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Research Article

Young Students' Ability on Understanding and Constructing Geometric Proofs

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Abstract: The present study investigated early secondary education students' ability to understand and construct geometric proofs before and after typical instruction in Euclidean Geometry. At the primary education level, proof is related to reasoning, while at the secondary level, the formation of mathematical proof is introduced. Students' difficulties can be examined within the framework of a possible gap. The research tools were designed to investigate the impact of students' conceptions about the structure of proof (experimental, semi-experimental, and formal) on their ability to construct geometric proofs and to identify errors in proofs presented to them. There were two main phases of measurement: before and after the teaching of Euclidean Geometry for the first time in the early grades of secondary education. Results indicated that the majority of students recognized the value of using mathematical symbols and the necessity of presenting a logical structure of arguments in order to construct a proof, while many students also preferred the semi-empirical proof as an acceptable form of constructed mathematical proof. Additionally, results indicated that students experienced considerable difficulties in solving tasks related to geometric proofs that were presented verbally and without figures. Based on the results of the present study, students' difficulties in studying and constructing geometric proofs are discussed in relation to the teaching practices of the concept of proof in the early grades of secondary education.

Keywords**:** proof, geometry, empirical, semi-empirical, formal

1. Introduction

Geometry is considered to be an important area of mathematics with great importance in peoples' necessity to solve everyday problems in space (Sunzuma et al., 2013). The problem-solving process under a realistic framework was the main of emphasis of the mathematics education during the last decades. The teaching of geometry, as a part of the teaching of Realistic Mathematics is a necessity even from the age of preschool education (Papadakis et al., 2016) in order to activate children's motivation to examine and investigate mathematical concepts. Undoubtedly at the age of secondary education, it is not easy to teach every mathematical concept in the context of the everyday life; however, the interdisciplinary framework of the contribution of mathematical concepts to the scientific development enables us to reveal the value of the science of mathematics. At secondary education the problem solving in geometry is related with Euclidean Geometry and the geometric proofs. The present work concentrates on the understanding and constructing

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of geometric proofs, the role of the teaching of Euclidean Geometry at secondary education and the impact of students' conceptions about the nature of mathematical proof on their ability to understand and construct geometric proofs. Specifically, at the present study three main research questions were posed:

(1) Which are students' conceptions about the accepted geometric form at the first grades of secondary education and which are the main difficulties they face?

(2) Which are the factors that construct students' general ability to handle the different types of geometric proofs they encounter?

(3) To what extent do students' experiences with the teaching of Euclidean Geometry at the first grades of secondary education, affect their ability to understand and construct geometric proofs?

2. Theoretical framework

The research about the development of appropriate teaching methods with or without technology is not recent (Papadakis et al., 2018), especially in the case of Geometry. It is one of the main emphases of mathematics education, as different methods will be always needed for students with inter-individual differences concerning their cognitive abilities and learning styles. In the case of the geometrical concepts the use of mobile learning (Panteli & Panaoura, 2020) and the use of technological tools (Tirkas & Panaoura, 2020) have increased the interest on the teaching methods for the improvement of geometrical competencies. However, the teaching of geometry, in respect to the development of geometrical thinking is related with the students' experiences derived by primary education teaching on reasoning as the presupposition for the construction of geometric proof.

2.1 *Geometrical thinking*

Students' mathematical skills, especially in Geometry, are closely related to the levels of development of their geometric thinking (Atebe & Scafer, 2008), which start from an intuitive understanding of the concepts, reasoning and justification and proceed to the proof method of documentation (Elchuck, 1992). Our teaching for decades indicates us that students very often face difficulties in working with geometric problems (Fischbein & Nachieli, 1998).

Several mathematics educators and researchers have investigated students' development of geometric thinking. For example, Van Hiele (1986) developed a model referring to the hierarchical levels of geometric thinking, Fischbein (1993) introduced the theory of figural concepts and Duval (1999) reported the cognitive analysis of geometric understanding. Van Hiele developed a theory involving levels of thinking in geometry that students pass, though, as they progress from recognizing a figure to being able to write a formal geometric proof (Mason, 2009). Based on Fischbein's theory, a geometric figure is an abstract ideal entity, a general representation of a category of objects and the age does not improve the control of the conceptual component of the interpretation of figures. Duval distinguishes four apprehensions, as he called them, for a geometric figure: perceptual, sequential, discursive and operative. According to his theory, to function as a geometric figure, a drawing must evoke perceptual apprehension and at least one of the other three dimensions (Panaoura, 2014).

Based on the abovementioned theories, there is a developmental transition from the intuitive understanding of the geometric concept and the respective reasoning which are characterized by an experimental format to the formal format of a geometric proof. The present study examined young students' ability to understand a presented proof and to construct a geometric proof by themselves in relation to their conceptions about the structure of the mathematical proof, before and after the teaching of a chapter on Euclidean Geometry at the first grades of secondary education. By this way we aimed to examine the impact of the teaching of geometric concepts at primary education and the role of reasoning on students' ability to understand and construct geometric proof. Suggestions for a fluent transmission from the experimental reasoning to the formal proof in secondary education have to be based on the students' initial conceptions and the Curriculum expectable learning outcomes.

2.2 *Geometry and geometric proof*

A special role in the teaching of Geometry is occupied by the understanding and the construction of mathematical proof. Proof should be at the core of doing and understanding mathematics and it has to be appreciated as an important component of students' mathematical education (NCTM, 2000). Traditionally, mathematical proof has been related with secondary mathematics teaching of Euclidean Geometry (Zeybek, 2016), while it is related with reasoning as a way of thinking from the elementary grades (Stylianides & Ball, 2008). Hanna (2000), Recio and Gobino (2001) emphasized that proof was the most important tool used in geometry and it has played a key role in the historical development of mathematics, considering the dominance of Euclidean Geometry.

Proving is a fundamental part of mathematical learning as conjecturing, generalizing and justifying and it requires students to think flexibly the mathematical ideas (Lesseig, 2016). The concept of proof refers to the justifications which are based on the previous statements that were accepted, for the use of acceptable forms of argumentation and acceptable ways of communication (Stylianides, 2007). According to Stylianides (2009) non-proof arguments including empirical arguments or other rationales are not valid mathematical proofs because they do not guarantee the truth of the assertion for all cases. For the mathematicians, general or inductive arguments are accepted as evidence (Rowland, 2002), while empirical ones are not (Morris, 2007). Empirical proof is based on observation or experience, while semi-empirical proof is based on observation and mathematical theorems. Examples after constructing proof are considered useful for improving sentences and proofs (Komatsu, 2017).

Despite the importance, the understanding and constructing of a proof, both of them are difficult processes for students in secondary education (Stylianides & Weber, 2017). Even undergraduate students in mathematics face difficulties in understanding and constructing mathematical proofs (Zazkis, 2013; Ericson & Herbst, 2016). According to Pedemonte (2007), although there is a cognitive unity or interrelation between argumentation and proof, there may be a gap between the reasoning types. Pedemonte and Reid (2011) claim that when students prove a mathematical statement, they make an inference that leads them to produce a claim based on their observation. Especially in the case of geometric problems, the tasks require students to relate their geometry and algebra knowledge, make assumption and verify those (Gulkilik et al., 2019). However, students need representations to manipulate symbols. They are expected to construct verbally-symbolic proofs.

At the ages of primary education, teachers relate the initial phase of mathematical proof with reasoning. The use of examples and counterexamples are introduced through an exploration or an investigation and construct the theoretical framework of an empirical proof. At secondary education the teaching of mathematics introduces the concept of mathematical proof through the teaching of Euclidean Geometry. Signs, symbols and geometric shapes acquire a significant role for the construction of geometric proof and there is an introduction to a more formal construction of the mathematical proof (Yang & Lin, 2008).

As a mathematical domain, Geometry is to a large extent concerned with specific mental entities, the geometric figures. Research by Komatsu (2016) has shown that problems and tasks with proofs solved more easily and in higher grade with the contribution of shapes as useful tools for empirical testing. At a mathematical level, geometric figures are mental entities, which exist only based on their definitions and properties (Panaoura, 2012). Gulkilik et al. (2019) analysed the relationship between argumentation and proof in terms of verbal, visual and algebraic representations of mathematical concepts. The study was conducted on undergraduate mathematics teachers. Their results indicated that the sample was able to transform the abductive and inductive arguments to deductive arguments in proofs if they could produce algebraic representations. In a study by Stylianides (2017) which evaluated the type of presentation that students use in constructing the arguments for mathematical proof, the results showed that the construction of claims for mathematical proof depended on the type of presentation, oral or written, that students examine allegations. Research by Ahmadpour et al. (2019) studied a learning model for students' understanding of mathematical proof, their transition from constructing empirical proof to mathematical proofs and the use of symbolic representations in the mathematical proofs they construct. The students that took part in the research were in the 7^{th} and 8^{th} grade of school (13-14 years old). The results showed that the students were able with the specific learning model to go from constructing empirical proof to constructing and understanding mathematical proof.

There are many studies on the understanding of mathematical proof at different ages and many others on the constructing of mathematical proof. Kunimune et al. (2010) indicated that the vast majority of the lower secondary students consider that experimental verification is enough to demonstrate the geometric statements are true, although they understand the necessity of the proof. We believe that the rationality of the present study concentrates on the stage before and after the teaching of Euclidean Geometry in the early stages of secondary education. It examined students' ability on a) understanding constructed geometric proofs, b) identifying mistakes at given geometric proofs and c) constructing geometric proofs by themselves which include or not a geometric shape.

3. Methodology

Sample: At the present study, 362 secondary school students (14 years old) participated during the school year 2018-2019 from various schools at Cyprus. The choice of the specific sample was based on the first name researcher's personalrelations with the teachers and their respective willingness to participate in the study.

Measurement tools: In order to examine the posed research questions four tests were constructed and administered to the students twice, before and after the teaching of the chapter on Euclidean Geometry. The construction of the tests and the selection of the mathematical proofs which were used, took into consideration the Curriculum and the textbooks which are used at the specific educational system. We describe the tests and present indicative tasks which were used:

1. Test A: The first test, Test A, referred to students' recognition and ability to recognize whether a proposed solution to a task is an acceptable form of geometric proof (there were differences on the level of the formality of the structure). It consisted of four tasks, each of which was solved in different forms of proof and students were asked to state whether they accepted each form as mathematical proof or not and to justify their answer. An indicative example is presented at the Figure 1, with a semi-empirical proof, two empirical and a formal proof about the sum of the angles of a triangle is equal to 180°.

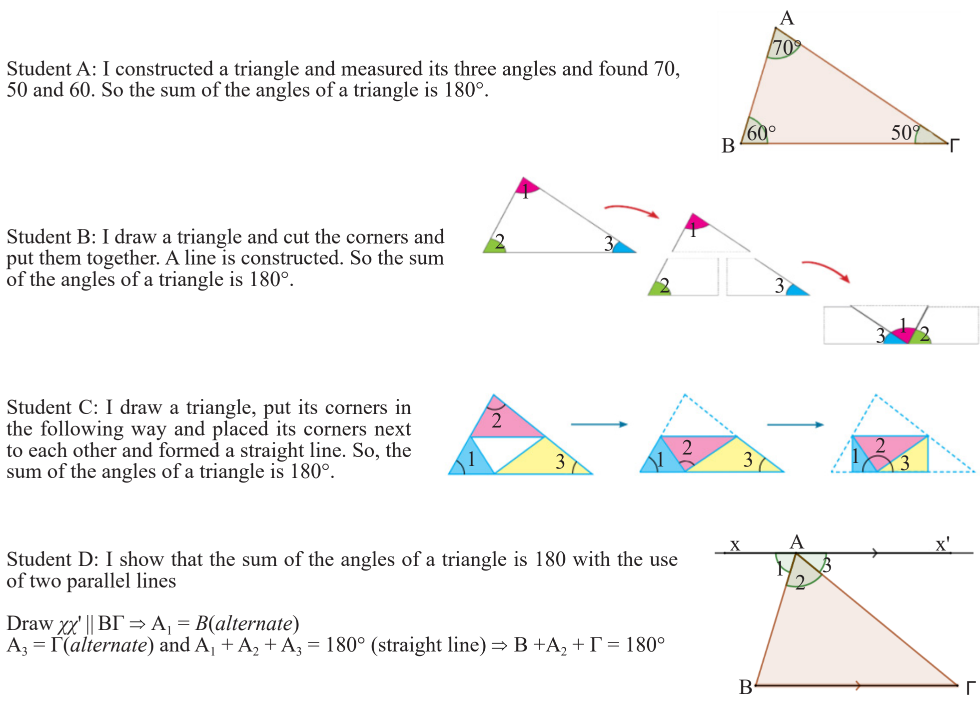

Figure 1. Indicative task. Question1 of Test A

Similarly, at the second task, there were three different forms of proof (a semi-empirical, an empirical and a formal mathematical proof). By the same way, the 3rd and the 4th tasks were consisted of two different forms of proof (semi-empirical and formal).

The second test, called Test B, referred to students' handling the different types of representations which were used for the construction of mathematical proofs. It was consisted of two tasks, with six different choices, and students were asked to indicate and justify which one was the most appropriate. The six choices were based on the following types of presentations: (A) verbal without shape, (B) verbal with shape, (C) verbal and symbolic without shape, (D) verbal and symbolic with shape, (E) symbols without shape and (F) symbols with shape. An indicative example with an indicative solution is presented in Figure 2.

The third test, called Test C, aimed to examine students' ability to detect errors and/or omissions in given geometric proofs. The test was consisted of two tasks which presented constructed mathematical proofs and students were asked to identify errors and/or omissions in the solutions. Also, they had to declare whether they would choose the same way of solving the task or a different problem-solving procedure. The first task, as an indicative example is presented in Figure 3.

The fourth test, called Test D, examined students' ability to construct geometric proofs. The test consisted of three tasks, each one presented differently, and students were asked to solve them. The first task was presented only verbally without shape, the second task was presented verbally and symbolically with the shape and the third task was presented only symbolically without shape. For example, the first task asked "Proof that each diagonal divides the parallelogram into two equal triangles".

(A) Prove that the heights we bring to the equal sides of an isosceles triangle are equal.

Solution
I draw an isosceles triangle ΑΒΓ (ΑΒ = ΑΓ) and bring the heights ΒΔ and ΓΕ.
I compare triangles ΒΔΓ and ΒΕΓ
i) ΒΓ common side
ii) Ε = Δ = 90° (ΒΔ and ΓΕ heights)
iii) Β = Γ (equals angles of an isosceles triangle)
⇒ ΒΔΓ = ΒΕΓ ⇒ all their corresponding elements are equal ⇒ ΓΕ = ΒΔ

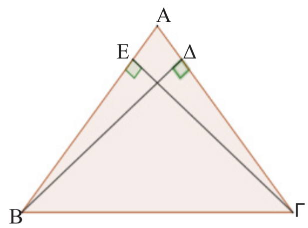

Figure 2. Indicative Task. Question 1 of test B

Prove that the bisectors of the opposite angles of a parallelogram are parallel. A) Identify errors or omissions in the given proof. B) Would you choose the same way of solving the task? Explain

Solution
I draw a parallelogram ΑΒΓΔ and bring the bisectors ΔΖ and ΒΕ.
I compare triangles ΑΔΖ and ΒΕΓ
i) $B_1 = \Delta_1$ (half of equal angles, Β = Δ, ΔΖ and ΒΕ bisectors)
ii) ΑΔ = ΒΓ (opposite sides of parallelogram are equal)
iii) Α = Γ (opposite angles of parallelogram are equal)
ΑΔΖ = ΒΕΓ (Γ – Π – Γ) ⇒ all their corresponding elements are equal
⇒ ΔΖ//ΒΕ

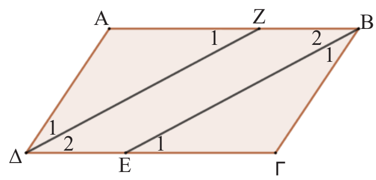

Figure 3. Indicative task. Question 1 of test C

In brief the structure of the four tests and their relation with the posed research questions are presented in Table 1. The four tests were administered at the end of October of the school year 2018-2019, before the teaching of Euclidean Geometry, to all the participants. The tests were administered in class either by the researcher or by other mathematicians. Tests A and B were completed together and with a break of a week from the tests C and D. Then the students were taught the chapter of geometry based on the objectives and teaching processes which were posed by the Curriculum. The lessons of the geometry lasted almost 18 teaching periods (40 or 45 minutes each period) and were conducted by the mathematic teachers of each class. Finally, the same tests were administered beginning February of the same school year after the teaching of Euclidean Geometry.

Table 1. The structure of the four tests

Test A	Test B	Test C	Test D
Conceptions about the form of geometric proof	Conceptions about the different types of representations	Ability to detect errors	Ability to construct geometric proof
4 tasks	2 tasks	2 tasks	3 tasks
RQ1 and RQ2	RQ2	RQ1	RQ3

Teaching methodology: The teaching of mathematics at the public schools of Cyprus is based on a formal Curriculum published by the Ministry of Education (Ministry of Education, 2015) and the textbooks are common. The teaching of Euclidean Geometry is introduced at the third grade of secondary education in relation to the teaching of the structure of the formal mathematical proof. Previously, at primary education students come up against the presentation of experimental processes for understanding geometric concepts. For example, they cut the angles of a triangle, transfer them on a line, and by this way accept the statement that the sum of them is 180°. At the 1st grade of the secondary education, they are taught about the geometric proof that the sum of the angles of a triangle is 180°, and they learn the elements of the triangle and circle. At the 2nd grade they learn the terms of quadrilaterals. At the 3rd grade of the secondary education, they are taught about the equality of triangles and quadrilaterals. According to the Curriculum the aims at the specific age concentrated on defining, proving and applying the concept of equality of shapes (equal shapes, equal triangles, criteria of right triangles) and on recognizing, constructing basic quadrilaterals (parallelogram, rectangle, rhombus, square, trapezium), proving and applying their properties in the problem solving.

According to the Ministry of Education guidelines, the teaching of mathematics has to be based on inquiry-based teaching processes. In the first phase, students are involved in situations that arouse their interest and attract their attention. These situations are effective if they raise questions that make sense to the students and which can be answered based on students' observations and interpretations. Through this process it is possible to relate the new concept with pre-existing knowledge and at the same time with students' misunderstandings. Explorations are activities in which students explore mathematical concepts through open-ended mathematical problems. Investigations are activities in which students explore mathematical ideas in specific context and have opportunity to check the validity of their cases, and to justify their answers with the aim of drawing conclusions. The procedures can be performed using examples, with appropriate problems or digital tools.

Data Analysis: Descriptive analysis was used in order to examine the posed research questions. For the first and second research questions, X^2 was performed to check the independence of the quality variables in both measurements of the tests and to check whether the quantitative variables followed the normal distribution. For the comparison of quantitative variables of repeated measurements, for the third research question, in tests C and D, the non-parametric Wilcoxon sign test method was used because the variables were not normally distributed.

4. Results

The emphasis of our analyses was on finding the students' difficulties derived by their conceptions on the accepted form of geometric proofs, their abilities to understand presented to them proofs and to identify the mistakes, and their abilities to construct a geometric proof with or without a given shape. Their performance was examined before and after the teaching of the section of geometric proof at secondary education.

The first task at the test A presented four different forms of proof. Table 2 presents the percentages of answers for the first task, before and after the teaching of Euclidean Geometry. It is noted that "Yes" means accepting the form of the given proof and in a corresponding way "No" rejecting it.

Table 2. Percentages of answers Task 1 Test A

	Before the teaching of Euclidean Geometry		After the teaching of Euclidean Geometry	
	Yes	No	Yes	No
1A	85.4%	14.6%	80.7%	19.3%
1B	66.6%	33.4%	33.4%	66.6%
1C	58.8%	41.2%	28.2%	71.6%
1D	88.7%	11.3%	96.7%	3.3%

The 1st solution was a semi-empirical proof that was accepted by 85.4% of the students in the sample before the teaching of Euclidean Geometry and 80.7% after its teaching. The X^2 test showed that there was not any statistically significant difference before and after the teaching of Euclidean Geometry ($X^2(1) = 2.830$, $p = 0.092 > 0.05$). Students before and after the teaching of Euclidean Geometry, to a large extent, believed that semi-empirical proof for this task is an acceptable form of mathematical proof.

The 2nd and 3rd solutions were empirical proofs and the students stated that they accepted the forms of the specific proofs (66.6% and 58.8% respectively). After the teaching of the chapter of Euclidean Geometry, the students stated with percentage of 33.4% for the 2nd proof and 28.2% for the 3rd solution, indicating an important reduction, as they started rejecting the empirical form of a proof. There was a significant statistical difference before and after the teaching of Euclidean Geometry with $X^2(1) = 79.558$, $p < 0.05$ and $X^2(1) = 69.239$, $p < 0.05$ respectively. It seems that the majority of students, after teaching Euclidean Geometry understood that empirical argumentation is not an acceptable form of a mathematical proof.

The 4th solution was a formal mathematical proof for which 88.7% of the sample, before the teaching of the Euclidean Geometry and after the teaching of geometry, 96.7% accepted it as a mathematical proof. There was a statistically significant difference before and after the teaching of Euclidean Geometry with $X^2(1) = 17.121$, $p < 0.05$.

At the second task, three different forms of proof were given before and after the relevant teaching, the results of which are presented in Table 3.

Table 3. Percentages of answers Task 2 Test A

	Before the teaching of Euclidean Geometry		After the teaching of Euclidean Geometry	
	Yes	No	Yes	No
2A	83.8%	16.2%	76.8%	23.2%
2B	60.8%	39.2%	30.7%	69.3%
2C	80.4%	19.6%	94.5%	5.5%

The 1st solution was a semi-empirical proof, which 83.8% of the sample, before the teaching of Euclidean Geometry, accepted it as a mathematical proof. After the teaching of geometry, 76.8% of the sample accepted it as a mathematical proof. From X^2 test it was found that there was a significant statistical difference before and after the teaching of Euclidean Geometry with $X^2(1) = 5.662$ and $p = 0.017 < 0.05$. The 2nd solution was an empirical proof, which 60.8% of the sample of students answered, before the teaching of Euclidean Geometry, that they accepted it as a mathematical proof, while after the teaching of geometry they did not accept it with a percentage of 69.3%. There was a statistically significant difference before and after the teaching of Euclidean Geometry with $X^2(1) = 66.126$ and $p < 0.05$. The 3rd solution was a formal mathematical proof for which 80.4% of the sample, before the teaching of Euclidean Geometry, accepted it as a mathematical proof, while after the teaching of geometry the percentage of the sample that did not accept it as a mathematical proof was only 5.5%. There was statistically significant difference in this solution $X^2(1) = 32.537$ and $p < 0.05$, showing their correct understanding of the concept of proof.

At the 3rd and 4th tasks students were given two different forms of proof. The 1st solution of the tasks was a semi-empirical proof and the 2nd one presented a solution of a formal mathematical proof. The results of the 3rd task are presented in Table 4.

Table 4. Percentages of answers Task 3 Test A

	Before the teaching of Euclidean Geometry		After the teaching of Euclidean Geometry	
	Yes	No	Yes	No
3A	87.0%	13.0%	59.1%	40.9%
3B	74.9%	25.1%	95.0%	5.0%

In the case of the 1st solution, which was a semi-empirical proof, there was a statistically significant difference before and after the teaching of Euclidean Geometry with $X^2(1) = 71.596$ and $p < 0.05$. The 2nd solution was a formal mathematical proof which 74.9% of the sample, before the teaching of Euclidean Geometry, accepted it as a mathematical proof and after the teaching of geometry, only 5% did not accept it as a mathematical proof. There was a statistically significant difference in this solution $X^2(1) = 57.555$ and $p < 0.05$, showing their correct understanding of the concept of proof. The respective results of the 4th task are presented in Table 5.

Table 5. Percentages of answers Task 4 Test A

	Before the teaching of Euclidean Geometry		After the teaching of Euclidean Geometry	
	Yes	No	Yes	No
4A	89.0%	11.0%	62.4%	37.6%
4B	75.7%	24.3%	98.1%	1.9%

Before the teaching of Euclidean geometry, the students with percentages of 89.0% and 75.7% accepted the 1st and 2nd solutions respectively as mathematical proofs. After finishing the teaching of Euclidean Geometry, 62.4% accepted the semi-empirical proof as mathematical proof while the formal proof was accepted by almost the whole sample, 98.1% of the sample. In both solutions of the fourth task, there was a statistically significant difference before and after the teaching of Euclidean Geometry, $X^2(1) = 69.181$, $p < 0.05$ and $X^2(1) = 79.253$, $p < 0.05$ respectively.

All the tasks which were used in the Test A, revealed that the majority of students, after being taught Euclidean Geometry, understood the meaning of proof and recognized formal proof as the most acceptable type of proof. Nevertheless, there were many students who insisted on accepting the semi-empirical proof as an acceptable format. It seems that there is a tendency to reject the empirical form of a proof, to accept the formal form as the most appropriate and to doubt about the value of the semi-empirical form.

The second research question concerned the factors that compose the general students' ability regarding the handling of different types of proof they encounter and the different types of representations used to represent the mathematical proof. Table 6 presents the percentage of preference for the two tasks of the different types of representations before and after the teaching of Euclidean Geometry.

From the results presented in Table 6, it seems that many students before being taught Euclidean Geometry preferred the form presented in the exercises 1A and 2A, where the data and the questions were presented verbally and without the shape, while after the teaching of Euclidean Geometry there was a significant decrease in their preference. Also, in the exercises that did not present the geometric shape (1A, 1C, 1E, 2A, 2C, 2E) they had reduced the percentage of preference. The majority of students, both before and after the teaching of Euclidean Geometry, chose exercises 1F and 2F which used only mathematical symbols without shape. The difficulty of the majority of students to translate the verbal part in mathematical symbols or terms and to construct the shape was apparent. It seems that their tendency to accept the formal form of a mathematical proof is accompanied by their conception of the necessity of using symbols and a geometric shape as integrated parts of a geometric proof.

Table 6. Percentage of preference for each type of representation for the Tasks 1 and 2

	Before the teaching of Euclidean Geometry	After the teaching of Euclidean Geometry
1A	16.02%	1.93%
1B	9.12%	5.52%
1C	6.35%	1.38%
1D	17.96%	32.60%
1E	13.26%	3.87%
1F	37.29%	54.70%
2A	12.71%	0.83%
2B	5.80%	3.04%
2C	5.80%	0.83%
2D	17.96%	28.45%
2E	15.19%	1.93%
2F	42.54%	64.92%

A key question that concerned the present study was the degree to which students' learning experiences based on the teaching of Euclidean Geometry affected their ability to understand and construct mathematical proofs in Geometry. A part of this question was examined by providing test C, in which two solved mathematical proofs were given and students were asked to identify errors or omissions in them. The test was graded with 0 if they did not detect errors/omissions and 1 if they correctly identified them. Table 7 presents the means and the standard deviations of the tasks for the detection of errors and/or omissions of given mathematical proofs of test C.

Table 7. Mean and standard deviation Test C

	Before the teaching of Euclidean Geometry		After the teaching of Euclidean Geometry	
	Mean	St. Deviation	Mean	St. Deviation
Task 1	0.01	0.074	0.19	0.395
Task 2	0.15	0.354	0.41	0.493
Total	0.15	0.369	0.61	0.788

Based on Table 7, it seems that the students found it difficult to identify errors and/or omissions in the given proofs. The mean performance in both the tasks increased after the teaching of Euclidean Geometry and this is probably due to a better understanding of the mathematical proof. Nevertheless, the mean performance of the first task remained low indicating a difficulty that the students of the sample may have in identifying errors in given mathematical proofs, probably due to the lack of relevant experiences. Although the new textbooks often present the way of students' thinking and ask for interpretation or explanation, it is not common to present the wrong way of thinking and ask them to identify errors and/or omissions.

Subsequently, the Wilcoxon test was performed to examine whether the teaching of Euclidean Geometry influenced the students' ability to understand mathematical proof. The Wilcoxon test showed that there was a statistically significant difference before and after the teaching of Euclidean Geometry both in the two separate tasks and in total score of the test with $Z = -8.246$, $p < 0.05$ and $Z = -0.9749$, $p < 0.05$ for the task 1 and 2 and $Z = -11.103$, $p < 0.05$ for the total.

The second part of the third research question concerned the extent to which students' learning experience, with the teaching of Euclidean Geometry, affected their ability to construct mathematical proof. This question was addressed by providing test D where students were asked to construct three proofs. The test graded with 0-3. The Table 8 presents the students' mean performance and the standard deviations.

Table 8. Mean and standard deviation Test D

	Before the teaching of Euclidean Geometry		After the teaching of Euclidean Geometry	
	Mean	St. Deviation	Mean	St. Deviation
Task 1	0.36	0.661	2.07	0.975
Task 2	0.36	0.541	2.01	1.167
Task 3	0.13	0.498	1.73	1.094
Total	0.74	1.424	5.80	3.045

It is obvious that the students' mean performance at the tasks increased after the teaching of Euclidean Geometry showing that students understood that a geometric proof is constructed without using only empirical arguments. Also, in the overall grade of the test, there was a significant increase. Students before being taught Euclidean Geometry seemed to be unable to respond to the test, as the exercises were unknown to them in terms of content and the process for solving them was difficult. Wilcoxon was used to examine if the teaching of Euclidean Geometry influenced the ability to understand mathematical proofs. The test showed that there was a significant statistical difference before and after the teaching of Euclidean Geometry in all three exercises of the test with $Z = -15.785$, $p < 0.05$, $Z = -15.8$, $p < 0.05$ and $Z = -14.896$, $p < 0.05$ for the tasks 1, 2 and 3 respectively. There is also a statistically significant difference in the total of the test with $Z = -15.785$ and $p < 0.05$. Undoubtedly the present study did not examine the stability of any improvement in order to be able to judge in the long term the appropriateness of the teaching process.

5. Conclusions and discussion

The present study constituted an effort to get an insight to students' geometric proof apprehension. It examined the secondary school students' ability to understand, develop and construct geometric proof at a specific educational system, by giving emphasis on the forms of proofs which are presented at primary education and have to be replaced at secondary education. The results showed that the students after the relevant teaching at secondary education understood the meaning of the mathematical proof and the form of presentation it should have in order to be acceptable in Mathematics. There is a strong tendency to quit the empirical form and to adopt the formal form. However, it seems that at the early grades of secondary education there is a significant number of students who insist on accepting the empirical proofs as structured mathematical proofs, indicating the adhesion on the concept image constructed at primary education. The gap between the empirical reasoning and the mathematical proof has to be overcome gradually and probably by following a different growth rate based on inter-individual differences derived by the cognitive conflict between the experimental perspective and the formation in mathematics. According to Healy and Hoyles (2000) excellent students also tend to think empirically about a proof, since this is how the teaching of proof begins and this facilitates the connection with the already existing cognitive structures. Teachers need to get acquainted with students' conceptions on the specific domain in order to recognize them as the initial starting point of their experiences and based on them pose the short-term and long-term learning aims. The main challenge for the teaching process based on the results of the present study is the fluent transmission from the empirical to the formal form of geometric proof, by accepting the necessary "stations" at the semi-empirical form.

The results, after the teaching of geometric proof in secondary education agreed with the research by Stylianou et al. (2015) in which students accept the formal proof as mathematical proof and recognize that empirical proof is not always accepted. However, at the same time the results are in line with the research of Heinze and Reiss (2003) and Noto et al. (2019) in which students considered the empirical solution as mathematical proof as well as their difficulty to move from empirical to formal mathematical proof. The peak point is the transmission from the empirical reasoning to the mathematical proof by using teaching processes in respect to students' beliefs, conceptions, cognitive styles and learning styles.

Moreover, the results showed that the majority of students preferred the tasks which ask for a proof by presenting the data symbolically and with a given shape, indicating that they believe that those elements are necessary in order to construct and present a formal geometric proof. Their beliefs are not necessarily followed by a respective behavior when they are asked to construct a proof, as they face difficulties to express their thoughts symbolically or through using a shape. Their difficulty in transforming the verbal part into symbolic or into a geometric shape confirmed the findings by Ngirishi and Bansilal (2019) in which most students were unable to connect the verbal part of the mathematical proof to the construction of the shape. Similarly, research by Komatsu et al. (2017) showed that students who failed to solve the mathematical proof were those who failed to construct the shape. The process of connecting the verbal part with the construction of the shape is necessary for the overall understanding of a mathematical proof, and it seems that it needs to have a predominant role on the teaching process.

After the teaching of Euclidean Geometry at secondary education, as it was expected (Hein & Prediger, 2017), students had better results in their ability to construct mathematical proof. However, they had difficulty in identifying errors and/or omissions in given mathematical proofs. The results agree with the study by Komatsu et al. (2017) in which students after being taught mathematical proof in geometry correctly validated a proof, but only 37% of students identified errors. During teaching, through activities, students should be encouraged to evaluate mathematical proofs (Larsen & Zandieh, 2008), either constructed by them as a part of self-reflection or presented to them as a part of critical thinking process.

The present study indicated that the learning experiences affected the students' ability to understand and construct mathematical proof and they played an important role on establishing students' performance, confirming research by Miyazaki et al. (2016) which showed a positive development of students in mathematical proof after the courses for the construction of a geometric proof. However, it reveals at the same time the issues which have to be taken into further consideration during the teaching of geometric proof at secondary education based on the conceptions which were constructed at primary education. The present study did not examine the students' performance under a longitudinal perspective in order to be able to further discuss the cognitive levels of understanding and constructing geometric proofs. Probably, the geometric proof and more generally the mathematical proof could be introduced at the 1st grade of secondary education through a milder transition from the experimental to the semi-experimental and the formal format. Additionally, a future study could be conducted on the upper grades of secondary education in order to have the relevant comparison and mainly the construction of an integraded model about the students' conceptions, beliefs, self-efficacy beliefs and performance on the understanding and the construction of geometric proofs. Finally, it is important for a future study to consider how students with different cognitive characteristics may use different representations for understanding and constructing geometric proofs.

Limitations of the present study:

A limitation of the present study is the inability to analyse qualitatively the students' conceptions on the empirical, semi-empirical and formalistic perspective of geometric proof;

Undoubtedly the sample of the present study is not representative and the results cannot be generalized. However, we can identify a tendency which enables us to propose suggestions about the teaching of geometric proofs at the first grade of secondary education.

Conflict of interest

The authors declare no conflict of interest.

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