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Research Article

Multisoliton Bound States in the Fourth-Order Concatenation Model of the Nonlinear Schrödinger Equation Hierarchy

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Abstract: We present a comparative analysis of exact soliton solutions that form multisoliton bound states arising in the hierarchy of the even-order nonlinear Schrödinger equations. Specifically, we consider two-soliton bound states in three models: the nonlinear Schrödinger equation, the fourth-order nonlinear equation, and the Lakshmanan-Porsezian-Daniel equation (LPDE). The LPDE may be viewed as one example of an infinite concatenation hierarchy. The exact integrability of these equations is ensured by the Lax pairs constructed using the Ablowitz, Kaup, Newell, and Segur formalism of the Inverse Scattering Transform method. We confirm that the main property of solitons-to interact elastically and attract or repel each other in the collision region depending on the difference of their initial phases-is also preserved for LPDE solitons, and this property should take place in the entire hierarchy of concatenation models. We present the detailed dynamics of two-soliton bound states periodically breathing in space and time, the conditions of their formation, and analytical formulas for their oscillation periods in all the considered models.

***Keywords*:** nonlinear Schrödinger equation, fourth-order nonlinear equation, two-soliton bound states, Lakshmanan-Porsezian-Daniel equation, oscillation periods, concatenation model

1. Introduction

Last year, the scientific community celebrated the 50th anniversary of optical solitons [1], predicted in 1973 by Hasegawa and Tappert [2, 3], and realized in 1980 by Mollenauer et al. [4]. In nonlinear optics, these solitons are often called nonlinear Schrödinger equation (NLSE) solitons to emphasize the physical mechanism of their formation as a complete compensation for the dispersion broadening of optical wave packets due to nonlinear self-compression [4–13].

Historically, rapid progress in the theory of optical solitons has been associated with the development and generalization of the method of slowly varying amplitudes (SVA) taking into account the higher-order dispersion and nonlinear effects [4–10]. It should be emphasized that in this way, many practical problems of physics of extremely short optical solitons and soliton supercontinuum were completely solved in the framework of extended NLSE models. However, as a rule, all subsequent SVA mathematical models turned out to be nonintegrable by the Inverse Scattering Transform (IST) method [14–19]. Therefore, it became natural to turn attention to another possibility for developing the theory of optical solitons, namely, to the consideration of nonlinear equations of the soliton hierarchy of ever higher order.

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One of the famous examples of such soliton hierarchy arises in the Ablowitz-Kaup-Newell-Segur (AKNS) formalism of the IST method after expanding the matrix elements into an infinite series in powers of the complex spectral parameter Λ [19]. For example, the integrable fourth-order nonlinear equations in this hierarchy have attracted considerable interest because of their possible physical applications, in particular in ultrashort nonlinear optics. One of the fourth-order equations, known today as the Lakshmanan-Porsezian-Daniel equation (LPDE) [20–22], was proposed to describe nonlinear spin waves appearing in one-dimensional Heisenberg spin chains under various types of magnetic interactions in ferromagnetic metals. The LPDE is attracting much attention today (see, e.g., [23–46] and references therein) because of its possible applications to systems that are considered potential data carriers for nanometer-scale computing devices.

Recently, Anjan Biswas with collaborators proposed a surprisingly appropriate and figurative name for the chain of equations arising in various soliton hierarchies, namely, concatenation models [23, 24]. The first examples of concatenation models were studied by Lakshmanan et al. [20–22], Zhang et al. [25], Guo and Hao [26], Ankievich and Akhmediev [27]: a chain of three well-known evolution equations, NLSE, the Hirota equation, and the LPDE. Since then, there has been rapid progress in this field, with novel and fundamental results obtained by leading groups of Akhmediev and Ankievich, Biswas, Kudryashov, Guo, Wazwaz, and Zhow (see, e.g., [23–46] and references therein).

The main objective of our paper is the comparative analysis of two-soliton bound states in the simplest concatenation model formed by the chain of the NLSE, the purely fourth-order equation (FONE), and the LPDE. The exact integrability of these equations is ensured by the Lax pair constructed using the AKNS formalism of the IST method. We present the detailed dynamics of two-soliton bound states periodically breathing in space and time, the conditions of their formation, and analytical formulas for their oscillation periods in all the considered models.

2. Soliton solutions of the nonlinear Schrödinger, the fourth-order, and the LDP equations

The IST method [14–19, 47–50] was developed more than sixty years ago and allows one to construct completely integrable nonlinear evolution equations of arbitrary order. The first even terms of the NLSE hierarchy are represented by the second- and fourth-order equations that have solitary wave solutions: the NLSE

$$iq_t + \frac{1}{2}D_2 q_{xx} + R_2|q|^2 q = 0, \tag{1}$$

the FONE

$$iq_t + D_4 q_{xxxx} + 6\frac{R_4^2}{D_4}|q|^4 q + R_4\left(8q_{xx}|q|^2 + 2q^2 q_{xx}^* + 4|q_x|^2 q + 6(q_x)^2 q^*\right) = 0, \tag{2}$$

and the LPDE

$$\begin{aligned} &iq_t + \frac{1}{2}D_2 q_{xx} + R_2|q|^2 q + D_4 q_{xxxx} + 6\frac{R_4^2}{D_4}|q|^4 q \\ &+ R_4\left(8q_{xx}|q|^2 + 2q^2 q_{xx}^* + 4|q_x|^2 q + 6(q_x)^2 q^*\right) = 0, \end{aligned} \tag{3}$$

with constant coefficients of dispersion D_2 and D_4, and nonlinearity R_2 and R_4.

Equations (1)-(3) with arbitrary constant coefficients arise as the compatibility condition

$$\widehat{\mathcal{F}}_t - \widehat{\mathcal{G}}_x + [\widehat{\mathcal{F}},\ \widehat{\mathcal{G}}] = 0 \tag{4}$$

of the system of the linear matrix differential equations

$$\Psi_x = \widehat{\mathcal{F}}\Psi(x,\ t), \quad \Psi_t = \widehat{\mathcal{G}}\Psi(x,\ t) \tag{5}$$

where $\Psi(x,\ t) = \{\psi_1,\ \psi_2\}^T$ is a 2-component complex function and the complex-valued (2×2) matrices are written as

$$\widehat{\mathcal{F}} = \begin{pmatrix} -i\Lambda & F^{1/2}q(x,\ t) \\ -F^{1/2}q^*(x,\ t) & i\Lambda \end{pmatrix}, \quad \widehat{\mathcal{G}} = \begin{pmatrix} A & B \\ -B^* & -A \end{pmatrix}, \tag{6}$$

with

$$A = \frac{i}{2}R_2|q|^2 + 3iF^2|q|^4 + iR_4\left(qq^*_{xx} + q_{xx}q^* - q_xq^*_x\right) \tag{7}$$

$$+ 2i\Lambda R_4\left(qq^*_x - q_xq^*\right) - i\Lambda^2\left(4R_4|q|^2 + D_2\right) + 8iD_4\Lambda^4,$$

$$B = iF^{1/2}\Big\{D_4q_{xxx} + 3R_4|q|^2q_x + \frac{1}{2}D_2q_x \tag{8}$$

$$+ \left[-iD_2q - 4iD_4|q|^2q - 2iD_4q_{xx}\right]\Lambda - 4D_4q_x\Lambda^2 + 8iD_4q\Lambda^3\Big\},$$

and time-independent spectral parameter $\Lambda = \kappa + i\eta$, which defines the velocity $V = 2\kappa$ and the amplitude $a = 2\eta$ of the soliton. The isospectral AKNS hierarchy corresponds to the $d\Lambda/dt = 0$.

The integrability condition of the LPDE as well as the FONE and NLSE is given by Eq. (4). The solution of this matrix equation leads to the condition

$$F = \frac{R_2}{D_2} = \frac{R_4}{D_4} \tag{9}$$

that is a generalization of the constraint found by Hirota in his work [47] for the third-order nonlinear equation of the NLSE hierarchy, which actually has his name-the Hirota equation. Hirota's condition states that $R_2/D_2 = R_3/D_3$ as is shown in his work [47], where R_3 and D_3 are the nonlinear and dispersion coefficients of the third order. In our case, this integrability condition is accompanied by the term R_4/D_4 instead of the R_3/D_3, as is clarified in Eq. (9).

Soliton solutions of order N of equations (1)-(3) can be obtained by applying the auto-Bäcklund transformation and the recurrent relation [48–50]

$$q_N(x,\ t) = -q_{N-1}(x,\ t) - 4F^{-1/2}\frac{\eta_N\widetilde{\boldsymbol{\Gamma}}_{N-1}(x,\ t)}{1+\left|\widetilde{\boldsymbol{\Gamma}}_{N-1}(x,\ t)\right|^2}, \tag{10}$$

which connects the $(N-1)$-th and N-th soliton solutions via the function $\widetilde{\boldsymbol{\Gamma}}_{N-1}(x,\ t) = \psi_1(x,\ t)/\psi_2(x,\ t)$ for the $(N-1)$-th soliton scattering functions $\Psi(x,\ t) = (\psi_1\,\psi_2)^T$. We begin the recurrent process at the zero-valued potential $q(x,\ t) = 0$ and write here the one soliton solution

$$q_1(x,\ t) = 2\eta_1 F^{-1/2}\,\text{sech}\,[\xi_1(x,\ t)]\exp[-i\chi_1(x,\ t)], \tag{11}$$

and the two-soliton solution of Eqs. (1)-(3)

$$q_2(x,\ t) = 4F^{-1/2}\frac{\mathbb{N}(x,\ t)}{\mathbb{D}(x,\ t)}, \tag{12}$$

where

$$\begin{aligned}\mathbb{N} =&\eta_1\cosh\xi_2\exp(-i\chi_1)\left[(\kappa_2-\kappa_1)^2+2i\eta_2(\kappa_2-\kappa_1)\tanh\xi_2+\left(\eta_1^2-\eta_2^2\right)\right]\\ &+\eta_2\cosh\xi_1\exp(-i\chi_2)\left[(\kappa_2-\kappa_1)^2-2i\eta_1(\kappa_2-\kappa_1)\tanh\xi_1-\left(\eta_1^2-\eta_2^2\right)\right],\end{aligned} \tag{13}$$

and

$$\begin{aligned}\mathbb{D} =&\cosh(\xi_1+\xi_2)\left[(\kappa_2-\kappa_1)^2+(\eta_2-\eta_1)^2\right]\\ &+\cosh(\xi_1-\xi_2)\left[(\kappa_2-\kappa_1)^2+(\eta_2+\eta_1)^2\right]\\ &-4\eta_1\eta_2\cos(\chi_2-\chi_1).\end{aligned} \tag{14}$$

The main parameters of the soliton solutions (11)-(14) are determined by the following expressions for each soliton with $i = 1$ and $i = 2$.

$$\xi_i(x,\ t) = 2(x-x_{0i})\,\eta_i + \left[4\eta_i\kappa_i D_2 + 64\eta_i\kappa_i\left(\eta_i^2-\kappa_i^2\right)D_4\right]t, \tag{15}$$

$$\chi_i(x,\ t) = 2(x-x_{0i})\,\kappa_i + \left[2\left(\kappa_i^2-\eta_i^2\right)D_2 - 16\left(\kappa_i^4-6\kappa_i^2\eta_i^2+\eta_i^4\right)D_4\right]t + \chi_{0i}, \tag{16}$$

where η_i and κ_i denote the amplitude and velocity of each soliton, x_{0i} and χ_{0i} denote their initial spatial position and phase. Two features of the exact solutions (11)-(16) are noteworthy:

(1) In all three models (1)-(3), the peak intensities and energies of the fundamental solitons, given by expression (11), remain the same. This means that both the second-order dispersion and nonlinearity and the higher-order dispersion and nonlinear effects completely compensate each other in the NLSE soliton hierarchies (concatenation models). Just as higher-order equations of the NLSE hierarchy are constructed by adding integrable equations of the next order, the argument (15) and phase (16) of the solutions are constructed by adding successive terms resulting from the Lax pair (6).

(2) In nonlinear fiber-optics applications, the coordinates t and x, are obviously, the fiber length Z and time T in the moving reference system. It is important to emphasize that equations (2) and (3) do not follow from Maxwell's equations in the slowly varying amplitudes method [4–11]. Therefore, strictly speaking, the considered concatenation models (2) and (3) can only predict possible effects and only in the case when the underlying high-order optical models are close to exactly integrable models.

3. Phase dependence of the soliton interaction

It is well known that the details and nature of the NLSE soliton interaction, attractive or repulsive, depend on the phase difference of the two injected solitons [51]. The NLSE solitons with different velocities will collide, but since their collisions are elastic, the resulting amplitudes and velocities remain the same before and after their elastic collisions. We confirm that this property is conserved for solitons given by equations (2) and (3). In Figure 1 we present a comparison of the interaction of solitons governed by NLSE (1) and LPDE (3) when the solitons are initially in phase or out of phase. Note that we have chosen the parameter set such that the soliton collision point is the same in Figures 1a-d. Two "in-phase" solitons attract each other, as shown in Figure 1a, c, forming a high maximum at the collision point. If the relative phase among the solitons is equal to π, then two solitons of opposite phases 4 repel each other as is shown in Figure 1b, d. One can see that the velocities of moving NLSE and LPDE solitons are different, as it follows from time-dependent parts of their arguments $\xi(x, t)$ (15).

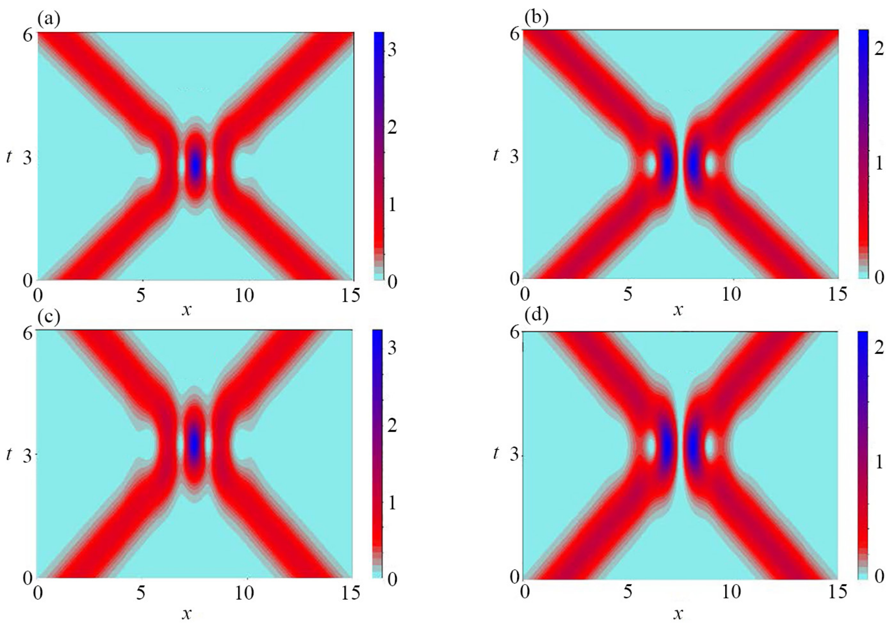


Figure 1. Comparative dynamics of two-soliton solutions of equations (1) and (3) with $D_2 = R_2 = 1.0,\ D_4 = R_4 = 1/16$. Initial soliton amplitudes and velocities are: $\eta_1 = \eta_2 = 0.5,\ \kappa_1 = -1.0,\ \kappa_2 = 1.0$. (a, c) In-phase solitons of Eqs. (1), (3), respectively; (b, d) Out-of-phase solitons of Eq. (1), (3), respectively. The color scales show the level of intensity of the solutions

4. Dynamics of the soliton bound states

The bound states of N solitons (known also as the higher-order solitons or the breather solutions on a zero background) can be constructed using the following conditions

$$\xi_1(x,\, t) = P_1\xi_2(x,\, t) = \ldots = P_N\xi_N(x,\, t), \tag{17}$$

where P_i are arbitrary positive constants. It is well known [4–13] that the bound states of fundamental NLSE solitons are formed with purely imaginary spectral parameter Λ. Let us consider how the bound states of soliton with a purely imaginary spectral parameter Λ are constructed in equations (1)-(3).

The two-soliton bound states of the NLSE (1) with purely imaginary spectral parameter $\Lambda = \kappa + i\eta$ correspond to zero initial soliton velocities,

$$\kappa_1 = \kappa_2 = 0, \tag{18}$$

and the amplitudes satisfying the condition: $\eta_1 = P_1\eta_2,\ P_1 \neq 1$, which implies that the amplitudes can not have the same values. The formation and properties of the bound states of NLSE solitons with equal amplitudes have been studied applying the soliton perturbation theory [51]. The same conditions are valid for the two-soliton bound states satisfying the FONE (2) and the LPDE (3). The two-soliton solution (12)-(16) of Eqs. (1)-(3) in the case of purely imaginary spectral parameter has the form

$$q_2(x,\, t) = 4F^{-1/2}\frac{\eta_2\left(P_1^2-1\right)\left[P_1\cosh\xi_2\exp\left(-i\chi_1\right)-\cosh\xi_1\exp\left(-i\chi_2\right)\right]}{\left(P_1-1\right)^2\cosh\left(\xi_1+\xi_2\right)+\left(P_1+1\right)^2\cosh\left(\xi_1-\xi_2\right)-4P_1\cos\left(\chi_2-\chi_1\right)}, \tag{19}$$

where

$$\xi_i(x,\, t) = 2\left(x-x_{0i}\right)\eta_i, \tag{20}$$

$$\chi_i(x,\, t) = -\left(2D_2\eta_i^2+16D_4\eta_i^4\right)t+\chi_{0i}, \tag{21}$$

for each soliton with $i = 1$ and $i = 2$.

The expression (19) describes the bound-state solutions with zero velocities of two constituent solitons for all concatenation models of the NLSE hierarchy and, in partucular, the models of the NLSE, FONE, and LPDE (1)-(3). All these solutions have the same amplitudes, but different phases dependent on the corresponding coeffitients of dispersion D_2 and D_4.

Let us calculate the period

$$T = \frac{2\pi}{d\left(\chi_2-\chi_1\right)/dt}, \tag{22}$$

of time oscillations of the solution (19)-(21) for the models (1)-(3).

The oscillation period of the LPDE bound-state solution with zero velocities of two constituent in-phase solitons is written as

$$T_{LPDE}\left(\kappa_{1,\,2}=0\right)=\frac{\pi}{D_2\eta_2^2\left(P_1^2-1\right)+8D_4\eta_2^4\left(P_1^4-1\right)}=\frac{\pi}{D_2\left(\eta_1^2-\eta_2^2\right)+8D_4\left(\eta_1^4-\eta_2^4\right)}. \tag{23}$$

This expression can be immediately transformed into the period of the NLSE two-soliton bound-state solution

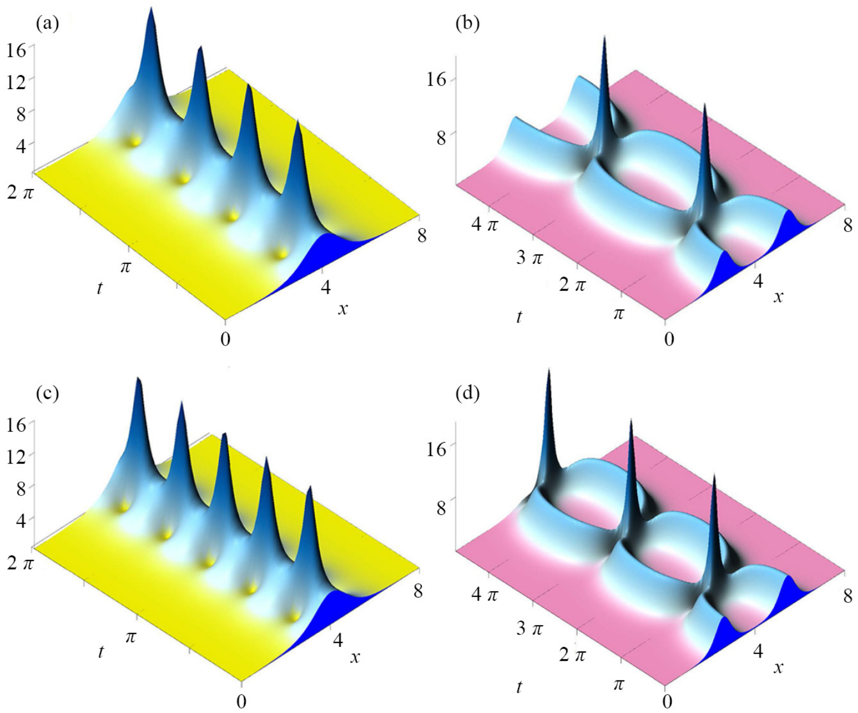
$$T_{NLSE}\left(\kappa_{1,\,2}=0\right)=\frac{\pi}{D_2\eta_2^2\left(P_1^2-1\right)}=\frac{\pi}{D_2\left(\eta_1^2-\eta_2^2\right)}, \tag{24}$$

in the case when the coefficient D_4 becomes zero.

Similarly, Eq. (23) is transformed into the breather period for the FONE

$$T_{FONE}\left(\kappa_{1,\,2}=0\right)=\frac{\pi}{8D_4\eta_2^4\left(P_1^4-1\right)}=\frac{\pi}{8D_4\left(\eta_1^4-\eta_2^4\right)}, \tag{25}$$

in the case when the coefficient D_2 becomes zero.



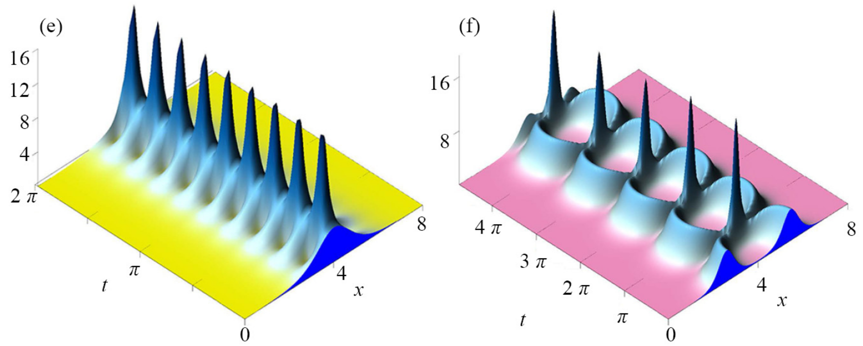


Figure 2. Comparison of the $|q(x, t)|^2$ intensities of the two-soliton bound state solutions of (a, b) Eq. (1) with $D_2 = 1.0$ and $D_4 = 0$; (c, d) Eq. (2) with $D_2 = 0.0$ and $D_4 = 1/16$; and (e, f) Eq. (3) with $D_2 = 1.0$ and $D_4 = 1/16$. Initial soliton amplitudes are: (a, c, e) $\eta_1 = 1.5$ and $\eta_2 = 0.5$; (b, d, f) $\eta_1 = 1.2$ and $\eta_2 = 1.0$

Figure 2 shows examples of the two-soliton bound states of (a, b) the NLSE (1), (c, d) the FONE (2), and e, f the LPDE (3). The solutions shown in Figure 2a, c, e are formed with the initial soliton amplitudes $\eta_1 = 1.5$ and $\eta_2 = 0.5$ $(P_1 = 3)$, whereas Figure 2b, d, f shows the two-soliton bound states formed with the initial soliton amplitudes $\eta_1 = 1.2$ and $\eta_2 = 1.0$ $(P_1 = 1.2)$. One can see that the NLSE soliton bound state shown in Figure 2a corresponds to the classic Satsuma-Yajima solution [52] with $D_2 = 1.0$, $\eta_1 = 1.5$, and $\eta_2 = 0.5$, which oscillates with a period $T_{S-Y} = \frac{\pi}{2}$. Periods of the oscillations of the solutions for the LPDE and the FONE decrease accordingly to Eqs. (23), (25). Note that the initial form of the bound-state solutions for all models if $\eta_1 = 1.5$, $\eta_2 = 0.5$, and $P_1 = 3$ is the same, $q_2(x, t = 0) = 2\,\mathrm{sech}(x)$ (in Eq. (19), $\chi_{01} = 0$, $\chi_{02} = \pi$), as is shown in Figure 2a, c, e, and this form is periodically reproduced.

5. Conclusion

First of all, we have confirmed that the basic properties of solitons-to interact elastically so that the nature of their collision depends on the difference in their initial phases-are also preserved for the FONE and LPDE solitons; this property should take place in all concatenation models. Physically, this fact is related to the influence of the higher-order dispersion and nonlinear effects. One of our main goals was the comparison of the periodical breathing in space and time of the two-soliton bound states formed on the zero background for the LPDE model with those of the fourth-order nonlinear equation model and with the classical Satsuma-Yajima breather behavior known for the NLSE models. We have investigated the conditions for the formation of these two-soliton breathers in all the models considered and present analytical formulas for their oscillation periods.

It can be expected that within the framework of the next, higher orders of concatenation models, more and more soliton breathers with vanishing boundary conditions may appear. In conclusion, we would like to emphasize that the development of this direction is happening very quickly, for example, very interesting and important results of the study of various concatenation models have been published recently in works [53–70].

We note that the practical significance of the obtained results may have two aspects. First, our results show possible ways of further development of the theory of nonlinear waves and, in particular, we describe the strategy for searching new and nontrivial bound states of solitons in concatenation models of higher orders. Second, as is well known [1–13], the higher-order dispersion and nonlinear effectsd escribed by the concatenation models can play an important role in experiments with femtosecond laser pulses in special optical fibers with dispersion parameters varying in the spectral domain [1, 6, 7, 13].

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Conflict of interest

The authors declare no competing financial interest.

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