
8192	6.9625E-07	3.00	1.2840E-02	2.00	3.6719E-08	4.00	7.2181E-04	3.00

Example 4.4 In this example, we show three examples for the nonlinear problem (14) with a Dirichlet boundary condition. For the first example, we set $q(u)u = u^2(x)$, and the analytic solution is defined as below

$$u(x) = e^{2x}, \quad 0 < x < 1. \tag{37}$$

The force term is

$$f(x) = -4e^{2x} + e^{4x}.$$

For the second example, we set $q(u)u = u^3(x)$, and the analytic solution is defined as below

$$u(x) = \sin(5\pi x), \quad -1 < x < 1, \tag{38}$$

with the following force term

$$f(x) = 25\pi^2 \sin(5\pi x) + \sin^3(5\pi x).$$

For the third example, we set $q(u)u = ue^u$. The analytic solution is

$$u(x) = \cos(4\pi x), \quad -1 < x < 1, \tag{39}$$

with the following force term

$$f(x) = 16\pi^2 \cos(4\pi x) + \cos(4\pi x)e^{\cos(4\pi x)}.$$

In Table 9, we show grid refinement results of the third-order compact FEM for three nonlinear problems with diffusion coefficient $\beta = 1$. The second column and the third column are the error and the convergence order for the first example, the fourth column, and the fifth column are the error and the convergence order for the second example, and the sixth column and the seventh column are the error and the convergence order for the third example. From the grid refinement results of three nonlinear problems, we can see that the computed convergence order has three-order convergence.

Table 9. A grid refinement analysis of the third-order compact FEM for the nonlinear problem (14)

N	$\| E \|_{L^2}$	Order	$\| E \|_{L^2}$	Order	$\| E \|_{L^2}$	Order
8	2.3951E-03		9.6198E-01		7.7473E-01	
16	2.4333E-04	3.30	8.4723E-02	3.51	9.6420E-02	3.01
32	2.6477E-05	3.20	1.1357E-02	2.90	9.8243E-03	3.29
64	3.0507E-06	3.12	1.4514E-03	2.97	1.0062E-03	3.29
128	3.4406E-07	3.15	1.8074E-04	3.01	1.1633E-04	3.11
256	4.4552E-08	2.95	2.2808E-05	2.99	1.2724E-05	3.19
512	5.5022E-09	3.02	2.8523E-06	3.00	1.5259E-06	3.06

5. Conclusions

In this paper, new high-order FEMs are proposed for one-dimensional S-L BVPs. First, a new third-order compact FEM has been developed. For constant coefficient, the method is a simple posterior error estimate technique and the third-order convergence has been proved. For variable coefficients, the new third-order compact FEM has been proposed and confirmed numerically. The degree of freedom and the structure of the resulting linear system of equations of the new third-order compact FEM are the same as the traditional FEM using piecewise linear functions over the mesh. The novel idea is also generalized to obtain a fourth-order FEM based on the quadratic finite element space.

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Conflict of interest

The authors have no competing interests to declare that are relevant to the content of this article.

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