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Commentary

Teacher: Where are You? About a Scenario for Permanent Teacher Training in Which the Teaching of Geometry is Invited to Think Professionally

José Villella[*], **Rosa Ferragina**, **Leonardo Lupinacci**, **Victoria Güerci**, **Gema Fioriti, Alejandra Almirón**

Center for Studies in Specific Didactics, Human Sciences Research Laboratory, National University of San Martín, Campus Miguelete, San Martín, Buenos Aires, Argentina
E-mail: joavillella@gmail.com

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Abstract: In this article we share the story of a proposal for permanent training for mathematics teachers who are in charge of the training of students between 12 and 18 years old. We present a task and the fundamentals that lead us to develop it in this training scenario, for the collective construction of specialized knowledge of those who teach topics related to metric geometry and its relationship with optical illusions. We comment on the planning of the scenario and the results achieved from the perspective of the Mathematics Teacher's Specialized Knowledge Model (MTSK Model), which we relate to studies on the use of technologies in mathematics classrooms. We conclude with an analysis of the strengths and weaknesses, opportunities and threats that may arise in scenarios such as the one described for the permanent training of teachers considered teaching professionals.

Keywords: continuous teacher training, specialized knowledge of the mathematics teacher, teaching geometry, dynamic geometry

1. Introduction

Within the framework of the Mathematics Teacher's Specialized Knowledge Model (MTSK) (Carrillo-Yañez et al., 2018), we propose the task *Teacher: where are you?* (Figure 1) for the continuous training of teachers. The proposal, designed to work on the dialogue between metric geometry and optical illusions in the classroom, is oriented towards the mastery of geometric knowledge and the didactic mastery of this content. If we consider content knowledge, the reflection focuses on determining invariant characteristics of figures and using the notion of geometric place to make and justify constructions, as well as exploring and formulating conjectures about figures built with technological resources, and their validation through the properties of geometric objects. From the didactic perspective, the formative interest lies in the discussion around the didactic intentionality of the teaching proposals, so it is proposed to debate the answers that we suppose students between 12 and 18 years old will give and why it is a geometric challenge in the classroom.

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1-What are the geometric-not visual-reasons that allow the protagonist of the photo to appear as we see her and not with her full body?

2-If we copy the photo as an image and use it in GeoGebra: what can be one of the construction protocols that allows us to geometrically analyze what happens in it? Why is it asked about "one" and not about "the" construction protocol?

3-Explain in a newspaper article, no more than 1,000 words, what is read in the algebraic view of your screens when you answer question 2.

4-For a discussion in the classroom: which tools of the software used seem to be the most relevant to use in the proposed solution? Why?

5-Suppose there is no equipment available to work with: how can you answer question 2 using pen and paper? Justify each step you decided to take.

Figure 1. Task designed for teacher training. Own source

In the school for students between 12 and 18 years of age, the teaching groups responsible for pedagogical decision-making when teaching mathematics face professional dilemmas. These include the demand for personalization and adaptation and inclusion that conflicts with the need to maintain the social dimension of education; the use of technology in a way that meets the needs of an increasingly diverse student population; the conflict between inclusion and exclusion that the use of technology can bring within student groups; respect for the principle of equity in education and learning when using technologies as resources for the mathematics classroom; the scarce provision of freely accessible technological resources and the conversion of the Internet into a gateway to educational content; the cost of educational technology in terms of environmental sustainability and the tension involved in thinking about whether it increases the resilience of the educational systems in which it is used, among others (UNESCO, 2023).

In the professional discussions that are generated within schools, the idea of thinking that students attend it to receive mathematical knowledge mediated by digital technologies seems to have been overcome. The topic of discussion focuses on how they are helped to use this knowledge to achieve sustainable development in social, economic and environmental terms. Digital technology makes it possible to create interactive learning environments that dynamize the proposed activities through the simulation of situations and the expansion of the connections that can occur between the different contents that are used. Therefore, it can be said that its use in mathematics class invites discussion on deeper issues than the mere substitution of resources as it may require student groups to learn individually or collectively, online or offline, independently or interconnected, offering content, creating student communities and connecting teachers with students (Serres, 2013). It is a tool that favors productivity, creativity, communication, collaboration, design, and data management in a mathematics class where collaboration and teamwork prevail (Villella et al., 2021).

The school has always had various technologies that enhance teaching situations through which the practices associated with mathematical activity can be rethought. When these technologies refer to the digital world, it must be considered that many groups of students do not have opportunities to practice with digital technology in poorly equipped schools or do not have access to them from their homes, so their hegemonic use has to be rethought so as not to generate situations of exclusion.

The processing of mathematical information with digital means allows students to receive feedback instantly and, through interaction with software, adapt the pace and itinerary of their learning: students can reorganize the sequence of what they learn according to their context and characteristics since the way of presenting the content can be varied. Stimulate interaction and encourage collaboration.

In this article, we share the story of a scenario for continuous teacher training in which discussions about the use of technology in the mathematics classroom provide evidence of their specific knowledge about the teaching of geometry topics. We share the description of a space in which the aim is to promote reflection and awareness among teachers about the keys to their professional identity and the institutional conditions that have produced it, in order to provoke new conscious and explicit commitments. In this way, we guarantee the appreciation of their professional knowledge and work from the construction of our own references that generate concrete commitments. Thus, from collective reflection. We can help to enter into a process of deconstruction of professional practices and the meanings that sustain them. With regard to the incorporation of computer resources for the teaching of geometry, our proposal does not focus only on the discussion of the resource, but also covers the design of the class where it will be used, the content that will be studied and the anticipation of the different types of learning that it can provoke.

2. Theoretical framework

2.1 *Teacher: Where are you? A task to build geometric knowledge*

We believe that, when working on geometric content, it is a matter of breaking with the idea that studying them is reduced to work on configurations armed with tangible signs (drawings) and abstract mathematical objects (concepts): the understanding of geometric objects requires interaction between tangible and visible representations. Studying geometry at school involves making use of drawing tools (pencil and paper, dynamic geometry software -DGS-, etc.), analyzing the epistemic conflicts that underlie constructions when compared with the geometric properties of the object to be studied (Gascón, 2002; Laborde, 1998; Villella et al., 2018a).

This requires developing a work related to tools, techniques and properties that allows analyzing, describing and interpreting how some geometric tasks can be relevant to reach the desired one and to plan problems that make the epistemological aspect appear and the validation of the knowledge thus acquired (Trouche, 2005). *Teacher, where are you?* as a scenario for permanent teacher training, it is based on the conceptual framework graphed in Figure 2 which, when traversed, opens up different opportunities designed for the learning of the contents involved: the primitive ones in pencil and paper constructions and in dynamic environments; the figures of analysis and drawings; soft and rigid constructions (Laborde, 2005) in different technologies.

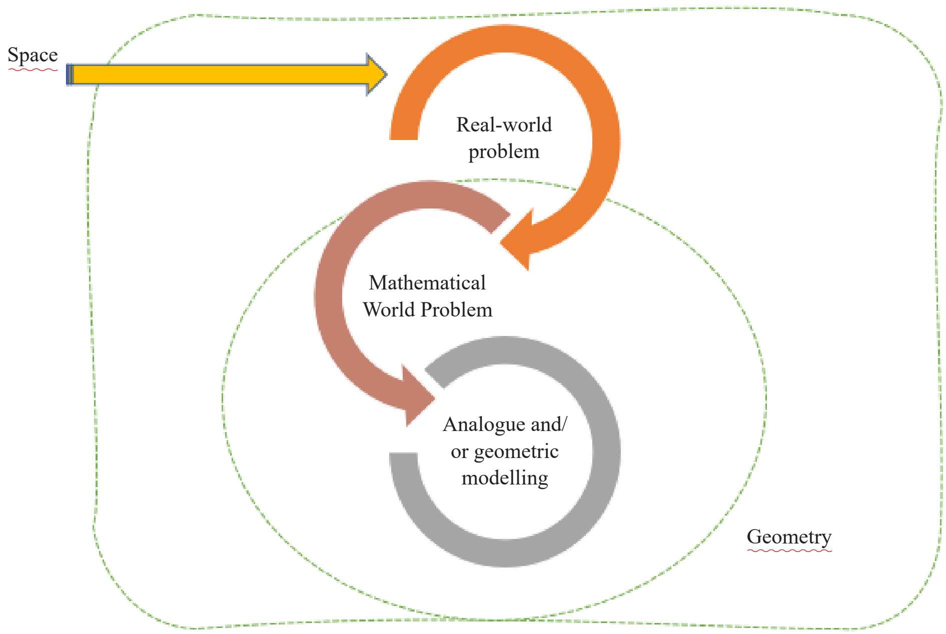


Figure 2. Conceptual map of the proposed geometric work. Own source

The task is planned to meet the objective that can be read in the table in the Figure 3.

ACTION...	… which refers to...	… by what...	… for what...
Procedural verb in infinitive.	Contents, mainly conceptual, on which the action will be carried out.	Concepts, procedures and attitudes that will participate in the action.	The purpose of contextualized action
Identify	the geometric properties of two- and three-dimensional figures	by analyzing optical illusions with a certain degree of hilarity	to justifiably argue the construction and use of geometric models

Figure 3. Construction of the objective to be achieved with the proposed activity in teacher training. Own source

We start from the idea that teachers have enough intellectual capacity and technical preparation to perform at work: our initiative, empathy, adaptability and persuasion help us to exercise our profession. The school demands that we be professionals who, in addition to developing technical capacities for our task, know how to listen to the needs of students and the community; let's communicate our ideas fluently; we adapt to the changing conditions of the environment by generating creative responses; let us use the opportunities that the environment offers us in order to minimize and transform the threats and weaknesses that manifest themselves in that same context; develop self-confidence, motivation to work towards a goal, desire to develop the profession and pride in what has been achieved; generate and develop scenarios for collaboration and teamwork, among others (Villella et al., 2018b). In other words, we would build our professional profile as a combination of common sense with specialized knowledge (Carrillo-Yañez et al., 2018) and the skill we acquire through professional practice: synergy between the faculties that make up our emotional intelligence (self-knowledge, motivation, self-regulation, empathy, relationship skills) and those that delineate our cognitive structure.

When working with digital technologies in the mathematics classroom, certain logical processes are used to interact with mathematical objects that are used both to justify and to represent properties. Along these lines, reference is made to the peers' particularization-generalization and materialization-idealization, which, according to Font & Contreras (2008), are associated with two other dualities: the extensive-intensive and ostensive-non-ostensive duality, respectively. For example, if in the input bar of a software (we choose GeoGebra) you write f(x) = 3x + 2, an extensive object is used as a particular case of a class of functions, those of the form f(x) = ax + b that, as a family of a certain type, are presented as intensive objects. While these grades can be associated with the way of defining sets, it can be stated that in the input bar a particularized object (algebraic expression of a curve) is proposed as an element to study a class (it goes from the extensive to the intensive) which allows building bridges between activities that refer to contexts of discovery and justification. When in mathematics class we move from the general to the particular, there is a contemplation of an individual object that generates the professional problem of seeking the answer to certain questions such as, for example:

Why is it decided to incorporate this phase that refers to a particular object into the demonstration of a proposition? Is it necessary to consider a concrete object so that intuition can act in the class?

Can this reasoning thus structured generate a universal conclusion? Is that particular element a generic element of another that can be considered as particular in the chain of propositions that are chained together in a reasoning? Is it intuition that allows us to grasp the general in the particular? (Duval, 2005; 2015)

Mathematical objects are not perceptible; however, they are presented through their associated ostensive (notations, signs, graphs). In class, what is shown in Figure 4 is usually presented as the bisector of the angle α.

When looking at the figure, it seems that the angle α is acute, that the bisector is a line of which only a semi-line was drawn originating from the vertex of the angle, that the amplitude of each of the angles into which it was divided is α the same. There is a process of idealization about this figure that supposes that the statements that were made about it are true (Abraira & Villella, 2002; Camargo et al., 2005; Villella, 2008). The figure in question is transformed into an ostensive and concrete figure (drawn with software and observable by all those in front of the screen) that by idealization alludes to a mathematical object (bisector of the angle α) that can only be presented to the class through associated ostensive objects. This object only refers to a particular case (it refers to the semi-straight line originating from vertex B of angle α and not to others) so through this process of idealization the passage from an ostensive that was extensive to a non-ostensive that continues to be extensive was worked on in class. This passage is the teaching problem since the teacher has to assume that he or she is offering his or her students' cultural artifacts (Radford, 2006) that materialize the elements of mathematical reasoning insofar as in it the mathematical objects are ideal and allow the construction of mental images as representations of the mathematical objects with which one works to, when using propositions, establish qualities of those objects (Anderson, 1983; D'Amore, 2010; D'Amore & Duval, 2019).

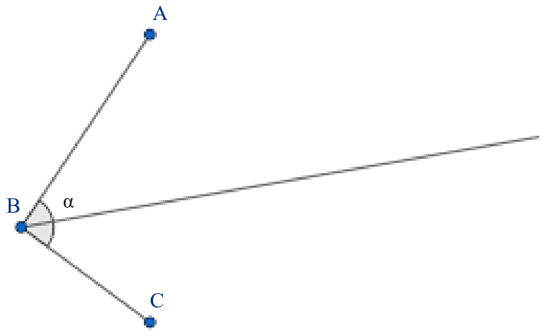


Figure 4. Representation of the bisector of an angle. Own elaboration using GeoGebra

The incorporation of computers into society generated a cultural change that modified the ways of seeing and being in the world; in the same way, the incorporation of computers in the classroom requires a cultural change in the way of studying and knowing. This change affects mathematical knowledge, the ways of studying it, the organization and management of the class and it is the teachers who are in the best position to manage it. Including technology in teaching is inevitable and presents teachers with the opportunity to rethink the activities and problems that give meaning to knowledge, knowing that they have powerful tools to solve the techniques (Fioriti, 2010). The incorporation of technology can be thought of in different ways: as a tool to do mathematics; as a way of expanding mathematical culture and consequently knowledge. It is a construct to be analyzed as part of the specialized training that the teacher acquires during his or her professional development. Technology evolves rapidly, a phenomenon that sometimes does not allow us to evaluate its efficiency with respect to the teaching and learning processes of mathematics in which we choose to use it. The issue of technological obsolescence installed by corporations sets the pace for change in the supports, software and ways in which interfaces are reorganized in the classroom, so in training spaces such as the one we share, it is necessary to rethink together with teachers that the professional discussion should focus on learning outcomes and not on the technological analysis of the resource. Thus, we propose to analyze how: the presence of electronic devices can lead students to be distracted by games, social networks or any other non-educational content available on the Internet; the ease of access to online resources can be overwhelming, making it difficult to select and process relevant information; The excessive use of technological tools can help reduce students' ability to perform tasks without digital help.

In the classroom, there are activities that focus on the link between mathematics and situations of reality, allowing work with models and intra-mathematical problems that require the use of properties to justify the solution found. This dialogue between modelling and the study of mathematics as a scientific discipline in itself allows each teacher to reflect on the qualities of the mathematical objects studied in school (Villella & Ferragina, 2022). In this reflective process (Figure 2):

- a model is built and the appearance of theoretical developments and the properties that they have in terms of how they relate to the situation that gives rise to them makes sense, and

- the properties and characterization of mathematical objects are studied according to the arguments and procedures of the discipline.

These discussions within what happens in the classroom account for a specific knowledge of the one who teaches

(Climent & Martin-Díaz, 2022).

In the field of teacher training, there was a shift from a vision focused on a logic of content to one located in a logic of professional situations (Villella & Steiman, 2021). For this change to be satisfactory, a perspective is required in the theoretical frameworks that helps to look at the situated action of the teacher who teaches mathematics in the classroom. To this end, we can work on the conceptualization of teaching action from the perspective of the reflective professional and professionalization, assuming that teaching does not legitimize itself by the implementation of routines focused on the presentation of knowledge, but rather makes sense in the awareness of why and how students learn. Thus, this practical dimension of the teaching activity comprises a set of repertoires and assumptions, which are sometimes contradictory to the theories proclaimed to explain the foundations of the decisions made in the classroom and require a scenario to identify the learning that occurs in teaching situations (Villella & Steiman, 2021).

This training scenarios that we are discussing was designed with tasks that have a function, a form and a focus (Grevholm et al., 2009). The task that we share and call *Teacher: where are you?* Fulfills the function of building specialized knowledge in teachers who teach geometry to 15-year-old students, based on the MTSK model (Carrillo-Yañez et al., 2018). Its form is identified in the resolution, analysis and reformulation of geometry activities mediated by DGS GeoGebra. Sometimes, technological resources are recommended without taking into account the long-term costs for economic budgets: the provision, operation and maintenance of technological infrastructure (electricity, computers and Internet connection) generates the tension between using them or not, causing unequal access to connectivity. Therefore, in our proposal we recommend using free and open source software that generates an adequate digital infrastructure so that teachers can integrate technology into their practice, regulating the time spent in front of screens so as not to generate adverse effects on the physical and mental health of students

Its focus, in order to specify the intended function, is to develop some of the subdomains of the MTSK (we develop them in the next section), in relation to the contents of metric geometry in the context of optical illusions through the art of photography (Duval, 2018).

2.2 *Teacher: Where are you? A task to think professionally about the classroom*

The answer to the question with which the task and the teacher training scenarios are titled, is not in the image with which it is presented, but in what that image connotes. That is: thinking about the problems of teaching geometry. Those of us who teach mathematics have sufficient technical knowledge that allows us to solve the activities we assign to our students more efficiently. We have specialized knowledge-different from that of students or other professionals-constituted by mathematical knowledge and skills that are typical of the teaching profession (Carrillo-Yañez et al., 2018). This gives us the possibility of investigating the mathematical aetiology of student errors in the resolutions of the activities we propose and reflecting on whether any of these solutions are generalizable or not. In addition, distinguish, find out, evaluate and interpret the validity of the diverse and sometimes unexpected student responses. It also characterizes the trajectory of mathematical content throughout the various educational stages, as well as its intra- and extra-mathematical connections. We can predict what will seem interesting, motivating, easy, difficult, boring or overwhelming for student groups as elements of the planning of the teaching situation. As well as identifying the pertinent previous concepts, learning difficulties and misconceptions of students about a particular geometric content; select which representations of the geometric objects to be worked on in the classroom are most appropriate to teach a specific content, thus adapting different methods, procedures, resources, and artifacts for their teaching (Villella et al., 2021).

Reflection on teaching and its professional problems involves analyzing student performance in the classroom, the use of resources in the organization of workspaces, the sense of knowledge, the social and institutional aspects that stress this scenario, the design and execution of teaching situations. In this way, we analyze the pedagogical sense of our actions, in correspondence with the intentionality manifested by the purposes of our class: there is something that we put in the focus, in the center of the didactic intention and although at each different moment there are particular intentions, there is a core intentionality that is the one that guides the decisions of the class and that defines its unity of meaning (the one that refers to the meanings with which it is assumes the raison d'être of a geometry class). Likewise, we think about the topic of the class-the issue to be taught-without implications referring to the approach from which we will develop it. We differentiate it from the content, which we consider to be a thematic specification that also includes some conceptual reference that allows us to identify a particular way of developing it in the classroom. This generates the answer to a key question for our analysis: how do we want students to work cognitively and affectively as they appropriate the conceptual categories of geometry that we set out to teach? (Villella & Steiman, 2021; Villella & Ferragina, 2022). If we start from the shared photo, we will think about the activity format to, without falling only into activism, make explicit: the pedagogical reason why we present it; the conceptual categories that are objects of teaching; the cognitive and affective challenges that we intend to promote as the situation is worked on. When we think of mathematical activity in the classroom, genuinely represented by problem solving, we do so in relation to the development of workspaces that allow the full involvement of each student in the resolution with our guidance and supervision (Kuzniak, 2011). To do this, it is necessary to take into account: the epistemological conception we have about mathematics; the fundamentals of the choice and organization of the sequences of activities; the analysis of the set of tasks developed by each student as a way of solving these sequences and the justifications they make about them.

Thinking about the class is a process in which two types of reflections converge. The prospective reflection by which, even when the class has not happened, we anticipate it mentally to move towards a practice more coherent with what we declaim. The retrospective reflection that helps us to analyze what has really happened (if we can reveal it) in that class, to question it, to unveil the implicit, the assumptions, the representations that motivated the decisions we made in an exploratory framework that tries to edit what at the moment of thinking about the class perhaps we naturalize and do not question and that, once the class has happened, we could stress based on the decisions made (Perrenoud, 2010).

The MTSK model (Carrillo-Yañez et al., 2018) allows interpreting the knowledge that teachers show in mathematics teaching practices. This model preserves, for analytical purposes, the division between the domains of content knowledge and didactic content knowledge proposed by Shulman (1986), as well as contributions made by Ball et al. (2008). It works with two domains that, crossed by the conceptions and beliefs that the teacher has about mathematics, are subdivided into subdomains:

1-mathematical knowledge comprising:

• Knowledge of Topics (KOT): reference to mathematical knowledge as an object of teaching, encompassing its phenomenological perspective and its applications, the various registers of representation, the definitions, properties, foundations and procedures to construct them.

• Knowledge of the Structure of Mathematics (KSM): refers to the conceptual networks that make up school mathematics.

• Knowledge of Practice in Mathematics (KPM): it contemplates elements of mathematical work, how it is defined, how it is justified, how it is drawn.

2-the didactic knowledge of the content that comprises:

• Knowledge of Mathematics Teaching (KMT): focuses on the teacher, contemplating knowledge about theories (personal or formal) of teaching, of the material and virtual resources to develop it, and knowledge of strategies and resources and how to orchestrate them in the mathematics classroom.

• Knowledge of Features of Learning Mathematics (KFLM): focuses on the learner, contemplating the knowledge of personal or formal theories about learning, the ways in which students interact with mathematical content and their interests and expectations.

• Knowledge of Mathematics Learning Standards (KMLS) refers to a holistic view of the curriculum, including the learning expectations at a given stage of the education system, the level of conceptual or procedural development that is expected at that stage with respect to mathematical content.

In this training scenario, the questions that form the backbone of the proposed task are graphically related to these subdomains, as shown in Figure 5 that intervenes the representation of the MTSK model with the numbers of the subtasks proposed in the shared activity in Figure 1:

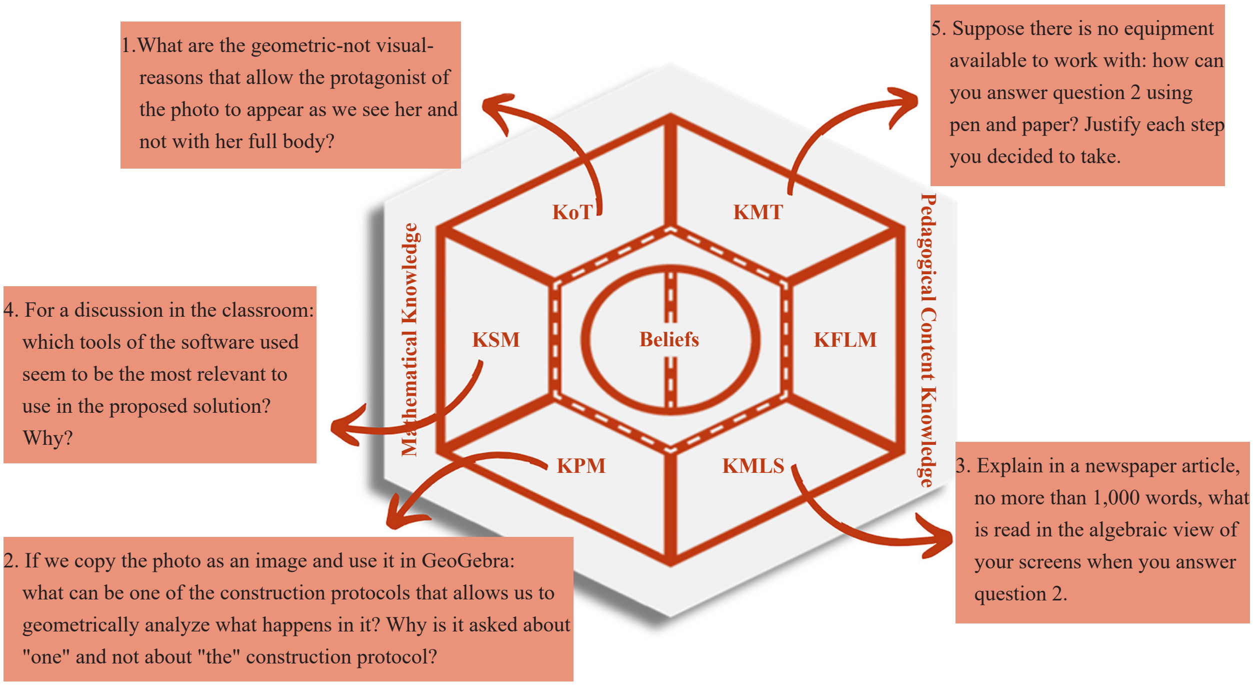


Figure 5. The MTSK model and the training scenarios Teacher: Where are you? Own source

2.3 *Teacher: Where are you? An activity designed for a scenario designed for the permanent training of teachers*

The classroom of permanent teacher training can be compared to a didactic device that stands as a moment of encounter of knowledge-that of the teachers who attend it as students and that of the teachers who officiate as such in these meetings-whose development is unpredictable insofar as the actions that take place in it cannot be prescribed in their entirety (Villella et al., 2018). It is the institutional scenario shared by colleagues who, possessing different knowledge about mathematics and its teaching, dialogue in order to achieve a greater understanding of them. Whoever officiates as a teacher in these meetings develops professional aspects that exceed the limits of the preparation and management of a class while deploying strategies for the development of emotional and social aspects that transcend the merely disciplinary. The action of permanent training is a pedagogical task that is organized in order to accompany and follow the process of study of mathematics and its didactics by other teachers who intend to achieve a more comprehensive training in these areas, taking into consideration their characteristics and personal needs.

The permanent training meeting is an instrument at the service of professional learning that seeks to contribute to each teacher who goes through it to learn more and better about the topics they have to teach, build specialized knowledge about teaching. During its development, it seeks to collaborate in the comprehensive training of teachers as a person, making the scenario a place with personalized actions that respond to the needs of each participant. To achieve this, trainers have to generate a good climate of coexistence and learning; to develop a group and personal follow-up of each participating teacher; channel teachers' concerns and demands and mediate in conflicts that may arise. These trainers will seek alternatives to guarantee the institutional conditions for the accompaniment, support and support of the professional trajectories of teachers so that they can make the most of their learning possibilities and thus expand their specific knowledge (Berenstein, 2007; Bolívar, 2004; Ferrata et al., 2005; García, 2010).

As an institutional scenario, the scenario of permanent learning is a collective project, a situated, flexible scenario insofar as it is conducive to thinking about and developing the unprecedented that arises in institutional life, in coexistence and in the social dynamics that impact the schools where the teachers who attend carry out their work.

The task of training involves the academic and bonding dimensions of the work in the school, leading to analyze and reflect on the teachers' interests, concerns, desires, achievements and difficulties that are perceived during teaching. To this end, it is necessary to plan the actions that will be developed with the teaching groups, in order to respect their characteristics, needs, concerns, interests and proposals. The central idea is that the trainer develops activities according to an accompaniment and management plan that allows him/her, with a didactic purpose, to pay attention to the demands that, according to the diagnosis, require his/her intervention for the sake of professional training of each teacher who attends his/her meetings. This involves the development of a process of empathy with the other as it is simultaneously a bridge and channel for the transmission of suggestions, concerns and proposals that are collected in their own studies on mathematics and its didactics and its professional demands (Climent & Martin-Diaz, 2022; Villella et al., 2018a; 2018b).

The problems and activities proposed in this scenario for permanent training are inputs for the didactic analysis and seek to discuss with the teachers:

- how to manage the class to encourage students to rehearse, produce different solutions, discuss, argue in their journeys through the networks of concepts that involve the entry into deductive reasoning as a way of validating (Camargo et al., 2006; Montes et al., 2022),

- how to use the different forms of representation, equivalent to each other, as some of the activities of mathematics that students must build (Asenova, 2018; Duval, 2018),

- how to organize interactions in the classroom to reflect on the validity, precision, clarity, generality of what is produced in them (García et al., 2010; Hargreaves et al., 2000).

From what has been described, the design of this permanent training scenarios is inscribed in the paradigm of collaborative didactic production (Bednarz, 2004). With its realization, the aim was to produce relevant knowledge about professional practice and to find channels of dialogue between the so-called world of research and the world of practice. Thus, a process of knowledge production on the teaching and learning of geometry was favored jointly between a group of researchers and teachers who interacted to contribute from their specificities to the construction of knowledge. This knowledge arose from reflection on the practice of teaching geometry and returned to it to give meaning to the decisions that each teacher makes daily in the teaching process.

The implementation of this methodology involved interventions in the geometry that is taught that were not only fertile in terms of learning for students between 12 and 18 years old, but also for the reflection of each teacher on how he or she teaches. In this process, the teacher's voice was recovered based on the understanding of their practice (Villella & Steiman, 2021). Under these conditions, the teaching scenario and the activities developed to be put into practice in the class appeared as a legitimate way of finding meaning in the teaching experience: the complexity of teaching practices allows us to highlight the limitations and margins of maneuver that the teacher has in his class, as well as the unexpected in terms of the work of the groups of students.

Epistemologically, the construction of specialized knowledge occurred in the real context where the practice of teaching geometry is carried out, assuming that the components of this context are subject to restructuring based on the assumption that teaching groups are actors in context (Herbst & Chazan, 2009; Villella & Ferragina, 2022). The activity that is the axis of the professional training scenario was arranged in such a way that it promotes and maintains a kind of conversation between teaching practice and reflection, between teaching and didactics. In the development of the work meetings, the activity took shape through periodic meetings, which allowed the creation of an interpretation area around situations that are the object of exploration. This reciprocal scenario of argumentation developed a series of discussions around how each one gives meanings in the context of teaching; it gives meaning to the co-construction of knowledge.

This work methodology shows the reconciliation of two communities of practice: that of research and teaching, which are, in turn, part of a certain community of practice that negotiate a certain way of acting and thinking about the limitations and resources available for teaching. In this sense, the scenarios for lifelong learning did not intend to unite these two worlds but rather to interact with them to give rise to the emergence of a new community: a reconciliation between research and teaching practice for the construction of knowledge at the service of informed practice (Bednarz, 2004).

3. Implementation and results achieved during the permanent training scenario

We assume (and invite to do so) a theoretical position that considers didactic action as a complex and joint action where meanings and intentions of teaching and meanings and possibilities of learning are exchanged. The practice of teaching geometry is associated with a situated activity in which the implementation of knowledge, processes and teaching skills place us in a professional type of action. The semantic analysis of an activity gives us the possibility of building, together with colleagues, a material for use in the classroom that may have gone unnoticed: we thus act out the dynamics of the equation document = resource + schemes of use, which makes collaborative work an unavoidable requirement (Trouche, 2005).

The mathematics classroom studied is transformed into an experimental scenario when the resolution of problems with the use of technology is proposed: the person who solves the problem to be experimented is enabled to make conjectures, explore several times the steps and solutions found to improve them or adjust them to what was requested, discuss the findings with the group and the teachers in charge (Ferragina, 2012; Laborde, 2005; Sokna & Trouche, 2010). At the beginning of the activity, it is possible that the action of the solver is related to processes of trial and error. In the dialogues for reflection on professional practice, it was discussed whether this quality of problem solving is maintained throughout the time that technological resources are used or whether, after an exploratory period, the resolutions cease to have a random character, to be transformed into processes of evidence with argumentation (Camargo et al., 2006).

Under the conditions described, the work of solving and discussing each of the questions of the training activity (Figure 1) resulted in what is shown in Figure 6. The recording of what was done (where each participating teacher is identified with the letter D and a reference number), the discussion of what was noted, the exchange of points of view on the steps taken in a possible answer to question 2 (Figure 1), highlighted the need to know in depth the geometric contents (symmetry, parallelism, angles...) used and the didactic decisions that focus on the analysis of the role of students, as producers of geometric knowledge in school (relationship between the construction protocol and the algebraic view).

In agreement with Duval (2018), we affirm that when faced with a geometric figure constructed from the use of certain tools (compass, software, various resources), that is, those that are not drawn freehand, the same thing happens as when you are in front of a pictorial work: it is seen or recognized by making use of the same cognitive processes of visual recognition that control the gaze. This means that when you are in front of a geometric figure, you first recognize it visually and you can also, through those processes that involve the gaze, spontaneously convert that representation of one register into another. This debate, applied to the resolution of the activity, allowed us to conclude that in order for students to enter the world of geometry, teachers must begin by educating their gaze since, when faced with a figure, they do not see the same as teachers or those who produce mathematics.

Thus, we analyze that the professional problem with respect to the teaching of geometry in school lies in the fact that its specificity is not to present figures that can be seen and can be constructed, but to use defined concepts to know what they represent and to resort to them to name and describe: the figures are not seen with the naked eye, special lenses must be used to decode what they contain, mean, connote (Laborde, 2005; Villella et al., 2018a; 2018b). The professional challenge to the teacher occurs insofar as the relationship between seeing and telling what is seen is manifested through the impossibility of words that no longer designate what is seen (Villella & Ferragina, 2022). The teaching of geometry, it seems, is thus faced with a possible student attitude that is characterized by the fact that they can:

- sustain the perception of basic geometric figures: triangle, parallelogram, square, circle...,
- stay in the closed outline of the figure,
- not decomposing and reconfiguring the figure to find the solution to a problem
- confusing terms in the tasks of constructing figures and in those of writing or verbal explanation of the instructions they followed to construct them,
- construct figures by taking measurements in the drawing or using given numerical values,
- recognize formulas for calculating the perimeter and area of quadrilaterals, without extrapolating them to other polygons.

1- What are the geometric-not visual-reasons that allow the protagonist of the photo to appear as we see her and not with her full body?

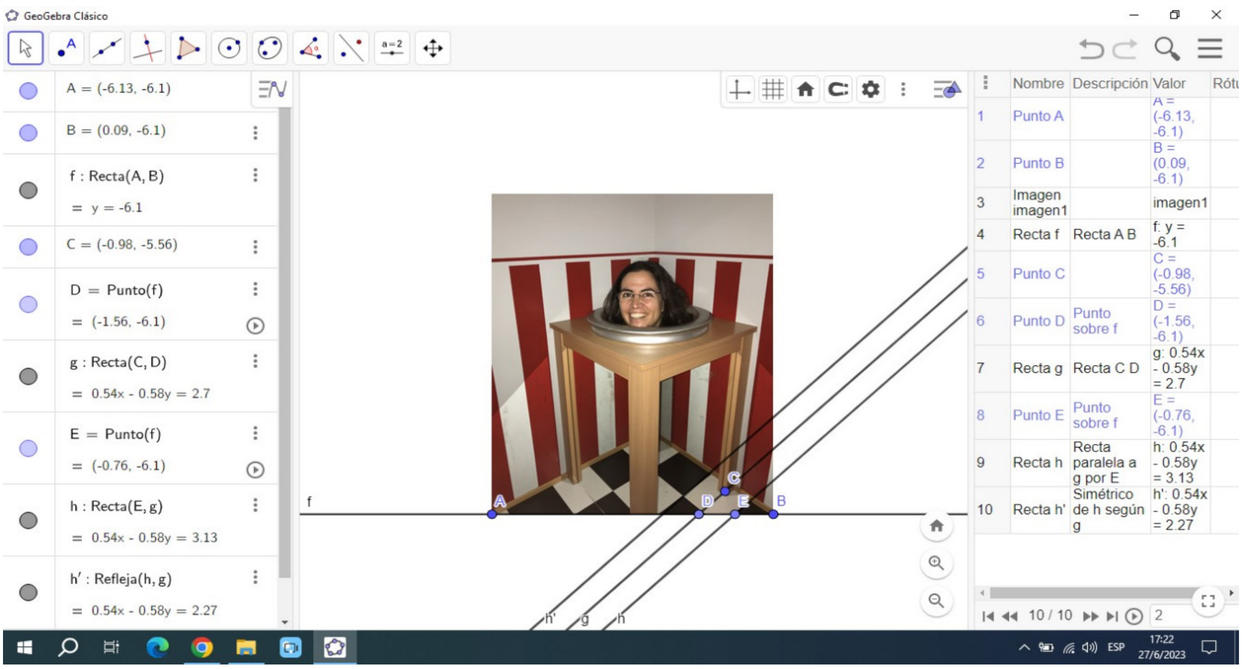
D1:-I suppose that in the classroom they will say the properties, I doubt they know which ones.

D2:-It seems to me that you are going to relate it to the idea of optical illusion.

D3:-I think I would invite you to say if the answer you thought has to do with the idea that the photo is an image that leads to perceiving reality in a distorted way, although the idea is to study geometric properties...

2- If we copy the photo as an image and use it in GeoGebra: what can be one of the construction protocols that allows us to geometrically analyze what happens in it? Why is it asked about "one" and not about "the" construction protocol?

D1:-One can be…



3- Explain in a newspaper article, no more than 1,000 words, what is read in the algebraic view of your screens when you answer question 2.

D1:-They are going to write colloquially that it is one of the possible ways to solve. They are going to translate symbols into words.

D2:-I don't think so, that they can relate what they did using the definitions of lines, parallels, perpendiculars, angles between parallels... but I don't know if with the visible protocol they would do it from the algebraic point of view. It is easier from the protocol that has the decisions that are seen on the screen than from the algebraic view that has them more abstract.

D3:-But that's interesting. I imagine giving them in the classroom the photocopy of the algebraic sight, pencil and paper, and writing. I would take them off the screens so that they can't experiment, but have to evoke, you understand?

4- For a discussion in the classroom: which tools of the software used seem to be the most relevant to use in the proposed solution? Why?

D1:-Here you will go to the toolbar and list the ones you used. I don't know if I'm interested in asking that.

D2:-But it seems to me that, emphasizing the idea of relevant, we can make them compare forms of resolution and geometric decisions that they were making. I would go from tools to properties, i.e. did you use perpendicular lines of the bar or did you build perpendicular lines with the 90° angle plot?

D3:-But not that they do it again, I mean, using pencil and paper and comparing doesn't seem to me. The discussion about the decision, right?

5- Suppose there is no equipment available to work with: how can you answer question 2 using pen and paper? Justify each step you decided to take.

D1:-Here, yes, making them use other materials seems interesting to me because it moves them away from dependence on machines.

D2:-But for that to happen, nothing about computers has to be in the classroom. I mean, only sheets, papers, books for consultations. Neither the protocols, nor the computers, nor the internet...

D3:-It seems to me that they limit them to use another resource that they are given. I would just restrict that there are no teams like in my school. There are many broken. It is a common thing for them to look for a way: to explore with what is at hand and explain how they did it, why in this way and not in another.

Figure 6. Teaching interventions in the answers to the questions of the task. Elaboration in the training space

We come to a first assertion about the teaching of geometry: taking into account that seeing is, in general, a purely sensory act and in geometry, generates a professional problem since in front of a figure you not only have to see it, but you have to know how to see it, which means distinguishing, recognizing, establishing, relating. To form the notion of seeing in geometry, which is not a notion linked to iconicity, the provision of teaching environments must be favored (Villella et al., 2018a) in which students can evolve through certain stages that allow them to:

- recognize all possible closed contours of a configuration. This must occur both in those that are recognizable by juxtaposition and by superimposition and that, in addition, can be recombined to obtain different configurations. This excludes the consideration of dimensions, measurement activities and all length indications relating to recognized figured units,
- consider the third dimension, perspective. This geometric construction organizes the field of view by subordinating all closed contours to magnitude relations, allowing the visualization of a relief surface or cavity, including the possibility of seeing an object in space depending on the side from which it is viewed, and regardless of its position with respect to all other objects viewed at the same time: The visualization of geometry in space is independent of that based on perspective,
- answer the question: How can you visualize geometric properties that cannot be visually perceived in a figure? This question is not answered by constructing figures, but by deconstructing those constructed (Laborde, 2005; Villella et al., 2018b).

With the proposed task, we were able to discuss in the training scenario together with the assistant teachers that, when looking at a figure to solve a problem, the type of tool that was used to build it (ruler and compass or software instructions) is downplayed and relevance is given to the way in which that figure is seen. For mathematicians and teachers, seeing is based on the dimensional deconstruction of forms, which leads to a silent verbalization as defined by Vygotsky (1995). At school, students arrive at a visual heuristic exploration of shapes, without necessarily requiring the logical chain of words with which what is seen is argued.

4. Conclusions and prospects

From the point of view of the relationship between the teaching and learning of geometry, it seemed necessary to work from a scientific approach to the problems generated by the communication of mathematical knowledge: professional problems of those who teach mathematics. This approach took into account the class in its broadest form, considering it as an object of study in which the interaction and interdependence that occurs between teachers, students, and the mathematical content to be taught so that they are learned is analyzed (Villella & Steiman, 2021) In these classes, the construction of meaning does not necessarily imply the appropriation of the mathematical knowledge that circulates in them. Under some conditions, this construction favors the structuring that allows the memorization of that knowledge: all the teaching work in the classroom is focused on the achievement of that objective.

When the Didactics of Mathematics is conceived as a science of design (Lesh & Sriramn, 2010), the question that underpins a scenario of permanent training such as the one described can be formulated as follows: Should teachers perceive themselves as engineers whose work is guided by the need to solve teaching problems? The answer is affirmative as we use the design and systematic analysis of teaching strategies and tools in close relation to didactic theory, so that they are interdependent. It was based on the premise that teacher training-which lasts a certain amount-includes the design phase of usually innovative curricular tasks and experimentation in classroom contexts.

This training scenario was nourished by interventions in the classroom to improve teaching practice through didactic engineering approaches (Artigue et al., 1995) and thus act on professional problems related to the teaching and learning of geometry. He tried to give more support to the theoretical understanding of the phenomena that are related to these problems (Stylianides & Stylianides, 2013). Its implementation gave rise to the possibility of generating records of practice that help to exemplify, from different representations of the practice of teaching mathematics, different theoretical ideas (Villella & Steiman, 2021), allowing attendees to relate educational practice and theoretical analyses, through situated reports on the learning of their students, directly relating the learning process to the way in which it has been promoted; working on the construction of knowledge; recognizing the limits of theory; capturing the specificities of practice and the potential advantages of adapting theory to a context in an iterative and refining way; studying the everyday problems in the classroom, schools and communities that influence teaching and learning, adapting teaching to these conditions (Kelly et al., 2008; Shavelson et al., 2003).

In this way, we were able to build the professional teaching knowledge that allows us to look meaningfully at our classrooms and thus identify practical problems related to teaching for the learning of geometry.

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Thanks, Carolina, for your original photo.

Conflict of interest

The article entitled "Teacher: where are you? About a scenario for permanent teacher training in which the teaching of geometry is invited to think professionally" declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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