



Research Article

On the interpretation of single and bilayer graphene minimal conductivity

Slavik V. Melkonyan

Department of Physics of Semiconductors & Microelectronics, Center of Semiconductor Devices & Nanotechnology, Yerevan State University, 1 Alex Manoogian Str., 0025 Yerevan, Armenia
E-mail: smelkonyan@ysu.am

Received: 1 September 2023; **Revised:** 28 November 2023; **Accepted:** 1 December 2023

Abstract: Within the Boltzmann kinetic theory minimal dc conductivity of single and bilayer graphene is studied. It is shown that the uncertainty principle plays a key role in graphene band structure. In the paper a new alternative interpretation for minimal universal conductivity of single and bilayer graphene is proposed. Minimal conductivity is determined by degenerate electrons and holes of the overlap range of electron and hole bands. It is established that the minimal conductivity of single and bilayer graphene can be explained in the framework of one definite physical approach based on the uncertainty relations. Within the proposed theoretical approach minimal conductivity of both single-layer and bilayer graphene equals $4e^2/h$. It is found that in crystals with parabolic dispersion quantum expansion of an energy level E is asymmetric with respect to E ; in a crystal with linear dispersion an energy level quantum expansion is symmetric.

Keywords: single and bilayer graphene; uncertainty principle; minimal conductivity

1. Introduction

Study of electronic transport in graphene is of great attention. The result of investigations has important theoretical and applicable meaning. Among transport characteristics the conductivity holds a special place [1–4]. Nature of single and bilayer graphene conductivity has been discussed experimentally [5–10] and theoretically [11–28]. As one of the remarkable properties of graphene nonzero minimal conductivity σ_{\min} is noted. Experimental observation shows that practically there is no difference between the minimal conductivity of the single and the bilayer graphene. Minimal conductivity of both single and bilayer graphene is universal value and equals $4e^2/h$ [5–10]. In the literature the existence of nonzero minimal conductivity at low temperatures ($T \sim 0K$) is presented as a unique, unusual, amazing or surprising feature of graphene. Authors believe that in the limit of zero temperature there are no scattering and no current carriers, i.e., $\sigma_{\min} \neq 0$ even at zero carrier concentration. Characterization of minimal conductivity as “carrier free” and “scattering free” conductivity it seems not a true interpretation of experimental data.

There are lots of theoretical papers which are devoted to the explanation of this feature. Graphene minimal conductivity is studied using various phenomenological models and theoretical approaches: Boltzmann’s [11, 12] and Landauer’s transport theories [13, 25], two-dimensional Drude model [11] and Kubo formalism [13–18], polarization tensor formalism [19, 20], Born approximation [21], current-current correlation functions, etc [22–24, 26–28]. For the qualitative perception, there has been discussed the possibility of the phenomenon to connect

Copyright ©2023 Slavik V. Melkonyan
DOI: <https://doi.org/10.37256/1220233609>

This is an open-access article distributed under a CC BY license
(Creative Commons Attribution 4.0 International License)
<https://creativecommons.org/licenses/by/4.0/>

with the Klein's paradox [1, 3, 14]. The results of different theoretical approaches can be presented (taking into account spin and valleys) as: $\sigma_{\min} = 4e^2\gamma/h$, where $\gamma = \pi/32; \pi/16; 1/4\pi; 1/\pi$, etc. [11-28]. Parameter γ is the factor characterising the difference between the theoretical results and experimental data. Note there is essential disagreement with the experiment, $\gamma \sim 0.1 \div 0.08$. Theoretical conductivity σ_{\min} for both single and bilayer graphene is a non-universal quantity and it is inferior to the experimental conductivity for several times. In contrast to experimental data, theoretical works predict that minimal conductivities of single and bilayer graphene are different.

Generalizing the results of the proposed theoretical approaches the following conclusions can be expressed:

- different theoretical approaches and calculation methods lead to different values of σ_{\min} for the same graphene (single or bilayer) system;
- same theoretical approaches lead to different values of σ_{\min} for the single and bilayer graphene;
- theoretical results are nonuniversal and practically are not in agreement with the measured data;
- nature of graphene minimal conductivity is not established finally.

The explanation of minimal conductivity of graphene systems within one definite physical approach still remains a theoretical interest. At discussing the task for single and bilayer graphene, as a rule, a model of zero-gap semiconductor with linear and quadratic low-energy dispersions is used, respectively. It is assumed that the conduction and valence bands touch at the K and K' critical points (Dirac points) of the Brillouin zone. This model underlies almost all theoretical approaches proposed so far (see, e.g., theoretical papers cited above). On the other hand, the value of the bandgap energy is one of the basic characteristics of a crystal. The concept of graphene as a crystal with a zero ($E_g = 0$) bandgap has a principle nature. Note that especially in crystals with a zero (or rather small) bandgap, influence of various well-known effects (for instance, temperature effects, spatial and temporal fluctuations of E_g due to the lattice chaotic vibrations, effects conditioned by uncertainty principle etc.) on E_g become decisive. Therefore, there are real physical reasons to study electronic, partly conductivity features of graphene within the uncertainty principle. Such investigations and detailed analyses practically have not been carried out yet.

1.1 Task outline

For better understanding graphene properties in the present work, "minimal conductivity-uncertainty principle" relationship is considered. Stating the task in this way is motivated by the view that graphene minimal conductivity is a fundamental phenomenon. Below dc conductivity of intrinsic (perfect) single and bilayer graphene at low temperatures ($T \sim 0K$) is analyzed. Consideration is carried out on the base of Boltzmann semiclassical kinetic theory with taking into account well-known general conclusions of the solid state band theory and the key principle of uncertainty (including energy-time as well as momentum-position uncertainties). As a result of the discussion a new alternative interpretation for minimal universal conductivity of single and bilayer intrinsic graphene is proposed.

2. Materials and methods

For the transparent physical understanding graphene conducting properties a semiclassical approach can be used. Below low-temperature conductivity of single and bilayer intrinsic (defect-free) graphene are considered on the base of Boltzmann's transport theory.

i) Single-layer graphene. According to Boltzmann's kinetic equation one valley electron conductivity σ_n of single-layer graphene within the τ -approximation [29, 30] is presented (taking into account spin) as follows:

$$\sigma_n = -\frac{e^2}{2\pi} \int_0^\infty k dk v_F^2 \tau \frac{df_0}{dE} \quad (1)$$

Here, k is the magnitude of the electron wave vector, τ is the electron transport scattering (or relaxation) time, E is the conduction electron energy, v_F is the Fermi group velocity of the Dirac electron and f_0 is the

equilibrium (Fermi) distribution function of the conduction electrons. In Eq.(1) and below, electron and hole conductivities are denoted by σ_n and σ_p , correspondingly.

The low-energy dispersion law of single-layer graphene is linear, $E = \pm v_F \hbar k$, where the \pm signs refer to the conduction and valence bands, respectively. Then converting integral over k to an integral over energy Eq. (1) can be written in the form

$$\sigma_n = -\frac{e^2}{2\pi\hbar^2} \int_{E_c}^{\infty} dE (E\tau) \frac{df_0}{dE}, \quad (2)$$

where E_c is the conduction band edge.

Such important features as magnitude and temperature dependence of σ_n are mainly determined by the relaxation time τ . At low ($T \sim 0K$) temperatures characteristic time τ is determined by the scattering of electrons off crystal boundaries (τ_s) and off iLA (in-plane longitudinal acoustic) phonons (τ_{ac}):

$$\frac{1}{\tau} = \frac{1}{\tau_{ac}} + \frac{1}{\tau_s},$$

where $\tau_s = L/v_F$, L is the size of a 2D crystal. Calculation of electron-iLA phonon scattering rate $1/\tau_{ac}$ in graphene at low temperature range is presented in Appendix.

Many physical properties of condensed matter are explained by band structure. At theoretical studies the following classical model of the band structure of single-layer intrinsic graphene is widely used. It is assumed that E_c and E_v edges of the conduction π^* (or electron) and valence π (or hole) bands touch at the Dirac point, $E_c = E_v$. At Dirac point is also the Fermi level, $E_c = E_v = E_F$. According to classic approach, a single isolated graphene sheet is a zero-gap semiconductor (or zero overlap semimetal). On the other hand, each model as a rule has certain limits of applicability. Therefore, before further estimation of the low-temperature conductivity, let us first clarify some important details about the low-energy spectrum of single-layer graphene. In a real crystal with finite sizes, motion of an electron is limited. As it is well-known, quantum confinement leads to an electron non-zero minimal energy. Quasimomentum and energy of an electron in finite crystal space are quantized. The effect of quantization on the physical properties is primarily determined by the sizes and temperature of the crystal. In the case of confinement by space with macroscopic sizes the separation between adjacent energy levels is extremely small. The adjacent values of allowed energy and quasimomentum are separated by albeit small but finite intervals. In Figure 1a low-energy spectrum of single layer graphene crystal with finite sizes is plotted near Dirac point. From a discrete set of energy levels in Figure 1a only the first (lowest) $E_{c,1}$, $E_{v,1}$ and second $E_{c,2}$, $E_{v,2}$ allowed levels of electron and hole bands, are represented, respectively. Note, that holes energy axis is directed opposite the direction of conduction electron energy axis E_k (Figure 1 and 2). For the convenience of further discussion, in Figure 1a formally presents also the conduction E_c and hole (valence) E_v band edges, which are not allowed levels. E_c and E_v energy levels are forbidden levels. Between the lowest levels $E_{v,1}$ and $E_{c,1}$ lies a range of forbidden energies $E_g = E_{c,1} - E_{v,1}$.

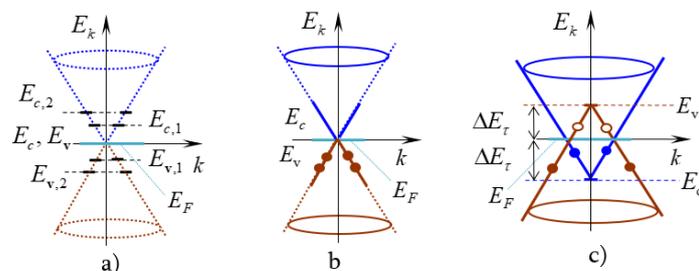


Figure 1. Low-energy band structure of single-layer intrinsic graphene K valley in the case of: a) - ignored energy uncertainty, b) – presence of energy uncertainty ΔE_p associated with quasimomentum-position uncertainty; c) - presence energy total uncertainty ΔE

Another important feature of an energy spectrum of a crystal with finite sizes is related to uncertainty relations. Energy level E is characterized by width, which is determined by the energy uncertainty ΔE as $2\Delta E$. Energy uncertainty ΔE (or standard deviation), in turn, is related to ΔE_p quasimomentum-position ($\Delta p_x \Delta x \approx \hbar / 2$) and ΔE_t energy-time ($\Delta E \tau \approx \hbar$) uncertainties. Here Δp_x and Δx are standard deviations of electron quasimomentum and position, respectively. So, in the direction of motion, an electron quasimomentum uncertainty Δp (where $\Delta p = \sqrt{p^2 - \bar{p}^2}$) leads to energy uncertainty ΔE_p due to $E(p)$ dependence. As well-known, on the base of momentum-position uncertainty the minimal energy of an electron can be evaluated. Minimal energy is calculated by the standard procedure: assuming that momentum uncertainty should not exceed the momentum itself $\Delta E_{pc,1}$ the condition $p_{\min} \approx \Delta p_{\min}$ is applied. In the case of single-layer graphene with linear dispersion $E = v_F p$ one obtains that conduction electron minimal energy ($E_{\min} = v_F p_{\min}$) and the $E_{v,1}$ component of the minimal energy uncertainty ($\Delta E_{\min,p} = v_F \Delta p_{\min}$) are equal quantities, $\Delta E_p = E_{\min}$. Taking into account that in Figure 1a the minimal energy of a conduction electron is denoted by $\Delta E_{pv,1} = E_{v,1}$ it can be stated that $\Delta E_{pc,1} = E_{c,1}$, i.e. the uncertainty $\Delta E_{pc,1}$, determined by quasimomentum-position uncertainty relation is equal to the energy $E_{c,1}$ (i.e. is equal to distance $E_c \div E_{c,1}$ or $E_F \div E_{c,1}$). Note, that this feature has a general character and does not depend on the graphene crystal size and temperature. Similarly the low-energy spectrum of the hole band can be considered and it can be stated that the hole minimal energy $E_{v,1}$ is equal to the uncertainty $\Delta E_{pv,1}$ of the energy level $E_{v,1}$, i.e. $\Delta E_{pv,1} = E_{v,1}$. Thus, if one takes into account quasimomentum uncertainty only, one can state that the single-layer graphene is a zero-gap semiconductor, $E_g = 0$, see Figure 1b. However, this conclusion cannot be considered as final yet; it is an intermediate conclusion. Presented in Figure 1b band structure significantly changes if one now takes also into account the second component of energy uncertainty ΔE_t , which is related to energy-time uncertainty relation $\Delta E_t \approx \hbar / \tau$ of energy levels $E_{c,1}$ and $E_{v,1}$. As shown in Figure 1c, the actual edge of the conduction band, which denoted by E_c , is located now inside the valence band. The actual edge of valence band, denoted by E_v , is located inside the conduction band. Thus, due to total uncertainty $\Delta E = \Delta E_p + \Delta E_t$, conduction and valence bands overlap. Therefore, it can be stated that within a more accurate approach single-layer intrinsic graphene is a semimetal with a non-standard (direct band) band structure (or a semiconductor with a negative band-gap $E_g = -2\Delta E_t$). In the given case, the Fermi level E_F is in conduction and valence bands simultaneously, $E_F = E_c + \Delta E_t$, $E_F = E_v - \Delta E_t$. Here it should be noted that the uncertainty principle has a significant meaning and is manifested in various physical phenomena. Particularly, the well-known effect of temperature dependence of a semiconductor band-gap width is explained by the energy uncertainty.

Let us now calculate the graphene electron one valley conductivity σ_n on the base of the band structure plotted in Figure 1c. For the calculation, the following equation should be used instead of Eq.(2)

$$\sigma_n = -\frac{e^2}{2\pi\hbar^2} \int_{E_c}^{\infty} dE (E\tau) \frac{df_0}{dE}, \quad (3)$$

where

$$f_0 = \left[\exp\left(\frac{E_c + E - E_F}{k_B T}\right) + 1 \right]^{-1}. \quad (4)$$

Using the relation $df_0 / dE|_{T=0} = -\delta(E_c + E - E_F)$ from Eq. (3) one obtains

$$\sigma_n(T=0) = \frac{e^2}{2\pi\hbar^2} (E\tau)_{T=0, E=E_F-E_c}. \quad (5)$$

Since $E_F = E_{c'} + \Delta E_\tau$, where $\Delta E_\tau = \hbar / \tau$ for the minimal conductivity of one valley electrons (taking into account spin) the following expression is obtained:

$$\sigma_{n,\min} = \sigma_n(T = 0) = e^2 / h. \quad (6)$$

In similar fashion, hole band should be considered and the minimal conductivity of holes $\sigma_{p,\min}$ should be calculated. According to Figure 1c band structure the Fermi level E_F is located inside the valence band at the distance ΔE_τ below the edge $E_{v'}$, $E_F = E_{v'} - \Delta E_\tau$. Minimal conductivity $\sigma_{p,\min}$ is due to degenerated gas of holes of the range $E_{v'} < E \leq E_F$. For the minimal conductivity of one valley holes (taking into account spin) $\sigma_{p,\min} = e^2 / h$ is obtained. Thus, for the minimal value of total conductivity of single-layer graphene the following result is obtained (taking into account spin and two valleys)

$$\sigma_{\min} = 2(\sigma_{p,\min} + \sigma_{n,\min}) = 4e^2 / h. \quad (7)$$

As we can see, σ_{\min} is determined by fundamental constants and does not depend on characteristic time τ .

ii) Bilayer graphene. Like single-layer, within the classical model an intrinsic bilayer graphene is a zero-gap semiconductor, but the conduction and hole bands of which are parabolic touching. Now let us apply presented in the previous section **i)** ΔE -approach to study bilayer graphene minimal conductivity. First, let us analyze the energy spectrum of bilayer graphene on the base of uncertainty principle. Energy uncertainty ΔE (or standard deviation $\Delta E = \sqrt{E^2 - \bar{E}^2}$) is interpreted as an energy level expansion. Due to the uncertainty the energy level E is transformed into an energy range $[E_-; E_+]$ with the width $2\Delta E$. Top E_+ and bottom E_- edges of the energy range $[E_-; E_+]$ are symmetric with respect to quantum mechanical average value \bar{E} : $E_- = \bar{E} - \Delta E$, $E_+ = \bar{E} + \Delta E$. However, for the calculation of $\sigma_{n,\min}$ the location of the edge E_- with respect to level E is important. In the case of linear dispersion, average value \bar{E} coincides with the energy E . Consequently, the energy level expansion in single-layer graphene is symmetric with respect to \bar{E} as well as to E . In bilayer graphene with low-energy parabolic dispersion one have other situation. So, quasimomentum uncertainty Δp through $E = p^2 / 2m$ dependence leads to the energy uncertainty. Top E_+ and bottom E_- edges of the energy range $[E_-; E_+]$ now are determined as follows

$$E_+ = \frac{(p + \Delta p)^2}{2m} = E + \frac{(\Delta p)^2 + 2p\Delta p}{2m}, \quad (8)$$

$$E_- = \frac{(p - \Delta p)^2}{2m} = E + \frac{(\Delta p)^2 - 2p\Delta p}{2m}. \quad (9)$$

Minimal width of energy level E is determined as

$$E_+ + E_- = \frac{2p|\Delta p|}{m}. \quad (10)$$

As one can see from Eqs.(8)-(9), E_+ and E_- edges are asymmetric with respect to E . Now let us this peculiarity apply to the lowest energy level $E_{c,1}$ of the bilayer graphene conduction band. Using standard condition $p_{\min} \approx \Delta p_{\min}$ from Eqs. (8), (9) and (10) for the expansion range $[E_{c,1-}; E_{c,1+}]$ of the energy level $E_{c,1}$ one have

$$E_{c,1+} = E_{c,1} + \frac{3(\Delta p_{\min})^2}{2m}, \quad E_{c,1-} = E_{c,1} - \frac{(\Delta p_{\min})^2}{2m}, \quad (11)$$

$$E_{c,1+} - E_{c,1-} = \frac{2(\Delta p_{\min})^2}{m}. \quad (12)$$

If one denotes the width $E_{c,1+} - E_{c,1-}$ by $2\Delta E_p$ (i.e. $E_{c,1+} - E_{c,1-} \equiv 2\Delta E_p$, where $\Delta E_p \equiv (\Delta p_{\min})^2 / m$), then one can state that the level $E_{c,1}$ expands upwards and downwards by a magnitudes $3\Delta E_p / 2$ and $\Delta E_p / 2$, respectively,

$$E_{c,1+} = E_{c,1} + \frac{3}{2}\Delta E_p, \quad E_{c,1-} = E_{c,1} - \frac{1}{2}\Delta E_p. \quad (13)$$

This ratio of asymmetric expansion has a general nature. The expansion $2\Delta E_\tau$ related with the energy-time uncertainty is asymmetric also - the level $E_{c,1}$ expands upwards and downwards by magnitudes $3\Delta E_\tau / 2$ and $\Delta E_\tau / 2$, respectively. Thus, in the case of parabolic dispersion the total uncertainty $\Delta E = \Delta E_p + \Delta E_\tau$ is distributed as follows

$$E_{c,1+} = E_{c,1} + \frac{3}{2}\Delta E, \quad E_{c,1-} = E_{c,1} - \frac{1}{2}\Delta E. \quad (14)$$

In particular, the second relation of Eq.(14) can be represented as follows

$$E_{c,1-} = E_{c,1} - \frac{1}{2}(\Delta E_p + \Delta E_\tau). \quad (15)$$

Using the standard condition $p_{\min} \approx \Delta p_{\min}$ and estimate the minimal energy $E_{c,1}$, one obtains $E_{c,1} = p_{\min}^2 / 2m = (\Delta p_{\min})^2 / 2m = \Delta E_p / 2$. Therefore, from relation (15) one will have

$$E_{c,1-} = -\Delta E_\tau / 2 = -\hbar\tau / 2. \quad (16)$$

In similar fashion, hole band edge $E_{v,1}$ should be considered. For the edge $E_{v,1-}$ of the expansion range $[E_{v,1-}; E_{v,1+}]$ of the lowest energy level $E_{v,1}$ of the hole band the similar expression is obtained

$$E_{v,1-} = -\Delta E_\tau / 2 = -\hbar\tau / 2. \quad (17)$$

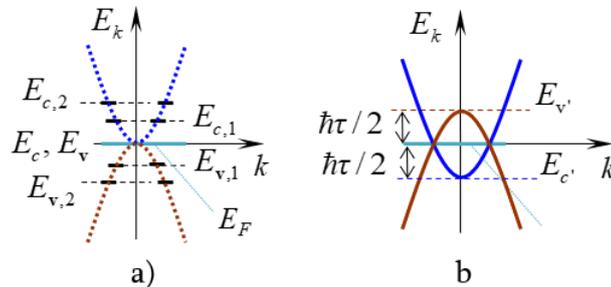


Figure 2. Band structure of an intrinsic bilayer graphene K valley at energy uncertainty ignored (a) and presence (b) cases

In Figure 2 shows a low-energy spectrum of an intrinsic bilayer graphene in the cases of ignored (a) and presence (b) of energy uncertainty. As shown in Figure 2b, the actual edge $E_{c,0}$ of the conduction band (i.e., the edge $E_{c,1-}$ of the range $[E_{c,1-}; E_{c,1+}]$, $E_{c,1-} \equiv E_{c,0}$) is located inside the valence band, at the distance $\hbar / 2\tau$ from the Fermi level, $E_{c,0} = E_F - \hbar / 2\tau$. The actual edge $E_{v,0}$ of the hole band (i.e., the edge $E_{v,1-}$ of the range $[E_{v,1-}; E_{v,1+}]$, $E_{v,1-} \equiv E_{v,0}$) is located inside the conduction band, at the distance $\hbar / 2\tau$ from the Fermi level, $E_{v,0} = E_F + \hbar / 2\tau$. Therefore, intrinsic bilayer graphene is a semimetal also whose conduction and valence bands are overlapped by magnitude \hbar / τ . Now based on the band structure presented in Figure 2b electron conductivity σ_n is calculated. According to Boltzmann's theory, one valley electron dc conductivity of a bilayer graphene can be represented by the following relation (taking into account the spin)

$$\sigma_n = -\frac{e^2}{\pi\hbar^2} \int_{E_c}^{\infty} dE (E\tau) \frac{df_0}{dE}, \quad (18)$$

where $E_c = E_F - \hbar/2\tau$.

At temperature $T = 0K$ from Eq.(18) $\sigma_n(T = 0) \equiv \sigma_{n,\min} = e^2/h$ is obtained. In a similar way for the hole minimal conductivity of a one valley (taking into account the spin) $\sigma_{p,\min} = e^2/h$ is obtained. Therefore, for the minimal conductivity of intrinsic bilayer graphene (taking into account spin and two valleys) one has $\sigma_{\min} = 4e^2/h$.

3. Results and Discussion

Minimal conductivity and other important transport or electronic properties of a crystal are primarily related to the energy spectrum. According to the above-presented approach, if one ignores uncertainties quantum effects, then conduction and valence bands of a single and bilayer graphene are separated by forbidden range, $E_g \neq 0$, see Figure 1a and 2a, $E_g = E_{c,1} - E_{v,1}$. If one take into account the quasimomentum-position uncertainty only, then conduction and valence bands are touching each other, $E_g = 0$. If one takes into account the energy-time uncertainty as well, then conduction and valence bands are overlapping, $E_g < 0$. Bands overlap is determined by the energy-time uncertainty ΔE_τ and consequently by characteristic time τ and the crystal sizes L (see, Eq. (A9)). Single-layer graphene bands overlap is $2\hbar/\tau$; bilayer graphene bands overlap is \hbar/τ . So, for example, the overlap of single-layer graphene with $L = 200$ nm is ~ 6 meV. For comparison, note the overlap of three-dimensional graphite bands is ~ 40 meV (experiment) or about $8 \div 30$ meV (theory). Thus, intrinsic single and bilayer graphene crystals are semimetals with a slightly overlap bands. Therefore, graphene must have and has the semimetallic behavior. Semimetals are characterized by a weak bands overlap; conductivity is always finite and non-zero; in the minimal conductivity electrons as well as holes have contributions.

Within the above-proposed Boltzmann's quasi-classical ΔE -model, minimal conductivity σ_{\min} of both single layer and bilayer graphene is equal to $4e^2/h$. It is a universal quantity and does not depend on the characteristic time τ and the 2D crystal sizes L . Note, first, the expression $\sigma_{\min} = 4e^2/h$ is differ from the expressions obtained in other theoretical papers by using various calculation methods. Second, which is very important, the value σ_{\min} is in complete agreement with the experimental data and it is not "carrier free" conductivity. The proposed theoretical approach predicts that the minimal conductivity of bilayer graphene is not different from that of single layer graphene.

Note that evaluations which were carried out on the base of uncertainty relations usually are considered as qualitative. On the other hand, the result $\sigma_{\min} = 4e^2/h$, which is also obtained on the base of the uncertainty principle, directly coincides with the experimental data. Therefore, it can also be considered as quantitative result. In solid state physics, there are cases when results of estimations based on the uncertainties coincide with the experimental data and/or with the results of more accurate calculations. Other important conclusion of the presented approach is the following. In crystals with parabolic dispersion, quantum expansion of an energy level E is asymmetric with respect to E . In particular, expansion $2\Delta E$ of the lowest energy level of the conduction and hole bands are distributed as follows – 1/4 part downward and 3/4 part upward. In the case of linear dispersion, the lowest energy level expansion $2\Delta E$ is symmetric – 1/2 part downward and 1/2 part upward.

4. Conclusions

So, the main conclusions of the paper are the following:

1. in crystals with parabolic dispersion, quantum expansion of an energy level is asymmetric;
2. in crystals with linear dispersion, quantum expansion of an energy level is symmetric;
3. single and bilayer intrinsic graphene are semimetals whose bands overlap determined by the uncertainties relations and equal $2\hbar/\tau$ and \hbar/τ , respectively;

4. minimal conductivity $4e^2/h$ of single and bilayer intrinsic graphene is determined by the degenerated electrons and holes of the overlap range of allowed bands.

Conflict of interest

There is no conflict of interest for this study.

Appendix

Rate of electron intravalley scattering off iLA phonons is given by [30, 31]

$$\frac{1}{\tau_{ac}} = \sum_{\mathbf{k}', \mathbf{q}} \frac{\pi q^2 D_{ac}^2}{2 \rho A \omega_q^{ac}} (1 - \cos^2 \theta) \left[N_q \delta_{\mathbf{k}', \mathbf{k} + \mathbf{q}} \delta(E(\mathbf{k}) - E(\mathbf{k}') + \hbar \omega_q^{ac}) + (N_q + 1) \delta_{\mathbf{k}', \mathbf{k} - \mathbf{q}} \delta(E(\mathbf{k}) - E(\mathbf{k}') - \hbar \omega_q^{ac}) \right]. \quad (\text{A1})$$

Here \mathbf{k} and \mathbf{q} are the electron and phonon wave vectors (boldface type is used to represent vectors), respectively, θ is the angle between \mathbf{k} and \mathbf{k}' vectors (scattering angle), N_q is the phonons equilibrium distribution function, ρ is the mass density of graphene, A is the area of graphene sheet, ω_q^{ac} is the iLA phonon frequency, D_{ac} is the acoustic phonon intravalley deformation potential. In expression (A1) the Kronecker delta $\delta_{\mathbf{k}', \mathbf{k} - \mathbf{q}}$ expresses quasimomentum conservation, and the delta function, energy conservation.

The relaxation rate $1/\tau_{ac}$ consists of two components, $1/\tau_{ac} = 1/\tau_{ac,ab} + 1/\tau_{ac,em}$. The first term on the r.h.s. of the sum (A1) describes the electron scattering with phonon induced absorption ($\tau_{ac,ab}$). The second term on the r.h.s. of the sum (A1) describes the electron scattering with induced and spontaneous emission of phonon ($\tau_{ac,em}$). At $T \sim 0K$ temperatures $N_q \xrightarrow{T \rightarrow 0} 0$ and electron-phonon scattering is due to phonon spontaneous emission only. Therefore, at low temperatures from Eq. (A1) for the relaxation rate $1/\tau_{ac}$ one has

$$\frac{1}{\tau_{ac}} = \frac{1}{\tau_{ac,em}} = \sum_{\mathbf{k}', \mathbf{q}} \frac{\pi q^2 D_{ac}^2}{2 \rho A \omega_q^{ac}} (1 - \cos^2 \theta) \delta_{\mathbf{k}', \mathbf{k} - \mathbf{q}} \delta(E(\mathbf{k}) - E(\mathbf{k}') - \hbar \omega_q^{ac}). \quad (\text{A2})$$

A simple summation over \mathbf{k}' is carried out using Kronecker delta $\delta_{\mathbf{k}', \mathbf{k} - \mathbf{q}}$. For further calculations it is taken into account that iLA phonons dispersion law of single-layer graphene is linear, $\omega_q^{ac} = v_{ac} q$, where v_{ac} is the acoustic phonon velocity [13, 30, 32]. Then, from Eq.(A2) one obtains

$$\frac{1}{\tau_{ac}} = \sum_{\mathbf{q}} \frac{\pi q D_{ac}^2}{2 \hbar \rho A v_F v_{ac}} (1 - \cos^2 \theta) \delta(k - |\mathbf{k} - \mathbf{q}| - q v_{ac} / v_F). \quad (\text{A3})$$

It is more convenient that the polar coordinate axis is directed along vector \mathbf{k} and the following transition from sum to integration is used

$$\sum_{\mathbf{q}} \rightarrow \frac{A}{(2\pi)^2} \int_{BZ} d\mathbf{q} = \frac{A}{(2\pi)^2} \int_{q_{\min}}^{q_{\max}} \int_0^{2\pi} q dq d\alpha. \quad (\text{A4})$$

where α is the angle between vectors \mathbf{k} and \mathbf{q} .

Converting the sum (A3) to an integral one have

$$\frac{1}{\tau_{ac}} = \frac{D_{ac}^2}{8\pi \hbar \rho v_F v_{ac}} \int_{q_{\min}}^{q_{\max}} \int_0^{2\pi} dq d\alpha (1 - \cos^2 \theta) q^2 \delta(k - |\mathbf{k} - \mathbf{q}| - q v_{ac} / v_F), \quad (\text{A5})$$

where δ -function expresses the energy conservation law. Limits of the integration over q are determined by the energy conservation as follows: $q_{\min} = 0$, $q_{\max} = 2k/(1+b)$, where $b = v_{ac}/v_F$. Using the well-known property of δ -function,

$$\delta(\varphi(x)) = \frac{\delta(x-x_0)}{|\varphi'(x_0)|}, \quad (\text{A6})$$

Eq.(A5) can be presented as

$$\frac{1}{\tau_{ac}} = \frac{D_{ac}^2}{8\pi\hbar\rho v_F v_{ac}} \int_{q_{\min}}^{q_{\max}} \int_0^{2\pi} dq d\alpha (1 - \cos^2 \theta) q \frac{|k-bq|}{k} \delta\left(\cos \alpha - \frac{2kb + (1-b^2)q}{2k}\right). \quad (\text{A7})$$

Here x_0 is the solution of equation $\varphi(x) = 0$

Using the following relationship between angles θ and α

$$\cos \theta = \frac{kk'}{kk'} = \frac{k(k-q)}{k|k-q|} \text{ or } \cos^2 \theta = \frac{(k^2 - kq \cos \alpha)^2}{k^2(k^2 + q^2 - 2kq \cos \alpha)},$$

integrations in Eq.(A7) is carried out first over α (δ -integration) and then over q . As result, taking into account $b = v_{ac}/v_F \ll 1$ inequality ($v_{ac} \approx 2 \cdot 10^4$ m/s, $v_F \approx 10^6$ m/s [13, 30, 32]) the following expression for low-temperature τ_{ac} is obtained

$$\frac{1}{\tau_{ac}} = \frac{4D_{ac}^2 E^2}{15\pi\rho\hbar^3 v_F^3 v_{ac}}. \quad (\text{A8})$$

This relation characterizes basic peculiarities of low-temperature electron-iLA phonon scattering rate in single-layer graphene. Thus, in the low-energy range the electron total scattering rate can be presented as

$$\frac{1}{\tau} = \frac{4D_{ac}^2 E^2}{15\pi\rho\hbar^3 v_F^3 v_{ac}} + \frac{v_F}{L}. \quad (\text{A9})$$

References

- [1] Castro Neto, A.H.; Guinea, F.; Peres, N.M.R.; Novoselov, K.S.; Geim, A.K. The electronic properties of graphene. *Rev. Mod. Phys.* **2009**, *81*, 109–162, <https://doi.org/10.1103/revmodphys.81.109>.
- [2] Das Sarma, S.; Adam, S.; Hwang, E.H.; Rossi, E. Electronic transport in two-dimensional graphene. *Rev. Mod. Phys.* **2011**, *83*, 407–470, <https://doi.org/10.1103/revmodphys.83.407>.
- [3] Katsnelson M.I. Graphene: Carbon in Two Dimensions. Cambridge: Cambridge University Press; N.Y., USA, 2012.
- [4] McCann, E.; Koshino, M. The electronic properties of bilayer graphene. *Rep. Prog. Phys.* **2013**, *76*, 056503, <https://doi.org/10.1088/0034-4885/76/5/056503>.
- [5] Novoselov, K.S.; Geim, A.K.; Morozov, S.V.; Jiang, D.; Katsnelson, M.I.; Grigorieva, I.V.; Dubonos, S.V.; Firsov, A.A. Two-dimensional gas of massless Dirac fermions in graphene. *Nature* **2005**, *438*, 197–200, doi:10.1038/nature04233.
- [6] Zhang, Y.; Tan, Y.-W.; Stormer, H.L.; Kim, P. Experimental observation of the quantum Hall effect and Berry's phase in graphene. *Nature* **2005**, *438*, 201–204, doi:10.1038/nature04235.
- [7] Novoselov, K.S.; McCann, E.; Morozov, S.V.; Fal'ko, V.I.; Katsnelson, M.I.; Zeitler, U.; Jiang, D.; Schedin, F.; Geim, A.K. Unconventional quantum Hall effect and Berry's phase of 2π in bilayer graphene. *Nat. Phys.* **2006**, *2*, 177–180, <https://doi.org/10.1038/nphys245>.
- [8] Tan, Y.-W.; Zhang, Y.; Bolotin, K.; Zhao, Y.; Adam, S.; Hwang, E.H.; Das Sarma, S.; Stormer, H.L.; Kim, P. Measurement of Scattering Rate and Minimum Conductivity in Graphene. *Phys. Rev. Lett.* **2007**, *99*, 246803–246803, <https://doi.org/10.1103/physrevlett.99.246803>.
- [9] Cho, S.; Fuhrer, M.S. Charge transport and inhomogeneity near the minimum conductivity point in graphene. *Phys. Rev. B* **2008**, *77*, 081402, <https://doi.org/10.1103/physrevb.77.081402>.
- [10] Chen, J.-H.; Jang, C.; Adam, S.; Fuhrer, M.S.; Williams, E.D.; Ishigami, M. Charged-impurity scattering in graphene. *Nat. Phys.* **2008**, *4*, 377–381, <https://doi.org/10.1038/nphys935>.

- [11] Moriconi, L.; Niemyer, D. Graphene conductivity near the charge neutral point. *Phys. Rev. B* **2011**, *84*, <https://doi.org/10.1103/physrevb.84.193401>.
- [12] Trushin, M.; Schliemann, J. Minimum Electrical and Thermal Conductivity of Graphene: A Quasiclassical Approach. *Phys. Rev. Lett.* **2007**, *99*, 216602, <https://doi.org/10.1103/physrevlett.99.216602>.
- [13] Kruglyak Yu A. Properties of Graphene in Landauer–Datta–Lundstrom Transport Model. *Nanosystems, Nanomaterials, Nanotechnologies*, 2015; *13*: 215-242.
- [14] Trushin, M.; Kailasvuori, J.; Schliemann, J.; MacDonald, A.H. Finite conductivity minimum in bilayer graphene without charge inhomogeneities. *Phys. Rev. B* **2010**, *82*, <https://doi.org/10.1103/physrevb.82.155308>.
- [15] Ziegler, K. Minimal conductivity of graphene: Nonuniversal values from the Kubo formula. *Phys. Rev. B* **2007**, *75*, 233407, <https://doi.org/10.1103/physrevb.75.233407>.
- [16] Bácsi, .; Virosztek, A. Low-frequency optical conductivity in graphene and in other scale-invariant two-band systems. *Phys. Rev. B* **2013**, *87*, <https://doi.org/10.1103/physrevb.87.125425>.
- [17] Dartora, C.A.; Cabrera, G.G. $U(1) \times SU(2)$ gauge invariance leading to charge and spin conductivity of Dirac fermions in graphene. *Phys. Rev. B* **2013**, *87*, 165416, <https://doi.org/10.1103/physrevb.87.165416>.
- [18] Cserti, J. Minimal longitudinal dc conductivity of perfect bilayer graphene. *Phys. Rev. B* **2007**, *75*, <https://doi.org/10.1103/physrevb.75.033405>.
- [19] Klimchitskaya, G.L.; Mostepanenko, V.M. Conductivity of pure graphene: Theoretical approach using the polarization tensor. *Phys. Rev. B* **2016**, *93*, 245419, <https://doi.org/10.1103/physrevb.93.245419>.
- [20] Klimchitskaya, G.L.; Mostepanenko, V.M.; Petrov, V.M. Conductivity of graphene in the framework of Dirac model: Interplay between nonzero mass gap and chemical potential. *Phys. Rev. B* **2017**, *96*, 235432, <https://doi.org/10.1103/physrevb.96.235432>.
- [21] Koshino, M.; Ando, T. Transport in bilayer graphene: Calculations within a self-consistent Born approximation. *Phys. Rev. B* **2006**, *73*, 245403, <https://doi.org/10.1103/physrevb.73.245403>.
- [22] Yokota, K.; Imoto, N. A weak-value model for virtual particles supplying the electric current in graphene: the minimal conductivity and the Schwinger mechanism. *New J. Phys.* **2014**, *16*, 73003, <https://doi.org/10.1088/1367-2630/16/7/073003>.
- [23] Palacios, J.J. Origin of the quasiuniversality of the minimal conductivity of graphene. *Phys. Rev. B* **2010**, *82*, <https://doi.org/10.1103/physrevb.82.165439>.
- [24] Dragoman, D. Low-energy conductivity of single- and double-layer graphene from the uncertainty principle. *Phys. Scr.* **2010**, *81*, <https://doi.org/10.1088/0031-8949/81/03/035702>.
- [25] Müller, M.; Bräuninger, M.; Trauzettel, B. Temperature Dependence of the Conductivity of Ballistic Graphene. *Phys. Rev. Lett.* **2009**, *103*, 196801–196801, <https://doi.org/10.1103/physrevlett.103.196801>.
- [26] Louvet, T.; Delplace, P.; Fedorenko, A.A.; Carpentier, D. On the origin of minimal conductivity at a band crossing. *Phys. Rev. B* **2015**, *92*, <https://doi.org/10.1103/physrevb.92.155116>.
- [27] Rani, L.; Singh, N. Dynamical electrical conductivity of graphene. *J. Physics: Condens. Matter* **2017**, *29*, 255602, <https://doi.org/10.1088/1361-648x/aa6ebf>.
- [28] Herrera, S.A.; Naumis, G.G. Electronic and optical conductivity of Kekulé-patterned graphene: Intravalley and intervalley transport. *Phys. Rev. B* **2020**, *101*, 205413, <https://doi.org/10.1103/physrevb.101.205413>.
- [29] Hamaguchi C. *Basic Semiconductor Physics*, Springer 2010.
- [30] Berdebes D, Low T and Lundstrom M. Low Bias Transport in Graphene: An Introduction. In Proc. NCN@Purdue Summer School: Electronics from the Bottom Up, 2009. Available: www.nanoHUB.org
- [31] Stauber, T.; Peres, N.M.R.; Guinea, F. Electronic transport in graphene: A semiclassical approach including midgap states. *Phys. Rev. B* **2007**, *76*, 205423, <https://doi.org/10.1103/physrevb.76.205423>.
- [32] Singh, D.; Murthy, J.Y.; Fisher, T.S. Spectral phonon conduction and dominant scattering pathways in graphene. *J. Appl. Phys.* **2011**, *110*, 094312, <https://doi.org/10.1063/1.3656451>.